

TITS ALTERNATIVE FOR THE 3-DIMENSIONAL TAME AUTOMORPHISM GROUP

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ABSTRACT. We prove the strong Tits alternative for the tame automorphism group of \mathbb{k}^3 , for \mathbb{k} of characteristic zero.

1. INTRODUCTION

A group G satisfies the *strong Tits alternative* (resp. *weak Tits alternative*) if each subgroup of G (resp. each finitely generated subgroup of G) is virtually solvable or contains a non-abelian free group. For each $n \geq 1$ and each field \mathbb{k} of characteristic zero (resp. of positive characteristic), Tits proved that the linear group $\mathbf{GL}_n(\mathbb{k})$ satisfies the strong (resp. weak) Tits alternative [Tit72]. This result was then extended to many classes of groups. We mention first some groups of polynomial or rational maps, also known as Cremona transformations. The group $\text{Aut}(\mathbb{C}^2)$ of polynomial automorphisms of the complex plane satisfies the strong Tits alternative by [Lam01]. For an arbitrary field \mathbb{k} , one can adapt the argument and obtain the (strong, or weak, according to the characteristic of \mathbb{k}) Tits alternative for $\text{Aut}(\mathbb{k}^2)$ [Lam22]. For birational maps, the weak and then strong Tits alternative for the 2-dimensional Cremona group $\text{Bir}(\mathbb{C}^2)$ was established in [Can11] and [Ure21]. Again, the argument can be extended to any field of characteristic zero [Lam22], while the case of positive characteristic remains unclear. In higher dimension, very little is known. However, the strong Tits alternative for $\text{Tame}(\mathbf{SL}_2(\mathbb{C}))$, a particular subgroup of the Cremona group $\text{Bir}(\mathbb{C}^3)$, was established in [BFL14].

The aim of this article is to establish the Tits alternative for the following subgroup of the 3-dimensional Cremona group. The *tame automorphism group* $\text{Tame}(\mathbb{k}^3)$ of the affine space \mathbb{k}^3 is the subgroup of the polynomial automorphism group generated by the affine automorphisms and by the automorphisms of the form

$$(x_1, x_2, x_3) \mapsto (x_1 + P(x_2, x_3), x_2, x_3), \quad P \in \mathbb{k}[x_2, x_3].$$

Main Theorem. *Let \mathbb{k} be a field of characteristic zero. Then $\text{Tame}(\mathbb{k}^3)$ satisfies the strong Tits alternative.*

A common feature of the proof of the Tits alternative for $\text{Aut}(\mathbb{C}^2)$, $\text{Bir}(\mathbb{C}^2)$ and $\text{Tame}(\mathbf{SL}_2(\mathbb{C}))$ is the use of an action on a CAT(0) metric space. Namely, $\text{Aut}(\mathbb{C}^2)$ acts on the Bass–Serre tree associated to its structure of amalgamated product, $\text{Bir}(\mathbb{C}^2)$ acts on an infinite dimensional hyperbolic space \mathbb{H}^∞ and $\text{Tame}(\mathbf{SL}_2(\mathbb{C}))$ acts on a CAT(0) square complex. None of these actions is proper.

In general, suppose that a group G acts by isometries on a Gromov-hyperbolic geodesic metric space X . By [GdlH90, §3.1], the group G (or its index 2 subgroup)

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fixes a point of X or the Gromov boundary of X , or contains a non-abelian free group. In particular, if G is word-hyperbolic, if we take X to be the Cayley graph of G , the stabilisers of points in the Gromov boundary of X are virtually cyclic and so G satisfies the strong Tits alternative. More generally, if G acts acylindrically on a Gromov-hyperbolic geodesic metric space X , and the subgroups of G with bounded orbits in X satisfy the weak (resp. strong) Tits alternative, then G satisfies the weak (resp. strong) Tits alternative [DGO17]. Taking for X the coned-off Deligne complex, this gives the strong Tits alternative for 2-dimensional Artin groups of hyperbolic type [MP21].

For groups G that are not word-hyperbolic, but possess some hyperbolic-like features, various strategies were employed. Strong Tits alternative was proved for mapping class groups of hyperbolic surfaces [Iva84, McC85] and outer automorphism groups of free groups [BFH00, BFH05]. This was generalised to outer automorphism groups of right-angled Artin groups [CV11, Hor14].

It is conjectured that if X is a CAT(0) metric space, and G acts on X properly and cocompactly, then it satisfies the strong Tits alternative. This conjecture was verified for 2-dimensional complexes X in [OP21b]. The methods can be extended to ‘recurrent’ spaces [OP21a], implying that the Artin groups of large type satisfy the strong Tits alternative. For CAT(0) cube complexes X the conjecture was established in [SW05]. More generally, if G acts on a finite-dimensional CAT(0) cube complex X with the stabilisers of points in $X \cup \partial_\infty X$ satisfying the weak (resp. strong) Tits alternative, then G also satisfies the weak (resp. strong) Tits alternative [CS11]. Applying this, one verifies the strong Tits alternative for Artin groups G of type FC [MP20], by computing the stabilisers of points in $\partial_\infty X$, where X is the (cubical) Deligne complex.

The group $\text{Tame}(\mathbb{k}^3)$ is the amalgamated product of three subgroups along their pairwise intersection [Wri15, Lam19]. This gives a natural action of $\text{Tame}(\mathbb{k}^3)$ on a 2-dimensional simplicial complex, which was proved in [LP19] to be contractible and Gromov-hyperbolic in order to find normal subgroups in $\text{Tame}(\mathbb{k}^3)$. However, this complex cannot be endowed with an equivariant CAT(0) metric, and so we do not know how to classify the subgroups of $\text{Tame}(\mathbb{k}^3)$ with bounded orbits in that complex.

In [LP21], a new 2-dimensional CAT(0) complex \mathbf{X} with an action of $\text{Tame}(\mathbb{k}^3)$ was constructed, leading to the description of the finite subgroups of $\text{Tame}(\mathbb{k}^3)$. However, this action is not proper and \mathbf{X} is not Gromov-hyperbolic. Therefore, in order to prove the Main Theorem, we need to classify and analyse all elements of $\text{Tame}(\mathbb{k}^3)$ that do not fix a point of \mathbf{X} and are not loxodromic of ‘rank 1’. In order to do this, we classify locally and globally nearly flat subcomplexes of \mathbf{X} .

We apply an original mixture of metric and combinatorial techniques, where we introduce ‘relative disc diagrams’, which are, roughly speaking, disc diagrams that are required to be combinatorial only away from the boundary. We often study the limit of relative disc diagrams $\phi_n: D \rightarrow \mathbf{X}$, whose domain D might degenerate in the limit into a ‘frilling’ D' . We believe that these concepts will be useful for studying the actions of other groups on non-proper CAT(0) complexes.

Organisation. We present now a simplified outline of the proof of the Main Theorem; see Section 12 for the full proof.

The group $\text{Tame}(\mathbb{k}^3)$ acts on the non-proper 2-dimensional CAT(0) complex \mathbf{X} , which we discuss in Section 3. An important property of \mathbf{X} is the existence of a proper quotient $\rho_+ : \mathbf{X} \rightarrow \nabla^+$, with ∇^+ isometric to a Weyl chamber in \mathbb{R}^2 .

Let $G < \text{Tame}(\mathbb{k}^3)$ and suppose that G does not stabilise a point in \mathbf{X} , or a ‘principal’ or an ‘antiprincipal’ point ζ in $\partial_\infty \mathbf{X}$. (Stabilisers of such points ζ are conjugate to subgroups denoted by $C, B' < \text{Tame}(\mathbb{k}^3)$, which satisfy the strong Tits alternative.) In particular, by [NOP21, Cor 2.5], there is an element of G that is not elliptic, i.e. has no fixed point in \mathbf{X} . If all the elements of G are elliptic or loxodromic of ‘rank 1’, one can find a non-abelian free group in G appealing to the work of Bestvina and Fujiwara. However, one needs to carry over the classical results of Ballmann and Ruane to the non-proper setup, which we do in Section 2.

It remains to classify the elements of $\text{Tame}(\mathbb{k}^3)$ that are parabolic or loxodromic not of rank 1. We do so eventually in Sections 8 and 9, where we show that all such elements are conjugate into C or B' . To do that, we first classify all cycles in the vertex links of \mathbf{X} of length $< 2\pi + \varepsilon_0$, for a uniform ε_0 (Sections 4 and 5). We then study nearly flat diagrams $\phi : D \rightarrow \mathbf{X}$, and we describe their ‘folding locus’ in D where the composition $\rho_+ \circ \phi : D \rightarrow \nabla^+$ is not a local isometry (Section 7). Note that the classification of g parabolic, which are all conjugate into C , is much easier and can be read without preparation.

In Section 10, we study diagrams for pairs (ξ, η) of limit points of distinct isometries $g, g' \in G$ that are parabolic or loxodromic not of rank 1. Since \mathbf{X} is not proper, we have to study the limits of disc diagrams ϕ_n by postcomposing them with ρ_+ . This requires introducing a procedure of passing to the limit with non-combinatorial ‘relative disc diagrams’ (see Section 6, portions of which could be ignored at a first reading by pretending that all the diagrams are combinatorial). Finally, in Section 11 we prove that since G does not stabilise a principal or an antiprincipal point in $\partial_\infty \mathbf{X}$, there exist $g, g' \in G$ with ξ and η ‘far’ enough so that some $g^n g'^n$ has rank 1, giving rise to the non-abelian free group.

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2. CONSTRUCTING RANK 1 ISOMETRIES

In this section, we consider an arbitrary complete CAT(0) space (X, d_X) . We allow X to be not proper. We generalise the methods of Ballmann and Ruane of constructing rank 1 isometries to this setting.

2.1. Boundary. Two geodesic rays $r, r' : [0, \infty) \rightarrow X$ are *asymptotic* if there is $d > 0$ with $d_X(r(t), r'(t)) \leq d$ for all t . This defines an equivalence relation, and we denote by $[r]$ the equivalence class of a geodesic ray r . The *visual boundary* $\partial_\infty X$ is the set of equivalence classes of geodesic rays. Note that for each geodesic ray r and each $y \in X$ there is a geodesic ray representing $[r]$ starting at y [BH99, II.8.2], which we call the *geodesic ray from y to $[r]$* . We extend the topology on X to the *cone topology* on $X \cup \partial_\infty X$ described in [BH99, II.8.6]. Namely, given a geodesic ray r , and $t, \varepsilon > 0$, a *basic neighbourhood of $[r]$, based at $r(0)$* , is the set of $x \in X \cup \partial_\infty X$ such that the geodesic (possibly a ray) r' from $r(0)$ to x has length $> t$ and satisfies $d_X(r'(t), r(t)) < \varepsilon$. Note that since X is not assumed to be proper, $\partial_\infty X$ might not be compact.

For $\xi, \eta \in \partial_\infty X$, the *angle* $\angle(\xi, \eta)$ is the supremum of the Alexandrov angle $\angle_x(r, r')$ (see [BH99, I.1.12]) over $x \in X$ and r representing ξ and r' representing η starting at x . The angle equips $\partial_\infty X$ with a complete CAT(1) metric [BH99, II.9.13]. However, the induced topology on $\partial_\infty X$ is stronger than the cone topology.

2.2. Isometries. Let g be an isometry of X . The *translation length* of g is $|g| = \inf_{x \in X} d_X(x, g(x))$. If the infimum is attained, we say that g is *elliptic* if $|g| = 0$ and *loxodromic* otherwise. If the infimum is not attained, we say that g is *parabolic*. If g is loxodromic, then by [BH99, II.6.8.(1)] it has (at least one) *axis*, which is a g -invariant geodesic line in X .

Let $\xi \in \partial_\infty X$. We say that ξ is the *positive* (resp. *negative*) *limit point* of g , if for some (hence any) point $x \in X$, we have $g^n(x) \rightarrow \xi$ for $n \rightarrow \infty$ (resp. $n \rightarrow -\infty$). Note that a limit point is fixed by g . If g is loxodromic, it has the positive limit point ξ^+ and the negative limit point ξ^- that are the endpoints of any of the axes for g . In particular $\angle(\xi^+, \xi^-) = \pi$. If g is parabolic with positive translation length, then by [Duc11, lem 23.2] g also has the positive and negative limit points ξ^+, ξ^- with $\angle(\xi^+, \xi^-) = \pi$. A parabolic isometry of translation length zero without a point $\xi \in \partial_\infty X$ that is its positive and negative limit point is called *vile*. (We do not know if vile isometries exist.)

2.3. Rank 1. Let $\xi, \eta \in \partial_\infty X$. In what follows we generalise the condition $d_T(\xi, \eta) > \pi$ used by Ruane in the setting of proper X (where d_T is the *Tits metric*, i.e. the path metric induced by the angle metric). In [Rua01, Prop 2.4] she proved that for proper X this condition is equivalent to ξ, η being the endpoints of a geodesic line that does not bound an isometrically embedded Euclidean half-plane.

Definition 2.1. Let $p \in X$ and let $\varepsilon, R > 0$. A geodesic quadrilateral $xyy'x'$ in X is a (p, ε, R) -*quadrilateral* if $p \in xy$, the concatenation $yy' \cdot y'x' \cdot x'x$ is at distance $> R$ from p , and the sum of the four Alexandrov angles of $xyy'x'$ is $> 2\pi - \varepsilon$.

Lemma 2.2. Let $\square = xyy'x'$ be a (p, ε, R) -quadrilateral in X . Then there is an embedded (p, ε, R) -quadrilateral $\bar{\square} = \bar{x}\bar{y}\bar{y}'\bar{x}'$ in X with $\bar{x}\bar{y} \subset xy$ and all the other sides of $\bar{\square}$ contained in some sides of \square .

Proof. If for some vertex v of \square the consecutive sides containing v intersect along a geodesic $v\bar{v}$, then we replace v by \bar{v} . Note that all the Alexandrov angles remain the same except that the angle 0 at v is replaced by a nonnegative angle at \bar{v} .

If opposite sides of \square intersect at a point z of X , then \square decomposes as a concatenation of two geodesic triangles with Alexandrov angle sum $> 2\pi - \varepsilon$. Let Δ be the triangle containing p . By [BH99, II.1.7(4)], the Alexandrov angle sum in the other triangle is $\leq \pi$, so the sum of the Alexandrov angles of Δ is $> \pi - \varepsilon$.

As in the first paragraph, we can assume that Δ is embedded. We introduce an extra vertex (of Alexandrov angle π) on a side of Δ not containing p to treat Δ as a (p, ε, R) -quadrilateral. \square

Definition 2.3. Let $\xi, \eta \in \partial_\infty X$. We say that ξ, η are (ε, R) -*far* (or, shortly, *far*), if there are neighbourhoods U, V of ξ, η in $X \cup \partial_\infty X$ and a closed metric ball $P \subset X$ such that for each $u \in U, v \in V$, with at least one of u, v in X ,

- the geodesic (possibly a ray) uv in X intersects P , and, furthermore,
- there is $p \in uv \cap P$ that does not lie in any segment of uv that is a side of a (p, ε, R) -quadrilateral.

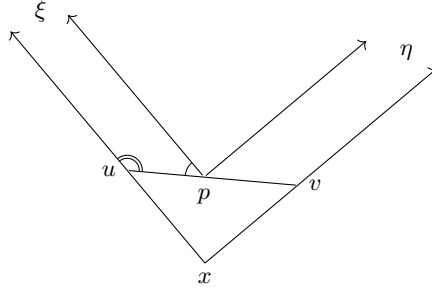


FIGURE 1.

Lemma 2.4. *Let $x \in X$, let $\xi, \eta \in \partial_\infty X$, and let $u, v \in X$ distinct from x lie on the geodesic rays from x to ξ, η , respectively. Then for any point p on the geodesic uv , distinct from u, v , we have $\angle_u(x, v) + \angle_v(x, u) \geq \pi - \angle_p(\xi, \eta)$.*

Proof. We have $\angle_u(x, v) \geq \pi - \angle_u(\xi, v)$, which is $\geq \angle_p(\xi, u)$ by [BH99, II.9.3(1)]. See Figure 1. Analogously, we have $\angle_v(x, u) \geq \angle_p(\eta, v)$ and the lemma follows. \square

Lemma 2.5. *Suppose that ξ, η are far and P is as in Definition 2.3. Then $\angle(\xi, \eta) = \pi$ and for each $\delta > 0$ there is $p \in P$ with $\angle_p(\xi, \eta) > \pi - \delta$.*

Proof. Let $x \in X$ and let $r, r': [0, \infty) \rightarrow X$ be the geodesic rays from x to ξ, η . Then for t sufficiently large, we have $r(t) \in U, r'(t) \in V$. If $\angle(\xi, \eta) < \pi$, then by [BH99, II.9.8(4)] the geodesic $r(t)r'(t)$ is arbitrarily far from x for t sufficiently large. Consequently, $r(t)r'(t)$ is disjoint from P , which is a contradiction.

Thus $\angle(\xi, \eta) = \pi$, and so we can assume $\angle_x(\xi, \eta) > \pi - \delta$. Then $\angle_{r(t)}(x, r'(t)) + \angle_{r'(t)}(r(t), x) < \delta$. Consequently, for $p \in P \cap r(t)r'(t)$, by Lemma 2.4 we have $\angle_p(\xi, \eta) > \pi - \delta$. \square

Following [BF09, Def 5.1], an isometry g of X has *rank 1* if for some (hence any) $x \in X$, there is $M > 0$ such that, for each $n > 0$,

- the geodesic $\beta_n = g^n(x)g^{-n}(x)$ is at Hausdorff distance $\leq M$ from the set $\{g^n(x), g^{n-1}(x), \dots, g^{-n}(x)\}$, and
- β_n is M -contracting, in the sense that the projection to β_n of any closed metric ball B disjoint from β_n has length $\leq M$.

It is not hard to prove (but we will not need it) that if X is proper, then ξ, η are (ε, R) -far (for some ε, R) if and only if ξ, η are the endpoints of a geodesic line that does not bound an isometrically embedded Euclidean half-plane. Thus the following theorem generalises [BF09, Thm 5.4].

Theorem 2.6. *Let g be an isometry of X with far limit points. Then g has rank 1.*

Note that if X has finite telescopic dimension, then such g is loxodromic, since by [CL10, Thm 1.1] (see also [Duc11, thm 6.1]), for parabolic g all of its fixed points in $\partial_\infty X$ are at Tits distance $\leq \pi$.

Before giving the proof, we record the following immediate consequence of Theorem 2.6 and [BF09, Prop 5.9].

Corollary 2.7. *Let g (resp. g') be isometries of X with far limit points ξ, η (resp. ξ', η'). If $\{\xi, \eta\}, \{\xi', \eta'\}$ are disjoint, then some powers of g, g' are a basis of a free subgroup in the isometry group of X .*

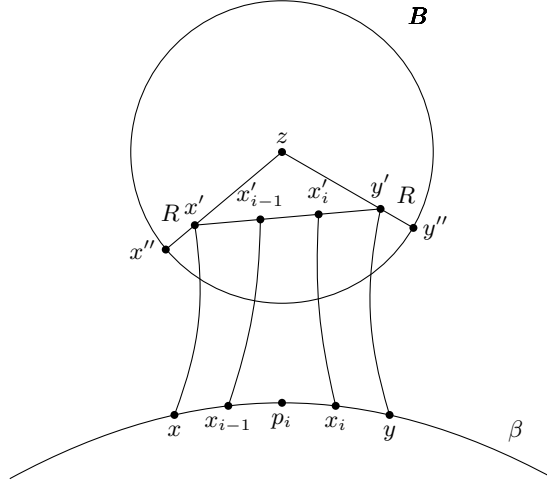


FIGURE 2.

To prove Theorem 2.6, we first establish the following.

Lemma 2.8. *Let $\varepsilon, R > 0$ and let $\beta \subset X$ be a geodesic with a family of points \mathcal{P} dividing it into segments of length $\leq D$. Then β is $2R + \lceil \frac{2\pi}{\varepsilon} \rceil (D + 2R)$ -contracting or there is $p \in \mathcal{P}$ and a (p, ε, R) -quadrilateral with a side on β .*

Proof. Suppose that β is not $2R + \lceil \frac{2\pi}{\varepsilon} \rceil (D + 2R)$ -contracting. Then there is a closed ball $B \subset X$ of radius $R' > R$ and centre z disjoint from β , and points $x'', y'' \in B$ with projections to β at distance $> 2R + \lceil \frac{2\pi}{\varepsilon} \rceil (D + 2R)$. Let x', y' be the point on the geodesic $x''z$ at distance $\min\{R, d_X(x'', z)\}$ from x'' . Define y' analogously. Then the geodesic $x'y'$ is contained in the closed ball of radius $R' - R$ and centre z and hence it is at distance $> R$ from β . Let x, y be the projections of x', y' to β . Note that x and y are at distance $\leq R$ from the projections of x'', y'' . Thus the geodesic xy has length $> \lceil \frac{2\pi}{\varepsilon} \rceil (D + 2R)$. Consequently, xy can be divided into $N = \lceil \frac{2\pi}{\varepsilon} \rceil$ segments $\gamma_1 = xx_1, \gamma_2 = x_1x_2, \dots, \gamma_N = x_{N-1}y$ of length $> D + 2R$.

Each γ_i contains some $p_i \in \mathcal{P}$ at distance $> R$ from the endpoints of γ_i . Choose x'_1, \dots, x'_{N-1} on $x'y'$ in that order so that the projection of each x'_i to β is x_i . See Figure 2. Denoting $x'_0 = x', x'_N = y', x_0 = x, x_N = y$, for $i = 1, \dots, N$ the geodesics $x'_{i-1}x_{i-1}, x'_ix_i$ are at distance $> R$ from p_i . Thus each $x_{i-1}x_ix'_ix'_{i-1}$ is either a (p_i, ε, R) -quadrilateral, or its sum of the Alexandrov angles is $\leq 2\pi - \varepsilon$. Since $\angle_{x_i}(x'_i, x_{i-1}) + \angle_{x_i}(x'_i, x_{i+1}) \geq \pi$ and $\angle_{x'_i}(x_i, x'_{i-1}) + \angle_{x'_i}(x_i, x'_{i+1}) \geq \pi$, the latter possibly would imply that the sum of the Alexandrov angles of $xyy'x'$ is $\leq 2\pi - N\varepsilon \leq 0$, which is a contradiction. \square

Proof of Theorem 2.6. Suppose that the limit points of g are (ε, R) -far. Let P, U, V be provided by Definition 2.3. Let $p \in P$. By possibly passing to a power of g , we can assume $g^n(p) \in U, g^{-n}(p) \in V$ for all $n \geq 1$. Consequently, given $n \geq 1$, for $-(n-1) \leq k \leq n-1$, we have $g^{n-k}(p) \in U, g^{-n-k}(p) \in V$, and so there is a point $p_k \in P$ on the geodesic $g^{n-k}(p)g^{-n-k}(p)$ that does not lie in a segment of $g^{n-k}(p)g^{-n-k}(p)$ that is a side of a (p_k, ε, R) -quadrilateral. Denote $D = \text{diam}(P \cup g(P))$. The points $g^{n-1}(p_{n-1}), g^{n-2}(p_{n-2}), \dots, g^{-(n-1)}(p_{-(n-1)})$ divide the geodesic $\beta_n = g^n(p)g^{-n}(p)$ into segments of length $\leq D$. In particular, β_n is at Hausdorff distance $\leq D$ from $\{g^n(p), g^{n-1}(p), \dots, g^{-n}(p)\}$.

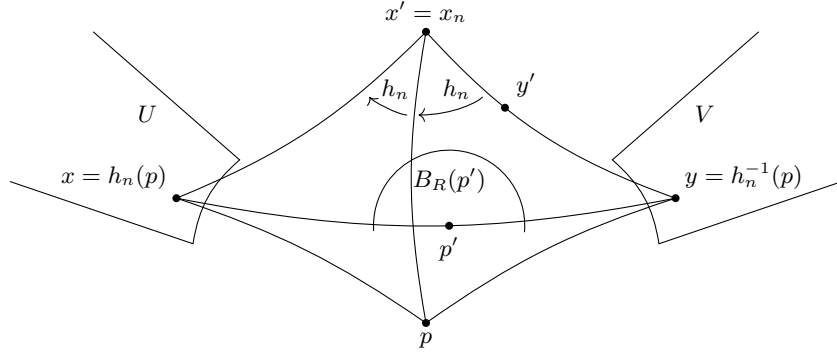


FIGURE 3.

Let $\beta = \beta_n$ and $\mathcal{P} = \{g^k(p_k)\}$. By Lemma 2.8, β_n is $2R + \lceil \frac{2\pi}{\varepsilon} \rceil (D + 2R)$ -contracting or there is $-(n-1) \leq k \leq n-1$ and a $(g^k(p_k), \varepsilon, R)$ -quadrilateral with a side on β_n . However, the latter possibility would imply that there is a (p_k, ε, R) -quadrilateral with a side on $g^{-k}(\beta_n) = g^{n-k}(p)g^{-n-k}(p)$, which would be a contradiction.

Thus each β_n is $2R + \lceil \frac{2\pi}{\varepsilon} \rceil (D + 2R)$ -contracting, as desired. \square

2.4. Construction. The main result of this section is the following construction of rank 1 isometries, which is a generalisation of [Rua01, Main Thm].

Proposition 2.9. *Let f, g be isometries of X that have positive limit points ξ_f^+, ξ_g^+ that are far and negative limit points ξ_f^-, ξ_g^- that are far. Then for any neighbourhoods U_0, V_0 of ξ_f^+, ξ_g^+ , for sufficiently large n , the isometry $f^n g^{-n}$ is vile or has positive translation length and far limit points in U_0, V_0 .*

To prove Proposition 2.9, we first need the following variant of [Bal95, Lem III.3.2].

Lemma 2.10. *Suppose that $\xi, \eta \in \partial_\infty X$ are far. Let h_n be a sequence of isometries of X such that for some (hence every) $p \in X$ we have $h_n(p) \rightarrow \xi, h_n^{-1}(p) \rightarrow \eta$. Then for any neighbourhoods U_0, V_0 of ξ, η , for sufficiently large n , the isometry h_n is vile or has positive translation length and far limit points in U_0, V_0 .*

Proof. Suppose that ξ, η are (ε, R) -far, and let P, U, V be provided by Definition 2.3. We can assume $U \subset U_0, V \subset V_0$. By Lemma 2.5, we can assume that for some $p \in P$ and any $u \in U, v \in V$ we have $\angle_p(u, v) > \pi - \frac{\varepsilon}{2}$. Furthermore, we can assume that U, V are basic neighbourhoods based at p and that for all $n > 0$ we have $h_n(B_R(P)) \subset U, h_n^{-1}(B_R(P)) \subset V$, where B_R denotes the closed R -neighbourhood.

We first justify that for sufficiently large n the isometry h_n has no fixed point $x_n \in X \cup \partial_\infty X$ outside $U \cup V$. Indeed, otherwise let $p' \in P$ be a point on the geodesic $h_n(p)h_n^{-1}(p)$ as in Definition 2.3. Since V is based at p , and $h_n^{-1}(B_R(P)) \subset V$ whereas $h_n^{-1}(x_n) = x_n \notin V$, we have that the geodesic $ph_n^{-1}(x_n)$ (which is possibly a ray) is disjoint from $h_n^{-1}(B_R(p'))$. Consequently, the geodesic $h_n(p)x_n$ is disjoint from $B_R(p')$. Analogously, the geodesic $h_n^{-1}(p)x_n$ is disjoint from $B_R(p')$.

If $x_n \in X$, then let $x = h_n(p), y = h_n^{-1}(p), x' = x_n$, and let y' be any point on the geodesic $x'y$ distinct from x', y . See Figure 3. We now compute the Alexandrov angles of the geodesic quadrilateral $xyy'x'$. The triangles $x'xp$ and $x'py$ are isometric and so $\angle_y(p, x') + \angle_x(x', p) = \angle_p(x, x') + \angle_p(x', y) > \pi - \frac{\varepsilon}{2}$. Consequently, $\angle_y(x, x') + \angle_x(x', y) > \pi - \frac{\varepsilon}{2} - \angle_y(p, x) - \angle_x(y, p) \geq \pi - \frac{\varepsilon}{2} - \frac{\varepsilon}{2}$. Thus the sum of the Alexandrov

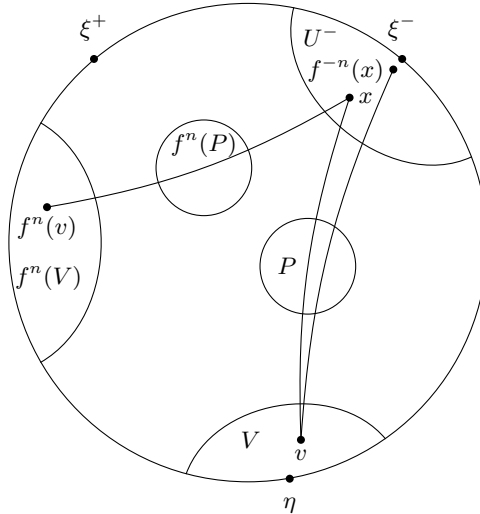


FIGURE 4.

angles of $xyy'x'$ at x and y exceeds $\pi - \varepsilon$, whereas at y' it equals π . Consequently, $xyy'x'$ is a (p', ε, R) -quadrilateral, which is a contradiction.

If $x_n \in \partial_\infty X$, then we take x' sufficiently far on the geodesic ray $h_n(p)x_n$, and we set $y' = h_n^{-2}(x')$. Then, since horoballs are limits of balls, the Alexandrov angles of $xyy'x'$ at x' and y' are $\geq \frac{\pi}{2}$. Thus an analogous calculation to the one in the case of $x_n \in X$ leads to a contradiction.

Suppose now that h_n has a fixed point $x_n \in X \cup \partial_\infty X$ in V . Then observe that the geodesic $h_n(p)x_n$ passes through P , which contains p . If $x_n \in X$, then for large n this contradicts $d_X(h_n(p), x_n) = d_X(p, x_n)$. If $x_n \in \partial_\infty X$ and x_n is the positive fixed point of h_n , then for large n this observation contradicts the fact that the Busemann function [BH99, II.8.17] satisfies $b_{x_n}(h_n(p)) \leq b_{x_n}(p)$. The only remaining possibility is that for large n the isometry h_n is vile or has positive translation length, with a positive limit point in U and a negative limit point in V . Choose neighbourhoods $U_n, V_n \subset X \cup \partial_\infty X$ of these limit points inside U, V . Then P, U_n, V_n satisfy Definition 2.3 for the limit points of h_n . \square

The following is essentially [Rua01, Lem 6.3].

Lemma 2.11. *Let f be an isometry of X with limit points $\xi^+, \xi^- \in \partial_\infty X$. Let $\eta \in \partial_\infty X$ with ξ^-, η far. Then there is a neighbourhood $V \subset \partial_\infty X \cup X$ of η such that for every neighbourhood $U^+ \subset \partial_\infty X \cup X$ of ξ^+ , for sufficiently large n we have $f^n(V) \subset U^+$.*

Proof. Let P, U^-, V be provided by Definition 2.3 for ξ^-, η . Choose a basepoint $x \in X$ such that $f^{-n}(x)$ lie in U^- for all $n \geq 0$. Then for each $v \in V$ and all $n \geq 0$ there is a point of the ball P on the geodesic $f^{-n}(x)v$ (which is possibly a ray). Consequently, each geodesic $xf^n(v)$ contains a point at a uniform distance from $f^n(P)$. See Figure 4. This proves that for every neighbourhood $U^+ \subset \partial_\infty X \cup X$ of ξ^+ we have $f^n(V) \subset U^+$ for n sufficiently large. \square

Proof of Proposition 2.9. Let U^+ be a neighbourhood of ξ_f^+ . Let V be a neighbourhood of ξ_g^- from Lemma 2.11. Choose $x \in X$. Then for n sufficiently large we have $g^{-n}(x) \in V$ and hence by Lemma 2.11 for n sufficiently large we have

$f^n g^{-n}(x) \in U^+$. Consequently for $h_n = f^n g^{-n}$ we have $h_n(x) \rightarrow \xi_f^+$. Analogously $h_n^{-1}(x) \rightarrow \xi_g^+$. By Lemma 2.10, for sufficiently large n the isometry h_n is vile or has positive translation length and far limit points in arbitrary neighbourhoods of ξ_f^+, ξ_g^+ . \square

3. BUILDING-LIKE SPACE \mathbf{X}

We recall the construction from [LP21] of a complete CAT(0) space \mathbf{X} with a natural action of $\text{Tame}(\mathbb{k}^3)$.

3.1. Metric. Let $\mathbb{R}_{>0}$ be the positive reals and let $\Pi = \mathbb{R}_{>0}^3$. The *weight space* ∇ is the projectivisation of Π . Let $\Pi^+ \subset \Pi$ consist of $(\alpha_1, \alpha_2, \alpha_3)$ satisfying $\alpha_1 \geq \alpha_2 \geq \alpha_3$, and let $\nabla^+ \subset \nabla$ be the projectivisation of Π^+ . We equip ∇ and ∇^+ with the topology induced from the projective plane. In particular, ∇^+ is a topological surface with boundary $\partial\nabla^+$ where $\alpha_1 = \alpha_2$ or $\alpha_2 = \alpha_3$. Then $\text{int } \nabla^+ = \nabla^+ \setminus \partial\nabla^+$ consists of $[\alpha]$ represented by α with $\alpha_1 > \alpha_2 > \alpha_3$.

For $\alpha \in \Pi$, we define the following map $\nu_{\text{id},\alpha}: \mathbb{k}[x_1, x_2, x_3] \rightarrow \mathbb{R}$. Let I stand for $(m_1, m_2, m_3) \in \mathbb{N}^3$. Namely, for $P = \sum_I c_I x_1^{m_1} x_2^{m_2} x_3^{m_3} \in \mathbb{k}[x_1, x_2, x_3]$, we set

$$\nu_{\text{id},\alpha}(P) = \min_{c_I \neq 0} \left(- \sum_{k=1}^3 \alpha_k m_k \right).$$

For $f \in \text{Tame}(\mathbb{k}^3)$, we define the *monomial valuation* $\nu_{f,\alpha}: \mathbb{k}[x_1, x_2, x_3] \rightarrow \mathbb{R}$ by $\nu_{f,\alpha}(P) = \nu_{\text{id},\alpha}(P \circ f)$. The group $\text{Tame}(\mathbb{k}^3)$ acts on monomial valuations by $g \cdot \nu(P) = \nu(P \circ g)$. The projective class of $\nu_{f,\alpha}$ depends only on $[\alpha]$ and is denoted by $\nu_{f,[\alpha]}$. Instead of $[(\alpha_1, \alpha_2, \alpha_3)]$ we shortly write $[\alpha_1, \alpha_2, \alpha_3]$.

We equip the space \mathbf{X} of projective classes of all monomial valuations with the following metric. First, consider the map $\nabla \rightarrow \mathbb{R}^3$ sending each $[\alpha] \in \nabla$ represented by α with $\prod_{k=1}^3 \alpha_k = 1$ to $(\log \alpha_1, \log \alpha_2, \log \alpha_3)$, with image in the Euclidean plane $\mathbb{R}^2 \subset \mathbb{R}^3$ of equation $\log \alpha_1 + \log \alpha_2 + \log \alpha_3 = 0$. We equip ∇ (and consequently ∇^+) with the Euclidean metric $|\cdot, \cdot|$ pulled back from \mathbb{R}^2 .

We view \mathbf{X} as the quotient (by an equivalence relation \sim) of the disjoint union of *chambers* $\mathbf{E}_f^+ = \{\nu_{f,[\alpha]} : [\alpha] \in \nabla^+\}$. We also consider *apartments* $\mathbf{E}_f = \{\nu_{f,[\alpha]} : [\alpha] \in \nabla\}$. The following proves in particular that the projective classes of monomial valuations $\nu_{\text{id},[\alpha]}$ and $\nu_{\text{id},[\alpha']}$ are distinct for $[\alpha] \neq [\alpha']$.

Corollary 3.1 ([LP21, cor 2.5(i)]). *The map $\bigsqcup \mathbf{E}_f^+ \rightarrow \nabla^+$ sending each $\nu_{f,[\alpha]}$ to $[\alpha]$ is well-defined and descends to a map $\rho_+: \mathbf{X} \rightarrow \nabla^+$.*

We equip each chamber with the Euclidean metric $|\cdot, \cdot|$ coming from ∇^+ . A *chain* is a sequence $(x'_0, x_1 \sim x'_1, x_2 \sim x'_2, \dots, x_{k-1} \sim x'_{k-1}, x_k)$ in $\bigsqcup \mathbf{E}_f^+$ such that there are $f_0, \dots, f_{k-1} \in \text{Tame}(\mathbb{k}^3)$ with $x'_i, x_{i+1} \in \mathbf{E}_{f_i}^+$. For $\bar{x}, \bar{y} \in \mathbf{X}$ we define the *quotient pseudo-metric* $d_{\mathbf{X}}(\bar{x}, \bar{y})$ to be the infimum of the *lengths* $\sum_{i=0}^{k-1} |x'_i, x_{i+1}|$ of chains with $x'_0 \in \bar{x}$ and $x_k \in \bar{y}$.

Theorem 3.2 ([LP21, Thm A]). *$(\mathbf{X}, d_{\mathbf{X}})$ is complete and CAT(0).*

Here are some other basic properties of $(\mathbf{X}, d_{\mathbf{X}})$ that we will be using. By $B_{\mathbf{X}}(\bar{x}, \epsilon)$ we denote the radius ϵ open ball in \mathbf{X} around $\bar{x} \in \mathbf{X}$.

Lemma 3.3. *(i) For each $\bar{x} \in \mathbf{X}$ there is $\epsilon = \epsilon(\bar{x}) > 0$ such that $B_{\mathbf{X}}(\bar{x}, \epsilon)$ coincides as a set with the union of radius ϵ open balls around x in $\bigsqcup \mathbf{E}_f^+$ for $x \in \bar{x}$.*

- (ii) Each \mathbf{E}_f^+ is isometrically embedded in \mathbf{X} .
- (iii) For each pair $\bar{x}, \bar{y} \in \mathbf{X}$, the infimum in the definition of the distance $d_{\mathbf{X}}(\bar{x}, \bar{y})$ is realised.

Proof. Item (i) is [LP21, lem 5.5(a)], and item (ii) is [LP21, lem 5.6(i)]. For item (iii), let γ be a geodesic from \bar{x} to \bar{y} guaranteed by Theorem 3.2. By the compactness of γ , we can find points $\bar{x}_0 = \bar{x}, \bar{x}_1, \dots, \bar{x}_k = \bar{y}$ such that the consecutive balls $B_{\mathbf{X}}\left(\bar{x}_i, \frac{\epsilon(\bar{x}_i)}{2}\right)$ intersect. Consequently, for each $0 \leq i < k$, we have $\bar{x}_i \in B_{\mathbf{X}}(\bar{x}_{i+1}, \epsilon(\bar{x}_{i+1}))$ or $\bar{x}_{i+1} \in B_{\mathbf{X}}(\bar{x}_i, \epsilon(\bar{x}_i))$. By item (i), this shows that \bar{x}_i, \bar{x}_{i+1} have representatives in a common $\mathbf{E}_{f_i}^+$. By item (ii), the length of γ coincides with the length of the chain determined by the \bar{x}_i . \square

3.2. Point stabilisers. Let $A < \text{Tame}(\mathbb{k}^3)$ be the subgroup consisting of the affine transformations of \mathbb{k}^3 . We will also consider the following subgroups of $\text{Tame}(\mathbb{k}^3)$:

$$B = \{(ax_1 + P(x_2, x_3), bx_2 + cx_3 + d, b'x_2 + c'x_3 + d')\};$$

$$P \in \mathbb{k}[x_2, x_3], a \neq 0, bc' - b'c \neq 0\},$$

$$H = \{(ax_1 + P_1(x_2, x_3), bx_2 + P_2(x_3), cx_3 + d)\};$$

$$P_1 \in \mathbb{k}[x_2, x_3], P_2 \in \mathbb{k}[x_3] \ a, b, c \neq 0\},$$

$$K = \{(ax_1 + bx_2 + P_1(x_3), a'x_1 + b'x_2 + P_2(x_3), cx_3 + d)\};$$

$$P_1, P_2 \in \mathbb{k}[x_3], ab' - a'b \neq 0, c \neq 0\}.$$

Let $C = \langle K, H \rangle$, where $C = K *_{K \cap H} H$ by [LP19, Prop 3.5]. Finally, let $B' = \langle B, H \rangle$, where $B' = B *_{B \cap H} H$, which corresponds to the splitting of $\text{Aut}(\mathbb{k}^2)$ in [vdK53].

Let $\alpha \in \Pi^+$. We consider the following subgroups $M_\alpha < H, L_\alpha < \mathbf{GL}_3(\mathbb{k}) < \text{Tame}(\mathbb{k}^3)$.

- (1) If $\alpha_1 = \alpha_2 = \alpha_3$, then $L_\alpha = \mathbf{GL}_3(\mathbb{k})$ and M_α is the group of translations, so that $M_\alpha \rtimes L_\alpha = A$.
- (2) If $\alpha_1 > \alpha_2 > \alpha_3$, then L_α is the group of the diagonal matrices and M_α is the group of the automorphisms

$$(x_1, x_2, x_3) \rightarrow (x_1 + P_1(x_2, x_3), x_2 + P_2(x_3), x_3 + d)$$

where P_i satisfy $\alpha_i \geq -\nu_{\text{id}, \alpha}(P_i)$. In particular, $M_\alpha \rtimes L_\alpha < H$.

- (3) If $\alpha_1 > \alpha_2 = \alpha_3$, then

$$L_\alpha = \{(ax_1, bx_2 + cx_3, b'x_2 + c'x_3)\},$$

$$M_\alpha = \{(x_1 + P(x_2, x_3), x_2 + d, x_3 + d'); \alpha_1 \geq \alpha_2 \deg P\}.$$

In particular, $M_\alpha \rtimes L_\alpha < B$.

- (4) If $\alpha_1 = \alpha_2 > \alpha_3$, then

$$L_\alpha = \{(ax_1 + bx_2, a'x_1 + b'x_2, cx_3)\},$$

$$M_\alpha = \{(x_1 + P_1(x_3), x_2 + P_2(x_3), x_3 + d); \alpha_1 \geq \alpha_3 \deg P_1, \alpha_3 \deg P_2\}.$$

In particular, $M_\alpha \rtimes L_\alpha < K$.

Note that L_α, M_α depend only on the class $[\alpha]$.

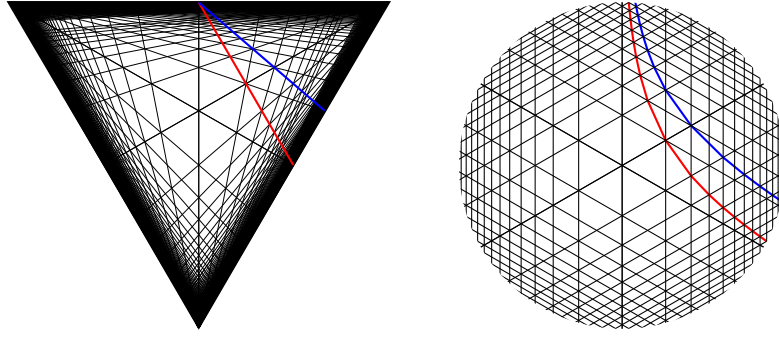


FIGURE 5.

Proposition 3.4 ([LP21, prop 3.2]). *Let $\alpha \in \Pi^+$. We have*

$$\text{Stab}(\nu_{\text{id},[\alpha]}) = M_\alpha \rtimes L_\alpha.$$

In particular, each element of the groups H , B , and K , has a fixed point in \mathbf{E}_{id}^+ .

The following result was proved in [Lam01] for $\mathbb{k} = \mathbb{C}$. In fact, the proof extends to any field of characteristic zero, see [Lam22].

Theorem 3.5. *Let \mathbb{k} be a field of characteristic zero. Then the polynomial automorphism group $\text{Aut}(\mathbb{k}^2)$ satisfies the strong Tits alternative.*

Below, the *step* of a solvable group is its derived length, that is, the length of its derived series. In particular, non-trivial abelian groups have step 1.

Remark 3.6. (i) The groups M_α are solvable and the groups L_α are linear. Thus, since the strong Tits alternative is preserved under semidirect products, by Proposition 3.4 all $\text{Stab}(\nu_{\text{id},[\alpha]})$ satisfy the strong Tits alternative.

(ii) Since M_α are solvable of step ≤ 3 , and all virtually solvable subgroups of all L_α contain a triangularisable subgroup of uniformly bounded index (see [Wan81] and [KM79, Thm 21.1.5]), which is solvable of step ≤ 3 , we have that all virtually solvable subgroups of all $\text{Stab}(\nu_{\text{id},[\alpha]})$ contain a solvable subgroup of uniformly bounded index and step ≤ 6 .

(iii) The group B' has a decomposition into a semidirect product of $\text{Aut}(\mathbb{k}^2)$ (with variables x_2, x_3) and the solvable normal subgroup consisting of automorphisms of the form $(ax_1 + P(x_2, x_3), x_2, x_3)$. Thus by Theorem 3.5, the group B' also satisfies the strong Tits alternative. Similarly, the group C is a semidirect product of the solvable subgroup consisting of $(x_1, x_2, cx_3 + d)$ and a normal subgroup that can be embedded in the automorphism group $\text{Aut}(\mathbb{k}(x_3)^2)$ with variables x_1, x_2 . Consequently, C also satisfies the strong Tits alternative.

Proposition 3.4 can be rephrased as follows.

Corollary 3.7 ([LP21, prop 4.4]). *Let $f \in \text{Tame}(\mathbb{k}^3)$ and let $\alpha \in \Pi^+$. Then $\nu_{\text{id},[\alpha]}$ is fixed by f if and only if α satisfies all the inequalities of the form*

$$\alpha_i \geq \sum_k m_k \alpha_k,$$

where $i = 1, 2, 3$ and $\prod_k x_k^{m_k}$ are the monomials appearing in f_i .

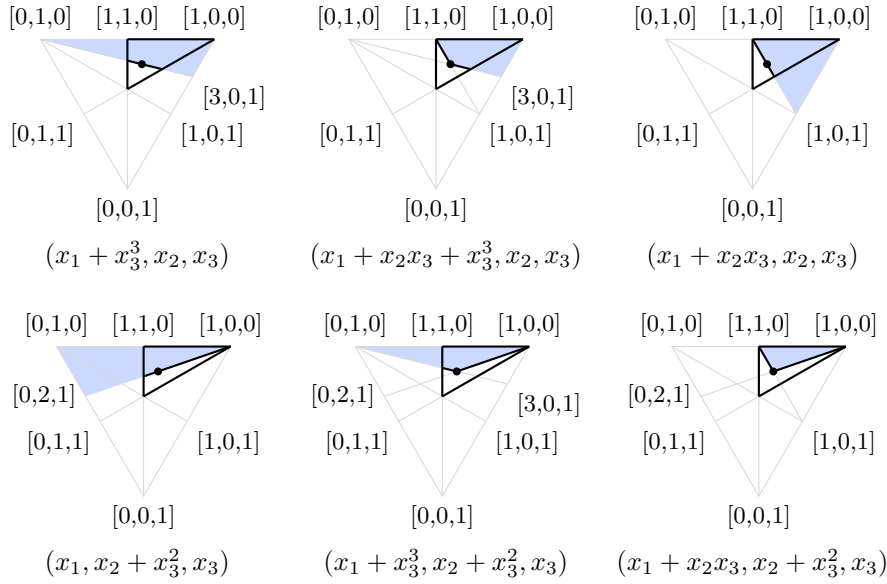


FIGURE 6. Intersections of apartments \mathbf{E}_{id} , \mathbf{E}_f and chambers \mathbf{E}_{id}^+ , \mathbf{E}_f^+ for particular f .

The subsets of ∇^+ and ∇ given by the equations of the form $\alpha_1 = m_2\alpha_2 + m_3\alpha_3$, up to interchanging α_i , are *admissible lines*. If additionally $m_2 = 0$ (or $m_3 = 0$), then such a line is *principal*. See Figure 5 for the set of the admissible lines in ∇ with the projective structure (on the left), and with the Euclidean metric $|\cdot, \cdot|$ (on the right). Note that the only admissible lines intersecting the interior of ∇^+ have the precise form $\alpha_1 = m_2\alpha_2 + m_3\alpha_3$, where $m_2 + m_3 \geq 2$, or $\alpha_2 = m_3\alpha_3$, where $m_3 \geq 2$.

The following Figure 6 illustrates some possible intersections between \mathbf{E}_{id} and \mathbf{E}_f (and \mathbf{E}_{id}^+ and \mathbf{E}_f^+ in bold), where f fixes the point $\nu_{\text{id},[3,2,1]}$.

Remark 3.8. Suppose in Corollary 3.7 that $\nu_{\text{id},[\alpha]}$ is fixed by f . If $\alpha_2 > \alpha_3$, then for each $\alpha' = (\alpha_1, \alpha_2, \alpha'_3)$ with $\alpha'_3 < \alpha_3$, the point $\nu_{\text{id},[\alpha']}$ is fixed by f . If $\alpha_1 > \alpha_2$, then for each $\alpha' = (\alpha'_1, \alpha_2, \alpha_3)$ with $\alpha'_1 > \alpha_1$, the point $\nu_{\text{id},[\alpha']}$ is fixed by f .

3.3. Combinatorial structure on \mathbf{X} . A map from a CW-complex X to a CW-complex Y is *combinatorial*, if its restriction to any open cell of X is a homeomorphism onto an open cell of Y . A CW-complex X is *combinatorial*, if the attaching map of each open cell of X is combinatorial for some subdivision of the sphere. The *star* of a cell σ in a CW-complex X is the union of all the open cells of X containing σ in their closure.

The space \mathbf{X} is equipped with the following structure of a combinatorial CW-complex (which induces a weaker topology than the metric $d_{\mathbf{X}}$ due to the lack of local finiteness). Namely, we first equip ∇^+ with the following structure of a combinatorial CW-complex. Its vertices are the intersection points of distinct admissible lines. Its open edges are the connected components of the complement of the vertices in the admissible lines. Finally, its open 2-cells are the connected components of the complement of the admissible lines in ∇^+ . In (the proof of) [LP21, prop 6.3], we identify \mathbf{X} with the universal development of a certain complex of groups with

underlying complex ∇^+ equipped with that CW-structure. This equips \mathbf{X} with a CW-complex structure. Equivalently, each open cell of \mathbf{X} is contained in a chamber \mathbf{E}_f^+ and it is the preimage in \mathbf{E}_f^+ under ρ_+ of a cell of ∇^+ .

4. TRIPLES OF POLYNOMIALS IN TWO VARIABLES

In this section we collect several estimates on particular exponents appearing in identities involving triples of polynomials in two variables. This will enable us to study the links of the vertices of \mathbf{X} in Section 5.

Lemma 4.1. *Let $Q \in \mathbb{k}[y, z]$ be a nonzero homogeneous polynomial of degree $d \geq 0$, let $c, c' \in \mathbb{k}^*$, and let $q \geq 1$ be an integer. Then the partial derivatives $\partial_y[(cy + c'z)^q Q]$ and $\partial_z[(cy + c'z)^q Q]$ are both nonzero homogeneous polynomials of degree $d + q - 1$ that are multiples of $(cy + c'z)^{q-1}$.*

Proof. Given a homogeneous polynomial $P \in \mathbb{k}[y, z]$, if $\deg_y P \geq 1$, then we have $\deg \partial_y P = \deg(P) - 1$. This gives the assertion about the degree. The assertion that $(cy + c'z)^{q-1}$ divides the partial derivatives is a straightforward computation. For instance, for the first one:

$$\partial_y[(cy + c'z)^q Q] = (cy + c'z)^{q-1}[cqQ + (cy + c'z)\partial_y Q]. \quad \square$$

Lemma 4.2. *Let $R, T, U \in \mathbb{k}[x_2, x_3]$ be nonzero homogeneous polynomials, and let $\ell_1, \ell_2, \ell_3 \in \mathbb{k}[x_2, x_3]$ be pairwise independent linear forms such that we have a relation*

$$\ell_1^r R + \ell_2^t T + \ell_3^u U = 0,$$

for some $r, t, u \in \mathbb{N}$. In particular, the three terms of the sum have the same degree $d = r + \deg R = t + \deg T = u + \deg U$. Then $\min\{r, t, u\} \leq \lfloor \frac{2d+1}{3} \rfloor$.

Proof. By a linear change of coordinates we can assume $\ell_1 = x_2$, $\ell_2 = cx_2 + c'x_3$ and $\ell_3 = x_3$, for some $c, c' \in \mathbb{k}^*$, so that the relation becomes

$$x_2^r R + (cx_2 + c'x_3)^t T + x_3^u U = 0.$$

Observe that for an integer a , the condition $a \leq \lfloor \frac{2d+1}{3} \rfloor$ is equivalent to $a < \frac{2}{3}(d+1)$. For contradiction, assume $r, t, u \geq \frac{2}{3}(d+1)$. Let n be the unique integer such that $\frac{1}{3}(d-2) < n \leq \frac{1}{3}(d+1)$. Then $2n \leq \frac{2}{3}(d+1) \leq t$. We have

$$\deg R = d - r \leq d - \frac{2}{3}(d+1) = \frac{1}{3}(d-2) < n,$$

so $\partial_{x_3}^n[x_2^r R] = x_2^r \partial_{x_3}^n R = 0$. Similarly, $\deg U < n$ and $\partial_{x_2}^n[x_3^u U] = 0$. Taking the partial derivatives $\partial_{x_2}^n \partial_{x_3}^n$ of the expression $E = x_2^r R + (cx_2 + c'x_3)^t T + x_3^u U$, we obtain $\partial_{x_2}^n \partial_{x_3}^n E = \partial_{x_2}^n \partial_{x_3}^n [(cx_2 + c'x_3)^t T]$. Since $2n \leq t$, we can apply $2n$ times Lemma 4.1 to obtain $\partial_{x_2}^n \partial_{x_3}^n [(cx_2 + c'x_3)^t T] \neq 0$, hence $E \neq 0$, contradiction. \square

We will need the following weighted versions of Lemma 4.2.

Lemma 4.3. *Let $p \geq 1$ be an integer, and let $R, T, U \in \mathbb{k}[x_2, x_3]$ be nonzero weighted homogeneous polynomials with x_2, x_3 of respective weights $p, 1$. Assume that there exist $r, t, u, m \in \mathbb{N}$ such that*

$$\deg R + rp = \deg T + tp = \deg U + up = m.$$

Suppose that for some distinct $c, c' \in \mathbb{k}^$ we have*

$$x_2^r R + (x_2 + cx_3)^t T + (x_2 + c'x_3)^u U = 0.$$

Let $m = \alpha p + \beta$ be the Euclidean division of m by p , and assume $\alpha \geq 2$. Then $\min\{rp, tp, up\} \leq \frac{3}{4}m$.

Remark 4.4. In the case $\alpha = r = t = u = 1$, up to scaling the only possibility for R, T, U satisfying the identity in Lemma 4.3 is

$$R = (c - c')x_3^\beta, \quad T = c'x_3^\beta, \quad U = -cx_3^\beta.$$

Proof of Lemma 4.3. There exist (ordinary) homogeneous polynomials $\tilde{R}, \tilde{T}, \tilde{U}$ satisfying

$$\begin{aligned} R(x_2, x_3) &= x_3^\beta \tilde{R}(x_2, x_3^p), \\ T(x_2, x_3) &= x_3^\beta \tilde{T}(x_2, x_3^p), \\ U(x_2, x_3) &= x_3^\beta \tilde{U}(x_2, x_3^p). \end{aligned}$$

By assumption, we have

$$x_2^r x_3^\beta \tilde{R}(x_2, x_3^p) + (x_2 + cx_3^p)^t x_3^\beta \tilde{T}(x_2, x_3^p) + (x_2 + c'x_3^p)^u x_3^\beta \tilde{U}(x_2, x_3^p) = 0.$$

Dividing by x_3^β and changing variables we get

$$x_2^r \tilde{R}(x_2, x_3) + (x_2 + cx_3)^t \tilde{T}(x_2, x_3) + (x_2 + c'x_3)^u \tilde{U}(x_2, x_3) = 0,$$

where each term of the sum is homogeneous of degree $\frac{m-\beta}{p} = \alpha$. By Lemma 4.2, we have $\min\{r, t, u\} \leq \lfloor \frac{2\alpha+1}{3} \rfloor$. Observe that

$$(\spadesuit) \quad \max_{\alpha \geq 2} \frac{\lfloor \frac{2\alpha+1}{3} \rfloor}{\alpha} = \frac{3}{4}, \text{ realised for } \alpha = 4.$$

We conclude that

$$\min\{rp, tp, up\} \leq \left\lfloor \frac{2\alpha+1}{3} \right\rfloor p \leq \frac{\lfloor \frac{2\alpha+1}{3} \rfloor}{\alpha} \alpha p \leq \frac{3}{4}m. \quad \square$$

Lemma 4.5. Let $p \geq 1$ be an integer, and let $R, T, U \in \mathbb{k}[x_2, x_3]$ be nonzero weighted homogeneous polynomials with x_2, x_3 of respective weights $p, 1$. Assume that there exist $r, t, u, m \in \mathbb{N}$ such that

$$\deg R + rp = \deg T + tp = \deg U + u = m.$$

Suppose that for some $c \in \mathbb{k}^*$ we have

$$x_2^r R + (x_2 + cx_3^p)^t T + x_3^u U = 0.$$

Let $m = \alpha p + \beta$ be the Euclidean division of m by p , and assume $\alpha \geq 2$. Then $\min\{rp, tp, u\} < \frac{4}{5}m$.

Remark 4.6. In the case $\alpha = r = t = 1$, and $u = m$, up to scaling the only possibility for R, T, U satisfying the identity in Lemma 4.5 is

$$R = -x_3^\beta, \quad T = x_3^\beta, \quad U = -c.$$

Proof of Lemma 4.5. We can assume that x_3 does not divide U . Write the Euclidean division $m - u = \alpha'p + \beta'$. Then there exist (ordinary) homogeneous polynomials

$\tilde{R}, \tilde{T}, \tilde{U}$ satisfying

$$\begin{aligned} R(x_2, x_3) &= x_3^\beta \tilde{R}(x_2, x_3^p), \\ T(x_2, x_3) &= x_3^\beta \tilde{T}(x_2, x_3^p), \\ U(x_2, x_3) &= x_3^{\beta'} \tilde{U}(x_2, x_3^p). \end{aligned}$$

By assumption, we have

$$x_2^r x_3^\beta \tilde{R}(x_2, x_3^p) + (x_2 + cx_3)^t x_3^\beta \tilde{T}(x_2, x_3^p) + x_3^u x_3^{\beta'} \tilde{U}(x_2, x_3^p) = 0.$$

We see that $u + \beta' - \beta$ must be a multiple of p , and writing $u + \beta' - \beta = pu'$ with $u' \geq 0$, we get

$$x_2^r \tilde{R}(x_2, x_3) + (x_2 + cx_3)^t \tilde{T}(x_2, x_3) + x_3^{u'} \tilde{U}(x_2, x_3) = 0.$$

By Lemma 4.2, we have $\min\{r, t, u'\} \leq \lfloor \frac{2\alpha+1}{3} \rfloor$. By definition, we have $u \leq pu' + \beta$, and so multiplying by p we obtain

$$\min\{rp, tp, u\} \leq \left\lfloor \frac{2\alpha+1}{3} \right\rfloor p + \beta.$$

Observe that for any real numbers $0 < c_1 < c_2$, we have $c_1 p + \beta < \frac{c_1+1}{c_2+1}(c_2 p + \beta)$. Applying this observation with $c_1 = \lfloor \frac{2\alpha+1}{3} \rfloor$ and $c_2 = \alpha$, we obtain

$$\min\{rp, tp, u\} \leq \left\lfloor \frac{2\alpha+1}{3} \right\rfloor p + \beta < C(\alpha p + \beta) = Cm$$

for

$$C = \max_{\alpha \geq 2} \frac{\lfloor \frac{2\alpha+1}{3} \rfloor + 1}{\alpha + 1} = \frac{4}{5}.$$

(The maximum is realised for $\alpha = 4$.) □

5. LINKS

In this section we will study the link of each vertex ν in \mathbf{X} , which is the following metric graph [BB95, page 176]. Namely, the vertices of the link correspond to the edges of \mathbf{X} containing ν and the edges of the link correspond to the corners of the 2-cells of \mathbf{X} containing ν . The length of each edge of the link is the angle in \mathbf{X} at the corresponding corner. Note that the link is not finite but has finitely many possible edge lengths.

Since $\text{Tame}(\mathbb{k}^3)$ maps any vertex into \mathbf{E}_{id}^+ , we will assume $\nu \in \mathbf{E}_{\text{id}}^+$. For example, by [LP21, lem 3.4] the link of $\nu_{\text{id}, [1,1,1]}$ in \mathbf{X} can be identified with the Bruhat–Tits building of $\mathbf{GL}_3(\mathbb{k})$. Consequently, each cycle in the link of length $< 2\pi + \frac{2\pi}{3}$ has length exactly 2π and is the link of ν in \mathbf{E}_f for $f \in A$.

We will now perform a similar classification for other weights. In [LP21, prop 7.9, 7.10 and 7.12] we proved that the link of ν is CAT(1), i.e. all of its cycles have length $\geq 2\pi$. Our aim is to classify all of its cycles of length $< 2\pi + \varepsilon_0$, for some small $\varepsilon_0 > 0$ independent of ν .

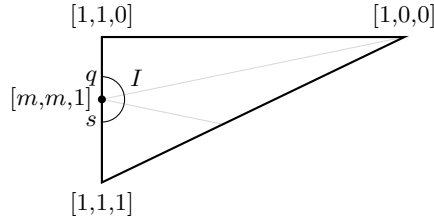


FIGURE 7.

5.1. Links of vertices with weight $[m, m, 1]$. Let $\nu = \nu_{\text{id}, [m, m, 1]}$ for an integer $m \geq 2$. Let Γ be the link of ν in \mathbf{X} . Let I be the link of $[m, m, 1]$ in ∇^+ , which is a simplicial path consisting of 3 edges of length $\frac{\pi}{3}$ [LP21, rem 5.2], see Figure 7. We keep the notation ρ_+ for the induced map from Γ to I . Let $s, q \in I$ be the directions towards $[1, 1, 1]$ and towards $[1, 1, 0]$, respectively. For $f \in \text{Stab}(\nu)$, let $\Gamma_f, \Gamma_f^+ \subset \Gamma$ be the cycle and the path, respectively, that are the links of ν in $\mathbf{E}_f, \mathbf{E}_f^+$. Let $s_f, q_f \in \Gamma_f^+$ be the preimages under ρ_+ of s, q , respectively.

Proposition 5.1. *Each cycle in Γ of length $< 2\pi + \frac{2\pi}{3}$ has length 2π and*

- equals Γ_f , or
- is the closure of $(\Gamma_f \cup \Gamma_g) \setminus (\Gamma_f \cap \Gamma_g)$, or
- intersects $\Gamma_f, \Gamma_g, \Gamma_h$ along length $\frac{2\pi}{3}$ paths with midpoints s_f, s_g, s_h .

In the proof we will need the following.

Lemma 5.2 ([LP21, lem 7.7(iii)]). *Each vertex $u \in \Gamma \setminus \rho_+^{-1}\{q, s\}$ is contained in exactly one edge e such that $\rho_+(e)$ separates $\rho_+(u)$ from q in I . Moreover, if $u \in \Gamma_f^+$ for some $f \in \text{Stab}(\nu)$, then $e \subset \Gamma_f^+$.*

Lemma 5.3 ([LP21, lem 7.8]). *Suppose that $s_f = s_{f'}$ for some $f, f' \in \text{Stab}(\nu)$. Then there exists $f'' \in \text{Stab}(\nu)$ with $\Gamma_f^+, \Gamma_{f'}^+ \subset \Gamma_{f''}$.*

Proof of Proposition 5.1. Since Γ is bipartite and all of its edges have length $\frac{\pi}{3}$, we only need to classify cycles γ of length 2π , each of which consists of six edges. We orient each edge of γ , so that its image in I under ρ_+ is oriented from q to s . By Lemma 5.2, γ decomposes into $j = 2, 4$, or 6 oriented paths each of which lies in an Γ_f^+ and ends with s_f . By Lemma 5.3, the cases $j = 2, 4, 6$, correspond to the respective bullet points of the proposition. \square

5.2. Links of vertices with weight $[m, 1, 1]$. Let $m \geq 2$ be an integer.

Remark 5.4. The weight $[\alpha] = [m, 1, 1]$ lies on exactly $m + 2$ admissible lines, namely $\alpha_2 = \alpha_3$ and $\alpha_1 = a\alpha_2 + (m - a)\alpha_3$, where $a = 0, \dots, m$. The latter lines contain points $[a, 1, 0]$, and are principal exactly for $a = 0$ and $a = m$. See Figure 8.

Let $\nu = \nu_{\text{id}, [\alpha]}$ with $[\alpha] = [m, 1, 1]$. Let Γ_0 be the link of ν in \mathbf{X} . We keep the notation ρ_+ for the induced map from Γ_0 to the link I of $[\alpha]$ in ∇^+ , which is a simplicial path. Let $s, q \in I$ be the directions towards $[1, 1, 1]$ and towards $[1, 0, 0]$, respectively. Let $e^s \subset I$ be the edge containing s . For $f \in \text{Stab}(\nu)$, let $\Gamma_f, \Gamma_f^+ \subset \Gamma_0$ be the cycle and the path, respectively, that are the links of ν in $\mathbf{E}_f, \mathbf{E}_f^+$. Let $s_f, q_f, e_f^s \subset \Gamma_f^+$ be the preimages under ρ_+ of s, q, e^s , respectively. Lemmas 5.2 and 5.3 remain true in this setting, with Γ replaced by Γ_0 (see the first paragraph of the proof of [LP21, prop 7.10]).

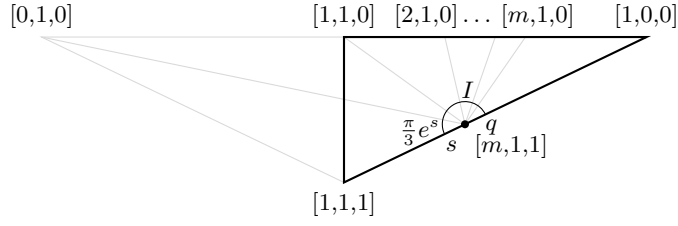


FIGURE 8.

Proposition 5.5. *There is $\varepsilon_0 > 0$ independent of m such that each cycle in Γ_0 of length $< 2\pi + \varepsilon_0$*

- equals Γ_f , or
- is the closure of $(\Gamma_f \cup \Gamma_g) \setminus (\Gamma_f \cap \Gamma_g)$.

Note that unlike in Proposition 5.1, here in the second bullet point we can have cycles of length $> 2\pi$ arbitrary close to 2π .

In Example 5.9 we will see that setting $\varepsilon_0 = \frac{2\pi}{3}$ does not suffice.

In the proof, we will need the following. For $f, g \in \text{Stab}(\nu)$, we write $f \sim_\nu g$ if $\Gamma_f^+ = \Gamma_g^+$. Let $\text{Stab}'(\nu) < \text{Stab}(\nu)$ be the subgroup of elements of form

$$(x_1 + P(x_2, x_3), bx_2 + cx_3 + d, b'x_2 + c'x_3 + d'),$$

which differs from $\text{Stab}(\nu)$ only by requiring the coefficient at x_1 to be equal 1. Let $N_\alpha < \text{Stab}'(\nu)$ be the subgroup of elements of form

$$(x_1 + P(x_2, x_3), x_2 + d, x_3 + d')$$

with $\deg P < m$.

Lemma 5.6. (i) *For each $f \in \text{Stab}(\nu)$, there is $g \in \text{Stab}'(\nu)$ with $f \sim_\nu g$.*

(ii) *Let $h \in N_\alpha$. Then $h \sim_\nu \text{id}$.*

(iii) *N_α is normal in $\text{Stab}'(\nu)$.*

Proof. Part (i) follows from the description of $\text{Stab}(\nu)$ in Proposition 3.4 and the observation [LP21, cor 3.3] that for $l = (ax_1, x_2, x_3)$ we have $l \sim_\nu \text{id}$. Part (ii) follows from Corollary 3.7. Part (iii) is an easy computation. \square

Lemma 5.7. *Let $f, g \in \text{Stab}'(\nu)$. If $\Gamma_f^+ \neq \Gamma_g^+$ and $\Gamma_f^+ \cap \Gamma_g^+ \neq \{q_f\}$, then either*

(i) *$\Gamma_f^+ \cap \Gamma_g^+ = \{s_f, q_f\}$ and $f^{-1}g \in hN_\alpha$ for some $h = (x_1, bx_2 + cx_3, b'x_2 + c'x_3)$ with $b' \neq 0$, or*

(ii) *$\Gamma_f^+ \cap \Gamma_g^+$ is the path from q_f to the direction mapping under ρ_+ to the direction towards $[a, 1, 0]$, where $0 \leq a \leq m$ and $f^{-1}g \in hN_\alpha$ for some*

$$h = (x_1 + \sum_{i=0}^a c_i x_2^i x_3^{m-i}, bx_2 + cx_3, c'x_3)$$

with $c_a \neq 0$.

Proof. After composing f and g with f^{-1} , we can assume $f = \text{id}$. Let $g = (x_1 + P(x_2, x_3), bx_2 + cx_3 + d, b'x_2 + c'x_3 + d')$. By Lemma 5.6(ii,iii), after possibly composing with an element of N_α , we can assume that P is homogeneous of degree m , and $d = d' = 0$. If P is trivial, then by Corollary 3.7 we have (i) for $b' \neq 0$ or $\Gamma_f^+ = \Gamma_g^+$ for $b' = 0$. Analogously, if P is nontrivial, then by Corollary 3.7 we have (ii) for $b' = 0$ or $\Gamma_f^+ \cap \Gamma_g^+ = \{q_f\}$ for $b' \neq 0$. \square

Lemma 5.8. *Let $0 \leq \delta \leq 1$. Let $[\alpha], [\alpha'] \in \nabla$ and let c, c' be the projective line segments joining $[\alpha], [\alpha']$ to $[\delta \frac{\alpha_1}{\alpha_2}, 1, 0], [\delta \frac{\alpha'_1}{\alpha'_2}, 1, 0]$, respectively. Then the translation of $(\nabla, |\cdot|)$ mapping $[\alpha]$ to $[\alpha']$ maps c to c' .*

Proof. In the proof of [LP21, lem 5.9] we showed that the reflections in the lines parallel to the principal lines in $(\nabla, |\cdot|)$ extend to projective maps of $P\mathbb{R}^3$ (preserving ∇). Thus the translation τ mapping $[\alpha]$ to $[\alpha']$ maps the projective line through $[\alpha]$ and $[0, 0, 1]$ to the projective line through $[\alpha']$ and $[0, 0, 1]$ and hence it maps $[\frac{\alpha_1}{\alpha_2}, 1, 0]$ to $[\frac{\alpha'_1}{\alpha'_2}, 1, 0]$. Since τ preserves $[0, 1, 0]$ and $[1, 0, 0]$, as well as the cross-ratio on the projective line through $[0, 1, 0]$ and $[1, 0, 0]$, we conclude that τ maps $[\delta \frac{\alpha_1}{\alpha_2}, 1, 0]$ to $[\delta \frac{\alpha'_1}{\alpha'_2}, 1, 0]$. \square

Proof of Proposition 5.5. Let $\gamma \subset \Gamma_0$ be a cycle of length $< 2\pi + \frac{2\pi}{3}$. We orient each edge of γ , so that its image in I under ρ_+ is oriented from q to s . By Lemma 5.2, γ decomposes into an even number j of oriented paths each of which lies in an Γ_f^+ and ends with s_f . By Remark 5.4, each such path contributes at least $\frac{\pi}{3}$ to the length of γ . Since γ has length $< 2\pi + \frac{2\pi}{3}$, we have $j < 8$ and so $j = 2, 4$, or 6 . The cases $j = 2, 4$ imply the first two bullet points of the proposition by Lemma 5.3.

Suppose now $j = 6$, with $\gamma = \gamma_0 \cdot \gamma_1^{-1} \cdot \gamma_2 \cdots \gamma_5^{-1}$, where each γ_i lies in $\Gamma_{f_i}^+$, for some $f_i \in \text{Stab}(\nu)$. Note that these paths contribute already 2π to the length of γ , and so for a contradiction we need to find ε_0 bounding the excess contribution from below. By Remark 5.4, points $\rho_+(\gamma_1 \cap \gamma_2), \rho_+(\gamma_3 \cap \gamma_4), \rho_+(\gamma_5 \cap \gamma_0)$ correspond to directions towards $[a, 1, 0], [a', 1, 0], [a'', 1, 0]$ for some $0 \leq a, a', a'' \leq m$. We need to find ε_0 independent of m bounding below at least one of the angles at $[m, 1, 1]$ between the admissible lines in these directions and the direction towards $[0, 1, 0]$. To do that, we will find $\delta > 0$ independent of m such that $\max\{a, a', a''\} \geq \delta m$. We can then assign ε_0 to be the angle at $[m, 1, 1]$ between the projective lines in the directions towards $[\delta m, 1, 0]$ and $[0, 1, 0]$, which is independent of m by Lemma 5.8.

By Lemma 5.6(i), we can suppose that all f_i belong to $\text{Stab}'(\nu)$. By Lemma 5.7, we have $f_0^{-1}f_1 \in h_0N_\alpha, \dots, f_5^{-1}f_0 \in h_5N_\alpha$, where the second and third components of $h_i = (x_1 + P_i, \cdot, \cdot)$ form a linear automorphism in x_2, x_3 . Furthermore,

- for $i = 1, 3, 5$, the polynomial P_i is homogeneous in x_2, x_3 of degree m , and the third component of h_i does not depend on x_2 , while
- for $i = 0, 2, 4$, we have $P_i = 0$ and the third component of h_i depends on x_2 .

By Lemma 5.6(iii), the group N_α is normal in $\text{Stab}'(\nu)$ and so by Lemma 5.6(ii) we have $h_0 \cdots h_5 \sim_\nu \text{id}$. Thus the first component of $h_0 \cdots h_5$ equals $x_1 + P$ for P a polynomial in x_2, x_3 of degree $< m$. Since P_i are homogeneous, we have $P = 0$. By Lemma 5.7, we have that P_1, P_3, P_5 are divisible by $x_3^{m-a}, x_3^{m-a'}, x_3^{m-a''}$, respectively. Since the first component of $h_0 \cdots h_5$ equals x_1 , we obtain

$$P_1(\cdot, \ell_1) + P_3(\cdot, \ell_2) + P_5(x_2, x_3) = 0,$$

where ℓ_1, ℓ_2, x_3 are pairwise independent linear forms in $\mathbb{k}[x_2, x_3]$. Now we apply Lemma 4.2 with

$$r = m - a, \ell_1^r R = P_1, t = m - a', \ell_2^t T = P_3, u = m - a'', x_3^u Q = P_5.$$

Thus one of a, a', a'' is $\geq m - \lfloor \frac{2m+1}{3} \rfloor$, and so by \spadesuit in Section 4 it is $\geq \frac{m}{4}$. We can then set $\delta = \frac{1}{4}$. \square

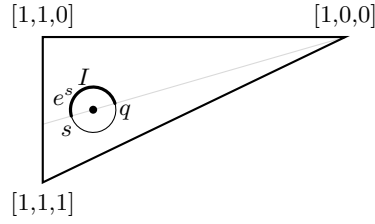


FIGURE 9.

Example 5.9. In the notation of the proof of Proposition 5.5, let $m = 2$, let

$$h_4 = (x_1, x_3, x_2), \quad h_2 = (x_1, x_3, x_2 + x_3), \quad h_0 = (x_1, x_2, x_3 - x_2),$$

$$P_5 = -x_3^2, \quad P_3 = x_3^2, \quad P_1 = x_3(x_3 - 2x_2).$$

Then the identity $x_2^2 - x_3^2 + (x_2 + x_3)(x_3 - x_2) = \text{id}$ implies $h_0 \cdots h_5 = \text{id}$. The paths γ_i for $i = 1, 2, 3, 4$ are of length $\frac{\pi}{3}$, while the paths γ_0, γ_5 are of length $\frac{\pi}{2}$ [LP21, lem 5.9]. Thus γ is a cycle in Γ of length $2\pi + \frac{\pi}{3}$.

Remark 5.10. Let L_0, L_1 be any non-principal admissible lines in ∇ . Then there is an isometry $(\nabla, |\cdot|)$ mapping L_0 to L_1 . Indeed, we can assume without loss of generality that the projective intervals L_0, L_1 have endpoints x_0^+, x_1^+ in the open projective interval from $[1, 0, 0]$ to $[0, 1, 0]$ and x_0^-, x_1^- in the open projective interval from $[1, 0, 0]$ to $[0, 0, 1]$. Let x_i be the intersection point of the projective line through $[0, 0, 1]$ and x_i^+ and the projective line through $[0, 1, 0]$ and x_i^- . Then as in the proof of Lemma 5.8, the translation of $(\nabla, |\cdot|)$ mapping x_0 to x_1 maps x_0^\pm to x_1^\pm , and hence maps L_0 to L_1 .

5.3. Links of vertices with weight in $\text{int } \nabla^+$. Let $\nu = \nu_{[\alpha]}$ with $[\alpha] = [\alpha_1, \alpha_2, \alpha_3] \in \text{int } \nabla^+$. Let Γ_0 be the link of ν in \mathbf{X} . We keep the notation ρ_+ for the induced map from Γ_0 to the cycle S that is the link of $[\alpha]$ in ∇^+ . Let $s, q \in S$ be the directions towards $[0, \alpha_2, \alpha_3]$ or towards $[1, 0, 0]$, respectively. Let $I \subset S$ be the path defined by $\frac{\alpha'_2}{\alpha'_3} \geq \frac{\alpha_2}{\alpha_3}$. If s is a vertex of S , then let $e^s \subset S$ be the edge containing s lying inside I . See Figure 9. For $f \in \text{Stab}(\nu)$, let $\Gamma_f \subset \Gamma_0$ be the cycle that is the link of ν in \mathbf{E}_f , and let $s_f, q_f, e_f^s \subset \Gamma_f$ be the preimages under ρ_+ of s, q, e^s , respectively.

Example 5.11. (a) Let

$$h_0 = (x_1, x_2 - cx_3^p, x_3), \quad h_2 = (x_1, x_2 + cx_3^p, x_3),$$

$$h_1 = (x_1 + x_2^{a'} P'(x_2, x_3), x_2, x_3), \quad h_3 = (x_1 + x_2^a P(x_2, x_3), x_2, x_3),$$

where $c, P, P' \neq 0$. Suppose that we have $h_0 \cdots h_3 = \text{id}$, and for $i = 0, \dots, 3$, let $f_i = h_0 \cdots h_i$, where $f_3 = \text{id}$. By Corollary 3.7, we obtain a cycle in the link of ν decomposing into paths in Γ_{f_i} , examples of which are illustrated in Figure 10(a), left and right. We proved in [LP21, page 40] that we have $p(a + a') \leq m$ and consequently the cycle has length $\geq 2\pi$ [LP21, lem 5.9]. Equality cases include, for example for $m = 10p$, the configurations where $c = 1$, and $a = 10, P = 1, a' = 0$ (illustrated on the right), or $a = 3, P = (x_2 + x_3^p)^7, a' = 7$, (illustrated on the left).

(b) Let $m = p + \beta$ with $0 < \beta < p$, and let $\alpha = (m, p, 1)$. Let

$$h_0 = (x_1, x_2 - c'x_3^p, x_3), \quad h_2 = (x_1, x_2 + (c' - c)x_3^p, x_3), \quad h_4 = (x_1, x_2 + cx_3^p, x_3),$$

$$h_1 = (x_1 - bcx_2x_3^\beta, x_2, x_3), \quad h_3 = (x_1 + bc'x_2x_3^\beta, x_2, x_3), \quad h_5 = (x_1 + b(c - c')x_2x_3^\beta, x_2, x_3),$$

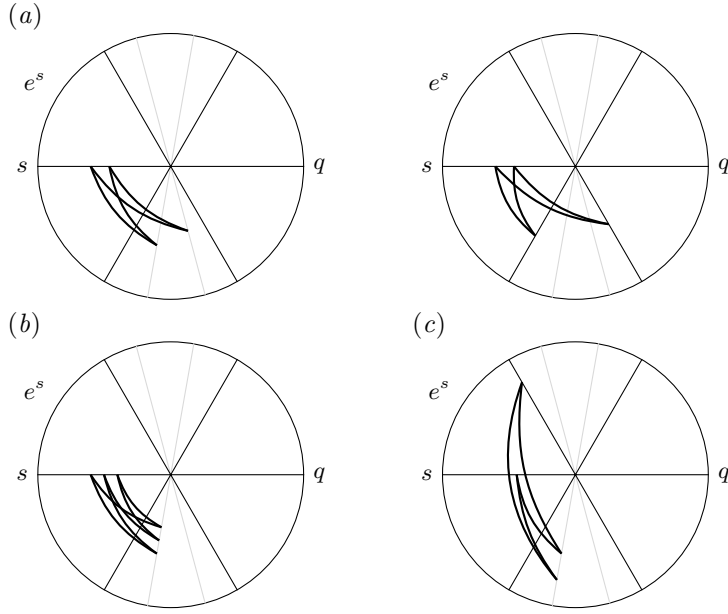


FIGURE 10.

where $c, c', c' - c, b \neq 0$. Note that we have $h_0 \cdots h_5 = \text{id}$. For $i = 0, \dots, 5$, let $f_i = h_0 \cdots h_i$, where $f_5 = \text{id}$. By Corollary 3.7, we obtain a cycle in the link of ν decomposing into paths in Γ_{f_i} illustrated in Figure 10(b).

(c) As in (b), let $m = p + \beta$ with $0 < \beta < p$, and let $\alpha = (m, p, 1)$. Let

$$h_0 = (x_1 - bcx_3^m, x_2 - cx_3^p, x_3), \quad h_2 = (x_1, x_2 + cx_3^p, x_3),$$

$$h_1 = (x_1 + bx_2x_3^\beta, x_2, x_3), \quad h_3 = (x_1 - bx_2x_3^\beta, x_2, x_3),$$

where $c, b \neq 0$. Note that we have $h_0 \cdots h_3 = \text{id}$. For $i = 0, \dots, 3$, let $f_i = h_0 \cdots h_i$, where $f_3 = \text{id}$. By Corollary 3.7, we obtain a cycle in the link of ν decomposing into paths in Γ_{f_i} illustrated in Figure 10(c).

Proposition 5.12. *There is $\varepsilon_0 > 0$ independent of α such that each cycle in Γ_0 of length $< 2\pi + \varepsilon_0$*

- equals Γ_f , or
- is the closure of $(\Gamma_f \cup \Gamma_g) \setminus (\Gamma_f \cap \Gamma_g)$, or
- is a cycle in Example 5.11(a), or
- is the cycle in Example 5.11(b), or
- is the cycle in Example 5.11(c),

up to a translation by an element in $\text{Stab}(\nu)$ in the last three bullet points.

Proof. Let $\gamma \subset \Gamma_0$ be a cycle of length $< 2\pi + \frac{2\pi}{3}$. We orient each edge of γ with interior not containing an s_f or a q_f , so that its image in S under ρ_+ is oriented from q to s . We divide each edge of γ containing an s_f (respectively, q_f) into two edges oriented towards s_f (respectively, away from q_f). As in the previous subsections, γ decomposes into an even number j of oriented paths each of which ends with an s_f . Each such path is contained in an Γ_g and contributes at least $\frac{\pi}{3}$ to the length of γ . Since γ has length $< 2\pi + \frac{2\pi}{3}$, we have $j < 8$ and so $j = 2, 4$, or 6 . The case $j = 2$

implies one of the first two bullet points of the proposition. In the remaining cases we have that $\rho_+(\gamma)$ is disjoint from q .

Consider now the case $j = 4$, with the paths of γ oriented towards s_f and s_g . In the second paragraph of the proof of [LP21, prop 7.12] we observed that if s is a vertex, then e_f^s is the only edge of Γ_0 containing s_f whose image under ρ_+ lies in I . Thus if γ contains e_f^s , then the restriction of ρ_+ to γ is locally injective at s_f . Since $\rho_+(\gamma)$ is disjoint from q , it follows that γ contains also e_g^s . This implies the second bullet point of the proposition. The same conclusion applies when s is not a vertex. If γ contains neither e_f^s nor e_g^s , then it is obtained by concatenating paths in four distinct Γ_f , each of which contains s_f as an endpoint. By [LP21, page 39], this brings us to Example 5.11(a).

Finally, we suppose $j = 6$, with the paths of γ oriented towards s_f, s_g , and s_h . Note that the six paths contribute already 2π to the length of γ , and so for a contradiction we need to find ε_0 bounding the excess contribution from below. Suppose first that γ contains none of e_f^s, e_g^s, e_h^s , as in Figure 10(b). Since s is a vertex, by [LP21, rem 4.9] we have $[\alpha] = [m, p, 1]$ for some integers $m > p > 1$. We have a decomposition $\gamma = \gamma_0 \cdot \gamma_1^{-1} \cdots \gamma_5^{-1}$, where each γ_i lies in Γ_{f_i} , for some $f_i \in \text{Stab}(\nu)$. By [LP21, rem 4.10], points $\rho_+(\gamma_1 \cap \gamma_2), \rho_+(\gamma_3 \cap \gamma_4), \rho_+(\gamma_5 \cap \gamma_0)$ correspond to directions towards $[m - pa, 0, 1], [m - pa', 0, 1], [m - pa'', 0, 1]$ for some $0 \leq a, a', a'' \leq \lfloor \frac{m}{p} \rfloor$.

For $\frac{m}{p} \geq 2$, we will find ε_0 bounding below at least one of the angles between the admissible lines in these directions and the direction towards $[0, 0, 1]$. These angles are equal to the opposite ones: the angles between the admissible lines in the directions towards $[a, 1, 0], [a', 1, 0], [a'', 1, 0]$ and the direction towards $[m, p, 0]$. Thus by Lemma 5.8, we will find appropriate ε_0 as long as there exists $\delta < 1$ independent of m such that $\min\{a, a', a''\} \leq \delta \frac{m}{p}$.

By [LP21, lem 4.11(i)], we can suppose that all f_i lie in M_α . By [LP21, lem 4.12], we have $f_0^{-1}f_1 \sim_\nu h_0, \dots, f_5^{-1}f_0 \sim_\nu h_5$, where $f \sim_\nu g$ means $\Gamma_f = \Gamma_g$, and h_i have the following form:

$$\begin{aligned} h_0 &= (x_1, x_2 - c'x_3^p, x_3), & h_2 &= (x_1, x_2 + c''x_3^p, x_3), & h_4 &= (x_1, x_2 + cx_3^p, x_3), \\ h_1 &= (x_1 + P_1, x_2, x_3), & h_3 &= (x_1 + P_3, x_2, x_3), & h_5 &= (x_1 + P_5, x_2, x_3). \end{aligned}$$

Here $c, c', c'' \neq 0$ and $P_i(x_2, x_3)$ are homogeneous polynomials of weighted degree m with x_2, x_3 of weights $p, 1$. By [LP21, lem 4.11(ii,iii)], the elements $\sim_\nu \text{id}$ of M_α are exactly the elements of M_α of form $(x_1 + P(x_2, x_3), x_2 + Q(x_3), x_3)$, where $-\nu(P) < m$, $\deg Q = -\nu(Q) < p$, and they form a normal subgroup $N_\alpha < M_\alpha$. Consequently, we have $h_0 \cdots h_5 \sim_\nu \text{id}$ and so $h_0 \cdots h_5$ has the above form. Since P_i are homogeneous of degree m , P is also homogeneous of degree m and so we have $c'' = c' - c, P = 0$ and

$$P_1(x_2 + c'x_3^p, x_3) + P_3(x_2 + cx_3^p, x_3) + P_5(x_2, x_3) = 0.$$

By [LP21, lem 4.12(i)], we have that P_1, P_3, P_5 are divisible by $x_2^a, x_2^{a'}, x_2^{a''}$, respectively. Thus we can apply Lemma 4.3 with $r = a'', t = a', u = a$, and set $\delta = \frac{3}{4}$.

For $\frac{m}{p} < 2$, if one of a, a', a'' equals 0, then $\delta = \frac{3}{4}$ works as well. Otherwise, we have $a = a' = a'' = 1$, which by Remark 4.4 brings us exactly to Example 5.11(b).

Since $\rho_+(\gamma)$ is disjoint from q , it passes through e^s an even number of times. Thus it remains to consider the case where γ contains e_f^s, e_g^s , but not e_h^s , as in Figure 10(c). Then we can assume $\gamma_0 \cdot \gamma_1^{-1} \subset \Gamma_{f_0}, \gamma_2 \cdot \gamma_3^{-1} \subset \Gamma_{f_1}, \gamma_4 \subset \Gamma_{f_2}, \gamma_5 \subset \Gamma_{f_3}$. By [LP21, rem 4.10], points $\rho_+(\gamma_1 \cap \gamma_2), \rho_+(\gamma_3 \cap \gamma_4), \rho_+(\gamma_5 \cap \gamma_0)$ correspond to the

directions towards $[a, 1, 0]$, $[m - pa', 0, 1]$, $[m - pa'', 0, 1]$ for some $0 \leq a, a', a'' \leq \lfloor \frac{m}{p} \rfloor$. By Lemma 5.8, for $\frac{m}{p} \geq 2$ we will find appropriate ε_0 as long as there exists $\delta < 1$ independent of m such that $\min\{\frac{m}{p} - a, a', a''\} \leq \delta \frac{m}{p}$.

By [LP21, lem 4.12], we have $f_0^{-1}f_1 \sim_\nu h_0, \dots, f_3^{-1}f_0 \sim_\nu h_3$, where

$$\begin{aligned} h_0 &= (x_1 + P_0, x_2 + c'x_3^p, x_3), & h_2 &= (x_1, x_2 + cx_3^p, x_3), \\ h_1 &= (x_1 + P_1, x_2, x_3), & h_3 &= (x_1 + P_3, x_2, x_3), \end{aligned}$$

where $c, c' \neq 0$ and $P_i(x_2, x_3)$ are homogeneous polynomials of weighted degree m with x_2, x_3 of weights $p, 1$. The identity $h_0 \cdots h_3 \sim_\nu \text{id}$ yields $c' = -c$ and

$$P_0(x_2 + cx_3^p, x_3) + P_1(x_2 + cx_3^p, x_3) + P_3(x_2, x_3) = 0.$$

By [LP21, lem 4.12(i,iii)] we have that P_0, P_1, P_3 are divisible by $x_3^{m-pa}, x_2^{a'}, x_2^{a''}$, respectively. Thus we can apply Lemma 4.5 with $r = a'', t = a', u = m - pa$, and set $\delta = \frac{4}{5}$.

For $\frac{m}{p} < 2$, if one of a', a'' equals 0 or $a = 1$, then $\delta = \frac{4}{5}$ works as well. Otherwise, we have $a' = a'' = 1$, and $a = 0$ which by Remark 4.6 brings us exactly to Example 5.11(c). \square

6. DIAGRAMS

In the entire section, let X be a simply connected combinatorial 2-complex (see Section 3.3 for the definition). A *disc* (resp. *half-plane*) *diagram* D is a combinatorial complex homeomorphic to a 2-disc (resp. a half-plane). A combinatorial map $\phi: D \rightarrow X$ from a disc (resp. half-plane) diagram to X is a *disc* (resp. *half-plane*) *diagram in* X . We say that ϕ is *reduced* if it is a local embedding at open edges. The restriction of ϕ to ∂D is the *boundary* of ϕ .

Remark 6.1. Given an embedded combinatorial closed path γ in X , there exists a reduced disc diagram $\phi: D \rightarrow X$ with boundary γ . This is essentially van Kampen's lemma [Kam33], see [JŠ06, Lem 1.6] for a complete proof for the case of X a simplicial complex, which implies the case of arbitrary X .

6.1. Relative disc diagrams. Often, we will consider the following diagrams that are in general not combinatorial.

Definition 6.2. Let D be a disc diagram with a subcomplex $I \subseteq \partial D$. A map $\phi: D \rightarrow X$ is a *disc diagram in* X *relative to* I (or, simply, a *relative disc diagram in* X), if:

- the restriction of ϕ to any open cell σ_D of D is an embedding into an open cell σ_X of X ,
- this restriction is a homeomorphism onto σ_X if the closure $\bar{\sigma}_D$ is disjoint from I ,
- if σ_D is an edge not contained in I , then σ_X is an edge.

Analogously we define a *half-plane diagram in* X *relative to* $I = \partial D$, or, simply, a *relative half-plane diagram in* X . Again, a relative disc or half-plane diagram ϕ is *reduced* if it is a local embedding at open edges.

Lemma 6.3. *Let γ be an embedded closed path in X , intersecting X^1 , and that can be subdivided into finitely many segments, the interior of each of which lies in a single open cell of X . Then there exists a reduced relative (to ∂D) disc diagram $\phi: D \rightarrow X$ with boundary γ .*

In the proof we will need:

Remark 6.4. Let σ be an open 2-disc, and let $g: \sigma \rightarrow \sigma$ be a local embedding (hence a local homeomorphism) that is proper, i.e. that extends to a continuous map on the one-point compactification S^2 of σ mapping the point $S^2 \setminus \sigma$ to itself. Then g is a covering map, and hence a homeomorphism. This implies the following.

Let $\bar{\sigma} = \sigma \cup \partial\sigma$ be a closed 2-disc, with $\partial\sigma$ a concatenation of paths $\beta_1 \cdots \beta_k$. Let $g': \sigma \rightarrow \sigma$ be a local embedding that extends to a continuous map \bar{g}' from $\bar{\sigma}$ to S^2 mapping only the endpoints of β_j to $S^2 \setminus \sigma$. Suppose that \bar{g}' is a local embedding at each interior point of each β_j and that the restriction of \bar{g}' to the interior of each β_j is an embedding. Then the map g' is an embedding. Indeed, we can extend g' to g above by attaching to σ additional discs along β_j and mapping them under g to the discs in S^2 bounded by $\bar{g}'|_{\beta_j}$.

Proof of Lemma 6.3. We subdivide X along γ . The vertices of the new combinatorial structure on X are contained in the original 1-skeleton. By Remark 6.1, there is a reduced disc diagram $\phi: D \rightarrow X$ with boundary γ , with respect to the new combinatorial structure. Assume that D has the minimal number of 2-cells among all such disc diagrams.

We will now discuss the original combinatorial structure on X . First, we remove from the combinatorial structure of D all the edges e of D outside ∂D with $\phi(e) \not\subset X^1$.

Second, let v be an interior vertex of D with $\phi(v) \notin X^0$, hence $\phi(v) \in X^1 \setminus X^0$. After possibly cutting and reglueing along distinct edges starting at v with the same image under ϕ (after which ϕ is still reduced by the minimality assumption) we can assume that ϕ is a local embedding at v . Consequently, there are exactly two edges of D containing v , and they combine to an edge embedding under ϕ in an edge of X . We then remove v from the combinatorial structure of D . We repeat this procedure for all such v .

The map ϕ is a local embedding on the resulting cells of D . Note that each cell of D is simply connected since otherwise it would have ∂D as a boundary component, which would contradict the assumption that γ intersected X^1 . By Remark 6.4, we have that ϕ is a disc diagram relative to ∂D . \square

Note that in this article we will only be applying Lemma 6.3 to γ satisfying Theorem 6.10, where the cutting and reglueing procedure above is not necessary.

Definition 6.5. Suppose that X has a *piecewise smooth Euclidean metric*. This means that each 2-cell of X is equipped with the Riemannian metric of a subset of the Euclidean plane bounded by a piecewise smooth closed path, with nonzero interior angles, and that these Riemannian metrics agree on common edges. We also assume that for each vertex v of X , all the cells of X containing v have only finitely many possible isometry types, and so in particular X is complete (see e.g. [NOP21, §1]).

We define the links of the vertices in X as in Section 5. Occasionally, we will discuss the links of points that are not vertices. The link of a point x in the interior of an edge e has two vertices corresponding to the two components of $e \setminus x$, and edges of length π corresponding to the 2-cells containing e . The link of a point in the interior of a 2-cell is a circle of length 2π .

Let $\phi: D \rightarrow X$ be a reduced disc or half-plane diagram relative to I . If the restriction of ϕ to each edge $\sigma_D \subset I$ with σ_X a 2-cell is a geodesic in σ_X , then

we can pull back the piecewise smooth Euclidean metric from X to a *degenerate* piecewise smooth Euclidean metric on D . This means that we allow the angles in D to be zero.

Furthermore, if X is CAT(0), then by [NOP21, §2], we have that D satisfies the local conditions of [BB96, Thm 7.1] implying CAT(0) in the non-degenerate case. Since the only angles in D that are possibly equal zero are formed by pairs of edges one of which lies in ∂D and has zero geodesic curvature, it is easy to find an embedding of D as a convex subspace of D with a (non-degenerate) piecewise smooth Euclidean metric also satisfying the conditions of [BB96, Thm 7.1]. Consequently, D is CAT(0) with both metrics.

We define the link of a vertex of such D as in the non-degenerate case, but we do not include the edges corresponding to the zero angles and we do not include their vertices corresponding to the edges in ∂D .

We say that a disc or half-plane diagram D is *locally Euclidean* if each interior point of D has a neighbourhood isometric to a Euclidean disc.

Remark 6.6. Suppose that X as in Definition 6.5 is CAT(0).

- (i) If $\phi: D \rightarrow X$ is a reduced relative disc diagram that is locally Euclidean, then ϕ is a local isometric embedding at each interior point of D (see e.g. the second to last paragraph of the proof of [OP21b, Lem 4.1]). In particular, if D is isometric to a convex subset of \mathbb{R}^2 , then by [BH99, II.4.14] ϕ is an isometric embedding, and so $\phi(D)$ is convex.
- (ii) Let $D^\pm \subset X$ be embedded relative disc diagrams, and let R be a connected component of $D^+ \cap D^-$. Suppose that D^\pm are isometric to convex subsets of \mathbb{R}^2 . Then by (i) we have that R is convex in each D^\pm .
- (iii) Generalising (i), let $R_1, R_2 \subset X$ be embedded relative disc diagrams isometric to convex subsets of \mathbb{R}^2 . Suppose that for each point x of $R_1 \cup R_2$, the set \mathcal{U} of the directions at x of the geodesics in $R_1 \cup R_2$ has diameter $\leq \pi$ in the path metric induced on \mathcal{U} from the link of x . (We refrain from calling \mathcal{U} the link of $R_1 \cup R_2$, since the piecewise Euclidean metric on $R_1 \cup R_2$ might be degenerate and $R_1 \cup R_2$ is not a disc diagram.) We claim that then $R_1 \cup R_2$ is locally convex (hence convex) in X .

Indeed, this follows from the fact that for $x_i \in R_i \setminus R_{3-i}$, the geodesic between x_1 and x_2 in $R_1 \cup R_2$, which passes through $x \in R_1 \cap R_2$, must be a geodesic in X , since otherwise the geodesic triangle $x_1 x x_2$ would have Alexandrov angle $< \pi$ at x . By the diameter hypothesis and [NOP21, Lem 2.1], the shortest path between the directions corresponding to $x x_1$ and $x x_2$ in the link of x in X would have to be contained in \mathcal{U} , contradicting the assumption that $x_1 x x_2$ is a geodesic in $R_1 \cup R_2$.

6.2. Flat lemma. A *flat* in X is an isometrically embedded Euclidean plane in X .

Lemma 6.7. *Suppose that X as in Definition 6.5 is CAT(0). Then for each circle ω of length 2π isometrically embedded in $\partial_\infty X$ with the angle metric, there is a flat $F \subset X$ with $\partial_\infty F = \omega$.*

Note that for X proper, Lemma 6.7 is [Lee00, Prop 2.1].

Proof. Choose a basepoint $x_0 \in X$ and seven geodesic rays $r_0, r_1, \dots, r_6: [0, \infty) \rightarrow X$ starting at x_0 representing consecutive points on ω at distance $\frac{2\pi}{7}$ in $\partial_\infty X$. Furthermore, for each integer $n \geq 0$, and $i = 0, \dots, 6$, let γ_i^n be the geodesic in X between $r_i(n)$ and $r_{i+1}(n)$, where $r_7 = r_0$.

Consider a closed path γ of the form $r_{i+1}[n+1, n] \cdot (\gamma_i^n)^{-1} \cdot r_i[n, n+1] \cdot \gamma_i^{n+1}$, for some $i = 0, \dots, 6$, and $n \geq 0$. Let B_i^n be the closed ball of radius $1 + |\gamma_i^n|/2$ centred at the midpoint of γ_i^n . Since X is CAT(0), we have that B_i^n is convex. Since B_i^n contains $r_{i+1}[n+1, n] \cdot (\gamma_i^n)^{-1} \cdot r_i[n, n+1]$, it contains the entire γ . By [OP21b, Rem 2.4], we have that γ is contained in the union of finitely many cells of X . Thus there is a union K_i^n of finitely many cells of X intersecting B_i^n , such that γ is contained in K_i^n and contractible in K_i^n .

By [BH99, II.9.8(4)], for each i we have $\frac{|\gamma_i^n|}{n} \rightarrow 2 \sin \frac{\pi}{7} < 1$. Consequently, the distance from B_i^n to x_0 becomes arbitrary large as $n \rightarrow \infty$. In particular, each vertex of X belongs to only finitely many K_i^n and so the union K of all K_i^n is proper. Since X is 2-dimensional, K is locally CAT(0) by [NOP21, §2] and [BB96, Thm 7.1].

Let $\tilde{K} \rightarrow K$ be the universal cover of K , which is a complete proper CAT(0) space. Let $\tilde{x}_0 \in \tilde{K}$ be a lift of x_0 , and for $i = 0, \dots, 6$, let \tilde{r}_i be the lift of r_i to \tilde{K} starting at \tilde{x}_0 . Since we included K_i^n in K , the geodesics γ_i^n lift as well to geodesics $\tilde{\gamma}_i^n$ connecting $\tilde{r}_i(n)$ and $\tilde{r}_{i+1}(n)$. Thus by [BH99, II.9.8(4)] we obtain that consecutive \tilde{r}_i represent points $[\tilde{r}_i] \in \partial_\infty \tilde{K}$ at distance $\frac{2\pi}{7}$. Note that since $\tilde{K} \rightarrow K \rightarrow X$ does not increase distances, we have that the distance between each $[\tilde{r}_i]$ and $[\tilde{r}_{i+2}]$ is $\frac{4\pi}{7}$. Consequently, the geodesics joining consecutive $[\tilde{r}_i]$ in $\partial_\infty \tilde{K}$ form a locally (hence globally, since $\partial_\infty \tilde{K}$ is CAT(1)) isometrically embedded circle $\omega_K \subset \partial_\infty \tilde{K}$ of length 2π . By [Lee00, Prop 2.1], there is a flat $F \subset \tilde{K}$ with boundary ω_K . By Remark 6.6(i), $F \rightarrow \tilde{K} \rightarrow K \rightarrow X$ is an isometric embedding. The boundary of the image of F in X contains $[r_i]$, and thus coincides with ω . \square

6.3. Overlaps and unions. Let $\phi^\pm: D^\pm \rightarrow X$ be two embedded (and hence reduced) relative disc diagrams. Suppose that we have a connected subset $\beta \subset \phi^+(D^+) \cap \phi^-(D^-)$. The *overlap of ϕ^\pm at β* is the connected component R of $\phi^+(D^+) \cap \phi^-(D^-)$ containing β . Denote $R^\pm = (\phi^\pm)^{-1}(R) \subseteq D^\pm$.

Let \hat{D} be the space obtained from the disjoint union of $\overline{D^+ \setminus R^+}$ and $\overline{D^- \setminus R^-}$ by identifying the subsets $P^+ \subseteq \text{fr } R^+$ (the topological boundary of R^+ in D^+) and $P^- \subseteq \text{fr } R^-$ consisting of points with the same image under ϕ^+ and ϕ^- . In other words, we have $P^\pm = \text{fr } R^\pm \cap (\phi^\pm)^{-1} \phi^\mp(\text{fr } R^\mp)$.

The *union of ϕ^\pm at β* is the map $\hat{\phi}: \hat{D} \rightarrow X$ whose restriction to $\overline{D^\pm \setminus R^\pm}$ equals ϕ^\pm . If each pair $(\overline{D^\pm \setminus R^\pm}, P^\pm)$ is *standard*, i.e. homeomorphic to the pair consisting of the closed unit disc and the closed unit half-circle, then \hat{D} is homeomorphic to a closed disc and $\hat{\phi}$ is a local embedding but possibly not an embedding. See Figures 11 and 12.

Note that the following lemma is trivial for ϕ^\pm combinatorial.

Lemma 6.8. *Let $\phi^\pm: D^\pm \rightarrow X$ be embedded disc diagrams relative to $I^\pm \subset \partial D^\pm$. Let β be a connected component of $\phi^+(\partial D^+) \cap \phi^-(\partial D^-)$ with $J^\pm = (\phi^\pm)^{-1}(\beta) \subseteq I^\pm$. Let $\hat{\phi}: \hat{D} \rightarrow X$ be the union of ϕ^\pm at β . Suppose that each pair $(\overline{D^\pm \setminus R^\pm}, P^\pm)$ is standard, that $\hat{\phi}(\partial \hat{D})$ intersects X^1 , and that the following condition (*) holds for w^\pm the first or the last points of P^\pm :*

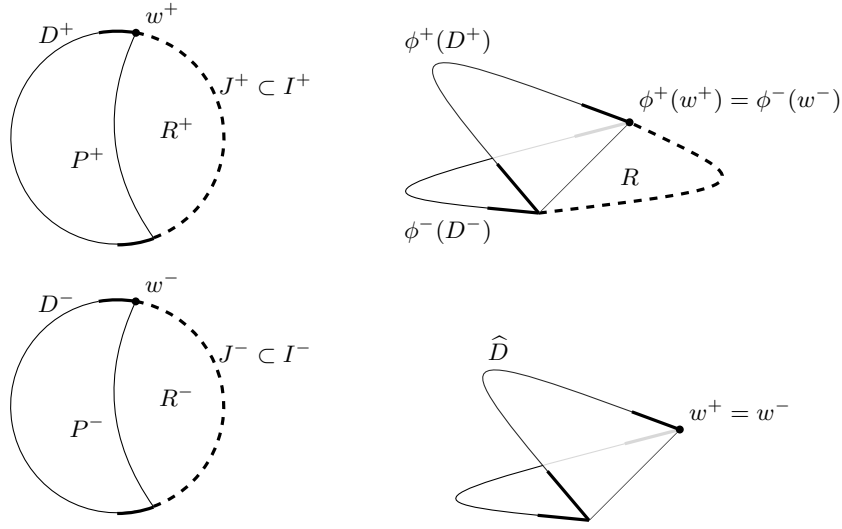


FIGURE 11. A first possibility for $\phi^+(D^+)$, $\phi^-(D^-)$, and \widehat{D} , where $P^\pm = \text{fr } R^\pm$. Examples of the sets I^\pm discussed in Lemma 6.8 are shown in bold.

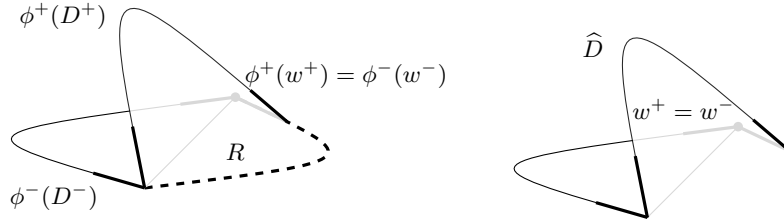


FIGURE 12. A second possibility for $\phi^+(D^+)$, $\phi^-(D^-)$, and \widehat{D} , where $P^+ \subsetneq \text{fr } R^+$. Here a subset of I^- becomes contained in $\overline{D^+ \setminus R^+} \subset \widehat{D}$.

We have $\phi^+(w^+) = \phi^-(w^-) \in X^1$, or, for the maximal subpaths f^\pm in $\partial(D^\pm \setminus R^\pm) \setminus P^\pm$ incident to w^\pm mapped under ϕ^\pm to the same open 2-cell of X as w^\pm , the path $\phi^+(f^+) \cdot \phi^-(f^-)$ is embedded in X .

Then \widehat{D} can be given a combinatorial structure such that $\widehat{\phi}$ is a disc diagram relative to the closure \widehat{I} of the union of $I^\pm \setminus J^\pm$.

Proof. We start with the combinatorial structure on each D^\pm subdivided along $\text{fr } R^\pm \subset D^\pm$, and at the endpoints of P^\pm . Let \widehat{D}_0 be the disc diagram obtained from glueing D^\pm along J^\pm , and let $\widehat{\phi}_0: \widehat{D}_0 \rightarrow X$ be the map with restrictions ϕ^\pm to D^\pm . Note that $\partial\widehat{D}_0$ is the closure of the union of $\partial D^\pm \setminus J^\pm$ and contains \widehat{I} . Furthermore, \widehat{D} and $\widehat{\phi}$ are obtained from \widehat{D}_0 and $\widehat{\phi}_0$ by cancelling the 2-cells of R^\pm , which gives a natural embedding $\widehat{I} \subseteq \partial\widehat{D}$. Let \widehat{J} be the subset of J^\pm remaining in \widehat{D} .

Note that the restrictions of ϕ^\pm to $\overline{D^\pm \setminus R^\pm}$, which are also restrictions of $\widehat{\phi}$, are disc diagrams in X relative to the intersection of their boundary with $\widehat{J} \cup \widehat{I}$. This

follows from the fact that the remaining part of $\overline{\partial D^\pm} \setminus R^\pm$ is mapped under ϕ^\pm into X^1 by the definition of R^\pm . However, $\widehat{\phi}$ might no longer be a relative disc diagram since the cells of \widehat{J} become interior in \widehat{D} . To correct this, we remove from the combinatorial structure of \widehat{D} the edges and the interior vertices contained in \widehat{J} that are not mapped under $\widehat{\phi}$ into X^1 .

After this correction, the restriction g' of the map $\widehat{\phi}$ to each open 2-cell $\sigma_{\widehat{D}}$ of \widehat{D} has still image in a single open cell of X . Note that $\sigma_{\widehat{D}}$ is simply connected since otherwise it would have $\partial\widehat{D}$ as a boundary component, which would contradict the assumption that $\widehat{\phi}(\partial\widehat{D})$ intersected X^1 . Since ϕ^\pm were relative disc diagrams, by the definition of R^\pm we have that g' is a local embedding. Since ϕ^\pm were embeddings, we can apply Remark 6.4 to g' , possibly using condition (*) if one of $\beta_j = f^+ \cdot f^-$. This implies that g' is an embedding, and even a homeomorphism for $\overline{\sigma_{\widehat{D}}}$ disjoint from \widehat{I} .

Finally, we remove from the combinatorial structure of \widehat{D} the interior vertices that are not mapped under $\widehat{\phi}$ into X^0 , and possibly the endpoints of P^\pm if they do not belong to \widehat{I} and are not mapped under $\widehat{\phi}$ into X^0 . Then $\widehat{\phi}$ becomes a disc diagram relative to \widehat{I} . \square

6.4. Curvature. Let D be a disc diagram with a degenerate piecewise smooth Euclidean metric. For a vertex $v \in D$, the *angle* at v is the sum of the Riemannian angles at v of all 2-cells of D containing v . We analogously define *angles* at v between a pair of edges containing v . If $v \in \text{int } D$, then its *curvature* $\kappa(v)$ is 2π minus its angle. If $v \in \partial D$, then its *boundary curvature* $\kappa_\partial(v)$ is π minus its angle. If e is an edge of D , then its *curvature* $\kappa(e)$ is the integral of its geodesic curvature in the (one or two) 2-cells of D containing it. In particular, if e is a Riemannian geodesic in the 2-cells containing it, then $\kappa(e) = 0$. From adding up the classical Gauss–Bonnet formula for the 2-cells of D , we obtain the following (see e.g. [BB96, §2.3]).

Theorem 6.9. *Let D be a disc diagram. Then*

$$\sum_{v \in \text{int } D} \kappa(v) + \sum_{v \in \partial D} \kappa_\partial(v) + \sum_{e \subset D} \kappa(e) = 2\pi.$$

We have the following consequence of [Sta21, Thm 2]. For the reader's convenience, we include the proof in our 2-dimensional setup. A similar proof was earlier provided in [Bad22, Thm 12.1] in the case where each edge e of X is a Riemannian geodesic in the 2-cells containing it.

Theorem 6.10. *Suppose that X is CAT(0). Let $\gamma: S^1 \rightarrow X$ be an embedded closed path consisting of 3 or 4 geodesic segments. Then each reduced relative disc diagram $\phi: D \rightarrow X$ with boundary γ is embedded.*

Proof. Step 1. ϕ is a local embedding.

Let $V \subset D$ denote the set of the (at most 4) endpoints of the geodesic segments in ∂D . Since X is CAT(0), each vertex $v \in \text{int } D$ and each edge $e \not\subset \partial D$ has nonpositive curvature. Since γ is locally geodesic outside V , each $e \subset \partial D$ or $v \in \partial D \setminus V$ has also nonpositive (boundary) curvature. Finally, each vertex of V has curvature $\leq \pi$. (In fact, it is $< \pi$ or one of the edges containing this vertex has negative curvature.) Let \mathcal{K} denote the sum of all these curvatures.

Let $e \neq f$ be oriented edges starting at a vertex v of D , and suppose $\phi(e) = \phi(f)$. If $v \in \text{int } D$, then, since X is CAT(0), both angles at v between e and f are $\geq 2\pi$, so the angle at v is $\geq 4\pi$. Thus $\kappa(v) \leq -2\pi$, and so $\mathcal{K} < 4\pi - 2\pi$, which contradicts Theorem 6.9. If $v \in \partial D \setminus V$, then let $\theta_1 + \theta_2 + \theta_3$ be the angle at v , decomposed to indicate the angles at v between $\partial D, e, f$, and ∂D again. Since X is CAT(0), we have $\theta_2 \geq 2\pi$. Since $v \in \partial D \setminus V$, we have $\theta_1 + \theta_3 \geq \pi$. Consequently, $\kappa_{\partial}(v) \leq -2\pi$, which is again a contradiction. Finally, if $v \in V$, then $\kappa_{\partial}(v) \leq -\pi$, and so $\mathcal{K} < 3\pi - \pi$, contradiction.

Step 2. ϕ is an embedding.

Suppose that $\beta \subset D$ is an embedded combinatorial path from w_1 to w_2 such that the restriction of ϕ to β is an embedding except that $\phi(w_1) = \phi(w_2)$. Let $\psi: D' \rightarrow X$ be a reduced relative disc diagram with boundary $\phi|_{\beta}$, guaranteed by Lemma 6.3. Suppose that among such β and ψ , the disc diagram D' has the minimal possible number of 2-cells.

Let $v' \in \partial D' \setminus \psi^{-1}\phi(w_1)$ be a vertex with $\kappa_{\partial}(v') > 0$ and let e', f' be the edges of $\partial D'$ containing v' . Let $e, v, f \subset \beta$ be such that $\phi(e) = \psi(e'), \phi(v) = \psi(v'), \phi(f) = \psi(f')$. By the minimality assumption, the images of the restrictions of ϕ and ψ to the open neighbourhoods of e, e' in any 2-cells containing e, e' , are distinct, and the same property holds for f, f' . Thus, if $v \in \text{int } D$, then since X is CAT(0), both angles in D between e and f are $\geq \pi + \kappa_{\partial}(v')$, and so $\kappa(v) \leq -2\kappa_{\partial}(v')$. If $v \in \partial D \setminus V$, then $\theta_2 \geq \pi + \kappa_{\partial}(v')$ and $\theta_1 + \theta_3 \geq \kappa_{\partial}(v')$. Consequently, $\kappa_{\partial}(v) \leq -2\kappa_{\partial}(v')$. Finally, if $v \in V$, then $\kappa_{\partial}(v) \leq -\kappa_{\partial}(v') \leq \pi - 2\kappa_{\partial}(v')$.

Let now $e' \subset \partial D'$ be an edge with $\kappa(e') > 0$, and let $e \subset \beta$ be such that $\phi(e) = \psi(e')$. Then $\phi(e)$ is not a geodesic and so $e \not\subset \partial D$. Moreover, since X is CAT(0), we have $\kappa(e) \leq -2\kappa(e')$.

By Theorem 6.9 applied to D' , the sum of all the curvatures of edges $e' \subset \partial D'$ and vertices $v' \in \partial D' \setminus \psi^{-1}\phi(w_1)$ is at least π . By the previous paragraph, each positive contribution $\kappa_{\partial}(v')$ or $\kappa(e')$ to that sum decreases the maximal possible curvature of v, e in D by at least $2\kappa_{\partial}(v')$ or $2\kappa(e')$. Thus $\mathcal{K} < 4\pi - 2\pi$, which is a contradiction. \square

6.5. Frillings. The following operation will be needed to discuss the limits of relative disc diagrams.

Definition 6.11. Let D be a half-plane diagram. A *frilling* of D is a combinatorial complex D^* obtained from D via the following subdivision and quotient map $D \rightarrow D^*$. First, we subdivide some 2-cells of D along new edges (and their vertices) each of which has exactly one endpoint in ∂D . Then, we quotient some of the edges intersecting ∂D to vertices, including all the new edges. Finally, we collapse some of the 2-cells.

Here a *collapse* of an open 2-cell $\sigma \subset D$, such that $\bar{\sigma} \cap \partial D$ is a nontrivial path τ , is a deformation retraction of D onto $D \setminus (\sigma \cup \text{int } \tau)$.

See Figure 13(a), where we subdivide D along the dashed edge, we quotient this edge along with the other red edges, and then we collapse the two indicated 2-cells σ , where the paths τ are indicated in green. The resulting frilling D^* is illustrated in Figure 13(b).

Remark 6.12. We denote by ∂D^* the image of ∂D in D^* . Note that if the preimage in ∂D of each point of ∂D^* is compact, then D^* is a half-plane diagram D' wedged

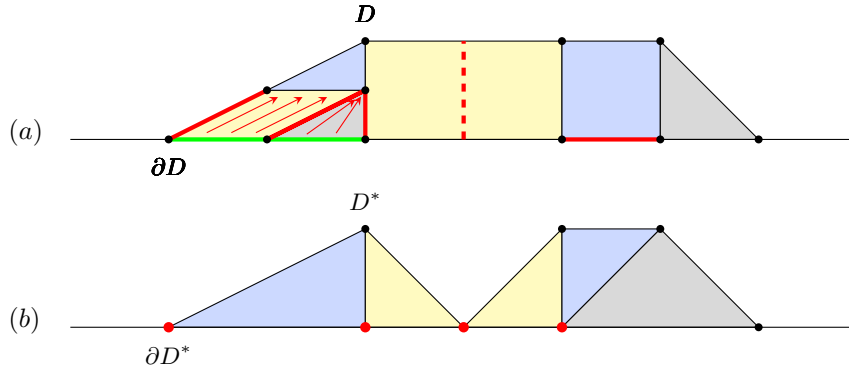


FIGURE 13. An example of a frilling.

at points of $\partial D'$ with combinatorial complexes that are contractible (which may exist only if ∂D^* is not an embedded line) or homeomorphic to 2-spheres. In our applications in further sections we will always have $D' = D^*$.

Construction 6.13. Let D be a half-plane diagram. Suppose that D is an increasing union of disc diagrams D_n with $I_n = D_n \cap \partial D$. We then say that (D_n, I_n) converge to $(D, \partial D)$.

Suppose that additionally we have relative disc diagrams $\phi_n: D_n \rightarrow X$ satisfying the assumptions (i–iii) below. In the construction below we define their *limit* $\phi^*: D^* \rightarrow X$, where D^* is a particular frilling of D . The reason for passing to the frilling is that it will turn out that if the hypothesis of Remark 6.12 holds, then the restriction ϕ' of $\phi^*: D^* \rightarrow X$ to D' is a relative half-plane diagram.

For future reference, whenever we say that $\phi^*: D^* \rightarrow X$ (or $\phi': D' \rightarrow X$) is the *limit* of ϕ_n , then in particular we mean that (D_n, I_n) can be identified (by combinatorial isomorphisms) with subcomplexes of a half-plane diagram D converging to $(D, \partial D)$, that ϕ_n satisfy the assumptions (i–iii), and that D^* (or D') is the particular frilling described below.

The assumptions that we make on ϕ_n are the following.

- (i) The attaching map of each cell in X is injective, and the boundary of each ϕ_n intersects X^1 at least twice (which implies that the attaching map of each cell in D_n is injective, since otherwise the boundary of a 2-cell would contain ∂D and would map into $X \setminus X^1$ except for one point).
- (ii) For each cell σ_D in D , the cell σ_X in Definition 6.2 is the same for all ϕ_n such that σ_D lies in D_n .
- (iii) For each 2-cell σ_D whose closure intersects ∂D , the closed paths $\phi_n|_{\partial\sigma_D}$ converge to a closed path $\phi_{\sigma_D}: \partial\sigma_D \rightarrow \bar{\sigma}_X$. Moreover, for $\phi_{\sigma_D}^*: \partial\sigma_D^* \rightarrow \bar{\sigma}_X$, obtained from ϕ_{σ_D} by quotienting to points the edges of $\partial\sigma_D$ on which ϕ_{σ_D} is constant, we have that
 - the restriction of $\phi_{\sigma_D}^*$ to the subset of $\partial\sigma_D^*$ that is the image of $\partial\sigma_D \cap \partial D$ is an embedding, and
 - $\phi_{\sigma_D}^*(\partial\sigma_D^*)$ bounds a finite (possibly empty) union of open discs in σ_X .

Then the *limit* $\phi^*: D^* \rightarrow X$ is the following map whose domain D^* is the following frilling D^* of D . To start, on each cell σ_D of D disjoint from ∂D (which is thus a cell of D^*) we define ϕ^* to be a homeomorphism onto σ_X . On each remaining edge σ_D

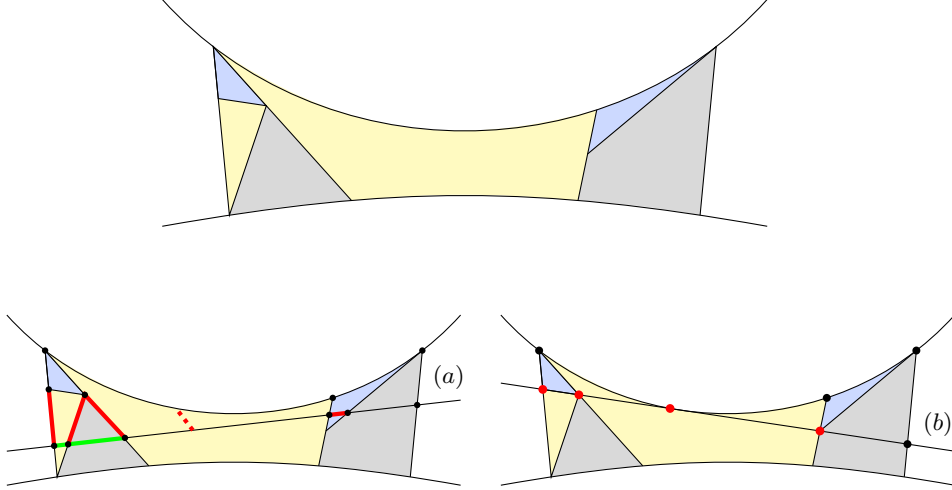


FIGURE 14.

of D not contained in ∂D , we define ϕ^* to be a homeomorphism onto the subset of the edge σ_X bounded by the limits of $\phi_n(\partial\sigma_D)$. In the case where these two limits coincide, we quotient σ_D to a vertex in D^* .

Consider now a 2-cell σ_D of D whose closure intersects ∂D . If $\phi_{\sigma_D}^*$ is an embedding on the entire $\partial\sigma_D^*$, then we define ϕ^* on σ_D to be a homeomorphism onto the subset of the 2-cell σ_X bounded by $\phi_{\sigma_D}^*(\partial\sigma_D^*)$. Then the 2-cell σ_D remains a 2-cell of D^* , and we identify $\partial\sigma_D^*$ with its boundary, and $\phi_{\sigma_D}^*$ with the restriction of ϕ^* to its boundary. Otherwise, we denote by τ_i, τ'_i the maximal subpaths of $\partial\sigma_D \cap \partial D, \overline{\partial\sigma_D} \setminus \overline{\partial D}$ with the same image under ϕ_{σ_D} , for τ_i not an endpoint of a connected component of $\partial\sigma_D \cap \partial D$. We

- subdivide the 2-cell σ_D along edges starting (resp. ending) at an endpoint of τ_i (resp. the corresponding endpoint of τ'_i),
- quotient these edges to vertices, and
- collapse the resulting 2-cells bounded by nontrivial τ_i, τ'_i .

This allows us to extend ϕ^* to an embedding on each of the 2-cells of D^* resulting from σ_D and on the image in $\partial\sigma_D^*$ of each edge of $\partial\sigma_D \cap \partial D$ that was not removed during a collapse. Consequently, if the hypothesis of Remark 6.12 holds, then the restriction ϕ' of $\phi^*: D^* \rightarrow X$ to D' is a relative half-plane diagram.

See Figure 14, where on the top we illustrate a part of the complex X , and where $\phi_1(D_1)$ is located above the straight line $\phi_1(I_1)$ in (a). Choosing ϕ_n so that $\phi_n(I_n)$ converges to the straight line in (b), we have that (D_n, I_n) converge to $(D, \partial D)$ from Figure 13(a), with limit $\phi^* = \phi'$ whose domain is the frilling D^* from Figure 13(b).

7. DIAGRAMS IN \mathbf{X}

In this section we describe the additional structure of the relative disc or half-plane diagrams $D \rightarrow X$ for $X = \mathbf{X}$ or $X = \nabla^+$.

Note that by Lemma 3.3(iii), each geodesic segment γ in \mathbf{X} passes through only finitely many chambers. Since admissible lines form a locally finite set in $(\nabla^+, |\cdot, \cdot|)$ [LP21, lem 4.1], and bound convex half-spaces [LP21, lem 5.1], each geodesic segment in ∇^+ intersects the union of admissible lines in a union of a finite set of points and a finite set of subsegments. Thus there are finitely many connected components

of intersections of γ with the open cells of \mathbf{X} , and Lemma 6.3 can be applied to piecewise geodesic closed paths in \mathbf{X} . Furthermore, the attaching maps of the cells in \mathbf{X} and ∇^+ are injective as required in Construction 6.13(i).

7.1. Limits in ∇^+ .

Remark 7.1. Suppose in Construction 6.13 that $X = \nabla^+$ and all ϕ_n are the compositions of reduced relative disc diagrams $\tilde{\phi}_n$ in \mathbf{X} with ρ_+ . Furthermore, suppose that ∂D is a concatenation $I_a \cdot I_b \cdot I_c$ and we have $\epsilon > 0$ and a positive function $\delta: I_a \cup I_c \rightarrow \mathbb{R}$ such that

- the restrictions of $\tilde{\phi}_n$ to the intersections of I_a, I_b, I_c with I_n are geodesics concatenated at an Alexandrov angle $\geq \epsilon$, and
- for each n , and each $t \in I_a \cap I_n$, we have $d_{\mathbf{X}}(\tilde{\phi}_n(t), \tilde{\phi}_n(I_c \cap I_n)) \geq \delta(t)$, and the same condition holds if we interchange I_a with I_c .

Then:

- For each 2-cell σ_D , the restriction of ϕ_n to $\partial\sigma_D \cap \partial D$ converges to a disjoint union of finitely many (possibly constant) piecewise geodesics. Thus the maps $\phi_{\sigma_D}^*$ satisfy the assumption of Construction 6.13(iii). Furthermore, the preimage of each point under $\partial D \rightarrow \partial D^*$ is connected.
- All D_n are locally CAT(0), and so D^* is locally CAT(0). Thus, by Theorem 6.9, we have that D^* contains no subcomplexes homeomorphic to 2-spheres.
- Consequently $D' = D^*$, $\phi' = \phi^*$, and $\partial D'$ is a path. Note that $\phi'_{|\partial D'}$ is obtained from $\phi_{|\partial D}$ by quotienting to points the segments on which $\phi_{|\partial D}$ is constant.
- Since there is a uniform bound on the number of 2-cells traversed by a geodesic in \mathbf{X} contained in the star of a vertex v of \mathbf{X} with given $\rho_+(v) \in \nabla^+$ [OP21b, Claim in §2], the preimage of each point under $\partial D \rightarrow \partial D'$ is compact, and so Remark 6.12 applies. We conclude that $\partial D'$ is a line and D' is a half-plane diagram.

Definition 7.2. Let $\phi': D' \rightarrow \nabla^+$ be a relative half-plane diagram that is the limit of ϕ_n that are the compositions with ρ_+ of reduced relative disc diagrams $\tilde{\phi}_n: D_n \rightarrow \mathbf{X}$. Let $J \subset \partial D$ be homeomorphic with $[0, \infty)$.

Let $r: J \rightarrow \mathbf{X}$ be a geodesic ray and suppose that the restrictions of $\tilde{\phi}_n$ to $\partial D_n \cap J$ coincide with the restrictions of r . We then say that ϕ' is *bordered* by r . This terminology is justified since then the sequence ϕ_n is constant on J , and so the map $D \rightarrow D'$ is the identity on J . We may thus identify J with a subset of $\partial D'$, and the restriction of ϕ' to J equals $\rho_+ \circ r$.

Let $r: J \rightarrow \mathbf{X}$ be a geodesic ray disjoint from the vertex set \mathbf{X}^0 and transverse to the 1-skeleton \mathbf{X}^1 . Suppose that the restrictions of $\tilde{\phi}_n$ to $J \cap \partial D_n$ are geodesics that converge to r . We then say that ϕ' is *weakly bordered* by r . In that case, we can arrange ϕ' to be bordered by r after modifying $\tilde{\phi}_n$ on the cells intersecting J .

Below we describe how does the operation of passing to the limit of relative disc diagrams behave with respect to the operation of the union.

Remark 7.3. Suppose that (D_n^\pm, I_n^\pm) converge to half-plane diagrams $(D^\pm, \partial D^\pm)$, with fillings $D^{*\pm} = D'^\pm$ and $\phi'^\pm: D'^\pm \rightarrow \nabla^+$ the limits of the compositions ϕ_n^\pm of

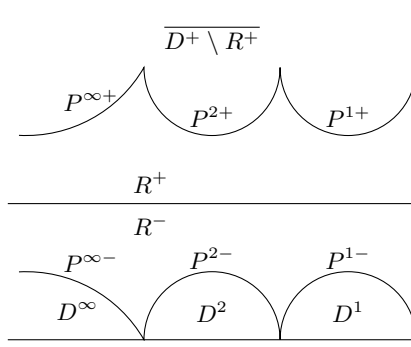


FIGURE 15.

embedded relative disc diagrams $\tilde{\phi}_n^\pm: D_n^\pm \rightarrow \mathbf{X}$ with ρ_+ . Let $J^\pm \subset \partial D^\pm$ be rays with $\beta_n = \tilde{\phi}_n^+(J^+ \cap \partial D_n^+) = \tilde{\phi}_n^-(J^- \cap \partial D_n^-)$ a connected component of the intersection of $\tilde{\phi}_n^\pm(\partial D_n^\pm)$. Let $\hat{\phi}_n: \hat{D}_n \rightarrow \mathbf{X}$ be the unions of $\tilde{\phi}_n^+$ and $\tilde{\phi}_n^-$ at β_n with overlaps R_n .

After passing to subdiagrams and a subsequence, for all n we have $(\tilde{\phi}_n^\pm)^{-1}(R_n) = R^\pm \cap D_n^\pm$ for some fixed $R^\pm \subseteq D^\pm$. Let \hat{D} be the space obtained from the disjoint union of $\overline{D^\pm \setminus R^\pm}$ by identifying the subsets $P^\pm \subseteq \text{fr } R^\pm$ consisting of points with the same image under $\tilde{\phi}_n^\pm$ (this does not depend on n).

Suppose that each pair $(\overline{D^\pm \setminus R^\pm}, P^\pm)$ is homeomorphic to the pair consisting of the closed upper half-plane in \mathbb{R}^2 and the negative real axis, so that we can assume that each $(\overline{D_n^\pm \setminus R_n^\pm}, P_n^\pm)$ is standard. Suppose that the condition $(*)$ of Lemma 6.8 holds for P_n^\pm , and so $\hat{\phi}_n$ are disc diagrams relative to appropriate \hat{I}_n .

Then \hat{D} has a combinatorial structure such that (\hat{D}_n, \hat{I}_n) converge to $(\hat{D}, \partial \hat{D})$. Moreover, we have a frilling $\hat{D} \rightarrow \hat{D}'$ and a relative half-plane diagram $\hat{\phi}': \hat{D}' \rightarrow \nabla^+$ that is the limit of $\rho_+ \circ \hat{\phi}_n$. Note that $\hat{D} \rightarrow \hat{D}'$ coincides with $D^\pm \rightarrow D'^\pm$ on the cells contained entirely in D^+ or D^- , and $\hat{\phi}'$ coincides with ϕ'^\pm on their quotients.

Furthermore, if ϕ'^\pm are both bordered by a geodesic ray with domains J^\pm , and $R^\pm = J^\pm$, then each cell σ of \hat{D}' with $\bar{\sigma}$ disjoint from $\partial \hat{D}'$, considered as a cell of \hat{D} , is a union of cells σ_\sharp of D^\pm , and $\hat{\phi}'|_\sigma = \hat{\phi}|_\sigma$ is the union of $\phi'_{|\sigma_\sharp}^\pm = \phi_{|\sigma_\sharp}^\pm$.

Remark 7.4. More generally, in Remark 7.3 we can suppose that (see Figure 15):

- P^\pm is a (possibly empty or infinite) union of bounded paths $P^{k\pm}$ and an unbounded path $P^{\infty\pm}$, pairwise disjoint except possibly at the endpoints, and
- $\overline{D^- \setminus R^-}$ is a union of disc diagrams D^k , and a half-plane diagram D^∞ , pairwise disjoint except possibly at the endpoints of P^{k-} .

We require that

- each $(\overline{D^+ \setminus R^+}, P^{\infty+}), (D^\infty, P^{\infty-})$ is homeomorphic to the pair consisting of the closed upper half-plane in \mathbb{R}^2 and the negative real axis, and
- each $(\overline{D^+ \setminus R^+}, P^{k+})$ is homeomorphic to the pair consisting of the closed upper half-plane in \mathbb{R}^2 and a bounded interval in its boundary, and
- each (D^k, P^{k-}) is standard.

We can then assume that $(\overline{D_n^\pm \setminus R_n^\pm}, P_n^\pm)$ have analogous structure, where $\overline{D_n^+ \setminus R_n^+}$ and D_n^∞ are disc diagrams. Suppose that the endpoints of P_n^\pm are mapped under $\phi'_{|\sigma}^\pm$.

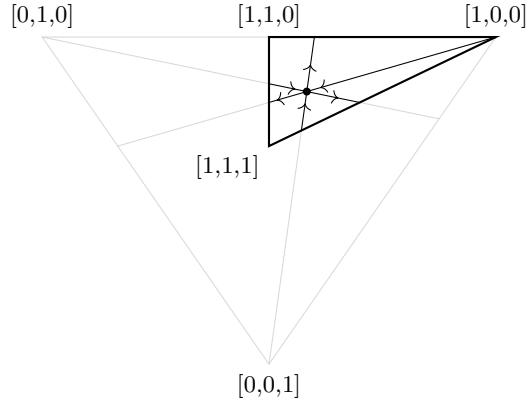


FIGURE 16.

into the 1-skeleton of ∇^+ . Then $\widehat{\phi}_n$ are disc diagrams relative to appropriate \widehat{I}_n and the last two paragraphs of Remark 7.3 hold as well.

7.2. Folding locus.

Definition 7.5. Let $\phi: D \rightarrow \nabla^+$ be a relative disc or half-plane diagram. We say that ϕ is **X-reduced** if at a neighbourhood of each vertex of D it is the composition of a reduced relative disc diagram in **X** and the map ρ_+ .

The *folding locus* $L \subset D$ of ϕ is the closure of the union of open edges e in $D \setminus \partial D$ that

- do not map to $\partial \nabla^+$ under ϕ , but
- at which ϕ is not a local embedding.

If such an edge e is mapped under ϕ into a principal line, then we *orient* e in the direction opposite to a vertex of ∇ (see Figure 16).

For example, since in Remark 7.3 the relative disc diagrams $\widehat{\phi}_n$ are local embeddings, we have that each $\rho^+ \circ \widehat{\phi}_n$, and consequently $\widehat{\phi}'$, is **X-reduced**.

From Propositions 5.1, 5.5, and 5.12 we deduce the following, where ε_0 is the minimum of the constants in Propositions 5.5 and 5.12.

Corollary 7.6. *Let $\phi: D \rightarrow \nabla^+$ be an **X-reduced** relative disc or half-plane diagram and let $v \in D \setminus \partial D$ be a vertex of the folding locus $L \subset D$. Suppose that the angle at v in D is $< 2\pi + \varepsilon_0$. Then the intersection of L with the star of v in D has one of the forms in Figure 17.*

The angles at v in (a),(b),(e) and the two top angles in (f) are the obvious multiples of $\frac{\pi}{3}$. Both angles in (c) are $> \pi$ and in (d) they are $\geq \pi$. The two bottom angles in (f), (g), the two top angles in (h), as well as all the angles in (i) are $> \frac{\pi}{3}$. The two bottom angles in (h) are $> \frac{2\pi}{3}$. The sum of the top and bottom angle at each side of (g) is $\geq \pi$.

Proof. The third and second bullet point in Proposition 5.1 correspond to (a) and (b), respectively. The second bullet point in Propositions 5.5 and 5.12 corresponds to (b), (c), and (d). The third bullet point in Proposition 5.12 corresponds to (e), (f), and (g). The fifth and fourth bullet points in Proposition 5.12 correspond to (h), (i), respectively. \square

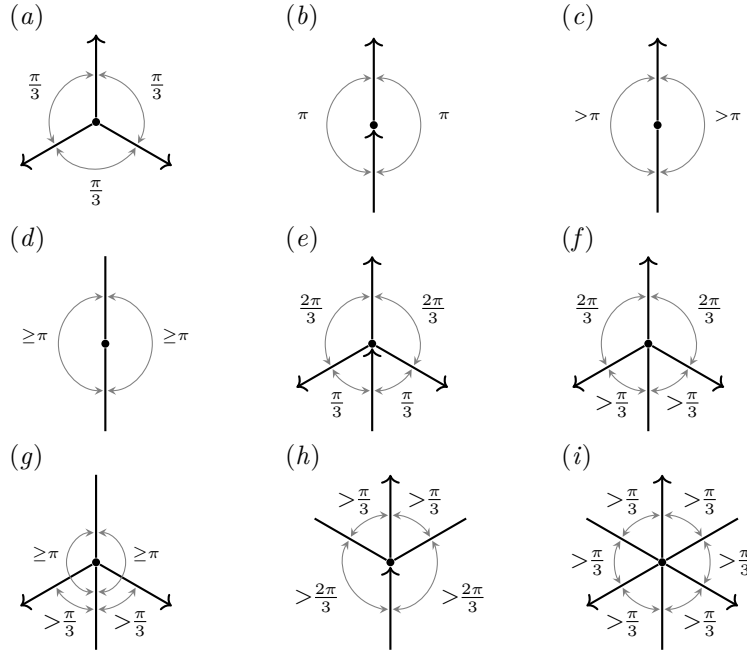


FIGURE 17.

- Remark 7.7.** (i) If in Corollary 7.6 we have an edge e of L containing v that is oriented towards v (resp. non-oriented but not as in Figure 17(h)), then there is an edge e' of L containing v at angle $\geq \pi$ from e that is oriented from v (resp. or non-oriented).
- (ii) Thus if at each interior vertex of D the angle is $< 2\pi + \varepsilon_0$, then each oriented edge of L is contained in a sequence of oriented edges of L forming an oriented geodesic ray or segment ending at ∂D . Furthermore, each non-oriented edge e of L is contained in a geodesic of L whose each end is unbounded or lies at ∂D or at v in Figure 17(h).
- (iii) Let $c \subset \nabla^+$ be the (typically non-geodesic) ray that is a parametrisation of an admissible line $\alpha_1 = m_2\alpha_2 + m_3\alpha_3$. Then there is a subray c' of c such that any other admissible line in ∇^+ with unbounded intersection with $\alpha_1 \geq m_2\alpha_2 + m_3\alpha_3$ is disjoint from c' or has form $\alpha_2 = m\alpha_3$.
- (iv) Consequently, if in part (i) in Figure 17(b,c,d) we have $\phi(e) \subset c'$, with $\phi(v)$ separating $\phi(e)$ from the unbounded end of c' , then $\phi(e')$ is the consecutive edge on c' , and we have Figure 17(b) or (d).

Definition 7.8. For any point $[\alpha] \in \nabla^+$, let $S_{[\alpha]}$ denote its unit tangent space in the Euclidean plane $(\nabla, |\cdot, \cdot|)$, which can be identified with the link of $[\alpha]$ in ∇ . Furthermore, let $S_{[\alpha]}^\angle$ denote the quotient of $S_{[\alpha]}$ by the action of the order 6 dihedral group generated by the reflections in the lines through $[\alpha]$ parallel to the principal lines. Parallel transporting in $(\nabla, |\cdot, \cdot|)$, we can identify all $S_{[\alpha]}$ and all $S_{[\alpha]}^\angle$, which we denote then shortly S and S^\angle . The direction towards (resp. away from) the vertices of ∇ determines the *principal point* (resp. *antiprincipal point*) of S^\angle . The union of these two points is denoted $\partial S^\angle \subset S^\angle$.

Consider now an \mathbf{X} -reduced relative disc or half-plane diagram $\phi: D \rightarrow \nabla^+ \subset \nabla$ with folding locus $L \subset D$. For $x \in D$, let S_x be its link in D , let $\phi_x: S_x \rightarrow S_{\phi(x)}$ be

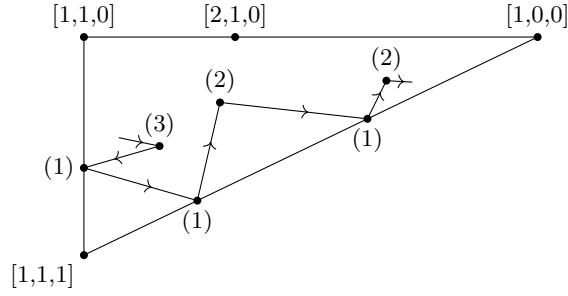


FIGURE 18. Image of a positively oriented ray in ∇^+ . Numbers (1), (2), (3) correspond to the cases in the proof of Lemma 7.9.

the map induced by ϕ between the links, and let ϕ_x^\angle be the composition of ϕ_x with $S_{\phi(x)} \rightarrow S_{\phi(x)}^\angle$.

Suppose now that D , with the degenerate piecewise Euclidean metric pulled back from ∇^+ , is a convex subset of the Euclidean plane \mathbb{R}^2 . For $x \in D$, let \mathbb{S}_x be its unit tangent space in \mathbb{R}^2 . Parallel transporting, we can identify all \mathbb{S}_x , which we denote then as \mathbb{S} . For $x \notin L \cup \phi^{-1}(\partial\nabla^+)$, the map ϕ_x extends to an isometry $\phi_x: \mathbb{S} = \mathbb{S}_x \rightarrow S_{\phi(x)} = S$. Since $D \subset \mathbb{R}^2$, all edges of L are mapped under ϕ to principal lines, and hence their directions are mapped to ∂S^\angle in S^\angle . Thus it can be easily seen that ϕ_x^\angle does not depend on x , and we can denote it ϕ^\angle .

Below, a *positively* (resp. *negatively*) *oriented* geodesic ray in the folding locus consists of edges that are all oriented towards (resp. away from) the unbounded end of the ray.

Lemma 7.9. *Let $\phi: D \rightarrow \nabla^+$ be an \mathbf{X} -reduced relative half-plane diagram with interior vertex angles $< 2\pi + \varepsilon_0$. Let r be a positively oriented geodesic ray in the folding locus $L \subset D$.*

- (i) *Then $\phi(r)$ is unbounded.*
- (ii) *Assume additionally that r is a connected component of L . Then there is M depending only on the first edge of $\phi(r)$, such that for the subray r' obtained from r by removing its initial length M segment, we have that $\phi(r')$ is contained in a principal line.*

The same statements hold for r negatively oriented.

Finally, if r is a geodesic ray in D disjoint from L , with the tangent direction of r at some (hence each) point x of r sent to ∂S^\angle under the map ϕ_x^\angle from Definition 7.8, then $\phi(r)$ is unbounded as well.

Proof. If r is positively oriented, then each edge of $\phi(r)$ is directed away from $[1, 0, 0]$, $[0, 1, 0]$, or $[0, 0, 1]$. The change of the direction might occur only at $\partial\nabla^+$ or at $\phi(v)$ for a vertex v described in Figure 17(e,h). We have the following possibilities, as illustrated in Figure 18:

- (1) At $\partial\nabla^+$, the direction away from $[1, 0, 0]$ changes to the direction away from $[0, 1, 0]$ or the direction away from $[0, 1, 0]$ changes to the direction away from $[0, 0, 1]$.
- (2) At $\phi(v)$ as in Figure 17(e) (which cannot occur under the additional assumption in (ii)), the direction away from $[0, 0, 1]$ changes to the direction away from $[0, 1, 0]$. In that case $\phi(v)$ has $\frac{\alpha_1}{\alpha_2}$ an integer ≥ 2 .

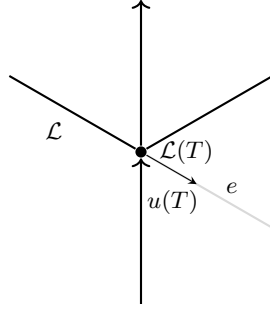


FIGURE 19.

- (3) At $\phi(v)$ as in Figure 17(h) (which also cannot occur in (ii)), the direction away from $[0, 1, 0]$ changes to the direction away from $[1, 0, 0]$. In that case $\phi(v)$ has $\frac{\alpha_1}{\alpha_2} < 2$.

Moving away from $[0, 1, 0]$ increases $\frac{\alpha_1}{\alpha_2}$. Consequently, for a subray $r' \subset r$ each edge of $\phi(r')$ is directed away from $[0, 1, 0]$ (which eventually does not happen in (ii)) or $[0, 0, 1]$. Thus the orthogonal projection w.r.t. $|\cdot, \cdot|$ of $\phi(r)$ onto the principal line $\alpha_2 = \alpha_3$ diverges to ∞ .

In r is negatively oriented, for a subray $r' \subset r$ each edge of $\phi(r')$ is directed towards $[0, 1, 0]$ (which eventually does not happen in (ii)) or $[1, 0, 0]$. Thus the orthogonal projection w.r.t. $|\cdot, \cdot|$ of $\phi(r)$ onto the principal line $\alpha_1 = \alpha_2$ diverges to ∞ .

The last assertion is proved similarly. \square

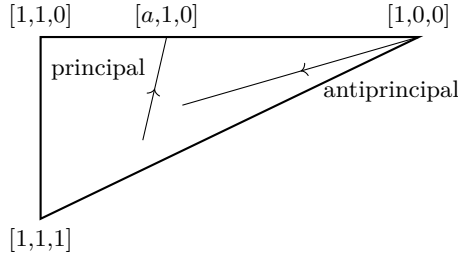
Below, a *non-oriented component* \mathcal{L} of the folding locus L is a connected component of the union of the open non-oriented edges of L and the vertices in Figure 17(d,g).

Lemma 7.10. *There is a continuous function $M: (0, \frac{\pi}{3}) \rightarrow \mathbb{R}$ such that the following holds. Let $\phi: D \rightarrow \nabla^+$ be an \mathbf{X} -reduced relative half-plane diagram with interior vertex angles $< 2\pi + \varepsilon_0$. Let $\mathcal{L} \subset L$ be a non-oriented component, let $x \in \mathcal{L}$, and let $u \in S_x$ be the direction of an edge of \mathcal{L} . If $\phi_x^\angle(u) \in S^\angle$ is at angle ε from the principal point, then the connected component of $\mathcal{L} \setminus x$, to which u is tangent*

- *has length $< M(\varepsilon)$ and ends at ∂D , or*
- *contributes $\leq -\varepsilon$ to the sum of the curvatures of the interior edges and the interior vertices of D .*

Proof. Consider the arc-length parametrisation $\mathcal{L}: [0, T) \rightarrow D$ (possibly $T = \infty$) of the union of x and the connected component of $\mathcal{L} \setminus x$ tangent to u , with $\mathcal{L}(0) = x$. Let $u(t) \in S_{\mathcal{L}(t)}$ be the direction of $\mathcal{L}[t, T)$. In the case where $T < \infty$ and $\lim_{t \rightarrow T} \mathcal{L}(t)$ is a vertex in the interior of D , which we call $\mathcal{L}(T)$, we have the configuration of Figure 17(c,f,h,i). For (c,f,i), we define additionally $u(T) \in S_{\mathcal{L}(T)}$ to be the direction of the edge e of L opposite at $\mathcal{L}(T)$ to the non-oriented edge of \mathcal{L} (Remark 7.7(i)). For (h) we denote by e the edge outside L indicated in Figure 19 whose direction $u(T) \in S_{\mathcal{L}(T)}$ has $\phi_{\mathcal{L}(T)}^\angle(u(T))$ principal.

We claim that $t \rightarrow \phi_{\mathcal{L}(t)}^\angle(u(t)) \in S^\angle$ is a decreasing function, where the interval S^\angle of length $\frac{\pi}{3}$ is parametrised by the angle from the principal point. Furthermore, the sum of the integral of the geodesic curvature and the exterior angles at each side of an interval $\mathcal{L}(t_1, t_2)$ (resp. $\mathcal{L}(t_1, t_2]$, including possibly the exterior angle at $\mathcal{L}(T)$


 FIGURE 20. Principal and antiprincipal rays in a chamber \mathbf{E}_f^+ .

of $\mathcal{L} \cdot e$ for $t_2 = T$), is at least as large as the angle between $\phi_{\mathcal{L}(t_1)}^{\leftarrow}(u(t_1))$ and $\lim_{t' \nearrow t_2} \phi_{\mathcal{L}(t')}^{\leftarrow}(u(t))$ (resp. $\phi_{\mathcal{L}(t_2)}^{\leftarrow}(u(t_2))$).

Indeed, for $\mathcal{L}(t_1, t_2)$ a single edge this follows from the definition of geodesic curvature and the position of non-principal admissible lines in ∇ . Furthermore, for $t \in (t_1, t_2]$ with $\mathcal{L}(t)$ a vertex, we have the configuration of Figure 17(d,g) for $t < T$ or Figure 17(c,f,h,i) for $t = T$. Except for (h,i), in all these cases the angle between $\lim_{t' \nearrow t} \phi_{\mathcal{L}(t')}^{\leftarrow}(u(t'))$ and $\phi_{\mathcal{L}(t)}^{\leftarrow}(u(t))$ is the exterior angle at $\mathcal{L}(t)$ of $\mathcal{L} \cdot e$. In case (i), and on one side in case (h) the exterior angle is even larger. This justifies the claim.

Let now $\mathcal{L}' : (-\infty, +\infty) \rightarrow \nabla$ be any non-principal admissible line, all of which are isometric by Remark 5.10. Let K be the path in \mathcal{L}' of points where the direction of \mathcal{L}' maps under $S \rightarrow S^{\leftarrow}$ to the interval $[\frac{\varepsilon}{2}, \varepsilon]$. Let $M = M(\varepsilon)$ be the length of K .

Without loss of generality, we can assume that $\phi(x)$ is the starting point of K . Furthermore, by the monotonicity in the claim and by Remark 5.10, consecutive edges of $\phi(\mathcal{L}[0, T])$ can be mapped by the translations of ∇ to disjoint (except at the endpoints) edges of \mathcal{L}' , ordered consistently. Consequently, if \mathcal{L} has length $\geq M$, then the minimal subpath K' of \mathcal{L}' containing all of these edges has also length $\geq M$. Then K' contains a point where the direction of \mathcal{L}' maps under $S \rightarrow S^{\leftarrow}$ into $[0, \frac{\varepsilon}{2}]$. By the claim, the sum of the curvatures at each side of \mathcal{L} is $\leq -\frac{\varepsilon}{2}$, as desired. If \mathcal{L} has length $< M$ but ends in the interior of D , then the image under $S \rightarrow S^{\leftarrow}$ of the direction of $\phi(e)$ above is the principal point and so by the claim the sum of the curvatures at each side of \mathcal{L} is $\leq -\varepsilon$. \square

8. RAYS

In this section, we discuss particular geodesic rays in \mathbf{X} that represent some points in $\partial_{\infty} \mathbf{X}$ fixed by elements conjugate into C or B' .

8.1. Principal rays. Let $a \geq 1$. An a -principal ray is a geodesic ray in a chamber \mathbf{E}_f^+ that is the reparametrisation of the curve $\nu_{f,[ta,t,1]}$, where $t \geq t_0$ for some $t_0 \geq 1$. An a -antiprincipal ray is a geodesic ray in a chamber \mathbf{E}_f^+ that is the reparametrisation of the curve $\nu_{f,[t,a,1]}$, where $t \geq t_0$ for some $t_0 \geq a$. A geodesic ray is *principal* (resp. *antiprincipal*) if it is an a -principal (resp. a -antiprincipal) ray for some $a \geq 1$. See Figure 20.

Principal rays r, r' are *elementarily parallel*, if there is a chamber containing them both in which r, r' are parallel with respect to the Euclidean metric $|\cdot, \cdot|$. This generates an equivalence relation on the set of principal rays, and we say that r, r' are *parallel* if they are in the same equivalence class. A class of parallel rays determines

a point ζ in the visual boundary $\partial_\infty \mathbf{X}$. We call such a point ζ *principal*. Analogously we define *antiprincipal* points in $\partial_\infty \mathbf{X}$.

Lemma 8.1. *Let $\zeta \in \partial_\infty \mathbf{X}$ be the principal (resp. antiprincipal) point that is the class of the principal (resp. antiprincipal) rays in \mathbf{E}_{id}^+ . Then the stabilizer $\text{Stab}(\zeta) < \text{Tame}(\mathbb{k}^3)$ equals C (resp. B').*

We need a preparatory result.

Lemma 8.2. *Let r be a principal (resp. antiprincipal) ray in \mathbf{E}_{id}^+ , and let r' be any geodesic ray in \mathbf{X} asymptotic to r . Then r' has a subray that is principal (resp. antiprincipal) and parallel to r .*

Proof. Suppose first that $r: [0, \infty) \rightarrow \mathbf{E}_{\text{id}}^+$ is principal. Let $d > 0$ be such that $d_{\mathbf{X}}(r(t), r'(t)) \leq d$ for all $t \geq 0$. After passing to subrays of r and r' , we can assume that $\rho_+(r)$ is at distance $> d$ from the set $\{\alpha_2 = \alpha_3\}$ in the metric $|\cdot, \cdot|$. Let $\bar{x}_0 = r(0), \bar{x}_1, \dots, \bar{x}_k = r'(0)$ be points guaranteed by Lemma 3.3(iii) such that for each $i = 0, \dots, k-1$ both \bar{x}_i and \bar{x}_{i+1} lie in a common chamber $\mathbf{E}_{f_i}^+$, and $\sum_{i=0}^{k-1} |\rho_+(\bar{x}_i), \rho_+(\bar{x}_{i+1})| = d$. Thus all $\rho_+(\bar{x}_i)$ lie outside $\{\alpha_2 = \alpha_3\}$. Hence, by Remark 3.8, the principal rays from \bar{x}_i in $\mathbf{E}_{f_i}^+$ and $\mathbf{E}_{f_{i-1}}^+$ coincide. Consequently r and r' are parallel, as desired. If r is antiprincipal, the argument is analogous. \square

Proof of Lemma 8.1. Recall from Section 3.2 that the group C is generated by K and H . Applying Corollary 3.7, we obtain the following. If $f \in K$, then \mathbf{E}_{id}^+ and \mathbf{E}_f^+ share a 1-principal ray. If $f \in H$, then \mathbf{E}_{id}^+ and \mathbf{E}_f^+ share an a -principal ray for a sufficiently large (and thus they have parallel 1-principal rays). In particular, $\text{Stab}(\zeta)$ contains C . Conversely, by Proposition 3.4, if for $f \in \text{Tame}(\mathbb{k}^3)$ the chambers \mathbf{E}_{id}^+ and \mathbf{E}_f^+ share a principal ray, then f belongs to H or K . Thus to show that for $f \in \text{Stab}(\zeta)$ we have $f \in C$ it remains to justify that \mathbf{E}_{id}^+ and \mathbf{E}_f^+ have parallel principal rays. This follows from applying Lemma 8.2 to $r' = f(r)$. The proof of the second assertion is analogous. \square

8.2. \mathbf{H} -generic rays.

Construction 8.3. (1) Let f be an element of B' not conjugate into B or H .

Then there is an isometrically embedded Euclidean half-plane \mathbf{H} in \mathbf{X} , invariant under f . Indeed, by Corollary 3.7, for $g, g' \in B'$ with $g^{-1}g' \in B \setminus H$, the chambers $\mathbf{E}_g^+, \mathbf{E}_{g'}^+$ intersect along a 1-antiprincipal ray, and for $g^{-1}g' \in H \setminus B$ they intersect along a region whose boundary contains an a -antiprincipal ray for some $a \geq 2$. After possibly conjugating f , we can assume that its normal form in $B' = B *_{B \cap H} H$ is $h_1 b_1 \cdots h_n b_n$. For $i = 1, \dots, n$, let r'_i, r_i be the above antiprincipal rays for $g = h_1 \cdots b_{i-1}, g' = gh_i$ and $g = h_1 \cdots h_i, g' = gb_i$, respectively. Choose a geodesic σ in $(\nabla, |\cdot, \cdot|)$ perpendicular to the line $\alpha_2 = \alpha_3$ that intersects the images under ρ_+ of all r_i, r'_i . Let $r_0 = f^{-1}(r_n)$. For $i = 1, \dots, n$, consider the geodesic σ_i in \mathbf{E}_g^+ for $g = h_1 \cdots b_{i-1}$ (respectively, σ'_i in $\mathbf{E}_{g'}^+$ for $g' = gh_i$) that joins r_{i-1} to r'_i (respectively, r'_i to r_i) and maps into σ under ρ_+ . Then the union of the concatenation $\sigma_1 \cdot \sigma'_1 \cdots \sigma_n \cdot \sigma'_n$ and its translates under $\langle f \rangle$ is an axis of f that bounds an isometrically embedded Euclidean half-plane \mathbf{H} in \mathbf{X} , which is the union of the half-strips bounded by σ_i, σ'_i and subrays of r_i, r'_i (Remark 6.6(i)).

- (2) We have a following similar construction of an invariant convex subset \mathbf{H} for an element $f \in C$ that is not conjugate into K or H . By Corollary 3.7, for $g, g' \in C$ with $g^{-1}g' \in K \setminus H$, the chambers $\mathbf{E}_g^+, \mathbf{E}_{g'}^+$ intersect along a 1-principal ray, and for $g^{-1}g' \in H \setminus K$ they intersect along a region whose boundary contains a curve projecting under ρ_+ inside the (possibly non-principal) admissible line $\alpha_1 = m_2\alpha_2 + m_3\alpha_3$ for some $m_2 \geq 1, m_2 + m_3 \geq 2$. After possibly conjugating f , we can assume that its normal form in $C = K *_{K \cap H} H$ is $h_1k_1 \cdots h_nk_n$. For $i = 1, \dots, n$, let c_i, r_i be the above curves and principal rays for $g = h_1 \cdots k_{i-1}, g' = gh_i$ and $g = h_1 \cdots h_i, g' = gk_i$, respectively. Choose a geodesic σ in $(\nabla, |\cdot, \cdot|)$ perpendicular to the line $\alpha_1 = \alpha_2$ that intersects the images under ρ_+ of all c_i, r_i . Let $r_0 = f^{-1}(r_n)$. For $i = 1, \dots, n$, consider the geodesic σ_i in \mathbf{E}_g^+ for $g = h_1 \cdots k_{i-1}$ (respectively, σ'_i in $\mathbf{E}_{g'}^+$ for $g' = gh_i$) that joins r_{i-1} to c_i (respectively, c_i to r_i) and maps into σ under ρ_+ . Let R_i, R'_i be the Euclidean half-strips bounded by σ_i, σ'_i and the principal rays issuing from their endpoints. Let $R'_0 = f^{-1}(R'_n)$. Note that for $i = 1, \dots, n$, the half-strips R'_{i-1}, R_i intersect along the 1-principal rays by which they are bounded, and R_i, R'_i intersect along the regions bounded by the other principal rays and the curves inside c_i . Thus the union of all R_i, R'_i and their translates under $\langle f \rangle$ form a subset \mathbf{H} that is, by Remark 6.6(iii), locally convex, and hence convex in \mathbf{X} .

We have the following immediate consequence of Construction 8.3.

- Corollary 8.4.** (1) *Let f be an element of B' not conjugate into B or H . Then f is loxodromic not of rank 1. Furthermore, the translation length $|f|$ is the sum of the distances from r_{i-1} to r'_i , and from r'_i to r_i , for $i = 1, \dots, n$, in the notation of Construction 8.3(1).*
- (2) *Let f be an element of C not conjugate into K or H . Then f is loxodromic not of rank 1, if all c_i are principal rays, or parabolic, otherwise. Furthermore, $|f|$ is the sum of the distances (that might be zero) from r_{i-1} to c_i , and from c_i to r_i , for $i = 1, \dots, n$, in the notation of Construction 8.3(2).*

Example 8.5. The map $f = (x_2, x_1 + x_2x_3, x_3) \in C$ is parabolic of translation length 0. The map $f = (x_2, x_1 + x_2^2x_3, x_3) \in C$ is parabolic with positive translation length.

Definition 8.6. Let f be an element of B' not conjugate into B or H or an element of C not conjugate into K or H , with positive translation length. Let $\xi \in \partial_\infty \mathbf{X}$ be a limit point of f . A geodesic ray $r: J \rightarrow \mathbf{H}$ representing ξ , for \mathbf{H} as in Construction 8.3, is \mathbf{H} -generic if it is disjoint from the vertex set \mathbf{X}^0 and transverse to the 1-skeleton \mathbf{X}^1 . The consecutive points $l_k \in J$ at which r passes from one chamber \mathbf{E}_g^+ to another are called *transition points*.

Remark 8.7. Let f be an element of B' not conjugate into B or H or an element of C not conjugate into K or H , with positive translation length, and let \mathbf{H} be as in Construction 8.3. Let $\zeta \in \partial_\infty \mathbf{X}$ be the antiprincipal or principal point fixed by B' or C , respectively. Let $\xi \in \partial_\infty \mathbf{X}$ be a limit point of f . Note that since \mathbf{H} is convex and f -invariant, each geodesic ray starting in \mathbf{H} and representing ξ is contained in \mathbf{H} . By [CL10, Thm 1.1] (see also [Duc11, thm 6.1]), we have that ξ is at Tits distance π in $\partial_\infty \mathbf{H} \subset \partial_\infty \mathbf{X}$ from the other limit point of f , and so $\angle(\xi, \zeta) = \frac{\pi}{2}$. In particular $\xi \neq \zeta$.

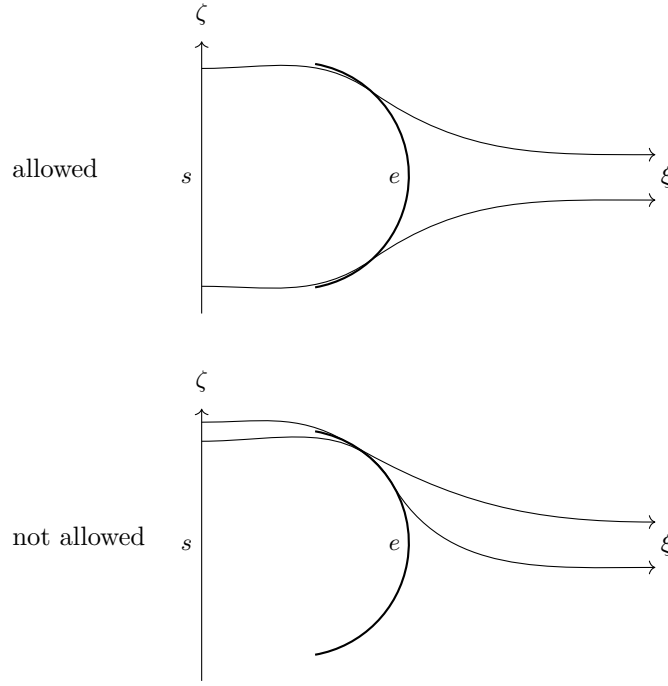


FIGURE 21. The solid curve e represents an edge to which two distinct rays in \mathcal{R} are tangent.

- (i) An \mathbf{H} -generic ray exists. Indeed, choose a ray s representing ζ contained in \mathbf{H} and disjoint from all the curves c_i . Let \mathcal{R} be the set of geodesic rays r representing ξ starting at the points of s . Note that distinct rays $r, r' \in \mathcal{R}$ are disjoint, because they are contained in a subspace of \mathbf{H} with no edges of degree 3, and because $r \subset r'$ would imply $\xi = \zeta$. Since there are uncountably many points in s , there exists $r \in \mathcal{R}$ disjoint from the vertices of \mathbf{X} in \mathbf{H} . Similarly, since the edges of \mathbf{X} have geodesic curvature of constant sign in each 2-cell, if two rays in \mathcal{R} are tangent to a given edge of \mathbf{X} intersecting \mathbf{H} , then we have one of the configurations in Figure 21. The bottom configuration is not allowed since the subspace bounded by the two rays and the edge, which is CAT(0) by the same argument as in Definition 6.2, would have two asymptotic geodesic rays starting at the same point. Consequently, at most two rays in \mathcal{R} are tangent to a given edge of \mathbf{X} . Thus, there exists a ray in \mathcal{R} transverse to \mathbf{X}^1 and disjoint from \mathbf{X}^0 .
- (ii) By [BH99, II.9.8(2)], the Alexandrov angle at $r(t)$ between r and the geodesic ray representing ζ converges to $\frac{\pi}{2}$ and so the same holds at the intersection with the curves c_i .
- (iii) Consequently, using [BH99, II.9.3], if at least one c_i projects to a non-principal admissible line in ∇^+ , then $\rho_+(r)$ leaves every compact set of ∇^+ .
- (iv) Conversely, suppose that all c_i project to principal lines in ∇^+ . Then \mathbf{H} is isometric to the Euclidean half-plane and all above intersection angles are $\frac{\pi}{2}$. Consequently, we have $f(r) \subset r$.
- (v) By (ii), the distances in J between the consecutive transition points of r are uniformly bounded from above.

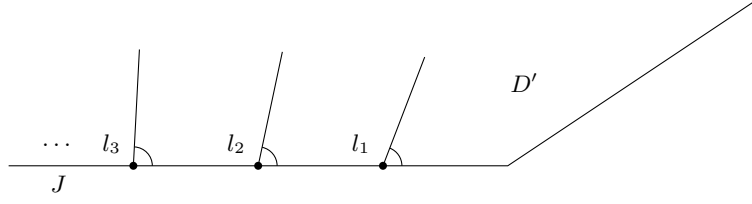


FIGURE 22.

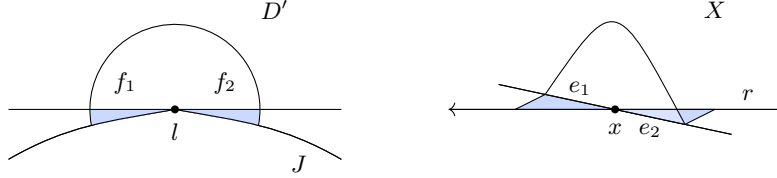


FIGURE 23.

Lemma 8.8. *Let $\tilde{\phi}_n: D_n \rightarrow \mathbf{X}$ be a sequence of reduced relative disc diagrams. Suppose that $\phi_n = \rho_+ \circ \tilde{\phi}_n$ have the limit half-plane diagram $\phi': D' \rightarrow \nabla^+$ with folding locus L . Furthermore, assume that ϕ' is bordered by an \mathbf{H} -generic ray $r: J \rightarrow \mathbf{X}$. Suppose that the sum of the curvatures of the vertices of $J \subset \partial D'$ is $\geq -\varepsilon_0$ from Corollary 7.6.*

Then the transition points l_k are contained in single edges of L at angle to J converging to $\frac{\pi}{2}$. See Figure 22.

Furthermore, for any $l \neq l_k$ in $L \cap J$, the angles between the edges of L at l and J are $< \frac{\varepsilon_0}{2}$.

Proof. Let $l = l_k$ be a transition point. Then $l \in L$. Moreover, since $x = r(l)$ is not a vertex of \mathbf{X} , there is an edge e in \mathbf{X} containing x in its interior. Since e lies in a translate of c_i, r_i , or r'_i , the Alexandrov angles between r and such e converge with k to $\frac{\pi}{2}$ by Remark 8.7(ii).

Let e_1, e_2 be the connected components of $e \setminus x$. If the point l is not contained in a single edge f of L , since all such edges map under $\tilde{\phi}_n$ into e_1 or e_2 , we have that the limit inferior (as $k \rightarrow \infty$) of the curvature of such l is $\leq -\pi$, which is a contradiction. Similarly, the angles between J and f have to coincide with the Alexandrov angles between r and e .

For the last assertion, let $x = r(l)$, where l is not a transition point. If $l \in L$, then l is contained in two edges f_1, f_2 mapping under $\tilde{\phi}_n$ into e_1 and e_2 . The angles that f_i make with J are equal, so if they are $\geq \frac{\varepsilon_0}{2}$, then, since the angle between f_1 and f_2 is π , this contradicts the assumption that the curvature of l is $> -\varepsilon_0$. See Figure 23. \square

Proposition 8.9. *Under the assumptions of Lemma 8.8, suppose additionally that the sum of the curvatures of the interior edges and the interior vertices of D' as well as of the vertices of J is $\geq -\varepsilon_0$.*

(1) *Suppose that l_k are contained in connected components L_k of L that are asymptotic geodesic rays. Then there is k and a subray $L'_k \subset L_k$ such that*

- $\phi'|_{L'_k}$ is an embedding into an admissible line diverging from $\{\alpha_2 = \alpha_3\}$ (resp. $\{\alpha_1 = \alpha_2\}$), and
 - for each $l'_k \in L'_k$, for n sufficiently large, the class in $\partial_\infty \mathbf{X}$ of the unique principal (resp. antiprincipal) ray in \mathbf{X} starting at $\tilde{\phi}_n(l'_k)$ is fixed by f from Definition 8.6.
- (2) Alternatively, suppose that L_k are geodesic segments, whose length becomes arbitrarily large when $k \rightarrow \infty$, ending at points $v_k \in \partial D'$ where $v_k v_{k+1} \subset \partial D'$ is geodesic and $\angle_{v_k}(L_k, v_{k+1}) + \angle_{v_{k+1}}(L_{k+1}, v_k) \geq \pi$. Then there is k such that for all $m \geq k$ we have that
- $\phi'(v_m)$ lies outside $\{\alpha_2 = \alpha_3\}$ (resp. $\{\alpha_1 = \alpha_2\}$), and
 - for n sufficiently large, the class in $\partial_\infty \mathbf{X}$ of the unique principal (resp. antiprincipal) ray in \mathbf{X} starting at $\tilde{\phi}_n(v_m)$ is fixed by f .

Proof. We focus on the assertion (1). Suppose first that $f \in C$ and there exists c_i in Construction 8.3 that projects to a non-principal admissible line in ∇^+ . By Remark 8.7(iii), the projection $\rho_+(r)$ leaves every compact set K of ∇^+ . Choose K to contain $c \setminus c'$ for $c = \rho_+(c_i)$ and c' as in Remark 7.7(iii). Let c'_i be the preimage in c_i of c' under ρ_+ . Then we can consider c'_k, c'_m defined as two consecutive translates of c'_i intersected by r in points $r(l_k), r(l_m)$. Let c''_k, c''_m be the unbounded connected components of $c'_k \setminus r(l_k), c'_m \setminus r(l_m)$. By [BH99, II.9.3] applied in the subspace of \mathbf{H} bounded by c''_k, c''_m and $r(l_k)r(l_m)$, we then have $\angle_{r(l_m)}(c''_m, r(l_k)) + \angle_{r(l_k)}(c''_k, r(l_m)) < \pi$.

Similarly, since D' is CAT(0) and L_k, L_m are asymptotic, we have $\angle_{l_m}(L_m, l_k) + \angle_{l_k}(L_k, l_m) < \pi$. Consequently, without loss of generality the first edge of L_k is mapped under each $\tilde{\phi}_n$ to c''_k . By Remark 7.7(iv), we have $\tilde{\phi}_n(L_k) \subset c''_k$, as desired.

Second, suppose that all c_i in Construction 8.3 project to principal lines in ∇^+ . Then we have $f(r) \subset r$ by Remark 8.7(iv). Note that since J intersects all L_k perpendicularly, and all L_k are asymptotic, by [BH99, II.9.3(2)], the regions between consecutive L_k are Euclidean half-strips. Cutting D along L_1 , we can assume that $D' = D$ is a quadrant in \mathbb{R}^2 with one boundary ray J , and $\phi' = \phi$. By Remark 6.6(i), we can assume that all $\tilde{\phi}_n$ are embeddings. Let $f_D: D \rightarrow D$ be the translation that acts on J as f does on r .

We apply Remark 7.3 with $D^+ = D^- = D, \phi^+ = \phi, \phi^- = f \circ \phi$ and $J^+ = f_D(J), J^- = J$. By Remark 6.6(ii), we have that R^\pm are convex in D . Note that $R^+ = f_D(R^-)$.

The path P^+ is asymptotic to a line in D determining a direction in $\partial_\infty D$ at angle θ from the direction of J . Consider first the case where $\theta = 0$. Then there is an infinite subcurve $\gamma \subset P^+$ with unit tangent directions mapped under ϕ^\angle (see Definition 7.8) arbitrarily close in S^\angle to the midpoint of S^\angle .

We claim that a neighbourhood N of γ in $\overline{D^+ \setminus R^+}$ can be identified with a subset of $\widehat{D} = \widehat{D}'$ on which $\phi^+, \widehat{\phi}$ coincide. If $\gamma = J^+$, then the claim follows from the last paragraph of Remark 7.3. Otherwise, γ maps to the 1-skeleton of \widehat{D} under $\overline{D^+ \setminus R^+} \subset \widehat{D}$, so N can be taken as a union of cells of \widehat{D} contained entirely in D^+ and the claim follows from the second to last paragraph of Remark 7.3.

We keep the notation γ, N for their image in \widehat{D} . Since the function ϕ^\angle is invariant under translation, and applying the claim to the neighbourhood $f_D^{-1}(N)$ of $f_D^{-1}(\gamma)$ in $\overline{D^- \setminus R^-}$, we have that the symmetry of $N \cup f_D^{-1}(N) \subset \widehat{D}$ interchanging N

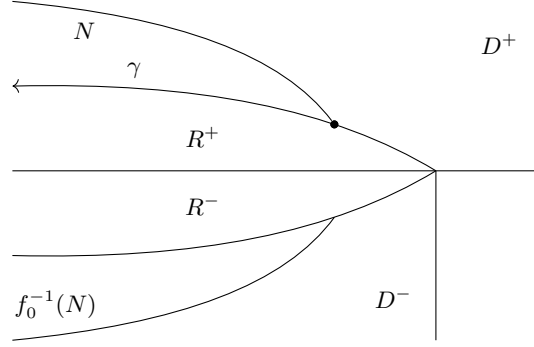


FIGURE 24.

and $f_D^{-1}(N)$ preserves $\widehat{\phi}^\angle$. See Figure 24. Since the unit tangent directions of γ are not mapped under ϕ^\angle to ∂S^\angle , we conclude that γ is contained in the folding locus \widehat{L} of \widehat{D} . This contradicts Lemma 7.10.

Second, consider the case where $\theta > 0$. We treat L_k as rays $L_k: [0, \infty) \rightarrow D$ parametrised by arc-length. By Lemma 7.9(ii), there is M such that each $\phi(L_k[M, \infty))$ is contained in a principal line. Furthermore, since l_k are at distance uniformly bounded from above (Remark 8.7(v)), we can assume that the geodesic path $L_k(M)L_{k+1}(M)$ in D maps under ϕ outside $\{\alpha_2 = \alpha_3\}$ (resp. $\{\alpha_1 = \alpha_2\}$).

Since $\theta > 0$, there are m and n with $l'_m = L_m(M)$ contained in R_n^+ . Then for $l'_k = f_D^{-1}(l'_m)$, and $x_k = \widetilde{\phi}_n(l'_k)$, $x_m = \widetilde{\phi}_n(l'_m)$, we have $f(x_k) = x_m$. Consequently, f maps the unique principal (resp. antiprincipal) ray starting at x_k to the one starting at x_m . These rays are parallel in \mathbf{X} since l'_k, l'_m are connected by a path in D_n mapping under ϕ_n to a piecewise geodesic path outside $\{\alpha_2 = \alpha_3\}$ (resp. $\{\alpha_1 = \alpha_2\}$), and by Remark 3.8.

The proof of assertion (2) is identical, with the following modifications. In the beginning of the proof, to justify $\angle_{l_m}(L_m, l_k) + \angle_{l_k}(L_k, l_m) < \pi$, we use the inequality in the hypothesis. Later, D might be no longer a quadrant but a possibly smaller subset of \mathbb{R}^2 . The last difference is that now L_k are bounded, but using the assumption that L_k become longer than M when $k \rightarrow \infty$, we can still find k such that for each $m \geq k$, for sufficiently large n , we have that $l'_m = L_m(M)$ belongs to R_n^+ . \square

9. CLASSIFICATION

In this section we finish the classification of parabolic isometries and loxodromic isometries not of rank 1, for the action of $\text{Tame}(\mathbb{k}^3)$ on \mathbf{X} . In particular, as a converse to Corollary 8.4, we show that such isometries always belong to the conjugates of the groups B', C introduced in Section 3.2.

9.1. Parabolic isometries.

Theorem 9.1. *If $f \in \text{Tame}(\mathbb{k}^3)$ is parabolic, then f belongs to a conjugate of C . Furthermore, such a conjugate is unique.*

Proof. Let $\mathbf{X}_l \subset \mathbf{X}$ be the subspace of points x with $d_{\mathbf{X}}(x, f(x)) \leq l$. We first prove that for each compact set $K \subset \nabla^+$ there is $l > |f|$ with $\rho_+(\mathbf{X}_l) \cap K = \emptyset$. Indeed, otherwise, there are $x_n \in \mathbf{X}_{|f| + \frac{1}{n}}$ with $\rho_+(x_n) \in K$. After passing to a subsequence, we can assume $\rho_+(x_n) \rightarrow [\alpha] \in K$. Choose ϵ so that each $[\alpha'] \in \nabla_+$ with $||[\alpha'], [\alpha]|| < \epsilon$ lies in the star of the cell containing $[\alpha]$ in its interior. Choose n such that $\frac{1}{n} < \frac{\epsilon}{3}$

and $|\rho_+(x_n), [\alpha]| < \frac{\epsilon}{3}$. Then there is $y \in \mathbf{X}$ with $\rho_+(y) = [\alpha]$ and $|x_n, y| < \frac{\epsilon}{3}$. Since f does not have a fixed point and acts without inversions on \mathbf{X} , we have $f(y) \neq y$ and $|y, f(y)| \geq \epsilon$. This contradicts $|y, f(y)| \leq |y, x_n| + |x_n, f(x_n)| + |f(x_n), f(y)| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3}$.

Let $\nabla_l^+ \subset \nabla^+$ be the set of points at distance $\leq l$ from $\partial\nabla^+$. We claim that $\rho_+(\mathbf{X}_l) \subset \nabla_l^+$. Indeed, for each $x \in \mathbf{X}_l$, by Lemma 3.3(iii), there is a chain of length $\leq l$ from x to $f(x)$. If $|\rho_+(x), \partial\nabla^+| > l$, then all the points in that chain map under ρ_+ outside $\partial\nabla^+$. Consequently, by Corollary 3.7, the chambers containing the points of that chain have nonempty intersection, which is fixed pointwise by f , contradiction.

Now, let K be the $4|f|$ neighbourhood of $[1, 1, 1]$ in ∇^+ , and choose $2|f| > l > |f|$ with $\rho_+(\mathbf{X}_l) \cap K = \emptyset$. Note that K separates ∇_l^+ into two connected components, which are contained in the l -neighbourhoods of the sets $\{\alpha_1 = \alpha_2\}$, or $\{\alpha_2 = \alpha_3\}$, respectively. Since \mathbf{X}_l is convex, hence connected, it is mapped under ρ_+ into one of these two connected components.

Consider first the former possibility. Let $x, f(x) \in \mathbf{X}_l$ and assume without loss of generality $x \in \mathbf{E}_{\text{id}}^+$. Then the minimal length chain from x to $f(x)$ in \mathbf{X}_l maps under ρ_+ outside the set $\{\alpha_2 = \alpha_3\}$. Then by Remark 3.8, the principal rays in $\mathbf{E}_{\text{id}}^+, \mathbf{E}_f^+$ starting at x and at $f(x)$ are parallel. By Lemma 8.1, we obtain $f \in C$, as desired. If \mathbf{X}_l maps under ρ_+ to the other connected component of $\nabla_l^+ \setminus K$, we analogously obtain $f \in B'$. However, this contradicts Corollary 8.4(1).

For the remaining assertion, consider parabolic $f \in C \cap gCg^{-1}$ for some $g \in \text{Tame}(\mathbb{k}^3)$. Let $r: [0, \infty) \rightarrow \mathbf{X}$ be the geodesic ray starting at $r(0) = x \in \mathbf{X}_l$ that represents the same point ζ in $\partial_\infty \mathbf{X}$ as the principal rays in \mathbf{E}_g^+ . Since $f \in gCg^{-1}$, we have $f(\zeta) = \zeta$. Consequently, for each $t \geq 0$, we have $d_{\mathbf{X}}(r(t), f(r(t))) \leq d_{\mathbf{X}}(r(0), f(r(0))) \leq l$, and so $r[0, \infty) \subset \mathbf{X}_l$. By Lemma 8.2, we have that a subray of r is parallel to a principal ray in \mathbf{E}_g^+ . Consequently, there is a chamber \mathbf{E}_h^+ with $h \in C$ with h simultaneously in gC , and so $g \in C$. \square

Proposition 9.2. *Let $f \in C$ be parabolic with translation length zero and let $\zeta \in \partial_\infty \mathbf{X}$ be the class of the principal rays in \mathbf{E}_{id}^+ . Then ζ is the positive (and the negative) limit point of f .*

In the language of Section 2, Proposition 9.2 states that f is not vile.

In the proof we will use the following. Let $a > 0$. An a -sector is the CAT(0) space obtained by glueing two Euclidean half-strips $[0, \infty) \times [0, R]$ along copies of the set of points (t, y) defined by $y \geq f(t)$ for some convex function $f: [0, \infty) \rightarrow [0, R]$ satisfying $\frac{1}{c}e^{-at} \leq f \leq ce^{-at}$ for some $c > 0$. We call the union of the two copies of the ray $[0, \infty) \times \{0\}$ the *frontier* of the sector. An a -quasicusp is the locally CAT(0) space glued of finitely many a -sectors by cyclically identifying them along their frontiers.

Lemma 9.3. *The universal cover \mathbf{H} of an a -quasicusp is quasi-isometric with a horodisc \mathbf{H}' in the rescaled hyperbolic plane $\frac{1}{a}\mathbb{H}^2$. Furthermore, the quasi-isometry conjugates the deck transformation group of \mathbf{H} with a group of parabolic isometries of \mathbf{H}' .*

Note that we could replace $\frac{1}{a}\mathbb{H}^2$ by \mathbb{H}^2 in the statement by composing with the obvious quasi-isometry between $\frac{1}{a}\mathbb{H}^2$ and \mathbb{H}^2 , but a constant a will appear naturally in the proof of Proposition 9.2.

Proof. Let f' be a parabolic isometry preserving \mathbf{H}' . Let r be the image of a geodesic ray in \mathbf{H}' starting on the horosphere bounding \mathbf{H}' . It suffices to show that the union of $f'^n(r)$, over all $n \in \mathbb{Z}$, with the metric induced from \mathbf{H}' , is quasi-isometric with the union of all the lifts to \mathbf{H} of all $[0, \infty) \times \{0\}$ and all $\{(t, f(t))\}$ in all the sectors of a quasicusp, with the metric induced from \mathbf{H} . To do that, it suffices to show that this metric on $r \cup f'(r)$ is bilipschitz to the metric on the union of $[0, \infty) \times \{0\}$ and $\{(t, f(t))\}$ in $[0, \infty) \times [0, R]$.

To do that, note first that the maps $t \rightarrow (t, 0), t \rightarrow (t, f(t))$ from $[0, \infty)$ to their image in $[0, \infty) \times [0, R]$ are bilipschitz (the first one is in fact an isometric embedding). Furthermore, the distance between $(t', 0)$ and $(t, f(t))$ is bounded from above by $|t' - t| + f(t)$ and from below by both $|t' - t|$ and $f(t)$. Consequently, up to a multiplicative constant, this distance equals $|t - t'| + e^{-at}$. The same formula holds in $\mathbf{H}' \subset \frac{1}{a}\mathbb{H}^2$ for the distances between the points on r and $f'(r)$. \square

Proof of Proposition 9.2. Let \mathbf{H} be the f -invariant convex subset of \mathbf{X} described in Construction 8.3(2). We claim that \mathbf{H} is the universal cover of a quasicusp with $\langle f \rangle$ the deck transformation group. Since $|f| = 0$ (Corollary 8.4(2)), for each $i = 1, \dots, n$, the curve c_i bounding the intersection $R_i \cap R'_i \subset \mathbf{H}$ maps under ρ_+ to the admissible line with equation $\alpha_1 = \alpha_2 + m\alpha_3$ for some integer $m \geq 1$. Passing to the Euclidean coordinates on the plane $\beta_1 + \beta_2 + \beta_3 = 0$ for $\beta_i = \log \alpha_i$ and $\alpha_1\alpha_2\alpha_3 = 1$ (see Section 3.1), this means

$$e^{\beta_1} = e^{\beta_2} + me^{\beta_3} = e^{\beta_2} + me^{-(\beta_1 + \beta_2)}.$$

On that plane we have orthogonal coordinates

$$y = \frac{\beta_1 - \beta_2}{\sqrt{2}}, t = \frac{\sqrt{3}(\beta_1 + \beta_2)}{\sqrt{2}}.$$

The equation above has then the form

$$e^{\frac{\sqrt{\frac{3}{2}}t + \sqrt{2}y}{2}} = e^{\frac{\sqrt{\frac{3}{2}}t - \sqrt{2}y}{2}} + me^{-\sqrt{\frac{2}{3}}t},$$

i.e. $2 \sinh \frac{y}{\sqrt{2}} = me^{-\sqrt{\frac{3}{2}}t}$. For y close to 0, up to a multiplicative constant, $\sinh \frac{y}{\sqrt{2}}$ equals y , and so $y = y(t)$ has the required asymptotics for an a -sector with $a = \sqrt{\frac{3}{2}}$. This justifies the claim.

By Lemma 9.3, we have an equivariant quasi-isometry $\mathbf{H} \rightarrow \mathbf{H}'$ for \mathbf{H}' a horodisc in a hyperbolic plane $\frac{1}{a}\mathbb{H}^2$ preserved by a parabolic isometry f' . The horodisc \mathbf{H}' is Gromov-hyperbolic and for any point $x' \in \mathbf{H}'$, for $n \rightarrow \pm\infty$, the sequence $f'^n(x')$ converges w.r.t. the Gromov product to the unique point in the Gromov boundary of \mathbf{H}' . Consequently, \mathbf{H} is Gromov-hyperbolic and for any point $x \in \mathbf{H}$, for $n \rightarrow \pm\infty$, the sequence $f^n(x)$ converges w.r.t. the Gromov product to the unique point in the Gromov boundary of \mathbf{H} . Since \mathbf{H} is proper, by [BH99, III.H.3.7(2) and 3.17(6)], for $n \rightarrow \pm\infty$, the sequence $f^n(x)$ converges in the cone topology to ζ . \square

9.2. Loxodromic isometries not of rank 1.

Theorem 9.4. *Let $f \in \text{Tame}(\mathbb{k}^3)$ be loxodromic not of rank 1. Then f belongs to a conjugate of C or B' . Furthermore, there is at most one such a conjugate of C and at most one such a conjugate of B' .*

Example 9.5. The map $f = (x_2, x_1 + x_2^2, x_3)$ belongs to both C and $gB'g^{-1}$ for $g = (x_3, x_2, x_1)$.

To prove the uniqueness in Theorem 9.4, it will be convenient to use the following, whose proof we postpone to Section 11.

Lemma 9.6. *Let $\xi \neq \eta \in \partial_\infty \mathbf{X}$ be principal and not far. Then $\angle(\xi, \eta) = \frac{2\pi}{3}$, the midpoint of the geodesic $\xi\eta$ in $\partial_\infty \mathbf{X}$ is antiprincipal, and there is a chamber containing geodesic rays representing ξ and that midpoint. The same statement holds if we interchange the words ‘principal’ and ‘antiprincipal’.*

Note that since $\partial_\infty \mathbf{X}$ is CAT(1), the geodesic $\xi\eta$ above is unique.

Remark 9.7. $\partial_\infty \mathbf{X}$ has geometric dimension 1 (see [Kle99]), which can be justified by the following argument proposed by Pierre-Emmanuel Caprace. Namely, $\partial_\infty \mathbf{X}$ embeds in the space of directions of any asymptotic cone of \mathbf{X} , where the sequence of basepoints is constant. Since \mathbf{X} has geometric dimension 2, we have that the asymptotic cone has also geometric dimension 2 [Lyt05, Lem 11.1], and so $\partial_\infty \mathbf{X}$ has geometric dimension 1.

Consequently, $\partial_\infty \mathbf{X}$ is locally an \mathbb{R} -tree, see e.g. the proof of [OP21b, Lem 6.3].

Proof of Theorem 9.4. Let γ be an axis of f in \mathbf{X} . If f is not of rank 1, then γ is not contracting. Let $p_0 \in \gamma$ and for $k \in \mathbb{Z}$ let $p_k = f^k(p_0)$. By Lemma 2.8 applied with $\beta = \gamma$ and $\mathcal{P} = \{p_k\}$, for each $n > 0$, there is a $(p_k, \frac{1}{n}, n)$ -quadrilateral with a side in γ . Thus its translate under f^{-k} is a $(p_0, \frac{1}{n}, n)$ -quadrilateral \square_n with a side in γ . By Lemma 2.2, we can assume that \square_n is embedded.

Let $\bar{\phi}_n: \bar{D}_n \rightarrow \mathbf{X}$ be a reduced relative disc diagram with boundary \square_n guaranteed by Lemma 6.3. Note that $\bar{\phi}_n$ is an embedding by Theorem 6.10. Then for each subcomplex $D_n \subset \bar{D}_n$ with $I_n = D_n \cap \partial \bar{D}_n$, the restriction $\tilde{\phi}_n$ of $\bar{\phi}_n$ to D_n is a disc diagram in \mathbf{X} relative to I_n .

Since ∇^+ is locally finite and since $\bar{\phi}_n$ are reduced disc diagrams, by Theorem 6.9 there is a uniform bound on the degree of a vertex in \bar{D}_n at a given distance from $q_n = \bar{\phi}_n^{-1}(p_0)$. Furthermore, since \square_n is a $(p_0, \frac{1}{n}, n)$ -quadrilateral, we can assume that the radius n closed ball in \bar{D}_n centred at q_n is disjoint from $\partial \bar{D}_n \setminus \bar{\phi}_n^{-1}(\gamma)$. Thus we can find D_n as above so that $I_n = \tilde{\phi}_n^{-1}(\gamma)$, the pairs (D_n, I_n) converge to a half-plane diagram $(D, \partial D)$, and all q_n coincide.

Furthermore, we can assume that we have the limit $\phi: D \rightarrow \nabla^+$ of $\rho_+ \circ \tilde{\phi}_n$ bordered by two rays with union γ . (Here $D^* = D' = D$.) Let $\gamma_n = \phi_n(I_n)$. Note that since \square_n were $(p_0, \frac{1}{n}, n)$ -quadrilaterals, by Theorem 6.9 we have that D , with the degenerate piecewise smooth Euclidean metric pulled back under ϕ from ∇^+ , is isometric to the Euclidean half-plane. Let $f_D: D \rightarrow D$ be the translation that preserves ∂D and acts on it as f does on γ .

We apply a variant of Remark 7.3 with $D^+ = D^- = D$, $\phi^+ = \phi$, $\phi^- = f \circ \phi$, identity fillings, and $J^+ = J^- = \partial D$, which are lines instead of rays. By Remark 6.6(ii), we have that R^\pm are convex in D^\pm . Thus $R^+ = R^-$ is a Euclidean strip (possibly $R^\pm = D^\pm$).

Consider first the case where $R^\pm \neq D^\pm$. Then \widehat{D} from Remark 7.3 becomes a combinatorial complex isometric to the Euclidean plane and $\widehat{\phi}: \widehat{D} \rightarrow \nabla^+$ is combinatorial and \mathbf{X} -reduced. Let $L^\pm \subset D^\pm$, $\widehat{L} \subset \widehat{D}$ denote the folding loci. We then have $\widehat{L} \cap (D^\pm \setminus R^\pm) = L^\pm \setminus R^\pm$. Observe that $\widehat{L} \cap (D^+ \setminus R^+)$ is sent to $\widehat{L} \cap (D^- \setminus R^-)$

by the glide reflection $f_{\widehat{D}}$ of \widehat{D} that is the composition of f_D with the reflection in $\text{fr } R^+ = \text{fr } R^-$.

All the edges of \widehat{L} are oriented, since \widehat{D} is locally Euclidean. By Remark 7.7(ii), for each oriented edge e of \widehat{L} , the entire oriented geodesic ray in \widehat{D} starting at e lies in \widehat{L} . By Lemma 7.9(i), such a geodesic ray cannot be parallel to $\text{fr } R^+$, in view of the fact that $\rho_+(\gamma)$ is bounded. Furthermore, by Corollary 7.6, none of such geodesics can intersect, except when one is contained in another. Thus by the observation above on $f_{\widehat{D}}$, we have that \widehat{L} has no vertices v of degree ≥ 3 , since otherwise some of the six oriented geodesics issuing from v and $f_{\widehat{D}}(v)$ would intersect.

If a connected component of \widehat{L} was a line l intersecting $\text{fr } R^+$, then either l would intersect $f_{\widehat{D}}(l)$, which is a contradiction, or l would be perpendicular to $\text{fr } R^+$. Then the function $\widehat{\phi}^\angle: \mathbb{S} \rightarrow S^\angle$ as in Definition 7.8 would simultaneously assign to the direction of l , which is the opposite of the direction of $f_{\widehat{D}}(l)$, the principal and the antiprincipal point, contradiction. Consequently, \widehat{L} is empty, which contradicts the last assertion of Lemma 7.9.

Second, consider the case where $R^+ = D$. Then f_D preserves the folding locus L of D . Thus, as in the previous case, by Remark 7.7(ii) and Corollary 7.6, all the vertices of order ≥ 3 in L are at a uniform distance from ∂D . Since L is again nonempty, and contains no line parallel to ∂D , there is a half-plane $D_0 \subseteq D$ such that $D_0 \cap L$ is a family of parallel oriented geodesic rays $(L_k)_{k=-\infty}^\infty$ starting or ending at ∂D_0 . By Lemma 7.9(ii), there is M such that each $\phi(L_k[M, \infty))$ is contained in a principal line. Choose m and n with $L_m(M)$ contained in R_n^+ . Then for $L_k(M) = f_D^{-1}(L_m(M))$, and $x_k = \widehat{\phi}_n(L_k(M))$, $x_m = \widehat{\phi}_n(L_m(M))$, we have $f(x_k) = x_m$. Consequently, f maps the unique (see Remark 3.8) principal or antiprincipal ray starting at x_k to the one starting at x_m . These rays are asymptotic in \mathbf{X} since L_k and L_m are asymptotic in D . By Lemma 8.1, f lies in a conjugate of C or B' .

For the uniqueness of the conjugate of B' , suppose $f \in B' \cap gB'g^{-1}$ with $g \notin B'$ and let $\zeta \neq g\zeta$ be the antiprincipal points represented by the antiprincipal rays in \mathbf{E}_{id}^+ , \mathbf{E}_g^+ . Let ξ, η be the limit points of f . By Construction 8.3(1), it follows that $\zeta, g\zeta$ are the midpoints of length π paths $\omega_{\text{id}} \neq \omega_g \subset \partial_\infty \mathbf{X}$ between ξ and η . Since $\partial_\infty \mathbf{X}$ is CAT(1) and is locally an \mathbb{R} -tree (Remark 9.7), we have that $\omega = \omega_{\text{id}} \cdot \omega_g$ is a circle of length 2π isometrically embedded in $\partial_\infty \mathbf{X}$. By Lemma 6.7, we obtain a flat $F \subset \mathbf{X}$ with $\partial_\infty F = \omega$. Thus ζ and $g\zeta$ are at distance π but they are not far. This contradicts Lemma 9.6.

The proof of the uniqueness of the conjugate of C is analogous. \square

10. (ξ, η) -DIAGRAMS

We will be using the following diagrams to describe pairs of points in $\partial_\infty \mathbf{X}$ that are not far (see Definition 2.3).

Construction 10.1. Let $\xi \neq \eta \in \partial_\infty \mathbf{X}$ be not far, and let $x \in \mathbf{X}$ be such that $\angle_x(\xi, \eta) > 0$. We construct a following \mathbf{X} -reduced relative half-plane diagram $\phi': D' \rightarrow \nabla^+$, which we call a (ξ, η) -*diagram*.

Consider $u_n, v_n \in \mathbf{X} \cup \partial_\infty \mathbf{X}$, with $u_n \rightarrow \xi, v_n \rightarrow \eta$, and at most one of each u_n, v_n in $\partial_\infty \mathbf{X}$, such that either:

- (i) for each closed metric ball P in \mathbf{X} , the geodesics $u_n v_n$ are disjoint from P for n large enough, or

- (ii) there is a closed metric ball P in \mathbf{X} containing a point p_n on each $u_n v_n$, such that there is a $(p_n, \frac{1}{n}, n)$ -quadrilateral \square_n with a side $\bar{u}_n \bar{v}_n$ in $u_n v_n$.

In case (i), by [BH99, II.9.8(4)], if $\angle(\xi, \eta) < \pi$, then we can assume that u_n, v_n lie on the geodesic rays from x to ξ, η , respectively. Let $\bar{u}_n \in x u_n \cap u_n v_n, \bar{v}_n \in x v_n \cap u_n v_n$ be closest possible to x . Let Δ_n be the geodesic triangle $x \bar{u}_n \bar{v}_n$, which is an embedded closed path. Let $\bar{\phi}_n: \bar{D}_n \rightarrow \mathbf{X}$ be a reduced relative disc diagram with boundary Δ_n guaranteed by Lemma 6.3. Note that $\bar{\phi}_n$ is an embedding by Theorem 6.10. Then for each subcomplex $D_n \subset \bar{D}_n$ with $I_n = D_n \cap \partial \bar{D}_n$, the restriction $\tilde{\phi}_n$ of $\bar{\phi}_n$ to D_n is a disc diagram in \mathbf{X} relative to I_n .

Since ∇^+ is locally finite and since $\bar{\phi}_n$ are reduced disc diagrams, by Theorem 6.9 there is a uniform bound on the degree of a vertex in \bar{D}_n at a given distance from $q_n = \bar{\phi}_n^{-1}(x)$. Furthermore, by the hypothesis in case (i), we can assume that the radius n closed ball in \bar{D}_n centred at q_n is disjoint from $\bar{\phi}_n^{-1}(\bar{u}_n \bar{v}_n)$. Thus, after possibly passing to a subsequence, we can find D_n as above so that (D_n, I_n) converge to a half-plane diagram $(D, \partial D)$ and all q_n coincide with a point $q \in \partial D$. Furthermore, we can assume Construction 6.13(ii). By Remark 7.1, where I_a, I_b are the connected components of $\partial D \setminus q$ and I_c is empty, we also have Construction 6.13(iii). This gives the frilling D' of D , and the limit map $\phi': D' \rightarrow \nabla^+$ of $\phi_n = \rho_+ \circ \tilde{\phi}_n$ that is an \mathbf{X} -reduced relative half-plane diagram. Note that by [BH99, II.9.8(4)] we have that I_a, I_b represent points in $\partial_\infty D$ at angle $\angle(\xi, \eta)$.

In case (ii), by Lemma 2.2, we can assume that each \square_n is an embedded closed path. Let $\bar{\phi}_n: \bar{D}_n \rightarrow \mathbf{X}$ be reduced relative disc diagrams with boundary \square_n guaranteed by Lemma 6.3. Again, $\bar{\phi}_n$ is an embedding by Theorem 6.10. All $\rho_+(p_n)$ lie in a finite set of cells of ∇^+ . Thus, after possibly passing to a subsequence, we have $D_n \subset \bar{D}_n$ with $I_n = D_n \cap \partial \bar{D}_n \subset \bar{\phi}_n^{-1}(\bar{u}_n \bar{v}_n)$ such that (D_n, I_n) converge to a half-plane diagram $(D, \partial D)$. Moreover, all $q_n = \bar{\phi}_n^{-1}(p_n)$ coincide with a point $q \in \partial D$. Furthermore, denoting $\tilde{\phi}_n = \bar{\phi}_n|_{D_n}$, we have the frilling D' of D , and the limit map $\phi': D' \rightarrow \nabla^+$ of $\phi_n = \rho_+ \circ \tilde{\phi}_n$ that is an \mathbf{X} -reduced relative half-plane diagram. Since \square_n were $(p_n, \frac{1}{n}, n)$ -quadrilaterals, D' is isometric to the Euclidean half-plane in \mathbb{R}^2 .

Lemma 10.2. *Let $\phi': D' \rightarrow \nabla^+$ be a (ξ, η) -diagram. If ξ is principal or antiprincipal, then there is $J' \subset \partial D'$ with $\phi'|_{J'}$ an embedding into a principal line.*

Note that if ϕ' is bordered by a geodesic ray representing ξ , which happens for example for $\angle(\xi, \eta) < \pi$, then Lemma 10.2 follows immediately from Lemma 8.2.

Proof. By Lemma 8.2, there is a point x' on the geodesic ray from x to ξ such that the geodesic ray r from x' to ξ is principal or antiprincipal. Let $J_\xi \subset \partial D$ be the ray starting with q with $J_\xi \cap I_n$ mapping into $x \bar{u}_n$ or $p_n \bar{u}_n$ under $\tilde{\phi}_n$.

In case (i) of Construction 10.1, since $u_n \rightarrow \xi$, the restrictions of $\tilde{\phi}_n$ to some $J \subseteq J_\xi$ converge to r (in C^1 piecewise, up to a parametrisation). Let l be the principal line containing $\rho_+(r)$.

In case (ii) of Construction 10.1, since $p_n \in P$, after possibly passing to a subray of r , each geodesic ray from p_n to ξ contains a subray r_n parallel to r with $\rho_+(r_n)$ in a uniform neighbourhood of $\rho_+(r)$. Furthermore, since $u_n \rightarrow \xi$, the restrictions of $\tilde{\phi}_n$ to each compact subset of some subray $J \subset J_\xi$, become arbitrarily close to r_n .

Since $\rho_+ \circ \tilde{\phi}_n$ have a limit, we conclude that $\rho_+(r_n)$ and $\rho_+ \circ \tilde{\phi}_n(J)$ converge into some principal line l .

In both cases, $\phi(J)$ lies in l , and so does $\phi'(J')$ for the image J' of J in $\partial D'$ by Remark 7.1. \square

Lemma 10.3. *Let $\phi': D' \rightarrow \nabla^+$ be a (ξ, η) -diagram from Construction 10.1(i), and assume $\angle_x(\xi, \eta) > \angle(\xi, \eta) - \varepsilon$, for some $\varepsilon > 0$. Suppose furthermore $u_n, v_n \in \mathbf{X}$ and $\angle_x(\xi, u_n), \angle_x(\eta, v_n) < \varepsilon$. Then for any D_n (and hence for D') the sum of the curvatures of vertices and edges in the interior of D_n and in the interior of $I_n \setminus q_n$ is $\geq -3\varepsilon$. Furthermore, the angle at q_n is $\leq \angle(\xi, \eta)$.*

Proof. Suppose first that $\angle(\xi, \eta) < \pi$, in which case we assumed that \bar{u}_n, \bar{v}_n lie on the geodesic rays from x to ξ, η , respectively. By Lemma 2.4, we observe $\angle_{\bar{u}_n}(x, \bar{v}_n) + \angle_{\bar{v}_n}(x, \bar{u}_n) \geq \pi - \angle(\xi, \eta)$. Thus by Theorem 6.9, the sum of the curvatures in question is $\geq -\varepsilon$. The assertion on the angle at q_n also follows from the above observation and Theorem 6.9.

Second, suppose $\angle(\xi, \eta) = \pi$, and so $\angle_x(\xi, \eta) > \pi - \varepsilon$. Thus $\angle_x(\bar{u}_n, \bar{v}_n) > \pi - 3\varepsilon$. Then again by Theorem 6.9, the sum of the curvatures in question is $\geq -3\varepsilon$. \square

We need a similar result for $\xi = \eta$. We start with the following.

Lemma 10.4. *Let X be a complete CAT(0) space, and let $d > 0$. Then there exists a function $\psi: [0, \infty) \rightarrow \mathbb{R}_{>0}$ with $\lim_{t \rightarrow \infty} \psi(t) = 0$, such that for any disjoint geodesic rays r, r' representing the same point $\xi \in \partial_\infty X$, with $d_X(r(0), r'(0)) \leq d$, the projection $p(t)$ of $r(t)$ to $r'[0, \infty)$ satisfies*

$$\left| \angle_{r(t)}(r(0), p(t)) - \frac{\pi}{2} \right| \leq \psi(t).$$

Proof. Without loss of generality, we can assume $b_\xi(r(t)) = b_\xi(r'(t)) = -t$ for a Busemann function b_ξ . Since the horoball $b_\xi \leq -t$ is a limit of balls, we have the inequality $\angle_{r(t)}(\xi, r'(t)) \leq \frac{\pi}{2}$. Consequently, $\alpha(t) = \angle_{r(t)}(r(0), r'(t)) \geq \frac{\pi}{2}$, and analogously $\beta(t) = \angle_{r'(t)}(r'(0), r(t)) \geq \frac{\pi}{2}$.

Let x be the last point on the geodesic $r(0)r'(0)$ whose projection to the geodesic $r(t)r'(t)$ equals $r(t)$. Then we have $\angle_{r(t)}(r'(t), x) = \frac{\pi}{2}$. Considering the triangle $r(0)r(t)x$, by [BH99, II.1.7(5)] we have

$$d \geq d_X(r(0), r'(0)) \geq d_X(r(0), x) \geq t \sin(\alpha(t) - \frac{\pi}{2}).$$

Consequently, there is a function ψ with $\lim_{t \rightarrow \infty} \psi(t) = 0$ satisfying for all r, r' the bounds $\frac{\psi(t)}{2} \geq \alpha(t) - \frac{\pi}{2}$, and analogously $\frac{\psi(t)}{2} \geq \beta(t) - \frac{\pi}{2}$. Since the sum of the Alexandrov angles in the triangle $r(t)r'(t)p(t)$ is $\leq \pi$, we obtain $\angle_{r(t)}(r'(t), p(t)) \leq \frac{\psi(t)}{2}$. Consequently,

$$\left| \angle_{r(t)}(r(0), p(t)) - \frac{\pi}{2} \right| \leq \left| \angle_{r(t)}(r(0), r'(t)) - \frac{\pi}{2} \right| + \left| \angle_{r(t)}(r'(t), p(t)) \right| \leq \psi(t). \quad \square$$

Corollary 10.5. *Let $r: [0, \infty) \rightarrow X$ be a geodesic ray in a complete CAT(0) space X . Then for any $\varepsilon > 0, d > 0$ and any non-increasing function $\delta: [0, \infty) \rightarrow \mathbb{R}_{>0}$, there is $T > 0$ such that*

- for any geodesic ray $r': [0, \infty) \rightarrow X$ asymptotic to r satisfying
 - $d_X(r(0), r'(0)) \leq d$, and
 - $d_X(r(t), r'[0, \infty)) \geq \delta(t)$ for each $t \in [0, \infty)$, and

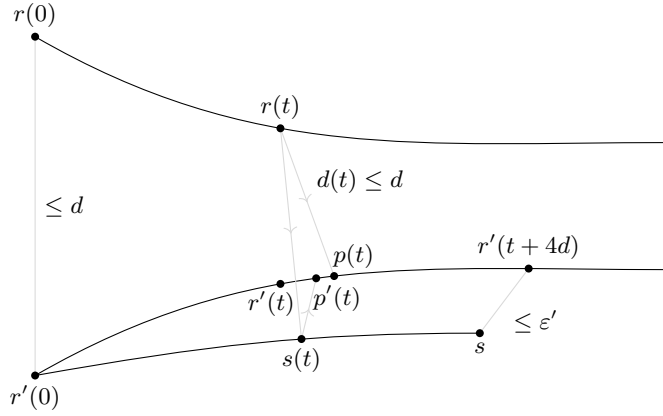


FIGURE 25. Proof of Corollary 10.5. Projections are denoted by arrows.

- for any $t \in (T, \infty)$, for any $0 < \varepsilon' < \min\{\delta(t)^4, d, \varepsilon\}$, and for any point $s \in X$ at distance $\leq \varepsilon'$ from $r'(t+4d)$ such that the projections $s(t), s(T)$ of $r(t), r(T)$ onto $sr'(0)$ are distinct,

we have that the Alexandrov angles of the geodesic quadrilateral $s(t)s(T)r(T)r(t)$ are $\geq \frac{\pi}{2}$ at $s(t)$ and $s(T)$ and $\geq \frac{\pi}{2} - 9\sqrt{d}\sqrt[4]{\varepsilon}$ at $r(t)$ and $r(T)$.

In particular, by Theorem 6.9, for any reduced relative disc diagram $\phi: D \rightarrow X$ with boundary path $s(t)s(T)r(T)r(t)$, the angles in D at the preimages under ϕ of $s(t), s(T), r(T), r(t)$ are nearly $\frac{\pi}{2}$.

Proof. Choose T so that the function ψ from Lemma 10.4 satisfies $\psi[T, \infty) < \sqrt{d}\sqrt[4]{\varepsilon}$ (this choice does not matter until the last line of the proof).

Let $p(t), p'(t)$ be the projections of $r(t), s(t)$ to r' , and let $d(t) = d_X(r(t), p(t))$. See Figure 25. Since $p(t)$ is at distance $\leq \varepsilon'$ from a point on $r'(0)s$, and $s(t)$ is the projection of $r(t)$ onto $r'(0)s$, we have $d_X(r(t), s(t)) \leq d(t) + \varepsilon'$. In particular, we have $d_X(r(t), s(t)) < 2d$, and so $d_X(r'(t), s(t)) < 3d$, which implies that $s(t)$ is distinct from s . Analogously, $s(T)$ is distinct from s . Similarly, taking $T > 3d$ we have that $s(t)$ and $s(T)$ are distinct from $r'(0)$. Then $\angle_{s(T)}(s(t), r(T)) \geq \frac{\pi}{2}$ and $\angle_{s(t)}(s(T), r(t)) \geq \frac{\pi}{2}$.

Furthermore, we have $d_X(r(t), p'(t)) \leq d(t) + 2\varepsilon'$ and so, considering the triangle $r(t)p(t)p'(t)$, we obtain $d_X(p(t), p'(t)) \leq \sqrt{4d(t)\varepsilon' + 4\varepsilon'^2}$. Using the assumptions $d(t) \leq d$ and $\varepsilon' < d$, we get $d_X(p(t), s(t)) \leq \varepsilon' + \sqrt{4d\varepsilon' + 4\varepsilon'^2} < 4\sqrt{d\varepsilon'}$. Analogously, the projection $p(T)$ of $r(T)$ onto r' is at distance $< 4\sqrt{d\varepsilon'}$ from $s(T)$. Consequently, by [BH99, II.1.7(5)] we have that $\sin \angle_{r(t)}(p(t), s(t))$ and $\sin \angle_{r(T)}(p(T), s(T))$ are $< 4\sqrt{d\varepsilon'}/\delta(t)$. Since $\theta \leq 2\sin \theta$ for $\theta \in [0, \frac{\pi}{2}]$, and $\sqrt{\varepsilon'}/\delta(t) < \sqrt[4]{\varepsilon'}$, these Alexandrov angles are $< 8\sqrt{d}\sqrt[4]{\varepsilon}$. By the definition of ψ , the differences between $\frac{\pi}{2}$ and $\angle_{r(t)}(p(t), r(T))$ and $\angle_{r(T)}(p(T), r(t))$ are $\leq \sqrt{d}\sqrt[4]{\varepsilon}$ and the corollary follows. \square

11. LIMIT POINTS THAT ARE NOT FAR

In this section we describe particular pairs of limit points that are not far.

11.1. Angles between principal (and antiprincipal) points in $\partial_\infty \mathbf{X}$. We first prepare the proof of Lemma 9.6.

Lemma 11.1. *Let $\phi: D \rightarrow \nabla^+$ be an \mathbf{X} -reduced half-plane diagram with $\partial D = I_a \cdot I_b \cdot I_c$, where I_a, I_c are geodesic rays, and I_b is a possibly trivial geodesic segment. Suppose that the sum of the curvatures of vertices and edges in the interior of D and in the interior of I_a, I_b, I_c is $\geq -\varepsilon \geq -\varepsilon_0$ from Corollary 7.6. Let θ be the sum of the curvatures of $I_a \cap I_b$ and $I_b \cap I_c$.*

Then for each compact subcomplex $F \subset D$, there is a compact subcomplex $F' \subseteq F' \subset D$, such that for any point $y_a \in I_a \setminus F'$ and a unit vector u_a tangent to I_a at y_a , there exists a point $y_c \in I_c \setminus F'$ such that the unit vector u_c tangent to I_c at y_c , directed towards the same end of ∂D as u_a , satisfies the following. Namely, there is an embedded path of length in $[\theta - 3\varepsilon, \theta + 3\varepsilon]$ in the unit circle S , whose endpoints project to $\phi'_{y_a}(u_a), \phi'_{y_c}(u_c) \in S^\angle$.

Proof. Choose F' to be the minimal subcomplex of D such that

- $D \setminus F'$ is simply connected, and
- F' contains F , and
- F' contains all the non-oriented components of the folding locus L of D with
 - both ends on ∂D , and
 - starting at I_b or intersecting F .

Let u_a be a unit vector tangent to I_a at a point y_a outside F' . Then there is a piecewise geodesic path β in $D \setminus D^0$ from y_a to a point $y_c \in I_c$ that intersects each non-oriented component \mathcal{L} of L at most once. Furthermore, by our choice of F' , we can assume that for each such \mathcal{L} , there is a connected component of $\mathcal{L} \setminus \beta$ that does not end on ∂D . Let u_c be the unit vector tangent to I_c at y_c , directed towards the same end of ∂D as u_a .

We consider the non-oriented edges f of L intersected by β at points z , and we denote by \vec{f} a unit tangent vector to f at z that is not directed towards ∂D . By Lemma 7.10, the total angle of all $\phi'_z(\vec{f})$ from the principal direction is $\leq \varepsilon$.

Let D_β be the disc diagram that is the bounded connected component of $D \setminus \beta$. The sum of the curvatures outside β in D_β lies in $[\theta - \varepsilon, \theta]$. By Theorem 6.9, the sum of the curvatures along β (including its endpoints) lies in $[2\pi - \theta, 2\pi - \theta + \varepsilon]$. Thus, if we parallel transport u_a along β , we obtain a vector u in the obvious extension of S_{y_c} to the unit circle, at angle from u_c in $[\theta - \varepsilon, \theta]$. Furthermore, $\phi'_{y_c}(u)$ and $\phi'_{y_a}(u_a)$ differ by an angle $\leq 2\varepsilon$ since $\phi'_z(\vec{f})$ were at total angle $\leq \varepsilon$ from ∂S^\angle . \square

Proof of Lemma 9.6. Suppose $\angle(\xi, \eta) \neq \frac{2\pi}{3}$. Choose $\varepsilon \leq \varepsilon_0$ from Corollary 7.6 such that the interval $[\angle(\xi, \eta) - 4\varepsilon, \angle(\xi, \eta) + 4\varepsilon]$ does not contain 0 and does not contain $\frac{2\pi}{3}$. Choose x so that $\angle_x(\xi, \eta) > \angle(\xi, \eta) - \frac{\varepsilon}{3}$. Let $\phi': D' \rightarrow \nabla^+$ be a (ξ, η) -diagram for that x .

Suppose first that we are in case (ii) of Construction 10.1, where D' is isometric to the Euclidean half-plane in \mathbb{R}^2 . Consider a unit vector u tangent to $\partial D'$. Since ξ, η are both principal, by Lemma 10.2 the function $\phi'^\angle: \mathbb{S} \rightarrow S^\angle$ as in Definition 7.8 simultaneously assigns to u the principal and the antiprincipal point, contradiction.

Now assume that we are in case (i) of Construction 10.1. Let q' be the image of q in D' . By Lemma 10.3, the sum of the curvatures of vertices and edges in the interior of D' and in the interior of $\partial D' \setminus q'$ is $\geq -\varepsilon$. Let $J_\xi, J_\eta \subset \partial D'$ be the rays J' from Lemma 10.2 applied to ξ and η . Then the closure F of $\partial D' \setminus (J_\xi \cup J_\eta)$ is compact. Choose $F' \supseteq F$ provided by Lemma 11.1, and let $y_a \in J_\xi \setminus F'$ with $\phi'_{y_a}(u_a)$ the principal point. Let $u_c \in S_{y_c}$ be the vector guaranteed by Lemma 11.1.

By Lemma 10.3, we have $\theta \in [\pi - \angle(\xi, \eta), \pi - \angle(\xi, \eta) + \varepsilon]$, which is at distance $> 3\varepsilon$ from $\frac{\pi}{3}$ and from π . Since $\phi'_{y_c}(u_c)$ is the antiprincipal point, there is no path in S of length in $[\theta - 3\varepsilon, \theta + 3\varepsilon]$ whose endpoints project to $\phi'_{y_a}(u_a), \phi'_{y_c}(u_c)$. This contradicts the conclusion of Lemma 11.1.

Consequently, we have $\angle(\xi, \eta) = \frac{2\pi}{3}$. Let ζ be the midpoint of the geodesic $\xi\eta$ in $\partial_\infty \mathbf{X}$. Let $\mathbf{X}_\xi, \mathbf{X}_\eta$ be the connected components of the preimage of $\nabla^+ \setminus \{\alpha_2 = \alpha_3\}$ under ρ_+ , containing the interior of some (hence all by Lemma 8.2) principal ray $r_\xi, r_\eta: [0, \infty) \rightarrow \mathbf{X}$ representing ξ, η , respectively. By [BH99, II.9.10] (applied with $\frac{a'}{a} = 2$), the geodesic ray r_ζ representing ζ starting at $r_\xi(1)$ satisfies

$$d_{\mathbf{X}}(r_\xi(2t), r_\zeta(t))/t \nearrow \sqrt{3}.$$

Consequently, $\rho_+(r_\zeta)$ is disjoint from $\{\alpha_2 = \alpha_3\}$ and so $r_\zeta \subset \mathbf{X}_\xi$. Furthermore, since $\sqrt{3} < 2$, for any compact subset $K \subset \nabla^+$ we have that $\rho_+(r_\zeta(t))$ is disjoint from K for t sufficiently large. We have analogous properties for the geodesic ray r'_ζ representing ζ starting at $r_\eta(1)$.

Since $\xi \neq \eta$, we have that $\mathbf{X}_\xi, \mathbf{X}_\eta$ are disjoint. Since r_ζ, r'_ζ are asymptotic, we have that $\rho_+(r_\zeta)$ is contained in the set $\{\alpha_2 \leq m\alpha_3\}$, for some $m \geq 2$. We set K above to $\{\alpha_1 = \alpha_2 \leq m\alpha_3\}$, and we conclude that a subray $r_\zeta[T, \infty)$ of r_ζ is contained in the preimage $\mathbf{X}'_\xi \subset \mathbf{X}_\xi$ under ρ_+ of $\{\alpha_3 < \alpha_2 \leq m\alpha_3\} \setminus K$. Each connected component of \mathbf{X}'_ξ is quasi-isometric to the connected component of the preimage under ρ_+ of $\{\alpha_2 = m\alpha_3\} \setminus K$, which is a tree with only one infinite path and that path is an antiprincipal ray. Consequently, $r_\zeta[T, \infty)$ is antiprincipal, and it lies in the same chamber as the principal ray representing ξ starting at $r_\zeta(T)$, as desired.

The last assertion of the lemma follows from interchanging in the proof the words ‘principal’ and ‘antiprincipal’, defining $K = \{\alpha_1 \leq m\alpha_2 = m\alpha_3\}$ and replacing $\{\alpha_3 < \alpha_2 \leq m\alpha_3\} \setminus K$ by $\{\alpha_2 < \alpha_1 \leq m\alpha_2\} \setminus K$. \square

The following application of Lemma 9.6 produces distinct principal points in $\partial_\infty \mathbf{X}$ at angle $\neq \frac{2\pi}{3}$ (and hence far).

Lemma 11.2. *Let $g \in C$ be parabolic and let ζ be the class of the principal rays in \mathbf{E}_{id}^+ . Let $\eta \in \partial_\infty \mathbf{X}$ be principal with $\angle(\zeta, \eta) = \frac{2\pi}{3}$. Then $\angle(\eta, g(\eta)) \neq \frac{2\pi}{3}$ or $\angle(\eta, g^2(\eta)) \neq \frac{2\pi}{3}$.*

Proof. Suppose by contradiction $\angle(\eta, g(\eta)) = \angle(\eta, g^2(\eta)) = \frac{2\pi}{3}$. Consider the unique geodesics $\omega_1, \omega_2, \omega_3 \subset \partial_\infty \mathbf{X}$ joining ζ, η , and $g(\eta)$. The midpoints ξ_i of ω_i are antiprincipal by Lemma 9.6. If $\omega_1 \cdot \omega_2$ is not a locally embedded path, then $\angle(\xi_1, \xi_2) < \frac{2\pi}{3}$, and so by Lemma 9.6 we have $\xi_1 = \xi_2 = \xi_3$. Consequently, g fixes the entire geodesic $\zeta\xi_1$, which by Lemma 9.6 constitutes the boundary of a chamber. Thus by Lemma 8.1, we have that g lies in a conjugate of $C \cap B' = H$. By Proposition 3.4, g is elliptic, which is a contradiction.

Since $\partial_\infty \mathbf{X}$ is CAT(1) and is locally an \mathbb{R} -tree (Remark 9.7), we have that $\omega = \omega_1 \cdot \omega_2 \cdot \omega_3$ is isometrically embedded and thus by Lemma 6.7 there is a flat $F \subset \mathbf{X}$ with $\partial_\infty F = \omega$. Since g^2 is also not elliptic, we have analogously a flat F' with $\partial_\infty F'$ containing $\zeta, \eta, g^2(\eta)$.

We will now show that $F \cap F'$ is nonempty. Choose $x \in F, x' \in F'$ and let $a = d_{\mathbf{X}}(x, x')$. Let $r_\zeta, r_\eta: [0, \infty) \rightarrow F, r'_\zeta, r'_\eta: [0, \infty) \rightarrow F'$ be the geodesic rays representing ζ, η starting at x, x' . Let y be a point in the sector bounded by r_ζ, r_η in F and at distance $> a$ from $r_\zeta \cup r_\eta$. We can assume that y lies in the interior

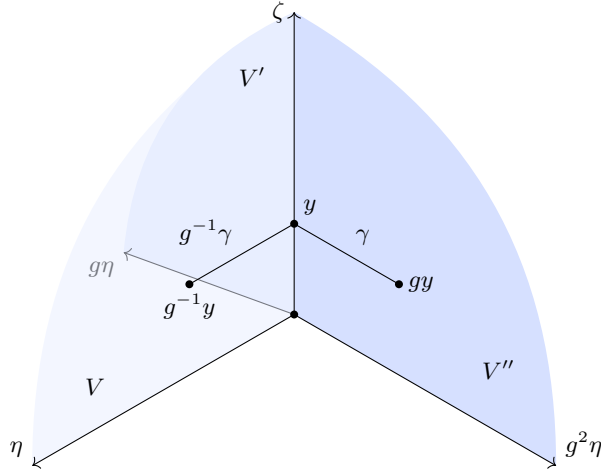


FIGURE 26.

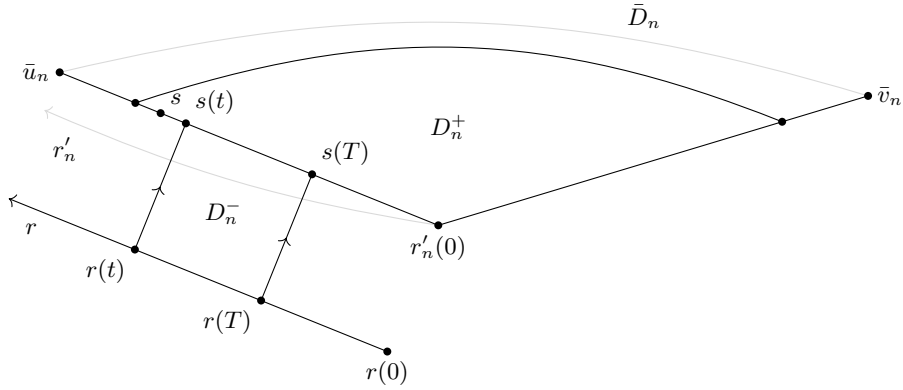
of a 2-cell. Let $R > d_{\mathbf{X}}(x, y) + a$, and let $\beta \subset F$ be the arc of the circle centred at x of radius R from $r_{\zeta}(R)$ to $r_{\eta}(R)$. Let $\beta' \subset F'$ be the arc of the circle centred at x' of radius R from $r'_{\zeta}(R)$ to $r'_{\eta}(R)$. Then the corresponding points of the closed paths $\alpha = r_{\zeta}[0, R] \cdot \beta \cdot r_{\eta}[R, 0]$ and $\alpha' = r'_{\zeta}[0, R] \cdot \beta' \cdot r'_{\eta}[R, 0]$ are at distance $\leq a$, since they are points of the corresponding asymptotic geodesic rays starting at x, x' . Consequently, α' is homotopic to α in $\mathbf{X} - y$. By projecting to a small metric ball around y homeomorphic to a disc, we see that α is homotopically nontrivial in $\mathbf{X} - y$. Thus the same is true for α' , and so $y \in F'$. Consequently, $F \cap F'$ contains the geodesic ray representing ζ starting at y . Analogously, $F \cap g(F)$ and $F' \cap g(F)$ contain geodesic rays representing ζ .

We will now prove that there is a geodesic ray r representing ζ in $F \cap F' \cap g(F)$. For contradiction, assume that all of $V = F \cap F', V' = F \cap g(F), V'' = g(F) \cap F'$ are disjoint. Let b_{ζ} be a Busemann function at ζ . Note that $V, V' \subset F$ are nonempty, closed, convex (Remark 6.6(i)) and invariant under positive translation in the direction ζ . Let $c, c': \mathbb{R} \rightarrow F$ parametrise $\partial V, \partial V'$, in a way that for t sufficiently large we have $b_{\zeta}(c(t)) = b_{\zeta}(c'(t)) = -t$. There is t such that for $T > t$ the curves $\partial V, \partial V'$ are at $c(T), c'(T)$ at Alexandrov angle $> \frac{\pi}{6}$ from the geodesic $c(T)c'(T)$. We can arrange that the same condition holds for $c'(T)c''(T) \subset gF$ joining $\partial V'$ with $\partial V''$ and for $c''(T)c(T) \subset F'$ joining $\partial V''$ with ∂V . Then the geodesic triangle $c(T)c'(T)c''(T)$ has all Alexandrov angles $> \frac{\pi}{3}$, which is a contradiction.

Consequently, there is a geodesic ray r representing ζ in $F \cap F' \cap g(F)$. Note that r lies in $\partial V, \partial V'$, and $\partial V''$. Choose $y \in r$ and let γ be the geodesic $yg(y)$. Note that $\gamma \cap V' = y$, since otherwise g would fix the midpoint of γ or would fix $g(\eta)$. See Figure 26. Consequently, $g^{-1}(\gamma) \cdot \gamma$ is a local geodesic. Thus $\bigcup_{k \in \mathbb{Z}} g^k(\gamma)$ is an axis of g , and so g is loxodromic, which is a contradiction. \square

11.2. Reaching far.

Proposition 11.3. *Let $g, g' \in \text{Tame}(\mathbb{k}^3)$ be conjugate and parabolic of nonzero translation length or loxodromic not of rank 1. Suppose that for principal points $\zeta, \zeta' \in \partial_{\infty} \mathbf{X}$ fixed by g, g' , respectively (if these points exist), we have $\zeta \neq \zeta'$, and the same holds for ζ, ζ' antiprincipal. (Equivalently, g, g' do not belong to a common conjugate of C or B'). Then any limit points ξ of g and η of g' are far.*

FIGURE 27. The images of D_n^\pm .

Proof. To begin with, we justify $\xi \neq \eta$. Otherwise, if, say, $g \in B'$ and $\zeta \neq \zeta' \in \partial_\infty \mathbf{X}$ are the antiprincipal points fixed by g, g' , then, by Construction 8.3(1), we have length $\frac{\pi}{2}$ paths $\omega, \omega' \subset \partial_\infty \mathbf{X}$ between $\xi = \eta$ and ζ, ζ' , respectively. By Lemma 9.6, we have that ζ is at distance $\frac{2\pi}{3}$ from ζ' . Since $\partial_\infty \mathbf{X}$ is CAT(1) and is locally an \mathbb{R} -tree (Remark 9.7), the geodesic $\zeta\zeta'$ is contained in the union of ω and ω' , and so in particular the midpoint of $\zeta\zeta'$, which is principal by Lemma 9.6, lies in ω and ω' . Consequently, this midpoint is represented by a geodesic ray in \mathbf{H} from Construction 8.3. However, such a ray does not satisfy Lemma 8.2, which is a contradiction.

The remaining proof is divided into five parts.

Part 1. *Defining D'^+ and D'^- .*

Choose x so that $\angle_x(\xi, \eta) > \angle(\xi, \eta) - \frac{\varepsilon}{3}$ for $\varepsilon < \varepsilon_0$ from Corollary 7.6. Let $\phi'^+ : D'^+ \rightarrow \nabla^+$ be a (ξ, η) -diagram from Construction 10.1.

Denote by $q' \in \partial D'^+$ the image of $q \in \partial D^+$, and by J_ξ^+, J_η^+ the closures of the appropriate connected components of $\partial D'^+ \setminus q'$. Recall that D'^+ is CAT(0), and J_ξ^+, J_η^+ are geodesic rays in D'^+ representing points in $\partial_\infty D'^+$ at angle $\angle(\xi, \eta)$. Let $\beta_n^+ = \tilde{\phi}_{n|J_\xi^+ \cap I_n}^+$.

Let r representing ξ be an \mathbf{H} -generic ray guaranteed by Remark 8.7(i). Let r'_n representing ξ be starting at p_n (in case (ii) of Construction 10.1) or x (in case (i)). Suppose first that:

(\heartsuit) for each t there is $\delta(t) > 0$ such that each r'_n is at distance $\geq \delta(t)$ from $r(t)$.

Let d be the maximal distance from $r(0)$ to x or a point of P . For ε above replaced by $\varepsilon^4/18^4 d^2$, let $T > 0$ be provided by Corollary 10.5, which applies to all r'_n . Then for each $t > T$, for sufficiently small $\varepsilon' > 0$ there is n with s in β_n^+ satisfying the second bullet point of Corollary 10.5. By Lemma 6.3, we have reduced relative disc diagrams $\tilde{\phi}_n^- : D_n^- \rightarrow \mathbf{X}$ (embedded by Theorem 6.10) bounded by geodesic quadrilaterals $r(t)r(T)s(T)s(t)$ with total curvature $\geq -\varepsilon$ outside the preimages under $\tilde{\phi}_n^-$ of $s(t), s(T), r(T), r(t)$. See Figure 27.

Let $I_n^- \subset \partial D_n^-$ be the union of the three sides not mapping under $\tilde{\phi}_n^-$ to $r(t)s(t)$. By passing to a subsequence and subdiagrams, we can assume that (D_n^-, I_n^-) converge to a half-plane diagram $(D^-, \partial D^-)$ with $\partial D^- = J \cdot I_b \cdot I_c$, where the intersections of J, I_b, I_c with ∂D_n^- map under $\tilde{\phi}_n^-$ into $r(t)r(T), r(T)s(T), s(T)s(t)$, respectively. Since $r(T)s(T)$ has uniformly bounded length, the first part of the second bullet point in Remark 7.1 implies the second part. Thus D^- has the frilling D'^- with the \mathbf{X} -reduced relative half-plane diagram $\phi'^- : D'^- \rightarrow \nabla^+$ that is the limit of $\phi_n^- = \rho_+ \circ \tilde{\phi}_n^-$ bordered by $r_0 = r_{|[T, \infty)}$, where we identify $[T, \infty)$ with $J \subset \partial D'^-$.

Part 2. *Calculating the angle.*

By Corollary 10.5 and Theorem 6.9, the angles of $J \cap I_b, I_b \cap J_c$ are at distance $\leq \varepsilon$ from $\frac{\pi}{2}$. We now apply three times Lemma 11.1, to D'^-, D'^+ (using Lemma 10.3), and to the analogue $D'^-(\eta)$ of D'^- with ξ replaced by η . For ε sufficiently small, we obtain vectors $u, u(\eta)$ tangent to $J, J(\eta)$ at points $y, y(\eta)$, with $\phi_y'^{-\angle}(u), \phi_y'^{-\angle}(u)(\eta)$ that are projections of the endpoints of an embedded path in S of length arbitrarily close to $\pi - \angle(\xi, \eta)$. Furthermore, we can arrange that $y, y(\eta)$ are in arbitrary subrays of $J, J(\eta)$. Since $r_0, r_0(\eta)$ are \mathbf{H} -generic, this implies that $\phi_y'^{-\angle}(u), \phi_y'^{-\angle}(u)(\eta)$ are arbitrarily close to the midpoint of S^\angle . It follows that $\angle(\xi, \eta) = \frac{\pi}{3}, \frac{2\pi}{3}$, or π .

Part 3. *Curvature in $Q \subset \widehat{D}'$.*

Let $\widehat{\phi}_n : \widehat{D}_n \rightarrow \mathbf{X}$ be the unions of $\tilde{\phi}_n^\pm$ at $s(T)s(t)$, to which we apply Remark 7.3 describing their limit $\widehat{\phi}' : \widehat{D}' \rightarrow \nabla^+$. In the case where $\overline{D'^- \setminus R^-}$ is not homeomorphic to a half-plane, we use the more general Remark 7.4. This covers the case of empty $\overline{D'^- \setminus R^-}$, where $\widehat{D}' = \overline{D'^+ \setminus R^+}$. After possibly increasing T , we can assume that R^+ is disjoint from J_η^+ , and so $J_\eta^+ \subset \widehat{D}'$. Note that $\widehat{\phi}'$ is bordered by r_0 . Observe that a geodesic ray in $D'^+ \setminus R^+$ asymptotic to J_ξ^+ in D'^+ is also asymptotic to J in \widehat{D}' . In particular, the rays J and J_η^+ represent points in $\partial_\infty \widehat{D}'$ at angle $\angle(\xi, \eta)$.

At this point, let us note that if (\heartsuit) was not satisfied, and so there is T such that a subsequence of r'_n becomes arbitrarily close to $r(T)$, then there are subrays of r'_n converging to $r_0 = r_{|[T, \infty)}$. This implies that β_n^+ have subpaths converging to r_0 and so ϕ'^+ is weakly bordered by r_0 (see Definition 7.2). After modifying ϕ_n^+ on the cells intersecting J , we have that the same ϕ'^+ is bordered by r_0 , and we set $\widehat{\phi}' = \phi'^+$ in the remaining discussion (where the following claim is trivial).

We claim that the sum of the curvatures of the vertices and edges of \widehat{D}' is finite. Indeed, if $\angle(\xi, \eta) < \pi$, then by [BH99, II.9.8(4)], the geodesics in \widehat{D}' joining the points on J and J_η^+ sufficiently far from a basepoint are disjoint from any given compact subcomplex of \widehat{D}' . Consequently, by Theorem 6.9, there is a uniform bound on the total curvature in any compact subcomplex of \widehat{D}' , as desired. For $\angle(\xi, \eta) = \pi$, one repeats this argument after subdividing \widehat{D}' into two half-plane diagrams along a ray representing at point in $\partial_\infty \widehat{D}'$ at distance $\frac{\pi}{2}$ from the points represented by J and J_η^+ . This justifies the claim.

By the claim, there exists a compact subcomplex $F \subset \widehat{D}'$, such that the sum of the curvatures in \widehat{D}' of the vertices and edges outside F is $\geq -\varepsilon_0$. Let $l_k \in J$ be the sequence of transition points of r_0 . There is k_0 such that the unbounded connected component of $J \setminus l_{k_0}$ is disjoint from F . Applying Lemma 8.8 to a neighbourhood of that component in \widehat{D}' , we obtain that for $k \geq k_0$ the points l_k are contained in single

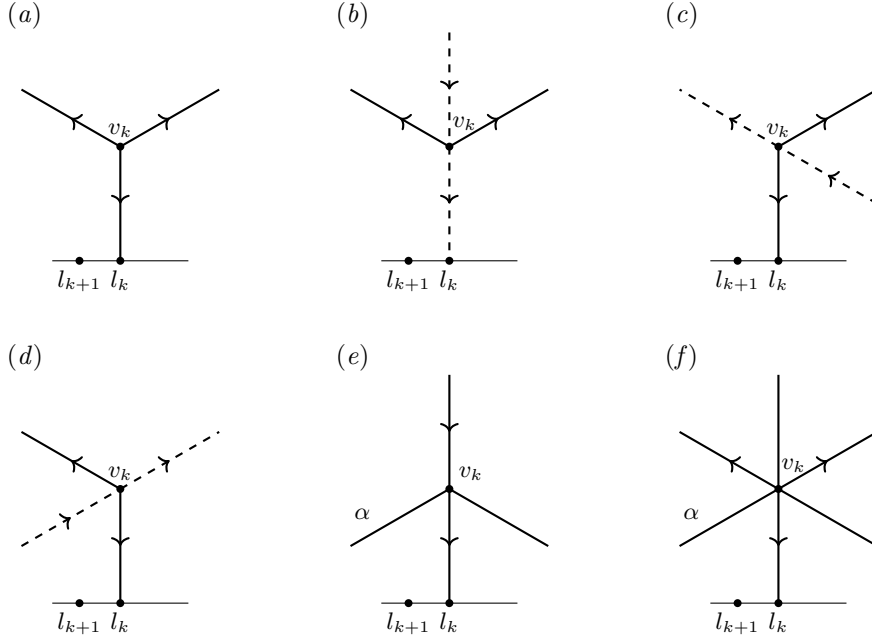


FIGURE 28.

edges of the folding locus \widehat{L} and at angle to J in $(\frac{\pi}{2} - \varepsilon_0, \frac{\pi}{2} + \varepsilon_0)$ and converging to $\frac{\pi}{2}$.

After possibly increasing k_0 , a geodesic γ in \widehat{D} , issuing from l_{k_0} at angle $\frac{\pi}{2} + \varepsilon_0$ to the unbounded connected component of $J \setminus l_{k_0}$, separates F from l_{k_0+1} . Cutting \widehat{D}' along γ , we obtain a half-plane diagram Q containing l_{k_0+1} , where the sum of all curvatures, except the curvature of l_{k_0} , and possibly except of the curvature of other endpoint of γ , if γ is bounded, is $\geq -\varepsilon_0$.

Part 4. Folding locus.

Let $L_k \subset \widehat{L}$ be the connected component of \widehat{L} minus the degree > 2 vertices of \widehat{L} , containing l_k . For each $k \geq k_0$, we have that $L_k \cap Q$ are geodesics in Q , as are $J \cap Q$ and γ . Considering the angles that L_k and γ make with J , we obtain that if $L_k \cap Q$ is bounded, then it does not end in $J \cup \gamma$. In particular, $L_k \cap Q = L_k$.

If L_k is contained in a non-oriented component \mathcal{L} of \widehat{L} , then analogously $\mathcal{L} \cap Q$ does not end in $J \cup \gamma$, and so $\mathcal{L} \cap Q = \mathcal{L}$. Furthermore, since J is not asymptotic to J_η^+ , after possibly increasing k_0 , we have that if \mathcal{L} has both ends in ∂Q , then it has arbitrarily large length. Thus, by Lemma 7.10, for each $x \in L_k$ and $\vec{f} \in S_x$ the direction of an edge of L_k , directed away from l_k , we have that $\widehat{\phi}_x^{\angle}(\vec{f}) \in S^\angle$ is at angle $\leq \varepsilon_0$ from a point in ∂S^\angle (same point for all k). The same would hold for other edges of the folding locus starting at J sufficiently far from γ , which is excluded by the last assertion of Lemma 8.8. Consequently, after possibly increasing k_0 , we have $\widehat{L} \cap J \cap Q = \{l_k\}_{k \geq k_0}$.

Suppose first that all $\widehat{\phi}_x^{\angle}(\vec{f}) \in S^\angle$ above are at angle $\leq \varepsilon_0$ from the antiprincipal point. If L_k is bounded and ends with an interior vertex v_k of Q , then by Corollary 7.6, we have one of the configurations in Figure 28, where the dashed edges might be oriented as indicated or non-oriented. However, configurations (e,f) can be excluded in the following way. If the non-oriented component α intersects ∂Q , then

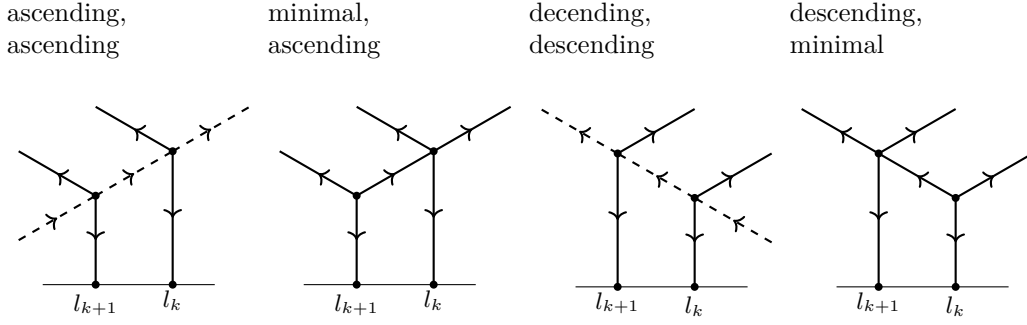


FIGURE 29.

this contradicts the previous paragraph. If α is disjoint from ∂Q , then cutting α into two and applying Lemma 7.10, we see that α contributes at least $\frac{2\pi}{3}$ to the curvature of Q , which is a contradiction. Analogously, the edges in configuration (d) must be oriented.

We call v_k as in configuration (c) *descending*, in (d) *ascending*, and in (a,b) *minimal*. Let α' be the connected component of \widehat{L} minus the degree > 2 vertices of \widehat{L} , containing the north-west edge in (a,b,c) or the south-west edge in (d). If α' is non-oriented (which might happen only for v_k descending), and ends at v in Figure 17(h), then we call v_k *wicked*.

Suppose that v_k is not wicked. Since J and J_η^+ represent points at $\partial_\infty \widehat{D}'$ at angle $\geq \frac{\pi}{3} > \frac{\pi}{6}$, we have that α' does not end on ∂Q . Consequently, L_{k+1} also ends with an interior vertex v_{k+1} of Q . Moreover, applying Corollary 7.6 to the endpoint of α' distinct from v_k , and recalling that the edges in configuration (d) must be oriented, we conclude that this endpoint coincides with v_{k+1} , and v_{k+1}, v_k are either

- ascending, ascending (see Figure 29), or
- minimal, ascending (see Figure 29 for the case that uses the minimal configuration from Figure 28(a)), or, symmetrically,
- descending, descending, or
- descending, minimal.

Note that for v_k wicked we could have other combinations. However, all $v_{k'}$ with $k > k' \geq k_0$ are then descending and not wicked. Consequently, after possibly replacing k_0 with $k + 1$, we can assume that none of v_k are wicked.

If the sequence v_{k+2}, v_{k+1}, v_k was descending, minimal, ascending, we would obtain a contradiction with Remark 7.7(ii) applied to the north-east edge in Figure 28(c) and the north-west edge in Figure 28(d). Furthermore, we cannot have an infinite sequence of ascending vertices. Consequently, after possibly increasing k_0 , all v_k are descending, see Figure 30, left. The remaining possibility is that all L_k are connected components of \widehat{L} , where L_k might be unbounded (see Figure 30, centre), or bounded (possibly non-oriented) if they end on ∂Q (see Figure 30, right).

Second, suppose that all $\widehat{\phi}_x^{\neq}(f) \in S^\angle$ are at angle $\leq \varepsilon_0$ from the principal point. If L_k is bounded and ends with an interior vertex v_k of Q , then we have one of the configurations in Figure 31, where the dashed edges in (a) might be oriented as indicated or non-oriented. In (a,b,c), the oriented geodesic α starting at v_k , given by Remark 7.7(ii), terminates at J at angle contradicting $\widehat{L} \cap J \cap Q = \{l_k\}_{k \geq k_0}$. In (d),

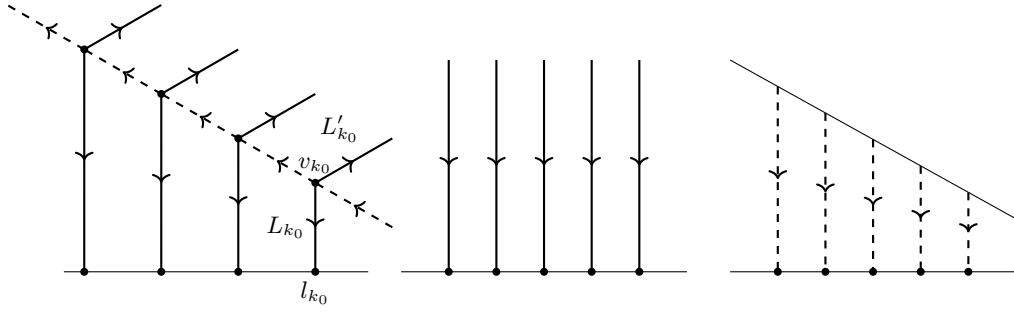


FIGURE 30.

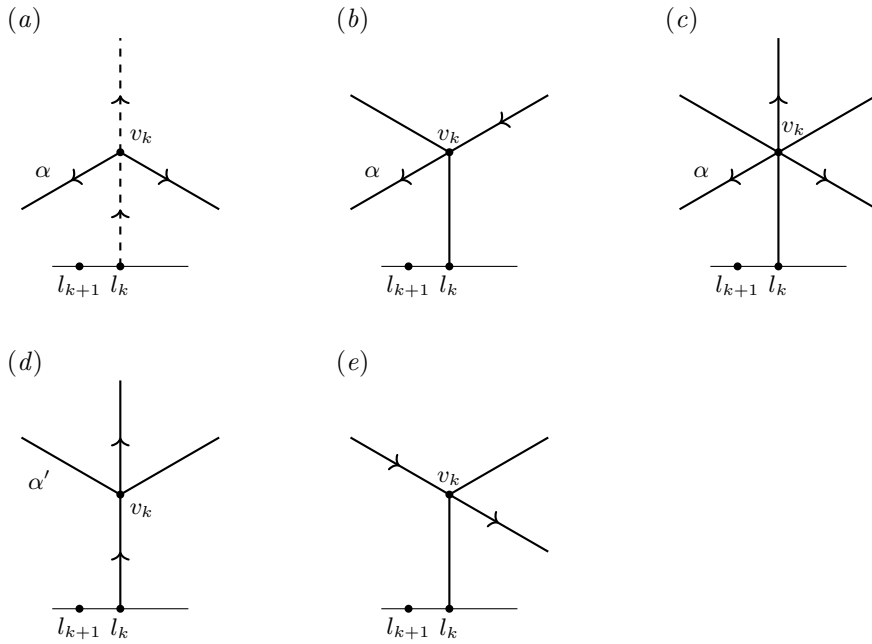


FIGURE 31.

the non-oriented component α' is disjoint from ∂Q , since J and J_η^+ represent points at $\partial_\infty \widehat{D}'$ at angle $> \frac{\pi}{6}$. Cutting α' into two and applying Lemma 7.10, we see that α' contributes at least $\frac{2\pi}{3}$ to the curvature of Q , which is a contradiction. It follows that all v_k are as in (e), see Figure 32, left, or all L_k are connected components of \widehat{L} , see Figure 32, centre and right (where some L_k are possibly non-oriented).

Part 5. Interaction.

To summarise all the possibilities for both ξ and η , consider first the situation where there is k_0 such that for all $k \geq k_0$ the L_k , denoted from now on by $L_k(\xi)$, are unbounded. Since they are at angle to J converging to $\frac{\pi}{2}$ as $k \rightarrow \infty$, they are geodesic rays representing points in $\partial_\infty \widehat{D}'$ at distance $\geq \frac{\pi}{2}$ from the point represented by J . Suppose that after replacing ξ by η we reach the same conclusion for $L_k(\eta)$. Then the subrays of $L_k(\xi)$ and $L_k(\eta)$ lying in D'^+ are disjoint. Since they represent points in $\partial_\infty D'^+$ at distance $\geq \frac{\pi}{2}$ from J_ξ^+, J_η^+ , they have to be asymptotic. Proposition 8.9(1),

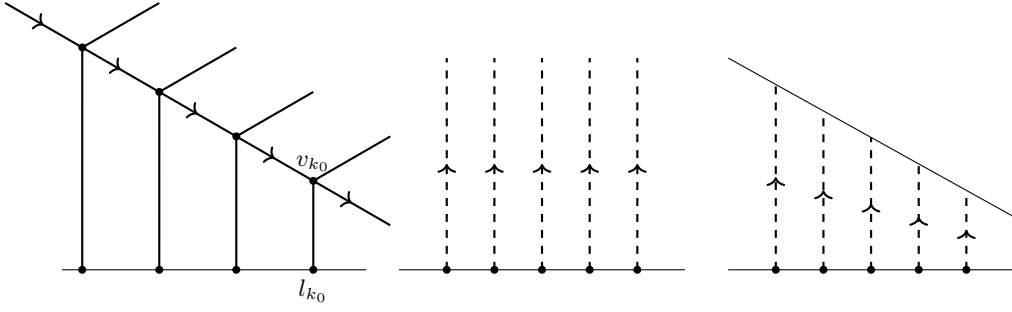


FIGURE 32.

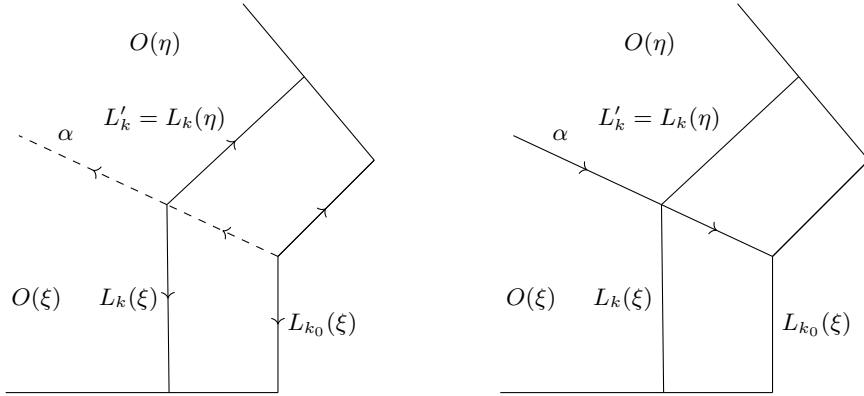


FIGURE 33.

applied twice, implies that g, g' fix a common principal or antiprincipal point in $\partial_\infty \mathbf{X}$, which is a contradiction.

Second, suppose that $L_k(\xi)$ are bounded but end on $\partial \widehat{D}'$. If $L_k(\eta)$ are unbounded, then subrays of $L_k(\eta)$ and $L_k(\xi)$ lying in D'^+ intersect, which is a contradiction. If $L_k(\eta)$ are bounded connected components of the appropriate folding locus, then, after renumbering, subrays of $L_k(\xi)$ and $L_k(\eta)$ lying in D'^+ coincide. This implies that J_ξ^+ and J_η^+ are asymptotic, which is a contradiction.

Finally, suppose that we have the configuration from Figure 30, left. Consider the north-east oriented geodesics $L'_k \subset \widehat{L}$ starting at v_k . In particular, if $L_k(\eta)$ are unbounded, then subrays of $L_k(\eta)$ and L'_k lying in D'^+ intersect, which is a contradiction. Otherwise, we have that subsegments of $L_k(\eta)$ coincide with subsegments of L'_k , since otherwise they would intersect in D'^+ . See Figure 33, left. The analogous remaining possibility is described in Figure 33, right.

In both of these configurations, Let $O(\xi)$ be the half-plane diagram in \widehat{D}' bounded by $L_{k_0}(\xi)$, and the ray $\alpha \subset \widehat{L}$ containing all v_k for $k \geq k_0$, and let $O(\eta)$ be defined analogously. Applying Proposition 8.9(2) to $O(\xi)$ and $O(\eta)$, we find k such that the unique antiprincipal or principal ray starting at v_k represents an antiprincipal or principal point in $\partial_\infty \mathbf{X}$ fixed by both g and g' , which is a contradiction. \square

12. PROOF OF THE TITS ALTERNATIVE

We start with the following preparatory lemmas. In the applications of the first one, the set ∂ will be the visual boundary $\partial_\infty \mathbf{X}$, which explains the notation.

Lemma 12.1. *Let G be a group acting on a set ∂ . Let Z be a finite subset of ∂ . Suppose that for each $u \in G$, the sets Z and $u(Z)$ intersect. Then there is a nonempty finite subset of the union of $u(Z)$ over $u \in G$ that is invariant under G .*

Let $|Z|$ denote the number of elements of Z . Before we start the proof, note that for $|Z| = 2$, the sets $u(Z)$ have either an element in common or take on only three values $\{\xi, \eta\}, \{\eta, \zeta\}, \{\zeta, \xi\}$. In particular, the invariant nonempty finite set can be taken as $\bigcap_{u \in G} u(Z)$ or $\bigcup_{u \in G} u(Z)$. For $|Z| = 3$, this is no longer true. Namely, consider the 2-dimensional simplicial complex Y built of a triangle with vertices ξ, η, ζ , and infinitely many triangles T_i^ζ (resp. T_i^η, T_i^ξ) sharing the edge $\xi\eta$ (resp. $\xi\zeta, \eta\zeta$). Let G be the automorphism group of Y , let ∂ be the vertex set of Y and let Z be the vertex set of T_1^ζ . Then $\bigcap_{u \in G} u(Z) = \emptyset, \bigcup_{u \in G} u(Z) = \partial$, and the required G -invariant finite set is $\{\xi, \eta, \zeta\}$.

We will be applying Lemma 12.1 only with $|Z| \leq 4$.

Proof of Lemma 12.1. We prove the lemma by induction on $|Z|$. If $|Z| = 1$, then Z is invariant under G . If $|Z| > 1$, then let $Z' \subsetneq Z$ be a maximal subset contained in infinitely many distinct translates $u(Z)$, where $u \in G$. If Z' is empty, then the union of all $u(Z)$ over $u \in G$ is finite and constitutes the desired finite set invariant under G . If Z' is nonempty, then we claim that for each $u \in G$ the set $u(Z')$ intersects Z' , which allows to appeal to the induction and finishes the proof.

To justify the claim, observe that by the maximality of Z' , there is a sequence $(u_i)_{i \geq 1}$ in G such that the sets $u_i(Z) \setminus Z'$ are pairwise disjoint. Consequently, the sets $uu_i(Z) \setminus u(Z')$ are pairwise disjoint. In particular, we can fix j such that $uu_j(Z) \setminus u(Z')$ is disjoint from Z' . Since $uu_j(Z)$ intersects all $u_i(Z)$, and all the sets $u_i(Z) \setminus Z'$ are pairwise disjoint, we have that $uu_j(Z)$ intersects Z' . By the choice of j , we have that $u(Z')$ intersects Z' , justifying the claim. \square

Lemma 12.2. *Let G be a group and let F be a finite group. Suppose that for each finitely generated subgroup $H < G$ we have a nonempty family \mathcal{F}_H of homomorphisms from H to F satisfying the following. For all finitely generated subgroups $H < H'$ of G , and each $f \in \mathcal{F}_{H'}$, we have $f|_H \in \mathcal{F}_H$. Then there is a homomorphism $f: G \rightarrow F$ such that for each finitely generated $H < G$ we have $f|_H \in \mathcal{F}_H$.*

Proof. The following proof was found by Marcin Sabok. Let $K = F^G$ be the compact space of all the functions from G to F . For each finitely generated subgroup $H < G$, let K_H be the nonempty subset of K consisting of all $f \in K$ satisfying $f|_H \in \mathcal{F}_H$. Since F is finite and H is finitely generated, we have that \mathcal{F}_H is finite, and so K_H is closed. Note that the family $\{K_H\}$ has the finite intersection property. Indeed, for finitely generated subgroups H_1, \dots, H_k of G , the group $H = \langle H_1, \dots, H_k \rangle$ is also finitely generated, and so any element of K_H belongs to all K_{H_i} . Since K is compact, there is $f \in K$ in the intersection of $\{K_H\}$, as desired. \square

Corollary 12.3. *Let $n, m > 0$ and let G be a group. Suppose that each finitely generated subgroup of G contains a non-abelian free group or a solvable subgroup of index $\leq n$ and step $\leq m$. Then G satisfies the strong Tits alternative.*

Proof. Suppose that G does not contain a non-abelian free group. Let F be the symmetric group on n elements. For each finitely generated subgroup $H < G$, let \mathcal{F}_H be the nonempty family of homomorphisms from H to F whose kernel is solvable of step $\leq m$. By Lemma 12.2, there is a homomorphism $f: G \rightarrow F$ such that for each finitely generated $H < G$ we have $f|_H \in \mathcal{F}_H$. Let G' be the kernel of f . Thus each finitely generated subgroup of G' is solvable of step $\leq m$. Consequently, G' is solvable of step $\leq m$. \square

Proof of the Main Theorem. Let $G < \text{Tame}(\mathbb{k}^3)$. We distinguish three cases according to the type of elements in G with respect to the action of $\text{Tame}(\mathbb{k}^3)$ on \mathbf{X} .

Case 1. All elements of G are elliptic.

Consider any finitely generated subgroup H of G . By [NOP21, Cor 2.6], the group H has a global fixed point in \mathbf{X} . By Remark 3.6(i,ii), the group H contains a non-abelian free group or a solvable subgroup of uniformly bounded step and index. Hence G satisfies the strong Tits alternative by Corollary 12.3.

Case 2. G contains a parabolic element g or a loxodromic element g that is not of rank 1.

By Remark 3.6(iii) and Lemma 8.1, we can suppose that G is not *elementary* in the sense that it is not virtually contained in the stabiliser of a principal or antiprincipal point in $\partial_\infty \mathbf{X}$. In other words, there is no finite subset of the union of the principal and antiprincipal points in $\partial_\infty \mathbf{X}$ that is invariant under G .

Suppose first that g has zero translation length. Let ζ be the unique principal point in $\partial_\infty \mathbf{X}$ fixed by g guaranteed by Theorem 9.1. Since G is not elementary, there is $h \in G$ with $h(\zeta) \neq \zeta$. If $\angle(\zeta, h(\zeta)) \neq \frac{2\pi}{3}$, then ζ and $h(\zeta)$ are far by Lemma 9.6. We set $h' = h$ in that case. If $\angle(\zeta, h(\zeta)) = \frac{2\pi}{3}$, then by Lemma 11.2, we have $\angle(h(\zeta), gh(\zeta)) \neq \frac{2\pi}{3}$ or $\angle(h(\zeta), g^2h(\zeta)) \neq \frac{2\pi}{3}$. Furthermore, $h(\zeta) \neq gh(\zeta), g^2h(\zeta)$ by the uniqueness in Theorem 9.1 applied to g and g^2 . Hence for $h' = h^{-1}gh$ or $h' = h^{-1}g^2h$, we have that ζ and $h'(\zeta)$ are far by Lemma 9.6.

Let $g' = h'gh'^{-1}$. By Proposition 2.9 and Proposition 9.2, for sufficiently large n we have that $f_n = (g')^n g^{-n}$ has far limit points, in arbitrary neighbourhoods of $\zeta, h'(\zeta)$. Since G is not elementary, by Lemma 12.1 there is $u \in G$ such that $Z = \{\zeta, h'(\zeta)\}$ is disjoint from $u(Z) = \{u(\zeta), uh'(\zeta)\}$. In particular, for n sufficiently large, f_n and $uf_n u^{-1}$ have distinct limit points. Thus by Corollary 2.7, there are powers of f_n and $uf_n u^{-1}$ that generate a non-abelian free group.

Second, suppose that g has nonzero translation length. By Theorems 9.1 and 9.4, there is a unique principal point ζ and/or a unique antiprincipal point ζ' fixed by g in $\partial_\infty \mathbf{X}$. Let $Z \subset \partial_\infty \mathbf{X}$ be $\{\zeta\}, \{\zeta'\}$ (if only one of the two points exists), or $\{\zeta, \zeta'\}$. Since G is not elementary, by Lemma 12.1 there is $h \in G$ with $Z \cap h(Z) = \emptyset$. By Proposition 11.3, the limit points ξ, η of g and $h(\xi), h(\eta)$ of $g' = hgh^{-1}$ are far. Again, by Proposition 2.9 and Proposition 9.2, for sufficiently large n we have that $f_n = (g')^n g^{-n}$ has far limit points, in arbitrary neighbourhoods of $\xi, h(\xi)$. Since G is not elementary, applying Lemma 12.1 to the set $Z \cup h(Z)$ in the place of Z , there is $u \in G$ such that $\zeta, \zeta', h(\zeta), h(\zeta')$ are distinct from $u(\zeta), u(\zeta'), uh(\zeta), uh(\zeta')$. By Proposition 11.3, the points $\xi, h(\xi), u(\xi), uh(\xi)$ are distinct. Thus for n sufficiently large, f_n and $uf_n u^{-1}$ have distinct limit points, and we conclude as before.

Case 3. G contains a loxodromic element g , and all its elements are loxodromic of rank 1 or elliptic.

If a finite index subgroup G' of G fixes one of the two limit points $\xi \in \partial_\infty \mathbf{X}$ of g , then let b_ξ be a Busemann function for ξ . The group G' acts on the level sets of b_ξ giving rise to a homomorphism $G' \rightarrow \mathbb{R}$, whose kernel H preserves the level sets. By Case 1, we can assume that there is $h \in H$ that is not elliptic. Thus h is loxodromic. Let $\sigma: \mathbb{R} \rightarrow \mathbf{X}$ be an axis of h . Since $h \in H$, the axis σ is contained in a level set, say $b_\xi = 0$. For each $s \in \mathbb{R}, t \geq 0$, let $\sigma(s, t)$ be the projection of $\sigma(s)$ onto the horoball $b_\xi \leq -t$. Thus for each $s, s' \in \mathbb{R}, t \geq 0$, we have $d_{\mathbf{X}}(\sigma(s, t), \sigma(s', t)) \leq d_{\mathbf{X}}(\sigma(s), \sigma(s'))$. On the other hand, since h acts on the curve $\sigma(\cdot, t)$ as a translation, and since σ was an axis of h , we have a converse inequality, and so $d_{\mathbf{X}}(\sigma(s, t), \sigma(s', t)) = d_{\mathbf{X}}(\sigma(s), \sigma(s'))$. Consequently, σ bounds an isometrically embedded Euclidean half-plane $\sigma(\cdot, \cdot)$, and so h is not of rank 1, which is a contradiction.

Thus, by Lemma 12.1, there is $u \in G$ such that g and ugu^{-1} have distinct limit points, and we conclude as before. \square

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