COXETER GROUPS ARE BIAUTOMATIC

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Abstract. We prove that Coxeter groups are biautomatic. From our construction of the biautomatic structure it follows that uniform lattices in isometry groups of buildings are biautomatic.

1. Introduction

Coxeter groups were introduced in 1934 as abstractions of reflection groups [Cox34]. They play a fundamental role in, among others, the theory of Lie groups and algebras, the representation theory, the geometry of Riemannian symmetric spaces, the topology of aspherical manifolds. Consequently, Coxeter groups are important in other areas of science, e.g. physics, chemistry, and biology. They are foundational objects for buildings — highly symmetric spaces having deep connections with algebraic groups. On the other hand, multiple existing ways of constructing them, make Coxeter groups a source of numerous important, often very exotic, examples of groups. Being studied thoroughly over decades, many important algebraic, geometric, and algorithmic properties of Coxeter groups have been established. Among few most important basic open problems concerning Coxeter groups, there has been the question of biautomaticity.

The notion of biautomaticity was introduced in the classical book by Epstein–Cannon–Holt–Levy–Paterson–Thurston [ECH+92] as a very powerful means of understanding a group. Having biautomaticity established for a finitely generated group, very roughly speaking, we know how to move, using the generators, between any two given elements of the group. Moreover, the resulting paths are determined by a finite state automaton, and are stable in the sense that changing slightly the endpoints does not perturb the paths too much. Such a property should be thought of as a strong form of controlling the structure of the group.

Main Theorem. *Every Coxeter group is biautomatic.*

Many partial results in this direction have been obtained in the past. Davis–Shapiro [DS91] showed a conjecturally weaker feature of all Coxeter groups — the automaticity, under the assumption of the Parallel Wall Theorem, and showed that their language does not provide a biautomatic structure. Brink–Howlett [BH93] proved the Parallel Wall Theorem and hence established the automaticity of all Coxeter groups using the same language as [DS91]. Biautomaticity has been established for a few subclasses of Coxeter groups in: [ECH+92] (Euclidean and Gromov hyperbolic), [NR98, NR03] (right-angled), [Bah06, CM05] (no Euclidean reflection triangles), [Cap09] (relatively hyperbolic), [MOP22] (2-dimensional).

Furthermore, there is an intensive research effort in deeper understanding languages in Coxeter groups, for example the Davis–Shapiro–Brink–Howlett language,

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see e.g. [Cas94, DH16, Yau21, PY22] and references therein. This is primarily inspired by computations in representation theory, and leads to theoretical results concerning algorithmic aspects of the languages, as well as to explicit computations and software implementations of corresponding algorithms.

For our proof of the Main Theorem we introduce a new geodesic language: the 'voracious' language $\mathcal{V}$. Besides providing a biautomatic structure, it has other interesting features compared to the previously considered languages, e.g. to the aforementioned Davis–Shapiro–Brink–Howlett language. In particular, $\mathcal{V}$ is preserved by the automorphisms of a Coxeter group preserving its given generating set. An immediate consequence of this property, together with a result by Świątkowski [Świ06, Thm 6.7] on geodesic languages for Coxeter groups, is the following.

Main Corollary. *Uniform lattices in isometry groups of buildings are biautomatic.*

A uniform lattice here means a group acting properly and cocompactly on the Davis realisation of a building associated to a Coxeter group. Previously, the biautomaticity of such lattices has been shown in few particular cases in: [ECH$^+$92, CS95] (Gromov hyperbolic and some Euclidean), [NR98, Dav98] (right-angled), [GS90, GS91, Nos00, Świ06] (some Euclidean cases), [MOP22] (2-dimensional).

Organisation. In Section 2 we recall the notions of a Coxeter group and a biautomatic structure, and we define the voracious projection and language $\mathcal{V}$ used to prove the Main Theorem. In Section 3, we show that the voracious projection is well defined. In Section 4, we prove that the distance between any element of W and its voracious projection is bounded above by a constant depending only on W. We verify parts (ii) and (iii) of the definition of biautomaticity in Section 5. In Section 6, we prove the regularity of $\mathcal{V}$.

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2. Preliminaries

We follow the notation adopted in [MOP22]. A *Coxeter group* W of *rank* k is a group generated by a finite set S of size k subject only to relations $s^2 = 1$ for $s \in S$ and $(st)^{m_{st}} = 1$ for $s \neq t \in S$, where $m_{st} = m_{ts} \in \{2, 3, \ldots, \infty\}$. Here the convention is that $m_{st} = \infty$ means that we do not impose a relation between s and t.

Consider an arbitrary group G with a finite symmetric generating set S. For $g \in G$, let $\ell(g)$ denote the *word length* of g, that is, the minimal number n such that $g = s_1 \cdots s_n$ with $s_i \in S$ for $i = 1, \ldots, n$. Let S^* denote the set of all words over S. If $v \in S^*$ is a word of length n, then by $v(i)$ we denote the prefix of v of length i for $i = 1, \ldots, n-1$, and the word v itself for $i \geq n$. For $1 \leq i \leq j \leq n$, by $v(i, j)$ we denote the subword of $v(j)$ obtained by removing $v(i-1)$. For a word $v \in S^*$, by $\ell(v)$ we denote the word length of the group element that v represents.

We say that G is *biautomatic* if there exists a regular language $\mathcal{L} \subseteq S^*$ (see Section 6 for the definition of regularity) and constants C, C' satisfying the following conditions.

(i) For each $g \in G$, there is a word in $\mathcal{L}$ representing g.

(ii) For each $s \in S$ and $g, g' \in G$ with $g' = gs$, and each $v, v' \in \mathcal{L}$ representing g, g', for all $i \geq 1$ we have $\ell\big(v(i)^{-1}v'(i)\big) \leq C$.
(iii) For each $s \in S$ and $g, g' \in G$ with $g' = sg$, and each $v, v' \in \mathcal{L}$ representing g, g', for all $i \geq 1$ we have $\ell\big(v(i)^{-1}s^{-1}v'(i)\big) \leq C'$.

This definition agrees with the characterisation of biautomaticity in [ECH+92, Lem 2.5.5], which is equivalent to the original definition of biautomaticity if in condition (i) the set of words in $\mathcal{L}$ representing each $g \in G$ is finite [Amr21, Thm 6]. Conditions (ii) and (iii) are called the 'fellow traveller property'.

To define the voracious language, we need the following. By X^1 we denote the *Cayley graph* of W, that is, the graph with vertex set $X^0 = W$ and with edges (of length 1) joining each $g \in W$ with gs, for $s \in S$. We consider the action of W on $X^0 = W$ by left multiplication. This induces an action of W on X^1. For $r \in W$ a conjugate of an element of S, the *wall* $\mathcal{W}_r$ of r is the fixed point set of r in X^1. We call r the *reflection* in $\mathcal{W}_r$ (for fixed $\mathcal{W}_r$ such r is unique). Each wall $\mathcal{W}$ separates X^1 into two components, called *half-spaces*, and a geodesic edge-path in X^1 intersects $\mathcal{W}$ at most once [Ron09, Lem 2.5]. Consequently, the distance in X^1 between $g, h \in W$ is the number of walls separating g and h.

For $g \in W$, let $\mathcal{W}(g)$ be the set of walls $\mathcal{W}$ in X^1 that separate g from the identity element $\mathrm{id} \in W$ and such that there is no wall $\mathcal{W}'$ separating g from $\mathcal{W}$.

We consider the partial order $\preceq$ on W, where $p \preceq g$ if p lies on a geodesic in X^1 from id to g. Equivalently, there is no wall separating p from both id and g.

For $g \in W$, let $P(g) \subset W$ be the set of elements $p \in W$ satisfying $p \preceq g$ and such that there is no wall in $\mathcal{W}(g)$ separating p from id. Note that $P(g)$ is nonempty, since $\mathrm{id} \in P(g)$. In Section 3 we will prove the following.

Theorem 2.1. *For every Coxeter group W, and each $g \in W$, the set $P(g)$ contains a largest element with respect to $\preceq$.*

This largest element will be called *the voracious projection* $p(g)$ of g. Note that $p(g) \neq g$ for $g \neq \mathrm{id}$.

We define the *voracious language* $\mathcal{V} \subset S^*$ for W inductively in the following way. Let $v \in S^*$ be a word of length n. If v represents the identity element of W, then $v \in \mathcal{V}$ if and only if v is the empty word. Otherwise, let $g \in W$ be the group element represented by v, let p be the voracious projection of g, and let $w = p^{-1}g \in W$ and $k = \ell(w)$. We declare $v \in \mathcal{V}$ if and only if $v(n-k) \in \mathcal{V}$ and $v(n-k+1, n)$ represents w. In particular, $v(n-k)$ represents p. It follows inductively that $n = \ell(g)$. Such a language is called *geodesic*. Note that the voracious language satisfies part (i) of the definition of biautomaticity, and the set of words in $\mathcal{V}$ representing each $g \in G$ is finite.

The paths in W formed by the words in the voracious language are inspired by the normal cube paths for CAT(0) cube complexes [NR98, §3] used to prove the biautomaticity for right-angled (or, more generally, cocompactly cubulated) Coxeter groups [NR03]. Namely, the voracious projection $p(g)$ of g is 'so' voracious that the geodesics from g to $p(g)$ intersect all the walls in $\mathcal{W}(g)$ (even if it means intersecting simultaneously other walls).

We will prove the Main Theorem with $\mathcal{L}$ the voracious language $\mathcal{V}$. It is clear from the definition that the voracious language is preserved by the automorphisms of W stabilising S, allowing us to apply [Świ06, Thm 6.7] on geodesic languages to obtain the Main Corollary concerning buildings.

3. Voracious projection is well defined

Definition 3.1. Let $r, q \in W$ be reflections. Distinct walls $\mathcal{W}_r, \mathcal{W}_q$ *intersect*, if $\mathcal{W}_r$ is not contained in a half-space for $\mathcal{W}_q$ (this relation is symmetric). Equivalently, $\langle r, q\rangle$ is a finite group. We say that such r, q are *sharp-angled*, if r and q do not commute and $\{r, q\}$ is conjugate into S. In particular, there is a component of $X^1 \setminus (\mathcal{W}_r \cup \mathcal{W}_q)$ whose intersection F with X^0 is a fundamental domain for the action of $\langle r, q\rangle$ on X^0. We call such F a *geometric fundamental domain for* $\langle r, q\rangle$.

Lemma 3.2. *Suppose that reflections $r, q \in W$ are sharp-angled, and that $g \in W$ lies in a geometric fundamental domain for $\langle r, q\rangle$. Assume that there is a wall $\mathcal{U}$ separating g from $\mathcal{W}_r$ or from $\mathcal{W}_q$. Let $\mathcal{W}'$ be a wall distinct from $\mathcal{W}_r, \mathcal{W}_q$ that is the translate of $\mathcal{W}_r$ or $\mathcal{W}_q$ under an element of $\langle r, q\rangle$. Then there is a wall $\mathcal{U}'$ separating g from $\mathcal{W}'$.*

Proof. Consider the group $W_0 < W$ generated by the 3 reflections: r, q, and the reflection in $\mathcal{U}$. By [Dye90], with the rank bound established in his Corollary 3.11 (see also [Deo89] or [Tit88, Prop 3]), we have that W_0 can be identified with a Coxeter group of rank 3 such that

- the reflections of W_0 are reflections of W, and
- the connected components of the complement in X^1 of the walls of W_0 correspond (equivariantly) to the elements of W_0, and
- pairs of such components with intersecting closure in X^1 correspond to pairs of elements of W_0 differing by a generator of W_0.

Let g_0 be the element of W_0 corresponding to the above component containing g. Then r, q are still sharp-angled in W_0, with g_0 in a geometric fundamental domain for $\langle r, q\rangle$. Thus to prove the lemma, it suffices to prove it for W of rank 3.

We can assume $S = \{r, q, s\}$, where id lies in the same geometric fundamental domain for $\langle r, q\rangle$ as g. Since $\mathcal{U}$ is disjoint from $\mathcal{W}_r$ or $\mathcal{W}_q$, the group W is infinite, so we can assume without loss of generality $m_{sr} \geq 3$. If $m_{sq} \geq 3$, or $m_{sq} = 2$ and

- $m_{sr} = \infty$, or
- $m_{sr} \geq 4$ and $\mathcal{W}' \neq q\mathcal{W}_r$, or
- $m_{sr} = 3$ and $\mathcal{W}' \notin \{q\mathcal{W}_r, r\mathcal{W}_q, qr\mathcal{W}_q\}$,

then $\mathcal{W}_s$ is disjoint from $\mathcal{W}'$. Since g is separated from $\mathcal{W}_r$ or $\mathcal{W}_q$ by $\mathcal{U}$, we have $g \neq \mathrm{id}$. Thus $\mathcal{U}' = \mathcal{W}_s$ separates g from $\mathcal{W}'$, as desired. See Figure 1(a).

If $m_{sq} = 2, m_{sr} < \infty$, and $\mathcal{W}' = q\mathcal{W}_r$, then let $\mathcal{U}' = s\mathcal{W}_r$. Note that $\mathcal{U}'$ is disjoint from $\mathcal{W}'$ since they are related by the point symmetry sq. Furthermore, since g is separated from $\mathcal{W}_r$ or $\mathcal{W}_q$ by $\mathcal{U}$, and $m_{sr} < \infty$, we have $g \neq \mathrm{id}, s$. Thus $\mathcal{U}'$ separates g from $\mathcal{W}'$, see Figure 1(b).

If $m_{sq} = 2, m_{sr} = 3$, and $\mathcal{W}' = qr\mathcal{W}_q$, then let $\mathcal{U}' = sr\mathcal{W}_q$. Again $\mathcal{U}'$ is disjoint from $\mathcal{W}'$ since they are related by the point symmetry sq. Furthermore, since g is separated from $\mathcal{W}_r$ or $\mathcal{W}_q$ by $\mathcal{U}$, we have $g \neq \mathrm{id}, s, sr$. Thus $\mathcal{U}'$ separates g from $\mathcal{W}'$, see Figure 1(c).

It remains to consider the case where $m_{sq} = 2, m_{sr} = 3$, and $\mathcal{W}' = r\mathcal{W}_q$. Suppose first the that wall $srq\mathcal{W}_r$ is disjoint from $\mathcal{W}'$, see Figure 2(a). Then we can set $\mathcal{U}' = srq\mathcal{W}_r$, since $g \neq \mathrm{id}, s, sr, srq$. Second, suppose that $srq\mathcal{W}_r$ intersects $\mathcal{W}'$. Then let $\mathcal{U}' = srqr\mathcal{W}_q$, see Figure 2(b). Note that $\mathcal{U}'$ is disjoint from $\mathcal{W}'$ since they are related by the point symmetry that is the composition of the reflections in $sr\mathcal{W}_q$ and in $\mathcal{W}_r$. If $\mathcal{U}'$ does not separate g from $\mathcal{W}'$, then $g \in \{\mathrm{id}, s, sr, srq, srqr, srqrs\}$.

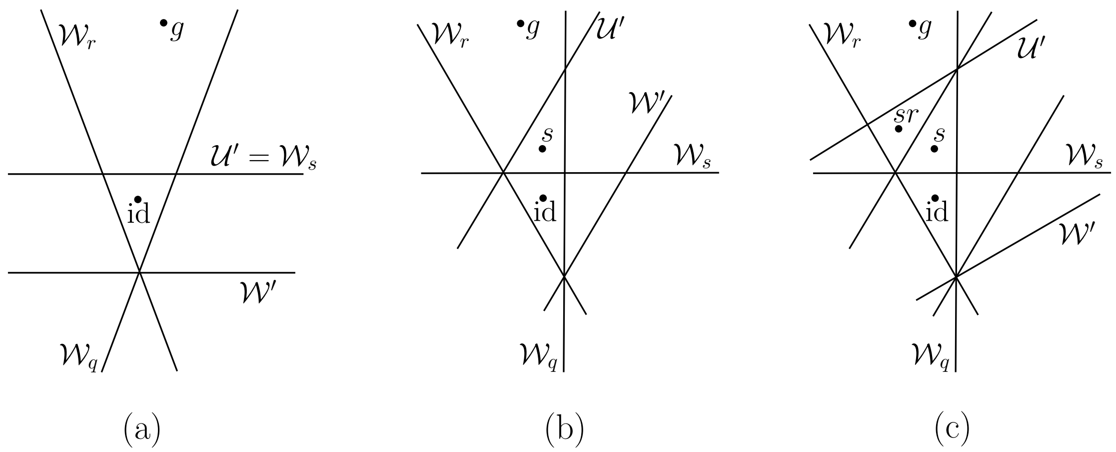

FIGURE 1. Proof of Lemma 3.2.

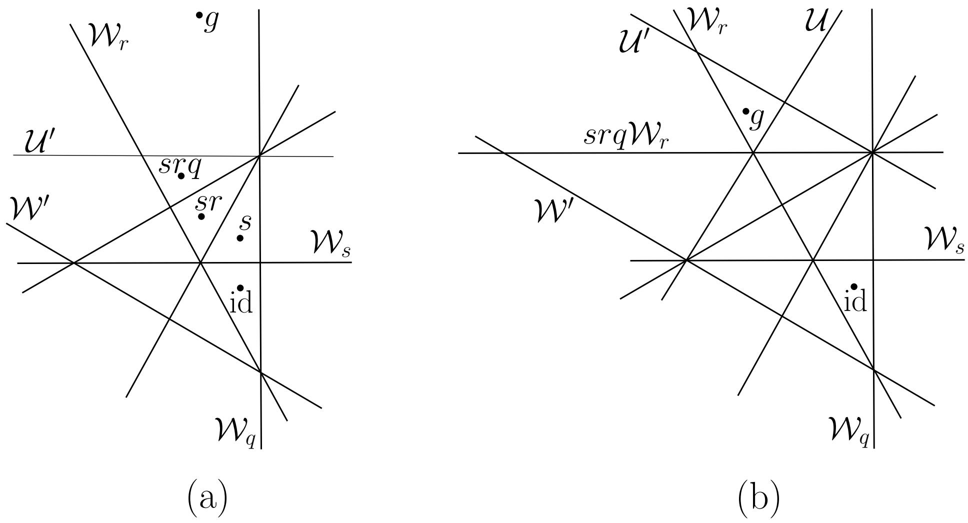

FIGURE 2. Proof of Lemma 3.2, the case $m_{sq} = 2, m_{sr} = 3$, and $\mathcal{W}' = r\mathcal{W}_q$.

Since g is separated from $\mathcal{W}_r$ or $\mathcal{W}_q$ by $\mathcal{U}$, we have that $g = srqrs$ is separated from $\mathcal{W}_q$ by $\mathcal{U} = srqr\mathcal{W}_s$. But the reflection r maps $\mathcal{W}_q$ and $\mathcal{U}$ to $\mathcal{W}'$ and $srq\mathcal{W}_r$, contradicting the assumption that the latter walls intersect. □

Proof of Theorem 2.1. It suffices to show that for each $p_0, p_n \in P(g)$ there is $p \in P(g)$ satisfying $p_0 \preceq p \succeq p_n$. Let $(p_0, p_1, \ldots, p_n)$ be the vertices of a geodesic edge-path π in X^1 from p_0 to p_n. Note that π does not intersect the walls in $\mathcal{W}(g)$, since p_0, p_n lie both in their half-spaces containing id. Furthermore, for any wall $\mathcal{W}$ containing id, g in the same half-space, we have that p_0, p_n lie in that half-space, and so do all p_i. Consequently, $p_i \preceq g$, and so $p_i \in P(g)$.

We will now modify π and replace it by another embedded edge-path (possibly not geodesic) from p_0 to p_n with vertices in $P(g)$, so that there is no p_i with $p_{i-1} \succeq p_i \preceq p_{i+1}$. Then we will be able to choose p to be the largest p_i with respect to $\preceq$.

If $p_{i-1} \succeq p_i \preceq p_{i+1}$, then let $\mathcal{W}_r, \mathcal{W}_q$ be the walls separating p_i from p_{i-1}, p_{i+1}, respectively. Since $p_{i-1} \preceq g, p_{i+1} \preceq g$, the walls $\mathcal{W}_r, \mathcal{W}_q$ intersect. Moreover, if r

and q do not commute, then r, q are sharp-angled, with g in a geometric fundamental domain for $\langle r, q\rangle$. We claim that all the elements of $R = \langle r, q\rangle(p_i)$ lie in $P(g)$.

By [Ron09, Thm 2.9], we have that all the elements of R lie on geodesics from p_i to g, and hence they are $\preceq g$. Since p_{i-1}, p_{i+1} are both in $P(g)$, we have that $\mathcal{W}_r, \mathcal{W}_q \notin \mathcal{W}(g)$. It remains to justify that each remaining wall $\mathcal{W}'$ that is the translate of $\mathcal{W}_r$ or $\mathcal{W}_q$ under an element of $\langle r, q\rangle$ does not belong to $\mathcal{W}(g)$. We can thus assume that r and q do not commute, since otherwise there is no such remaining $\mathcal{W}'$. Since $\mathcal{W}_r \notin \mathcal{W}(g)$, there is a wall $\mathcal{U}$ separating g from $\mathcal{W}_r$. By Lemma 3.2, there is a wall $\mathcal{U}'$ separating g from $\mathcal{W}'$, justifying the claim.

We now replace the subpath (p_{i-1}, p_i, p_{i+1}) of π by the second edge-path with vertices in R from p_{i-1} to p_{i+1}. This decreases the complexity of π defined as the tuple $(n_1, n_2, \ldots, n_{\ell(g)})$, where n_j is the number of p_i in π with $\ell(p_i) = j$, with lexicographic order. After possibly removing a subpath, we can assume that the new edge-path is embedded. After finitely many such modifications, we obtain the desired path. □

4. Bounding the voracious projection

Proposition 4.1. *Let W be a Coxeter group. There exists a constant $C = C(W)$ such that for each $g \in W$, we have $\ell\big(p(g)^{-1}g\big) \leq C$, where $p(g)$ is the voracious projection of g.*

In the proof we need the following Parallel Wall Theorem.

Theorem 4.2 ([BH93, Thm 2.8])**.** *Let W be a Coxeter group. There exists a constant $Q = Q(W)$ such that for each $g \in W$ and a wall $\mathcal{W}$ at distance $> Q$ from g in X^1, there is a wall $\mathcal{W}'$ separating g from $\mathcal{W}$.*

In particular, for $g \in W$, each of the walls in $\mathcal{W}(g)$ is at distance $\leq Q$ from g.

Lemma 4.3. *Let W be a Coxeter group. There exists a constant $C_0 = C_0(W)$ such that for each $g \in W$ and each $\mathcal{W} \in \mathcal{W}(g)$, there is $h \in W$ satisfying $h \preceq g$, at distance $\leq C_0$ from g in X^1, and separated from g by $\mathcal{W}$.*

Proof. For $g \in W$, and $\mathcal{W} \in \mathcal{W}(g)$, among elements $h \in W$ satisfying $h \preceq g$, and separated from g by $\mathcal{W}$, consider h with minimal distance C_0 from g in X^1. Our goal is to bound C_0 uniformly in g and $\mathcal{W}$. We can assume $h = \mathrm{id}$. Furthermore, by the minimality assumption, we have $\mathcal{W} = \mathcal{W}_s$ for $s \in S$.

Note that for $t \in S \setminus \{s\}$, the wall $\mathcal{W}_t$ does not separate id from g, since otherwise we could replace id by t contradicting the minimality assumption.

We now use the *Davis complex* X of W, which is obtained from X^1 by adding Euclidean polyhedra of edge length one spanned on all the cosets of finite $\langle T\rangle$ for $T \subseteq S$ (see [Dav08, Prop 7.3.4]). By [Mou88] (see also [Bow95]), we have that X is CAT(0). The fixed point sets of the reflections of W in X are still called *walls*, and they still separate X.

Let α be the minimal angle that can be formed between an intersection M of a wall with one such polyhedron σ, and a geodesic in σ joining a vertex of σ (all of which lie outside M) to a point of M. Let γ be the CAT(0) geodesic in X between id and g. Since X and X^1 are quasi-isometric, we need to find a uniform bound for the length c of γ. By Theorem 4.2, there is a uniform bound d for the CAT(0) distance from g to $\mathcal{W}_s$ in X. Let σ be the first polyhedron of X with interior intersected by γ, and let $M = \sigma \cap \mathcal{W}_s$. Note that γ intersects M (transversally at a point m) since γ

is disjoint from all $\mathcal{W}_t$, for $t \in S \setminus \{s\}$. The length c_2 of the second component of $\gamma \setminus \{m\}$ is $\geq c - \mathrm{diam}(\sigma)$.

Since X is CAT(0), by [BH99, II.1.7(5)], we have $c_2 \sin\alpha \leq d$, and so $c \leq \mathrm{diam}(\sigma) + d/\sin\alpha$, as desired. □

Proof of Proposition 4.1. By Theorem 4.2, there exists a constant $N = N(W)$ such that each $\mathcal{W}(g)$ has size $\leq N$. For each $g \in W$, applying at most N times Lemma 4.3, there is $h \in W$ satisfying $h \preceq g$, at distance $\leq C = C_0 N$ from g, and separated from g by all $\mathcal{W} \in \mathcal{W}(g)$. Consequently, $h \in P(g)$ and thus $h \preceq p(g) \preceq g$, implying $\ell\big(p(g)^{-1}g\big) \leq C$, as desired. □

5. Fellow traveller property

In this section we verify parts (ii) and (iii) of the definition of biautomaticity.

Lemma 5.1. *Suppose that for $g, g' \in W$, we have $g' \preceq g$. Then $p(g') \preceq p(g)$.*

Proof. It suffices to prove $p(g') \in P(g)$. We have $p(g') \preceq g' \preceq g$. If $\mathcal{W} \in \mathcal{W}(g)$, and $\mathcal{W}$ separates g' from id, then we have $\mathcal{W} \in \mathcal{W}(g')$. Consequently, $\mathcal{W}$ separates $p(g')$ from g', and hence from g. □

Corollary 5.2. *The voracious language satisfies part (ii) of the definition of biautomaticity with C replaced by $2C$ from Proposition 4.1.*

Proof. Let $g \in W$ and let $s \in S$ with $\ell(gs) < \ell(g)$. Let $g' = gs$. Since $p(g) \preceq g' \preceq g$, iterating Lemma 5.1, we obtain
$$\cdots \preceq p^2(g') \preceq p^2(g) \preceq p(g') \preceq p(g) \preceq g' \preceq g,$$
where p^k is defined inductively as $p^0(g) = g$ and $p^k(g) = p(p^{k-1}(g))$ for $k > 0$.

Let $v, v' \in \mathcal{V}$ represent g, gs, respectively. Let $1 \leq i \leq \ell(g)$, and let $h, h' \in W$ be the elements represented by $v(i), v'(i)$, respectively. We then have $\ell(p^k(g')) \leq i \leq \ell(p^k(g))$, or $\ell(p^{k+1}(g)) \leq i \leq \ell(p^k(g'))$, for some $k \geq 0$. Furthermore, by Proposition 4.1, we have that both h, h' are at distance $\leq C$ from $p^{k+1}(g)$ (respectively, $p^{k+1}(g')$) in X^1, and so $\ell(h^{-1}h') \leq 2C$. □

Lemma 5.3. *The voracious language satisfies part (iii) of the definition of biautomaticity.*

Proof of Lemma 5.3. Let $C' = 2C(C + 2Q) + 2Q$, where Q is the constant from Theorem 4.2 and C is the constant from Proposition 4.1.

We prove part (iii) of the definition of biautomaticity, with constant C', inductively on $\ell(g)$, where we assume without loss of generality $\ell(sg) > \ell(g)$. If $g = \mathrm{id}$, then there is nothing to prove. Suppose now $g \neq \mathrm{id}$. Let $v, v' \in \mathcal{V}$ represent g, sg, respectively.

Assume first $\mathcal{W}_s \notin \mathcal{W}(sg)$. Then we have $\mathcal{W}(sg) = s\mathcal{W}(g)$. Consequently, $p(sg) = sp(g)$. In particular, the words $v'\big(\ell(p(sg))\big)$ and $sv\big(\ell(p(g))\big)$ represent the same element $sp(g)$ of W. Then part (iii) of the definition of biautomaticity for g follows inductively from part (iii) for $p(g)$, for $i < \ell(p(sg))$, or from the definition of C, for $i \geq \ell(p(sg))$.

Second, assume $\mathcal{W}_s \in \mathcal{W}(sg)$. Then $p(sg)$ and id lie in the same half-space for $\mathcal{W}_s$. We claim that for any element $h \preceq p(sg)$, there is no wall $\mathcal{W}'$ separating h from $\mathcal{W}_s$. Indeed, otherwise a geodesic from sg to id passing through h would intersect $\mathcal{W}'$ twice. By the claim and Theorem 4.2, we have that h is at distance $\leq 2Q$ from sh, which holds in particular for $h = p(sg)$. By the triangle inequality, sg and $sp(sg)$ are at distance $\leq C + 2Q$ in X^1, and hence so are g and $p(sg)$. By Corollary 5.2, for $i < \ell(p(sg))$, we have that the elements of W represented by $v(i), v'(i)$ are at distance $\leq 2C(C + 2Q)$ in X^1. Setting h above to be the element of W represented by $v'(i)$, we obtain by the triangle inequality $\ell\big(v(i)^{-1}sv'(i)\big) \leq 2C(C + 2Q) + 2Q$, as desired. For $i \geq \ell(p(sg))$, we have obviously $\ell\big(v(i)^{-1}sv'(i)\big) \leq 2C$ as well. □

6. Regularity

A *finite state automaton over S* (or, shortly, *FSA*) is a finite directed graph Γ with:

- vertex set A, edge set $E \subseteq A \times A$,
- an edge labeling $\phi \colon E \to \mathcal{P}(S^*)$ (the power set of S^*), where each $\phi(e)$ is finite,
- a *start state* $a_0 \in A$, and
- a distinguished set of *accept states* $A_\infty \subseteq A$.

A word $v \in S^*$ is *accepted by* Γ if there exists a decomposition $v = v_0 \cdots v_m$ into subwords, and a directed edge-path $e_0 \cdots e_m$ in Γ such that e_0 has initial vertex a_0, e_m has terminal vertex in A_∞, and $v_i \in \phi(e_i)$ for each $i = 0, \ldots, m$. A subset of S^* is a *regular language* if it is the set of accepted words for some FSA over S.

Proposition 6.1. *The voracious language is regular.*

To prove Proposition 6.1, we define an FSA Γ over S that will accept exactly the voracious language.

Definition 6.2. Let $\mathcal{U}$ be the set of walls that are not separated from id by any other wall, which is finite by Theorem 4.2. The vertex set A of our FSA Γ is the power set $\mathcal{P}(\mathcal{U})$. Let $a_0 = \emptyset \subset \mathcal{U}$, and $A_\infty = A$.

To define the edges from $a \in A$, suppose that $w \in W$ satisfies:

(a) w is not separated from id by any wall in a, and
(b) w is separated from each wall in a by another wall, and
(c) $p(w) = \mathrm{id}$.

We then put en edge e in E between a and $w^{-1}\mathcal{W}(w)$ with $\phi(e)$ consisting of all the minimal length words representing w.

Proof of Proposition 6.1. Let Γ be the FSA from Definition 6.2, and let $\mathcal{V}$ be the voracious language. We argue inductively on $j \geq 0$ that, among the words $v \in S^*$ of length $\leq j$,

- Γ accepts exactly the words in $\mathcal{V}$, and
- the accept state of each such word v is $g^{-1}\mathcal{W}(g)$, where v represents $g \in W$.

This is true for $j = 0$ by our choice of a_0. Now let $n > 0$ and suppose that we have verified the inductive hypothesis for all $j < n$. Let v be a word in S^* of length n.

Suppose first that v is a word in $\mathcal{V}$ representing $g \in W$. Let $p = p(g), w = p^{-1}g$. By the definition of $\mathcal{V}$, we have $v\big(\ell(p)\big) \in \mathcal{V}$. Moreover, $v\big(\ell(p) + 1, n\big)$ represents w. By the inductive hypothesis, Γ accepts $v\big(\ell(p)\big)$. Furthermore, $v\big(\ell(p)\big)$ labels some directed edge-path in Γ from a_0 to $p^{-1}\mathcal{W}(p)$. We will now show that Γ has an edge e from $a = p^{-1}\mathcal{W}(p)$ to $g^{-1}\mathcal{W}(g)$, with $\phi(e)$ consisting of all minimal length words representing w. To do that, we verify the conditions for w from Definition 6.2. Condition (a) follows from the fact that $p \preceq g$ and so g is not separated from p by any wall in $\mathcal{W}(p)$. Since $p \in P(g)$, we have that $\mathcal{W}(p)$ is disjoint from $\mathcal{W}(g)$, which implies condition (b) and $g^{-1}\mathcal{W}(g) = w^{-1}\mathcal{W}(w)$. Consequently, $p(g)p(w) \in P(g)$ implying $p(w) = \mathrm{id}$, which is condition (c).

Conversely, let v be accepted by Γ and suppose that $v = v_0 \cdots v_m$ as in the definition of an accepted word. By the inductive hypothesis, the word $v_0 \cdots v_{m-1}$ belongs to $\mathcal{V}$ and represents $p \in W$ such that e_m starts at $a = p^{-1}\mathcal{W}(p)$. By the definition of the edges, v_m is a minimal length word representing an element w satisfying the conditions (a,b,c). By condition (a), we have $p \preceq g$ for $g = pw$. By condition (b), we have $w^{-1}\mathcal{W}(w) = g^{-1}\mathcal{W}(g)$. Thus by condition (c), we have $p = p(g)$. Consequently, $v \in \mathcal{V}$, as desired. □

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