Weak Tits alternative for uniform lattices in buildings

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Abstract

We show that if a group G acts geometrically by type-preserving automorphisms on a building, then G satisfies the weak Tits alternative, namely, that G is either virtually abelian or contains a non-abelian free group.

1 Introduction

Buildings were introduced by Jacques Tits in the 1950s as a tool to study semisimple algebraic groups. Since their inception, buildings have found diverse applications throughout mathematics, well beyond their roots in the theory of algebraic groups; see for instance the survey article [10].

An ongoing area of interest has been in the study of algebraic properties of groups acting on buildings. Buildings might be equipped with a structure of a nonpositively curved metric space (see e.g. [8, Chapter 18]), so it is believed that groups acting on them in a nice enough manner exhibit a property shared by many 'non-positively curved' groups: the *Tits alternative*. The Tits alternative is a dichotomy for groups and their subgroups, first studied by Tits in [15], where it was shown that every finitely generated linear group is either virtually solvable or contains the free group F_2 as a subgroup. We will consider a weaker version of the Tits alternative: we will say that a group satisfies the *weak Tits alternative* if it is either virtually abelian or contains F_2 as a subgroup. The weak Tits alternative has been shown to be satisfied for groups acting properly and cocompactly on Euclidean buildings in [2, Theorem 8.10]. Sageev and Wise show in [14] that groups acting properly on finite-dimensional CAT(0) cube complexes with a bound on the cardinality of finite subgroups satisfy the Tits alternative. In particular, this implies the Tits alternative for such groups acting properly on right-angled buildings. The Tits alternative was proved for groups acting properly with a bound on the cardinality of finite subgroups on 2-dimensional complexes with some 'non-positive curvature' features in [11, 12]. This covers the case of all 2-dimensional buildings.

In this paper, we extend the above results obtained for Euclidean, right-angled, and 2-dimensional buildings to actions on arbitrary finite rank buildings. Our main theorem is the following:

Theorem. *Let G be a group acting properly and cocompactly (i.e. geometrically) by type-preserving automorphisms on a finite rank building. Then G is either virtually abelian or contains a non-abelian free subgroup.*

Proof outline. Our proof consists of first removing a possible finite factor of the underlying Coxeter group W of the building (Lemma 3.4) and then splitting into the cases of whether or not the building is thin. In the case of the building being thin, the weak Tits alternative for G follows quickly by purely algebraic arguments from the classical Tits alternative for linear groups.

In the non-thin case, our proof relies on the construction of a tree of chambers in the building and group elements $g, g' \in G$ acting on this tree. Our construction relies on probabilistic arguments adapted from the proof of [11, Lemma 2.10], originally stemming from arguments in [2]. More precisely, basic idea of the proof in the non-thin case is the following.

We begin with a branching panel in some wall Ω in an apartment Σ of a building Δ. By Lemma 3.3, there exists a wall Ω' in Σ which is parallel to Ω. We then connect Ω to Ω' via a minimum length gallery γ between pairs of panels on these walls. Let σ be the panel in Ω containing the initial chamber of γ. By Lemma 4.1, we have that every panel in Ω branches, so that σ branches. Using Proposition 4.4 applied to pairs of three chambers in σ, we produce a "dumbbell graph" and group elements $g, g' \in G$ which act on the universal cover of this dumbbell graph. Using the universal cover of the dumbbell graph, we show that g, g' generate a free subgroup of G by examining the orbit of σ under $\langle g, g' \rangle$. This yields the desired F_2 subgroup of G.

Acknowledgements: We thank Marcin Sabok for providing helpful corrections and suggestions to previous drafts of this article. DO and PP were partially supported by (Polish) Narodowe Centrum Nauki, UMO-2018/30/M/ST1/00668.

2 Preliminaries

2.1 Chamber systems

The following definitions are from [13].

Definition 2.1. *A **chamber system** is a set C together with a set I such that each element i of I determines a partition of C. Two elements in the same part of C determined by $i \in I$ are called i-**adjacent**, and we will call two elements of C **adjacent** if they are i-adjacent for some $i \in I$. The elements of C are called **chambers** and we refer to I as the **index set**.*

A **gallery** is a finite sequence of chambers $(c_0, \ldots, c_k)$ such that each c_{j-1} is adjacent to c_j and $c_{j-1} \neq c_j$. A **subgallery** of a gallery $(c_0, \ldots, c_k)$ is a subsequence of $(c_0, \ldots, c_k)$ of the form $(c_i, c_{i+1}, \ldots, c_j)$ for some $0 \leq i \leq j \leq k$. Given a gallery $\gamma = (c_0, \ldots, c_k)$, the **inverse gallery** is the gallery $\gamma^{-1} := (c_k, c_{k-1}, \ldots, c_0)$. The gallery $(c_0, \ldots, c_k)$ has **type** $i_1 \cdots i_k \in I^*$ (where I^* denotes the set of all finite length words in elements of I) if c_{j-1} is i_j-adjacent to c_j. The **length** of a gallery γ, denoted $\ell(\gamma)$, is the length of its type as a word in I^*. A **geodesic gallery** is a gallery that has minimal length among all galleries with the same initial and terminal chambers. If each i_j belongs to a fixed subset $J \subseteq I$, then we call the gallery $(c_0, \ldots, c_k)$ a J**-gallery**.

A chamber system C over a set I is called **connected** (resp. J**-connected**) if any pair of chambers can be joined by a gallery (resp. J-gallery). The J-connected components are called J**-residues**. For $i \in I$, an $\{i\}$-residue is called a **panel**, whose **type** is i. If σ is a panel, we will say that each chamber $c \in \sigma$ **has** σ as a panel. By a gallery between panels α, σ, we mean a gallery between a chamber in α and a chamber in σ. The **rank** of a chamber system over a set I is the cardinality of I.

2.2 Coxeter groups

Definition 2.2. *A **Coxeter group** is a group W having a **Coxeter presentation**, that is, a presentation of the form:*

$$W = \langle S | s^2 = 1 = (rs)^{m_{rs}} \text{ for all } r \neq s \text{ in } S, m_{rs} \in \{2, 3, \ldots, \infty\} \text{ and } m_{rs} = m_{sr} \rangle$$

where $m_{rs} = \infty$ means that there is no relation between r, s.

Given a Coxeter presentation as above, we say that S is a **Coxeter generating set** of W and that (W, S) is a **Coxeter system**. The **rank** of a Coxeter system (W, S) is $|S|$. A conjugate $s^w := wsw^{-1}$ for $w \in W$ and $s \in S$ is called a **reflection**.

Given a Coxeter system (W, S), we denote by $|\cdot|_S$ the word length of an element of W with respect to S (i.e. for $w \in W$, $|w|_S$ represents the length of the shortest word over S representing w in W) and we denote d_S the word metric on W with respect to S (i.e. $d_S(u, v) = |u^{-1}v|_S$ for each $u, v \in W$). A **standard Coxeter subgroup** of a Coxeter group W with Coxeter generating set S is a subgroup generated by some $T \subseteq S$.

A key example of a chamber system is a Coxeter system (W, S). Here, the set of chambers is W and the index set is S. Two chambers $w_1, w_2 \in W$ are s-adjacent for $s \in S$ if $w_2 = w_1 s$ in W.

2.3 Buildings

For the background on buildings, we follow the books [1] and [13].

Definition 2.3. *A* ***building*** *of type (W, S) is a chamber system Δ over S such that each panel contains at least two chambers, equipped with a map $\delta : \Delta \times \Delta \to W$ such that if f is a geodesic word over S, then $\delta(x, y) = f \in W$ if and only if x, y can be joined by a gallery of type f. The map δ is called a* ***W-metric on*** *Δ.*

Note that in a building Δ, two adjacent chambers x, y are s-adjacent for a unique $s \in S$ ($s = \delta(x, y)$). We also have a metric d on Δ defined by $d(x, y) = |\delta(x, y)|_S$ for each $x, y \in \Delta$. We will refer to d as the **gallery metric** (note that the triangle inequality for d follows from [1, Lemma 5.28], so d is indeed a metric). We will use the notation (Δ, δ) to denote a building Δ with its associated W-metric δ.

A **type-preserving automorphism** ϕ of a building (Δ, δ) is a bijective map $\phi : \Delta \to \Delta$ that preserves the W-metric δ, i.e. $\delta(x, y) = \delta(\phi(x), \phi(y))$ for each $x, y \in \Delta$.

Given a panel σ in a building, the **degree** of σ, denoted $\deg \sigma$, is the number of chambers in the building having σ as a panel. A panel is **branching** if it has degree at least 3. A building is called **thin** if it has no branching panels, i.e. each panel has degree 2 and hence is a panel of exactly two chambers.

Given a building Δ of type (W, S) with W-metric δ, a subset $\Delta_2 \subseteq \Delta$ is a **subbuilding** if $(\Delta_2, \delta|_{\Delta_2})$ is a building (of possibly different type than Δ). A subset $\Delta_2 \subseteq \Delta$ is **convex** if for any $x, y \in \Delta_2$ and any geodesic gallery γ in Δ joining x to y, we have $\gamma \subseteq \Delta_2$.

Note that a Coxeter group W is an example of a building, where we take W as the set of chambers and a Coxeter generating set S as the index set, and equip W with W-metric δ_W defined by $\delta_W(x, y) = x^{-1}y$ for each $x, y \in W$. The corresponding gallery metric is the word metric d_S on W.

A Coxeter group W admits a type-preserving action on its associated building induced from the action on itself by left multiplication. When viewing a Coxeter group W as a building, we have the notion of **walls** that separate W into two connected components. Given a reflection $r = s^u \in W$ (for $s \in S$ and $u \in W$), the **wall** associated to r is the set $M_r = \{\{c_1, c_2\} : c_1, c_2 \in W$ are adjacent and $rc_1 = c_2\}$. Thus, a wall is the set of all panels fixed by r. The sets $\alpha_r^+ = \{w \in W : d_S(w, u) < d_S(w, us)\}$ and $\alpha_r^- = \{w \in W : d_S(w, u) > d_S(w, us)\}$ are called the **roots** of r. Two walls M_r and M_s are **parallel** if $\langle r, s\rangle \cong D_\infty$.

Buildings contain special subspaces, called **apartments**. Let (W, S) be a Coxeter system and let Δ be a building of type (W, S). For a subset $X \subseteq W$, a map $\alpha : X \to \Delta$ is an **isometry** if it preserves W-distance: $\delta_\Delta(\alpha(x), \alpha(y)) = \delta_W(x, y)$ for each $x, y \in X$. An **apartment** in a building Δ of type (W, S) is an image $\alpha(W)$ for an isometry $\alpha : W \to \Delta$.

By the characterization in [1, Section 5.5.2], the apartments of a building Δ are precisely the thin subbuildings of Δ. Every panel in an apartment is contained in a panel in the building. For a panel σ in an apartment, we say that σ is **branching** if the panel in the building containing σ is branching.

A **wall** (resp. **root**) in an apartment of Δ is an isometric image of a wall (resp. root) in W. A gallery α **crosses** a wall Ω if α passes through both chambers in a panel of Ω.

3 Proof of the main theorem

Recall that an action of a group G on a metric space (X, d) is called **geometric** if it is **proper** (i.e. for each $x \in X$, $r \geq 0$, we have $|\{g \in G : d(x, gx) \leq r\}| < \infty$) and **cocompact** (i.e. there is a compact fundamental domain for the action $G \curvearrowright X$). We fix a group G acting geometrically by type-preserving automorphisms on (Δ, d), where (Δ, δ) is a finite rank building of type (W, S), with d the gallery metric. Note that since Δ is finite rank and since $G \curvearrowright (\Delta, d)$ is geometric, we have that the metric space (Δ, d) is locally finite (i.e. closed balls of finite radius are finite).

We consider two cases on the building Δ: the case of Δ being thin (equivalently, consisting of a single apartment) and the complementary case of Δ not being thin, hence consisting of more than one apartment.

Proposition 3.1. *Let (Δ, δ) be a thin, finite rank building and let G act geometrically on Δ by type-preserving automorphisms. Then G is either virtually abelian or contains F_2 as a subgroup.*

Proof. Since Δ is thin, it consists of a single apartment. Since G acts by type-preserving automorphisms of Δ and since W is isomorphic to the group of all type-preserving automorphisms of Δ (by [1, Proposition 3.32]), we have a group homomorphism $\rho : G \to W$, given by fixing a chamber c and putting $g \mapsto \delta(c, gc)$. Since the action of G on Δ is proper, we have that the stabilizer $\mathrm{Stab}(c)$ is finite and hence that $\ker \rho$ is finite. Also, since the action of G on Δ is cocompact, we have that $\rho(G)$ is of finite index in W.

Since W is linear over $\mathbb{R}$ (see, for instance, [1, Section 2.5]), we have that W satisfies the classical Tits alternative: every subgroup of W is either virtually solvable or contains F_2. Therefore, $\rho(G)$ is either virtually solvable or contains F_2.

If $\rho(G)$ contains a subgroup $H \cong F_2$, then $\rho^{-1}(H) \leq G$ surjects onto $H \cong F_2$, and hence contains F_2, so G contains F_2.

If $\rho(G)$ is virtually solvable, then $\rho(G)$ is virtually abelian, since Coxeter groups are CAT(0) (since they act geometrically on their Davis complex, which is a CAT(0) space; see Chapters 7 and 12 of [8]), and solvable subgroups of CAT(0) groups are virtually abelian by [3, Theorem III.Γ.1.1(3)].

We show that G is also virtually abelian using that $\ker \rho$ is finite. Let $H \leq \rho(G)$ be finite index and abelian. Then $\widetilde{H} := \rho^{-1}(H) > \ker \rho$ has finite index in G. Since $\ker \rho$ is finite, it follows that $\widetilde{H}$ has finite commutator subgroup. We show that $Z(\widetilde{H})$ has finite index in $\widetilde{H}$.

Since $\widetilde{H}$ has finite commutator subgroup, we have that every conjugacy class in $\widetilde{H}$ is finite, since every conjugacy class is contained in a coset of the commutator subgroup. We have that $\widetilde{H}$ is finitely generated, since H is finitely generated (being of finite index in W, which is finitely generated) and $\widetilde{H}$ surjects onto H with finite kernel. Let $\{h_1, \ldots, h_n\}$ be a set of generators for $\widetilde{H}$. For each i, denote by $C_{\widetilde{H}}(h_i)$ the centralizer of h_i in $\widetilde{H}$ and by $[h_i]_{\widetilde{H}}$ the conjugacy class of h_i in $\widetilde{H}$.

We have that $|\widetilde{H} : C_{\widetilde{H}}(h_i)| = |[h_i]_{\widetilde{H}}| < \infty$. Thus, $Z(\widetilde{H}) = \bigcap_{i=1}^{n} C_{\widetilde{H}}(h_i)$ has finite index in $\widetilde{H}$, and hence in G. Since $Z(\widetilde{H})$ is abelian, we conclude that G is virtually abelian. □

We now move on to the case where Δ is not thin.

Proposition 3.2. *Let (Δ, δ) be a finite rank building of type (W, S) that is not thin and such that W does not decompose as $W \cong W_1 \times W_2$ for W_1, W_2 standard Coxeter subgroups of W, with W_1 finite and non-trivial. Let G act geometrically on Δ by type-preserving automorphisms. Then G contains F_2 as a subgroup.*

In the proof of Proposition 3.2, we will need the following lemma. It was first stated in [7, Lemma 4.1], and later in [6], where a different proof was given. The proof in [6] relies on [5, Lemma 8.2] and the parallel wall theorem ([4, Theorem 2.8]).

Lemma 3.3. *If (W, S) is a Coxeter system such that W does not decompose as the direct product of standard Coxeter subgroups W_1, W_2, where W_1 is a finite non-trivial Coxeter group, then for each wall Ω in W, there exists a wall Ω' in W which is parallel to Ω.*

The following lemma and Lemma 3.3 allow us to reduce to the case where we can find a wall disjoint from any given wall.

Lemma 3.4. *If G acts geometrically by type-preserving automorphisms on a building (Δ, δ) of type (W, S) and $W \cong W_1 \times W_2$, where $W_1 = \langle S_1 \rangle$ and $W_2 = \langle S_2 \rangle$ are standard Coxeter subgroups of W with $S_1 \coprod S_2 = S$ and with W_1 finite, then there exists a building Δ_2 of type (W_2, S_2) on which G acts geometrically by type-preserving automorphisms.*

Proof. We form Δ_2 by identifying chambers in Δ that are in the same S_1-residue i.e. for $x, y \in \Delta$, we put $x \sim y$ if $\delta(x, y) \in W_1$. Let q be the associated quotient map. Note that Δ_2 is a chamber system over S_2, where for each $s \in S_2$, we define $a, b \in \Delta_2$ to be s-adjacent if there exist lifts x, y of a, b, respectively, such that $\delta(x, y) \in W_1 s$, i.e. such that x, y are in the same $S_1 \cup \{s\}$-residue.

We show that Δ_2 is a building of type (W_2, S_2), and we show that G acts geometrically on Δ_2. We have already noted above that Δ_2 is a chamber system over S_2. Every panel in Δ_2 contains at least two chambers, since if $a \in \Delta_2$ and $s \in S_2$, then if $x \in \Delta$ is any lift of a, choosing any y that is s-adjacent to x yields $q(y) \neq a$ and $q(y)$ s-adjacent to a. We define a function $\delta_2 : \Delta_2 \times \Delta_2 \to W_2$ by $\delta_2(q(x), q(y)) = \mathrm{proj}_{W_2}(\delta(x, y))$ for any $x, y \in \Delta$, where proj_{W_2} denotes the projection $W \cong W_1 \times W_2 \to W_2$. We show that δ_2 is a W_2-metric. We begin by showing that δ_2 is well-defined. Let x', y' be such that $q(x) = q(x')$ and $q(y) = q(y')$, so that $\delta(x, x') \in W_1$ and $\delta(y, y') \in W_1$. By [1, Lemma 5.28(1)], we have $\delta(x, y) = s_{x,x'}\delta(x', y)$, where $s_{x,x'}$ is a word consisting of a subset of letters from $\delta(x, x')$ (hence $s_{x,x'} \in W_1$) and by [1, Lemma 5.28(2)], we have that $\delta(x', y) = \delta(x', y')s_{y',y}$, where $s_{y',y}$ is a word consisting of a subset of letters from $\delta(y', y)$ (hence $s_{y',y} \in W_1$). Therefore, we have $\delta(x, y) \in W_1\delta(x', y')W_1$, so that $\mathrm{proj}_{W_2}(\delta(x, y)) = \mathrm{proj}_{W_2}(\delta(x', y'))$. Thus, δ_2 is well-defined.

Next, we show that δ_2 satisfies the required property on galleries. Let $f = s_1 \cdots s_n \in (S_2)^*$ be a geodesic word over S_2. We need to show that for each $a, b \in \Delta_2$, we have $\delta_2(a, b) = f$ if and only if there exists a gallery from a to b of type f. Suppose $\delta_2(a, b) = f$. Choose any lifts x, y of a, b, respectively. Then by definition of δ_2, we have $\delta(x, y) = w_1 f$ for some $w_1 \in W_1$. Writing $w_1 = u_1 \cdots u_m$ as a geodesic word over S_1, we then have that $u_1 \cdots u_m s_1 \cdots s_n$ is a geodesic word over S representing $w_1 f$. Thus, by definition of a W-metric, there exists a gallery $\gamma = (c_0, c_1, \ldots, c_k)$ from x to y with type $u_1 \cdots u_m s_1 \cdots s_n$. Then $\widetilde{\gamma} := (q(c_m), q(c_{m+1}), \ldots, q(c_k))$ is a gallery of type f joining $a = q(x) = q(c_0) = q(c_m)$ to $b = q(y) = q(c_k)$.

Suppose there exists a gallery $\gamma = (c_0, \dots, c_n)$ of type f joining a to b. For each i, let $\tilde{c_i}$ be a lift of c_i. We show by induction that $\delta(\tilde{c_0}, \tilde{c_i}) \in W_1 s_1 \cdots s_i$ for each $i = 0, \dots, n$. For $i = 0$, we have that $\delta(\tilde{c_0}, \tilde{c_0}) = 1_W \in W_1$.

Suppose $\delta(\tilde{c_0}, \tilde{c_i}) = w_1 s_1 \cdots s_i$ for some word w_1 over S_1. Since $\delta_2(c_i, c_{i+1}) = s_{i+1}$, we have $\delta(\tilde{c_i}, \tilde{c}_{i+1}) = w_1' s_{i+1}$ for some word w_1' over S_1. Therefore, there exists a gallery of type $w_1 s_1 \cdots s_i w_1' s_{i+1}$ from c_0 to c_{i+1}, hence also a gallery of type $w_1 w_1' s_1 \cdots s_i s_{i+1}$ from c_0 to c_{i+1} (since the reduced words $s_1 \cdots s_i w_1'$ and $w_1' s_1 \cdots s_i$ are equal in W). By [1, Lemma 5.28(2)], since $w_1' s_1 \cdots s_{i+1}$ is a geodesic word over S, and since w_1 can only cancel letters in w_1', we have that $\delta(\tilde{c_0}, \tilde{c}_{i+1}) \in W_1 s_1 \cdots s_{i+1}$.

Thus, we conclude by induction on i that $\delta(\tilde{c_0}, \tilde{c_n}) \in W_1 s_1 \cdots s_n = W_1 f$. By definition of δ_2, we conclude that $\delta_2(a, b) = f$. Hence, δ_2 is a W_2-metric on Δ_2. Therefore, Δ_2 is a building of type (W_2, S_2).

We now define a type-preserving action of G on Δ_2 and show that this action is geometric. We define $gq(x) := q(gx)$ for every $x \in \Delta$ and $g \in G$. We immediately have that this action is well-defined and type-preserving since the action of G on Δ is type-preserving.

Next, we show that the action of G is geometric. The action of G on Δ_2 is cocompact since by the definition of the action of G on Δ_2, if F is a finite fundamental domain for the action of G on Δ, then the image $F' := q(F)$ of F is a finite fundamental domain for the action $G \curvearrowright \Delta_2$.

For properness of the action of G on Δ_2, note first that the quotient map q satisfies $d_\Delta(a, b) \le M + d_{\Delta_2}(q(a), q(b))$ for each $a, b \in \Delta$, where $M = \max\{|w_1|_{S_1} : w_1 \in W_1\}$ and where d_Δ (resp. d_{Δ_2}) is the gallery metric on Δ (resp. Δ_2). Indeed, given two chambers a, b in Δ, if $\delta_2(q(a), q(b)) = w_2$, then $\delta(a, b) = w_1 w_2$ for some $w_1 \in W_1$, so

$$d_\Delta(a, b) \le |w_1 w_2|_S \le |w_1|_S + |w_2|_S \le M + |w_2|_{S_2} = M + d_{\Delta_2}(q(a), q(b))$$

Now if $g \in G$ and $a' = q(a) \in \Delta_2$ (for $a \in \Delta$), then we have $d_\Delta(ga, a) \le M + d_{\Delta_2}(ga', a')$, so for any $R \ge 0$, we have $\{g \in G : d_{\Delta_2}(ga', a') \le R\} \subseteq \{g \in G : d_\Delta(ga, a) \le M + R\}$, and the latter set is finite by properness of the action of G on Δ.

Therefore, the action $G \curvearrowright \Delta_2$ is geometric. □

Combining the results of Proposition 3.1, Proposition 3.2 and Lemma 3.4, we conclude the proof of the main theorem. It remains to prove Proposition 3.2.

4 Proof of Proposition 3.2

In the proof of Proposition 3.2, we will need the following lemma.

Lemma 4.1. *Let Ω be a wall in an apartment Σ of a building (Δ, δ). Suppose that Ω has a branching panel. Then every panel of Ω branches.*

Proof. Let α be a panel with type s in $\Omega \subseteq \Sigma$ that branches and let $a, \overline{a}$ be a pair of chambers in Σ having α as a panel. Denote by Ω_a the root of Σ containing a and $\Omega_{\overline{a}}$ the root containing $\overline{a}$. Let β be any other panel in Ω, containing chambers $b, \overline{b}$ in Σ. Suppose that $b \in \Omega_a$. We will proceed by induction on $d(a, b)$ to show that β also branches.

For the base case $d(a, b) = 0$, we have that $a = b$ and so $\alpha = \beta$. Hence, β branches.

Now for the induction step, we will show that there exists a branching panel in Ω consisting of chambers $a'', \overline{a''}$ in Σ such that $d(a'', b) < d(a, b)$. Fix a geodesic gallery η between a and b. By convexity of roots (c.f. [13, Proposition 2.6(i)]), we have that $\eta \subseteq \Omega_a$.

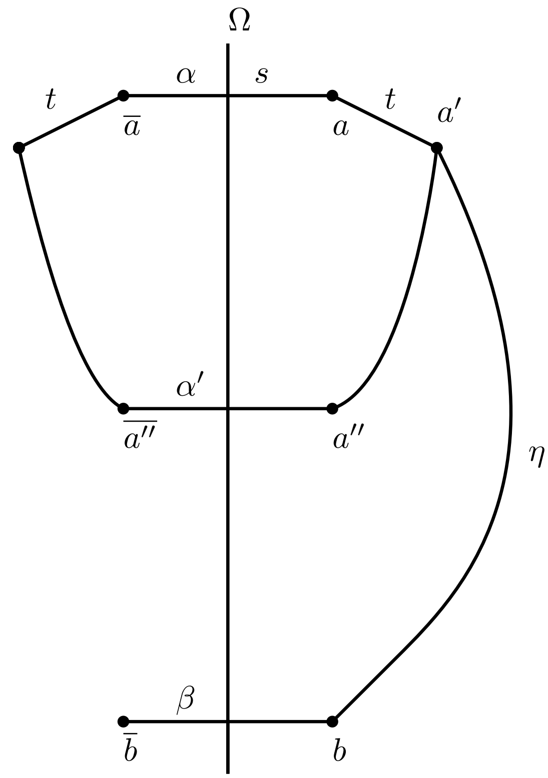

Figure 1: An illustration of the setup for the induction step.

Let a' be the second chamber of η and let t be the type of the panel of a and a' (see Figure 1). Since a' is on the geodesic gallery η from a to b, we have that $d(a', b) < d(a, b)$, and hence $d(a', \overline{b}) = d(a', b) + 1 < d(a, b) + 1 = d(a, \overline{b})$. Also, reflecting the geodesic gallery η across the wall Ω, we have that $d(\overline{a}, \overline{b}) = d(a, b) = d(a, \overline{b}) - 1 < d(a, \overline{b})$. Thus, we have that $d(\overline{a}, \overline{b}) < d(a, \overline{b})$ and $d(a', \overline{b}) < d(a, \overline{b})$. Letting $\mathcal{R}$ denote the $\{s, t\}$-residue of Δ containing a, by [13, Theorem 2.16], we obtain that the $\{s, t\}$-residue $\mathcal{R}_\Sigma := \mathcal{R} \cap \Sigma$ containing a in Σ is finite, and there is a unique chamber $\overline{a''}$ in $\mathcal{R}_\Sigma$ at minimal distance to $\overline{b}$ and opposite to a (i.e. at distance $\operatorname{diam}(\mathcal{R}_\Sigma)$ from a).

We have that $\overline{a''} \in \Omega_{\overline{a}}$ since if not then $d(\overline{a}, \overline{a''}) = d(a, \overline{a''}) + 1 > d(a, \overline{a''}) = \operatorname{diam}(\mathcal{R}_\Sigma)$, a contradiction. Furthermore, $\overline{a''}$ has a panel α' in Ω, since $\overline{a''}$ is adjacent to a chamber $a'' \in \mathcal{R}_\Sigma$ opposite to $\overline{a}$, and a'' must be in Ω_a by the same argument for why $\overline{a''} \in \Omega_{\overline{a}}$.

We show that α' branches and that $d(a'', b) < d(a, b)$. We will first show that α' branches. Denote $D = \langle s, t \rangle$. By [13, Theorem 3.5], we have that $\mathcal{R}$ is a subbuilding of Δ of type $(D, \{s, t\})$. Let $\delta_{\mathcal{R}}$ be the D-metric on $\mathcal{R}$. Note that $\delta_{\mathcal{R}}$ equals the restriction of δ to $\mathcal{R}$, since $\mathcal{R}$ is convex. By [1, Proposition 1.77(1)], there exists a unique longest element $w_{\mathcal{R}}$ in D (with respect to the word metric on D induced from the generating set $\{s, t\}$) which has word length equal to $\operatorname{diam}(\mathcal{R})$.

Let f be a chamber not in Σ having α as a panel. Then $\delta(f, \overline{a''}) = \delta(f, a'') = w_{\mathcal{R}}$, since concatenating the geodesic gallery from $\overline{a}$ to $\overline{a''}$ in $\mathcal{R}$ with f yields a gallery from f to $\overline{a''}$ with type $s\delta(\overline{a}, \overline{a''}) = w_{\mathcal{R}}$ (which is a geodesic word), and similarly concatenating the geodesic gallery from a to a'' in $\mathcal{R}$ with f yields a gallery of type $s\delta(a, a'') = w_{\mathcal{R}}$.

Since $\delta(f, a'') = \delta(f, \overline{a''}) = w_{\mathcal{R}}$, we have that f is opposite to both a'' and $\overline{a''}$, and so f and $\overline{a''}, a''$ cannot be contained in a common apartment in $\mathcal{R}$ since opposite chambers are unique in apartments (c.f. [13, Theorem 2.15(iii)]). Let $\mathcal{B}_1$ be an apartment of $\mathcal{R}$ containing $\overline{a''}$ and f. Then a'' is not in $\mathcal{B}_1$, hence there exists a chamber e different from $\overline{a''}$ and a'' having the panel α'. Thus, α' is a branching panel.

We lastly show that $d(a'', b) < d(a, b)$. By [13, Theorem 2.9], there exists a geodesic gallery γ in Σ from a to $\bar{b}$ containing $\overline{a''}$. By convexity of residues (c.f. [13, Lemma 2.10]), we have that the portion of γ from a to $\overline{a''}$ is contained in $\mathcal{R}_\Sigma$, and so we can assume that γ passes through the chambers $a'', \overline{a''}$ (since there are two geodesics in $\mathcal{R}_\Sigma$ from a to $\overline{a''}$: one through $\bar{a}$ and the other through a''). Thus, we obtain that $d(a'', \bar{b}) \leq d(a, \bar{b}) - 1 = d(a, b)$, and hence $d(a'', b) = d(a'', \bar{b}) - 1 \leq d(a, b) - 1 < d(a, b)$.

Since α' branches and $d(a'', b) < d(a, b)$, we conclude by induction that β branches. □

For the remainder of this section, we fix an apartment Σ containing a branching panel. Let $\Omega \subseteq \Sigma$ be any wall containing this branching panel. Invoking Lemma 3.3, there exists a wall Ω' which is parallel to Ω and in the same apartment Σ as Ω. Fix a geodesic gallery γ which has minimal length among all galleries joining a chamber inside a panel of Ω and a chamber inside a panel of Ω'. Let s_0 be the type of the panel in Ω containing the first chamber of γ and let s_k be the type of the panel in Ω' containing the last chamber of γ. Let $s_1 \cdots s_{k-1}$ be the type of γ for $s_i \in S$. For $j = 1, \ldots, k-1$, put $s_{k+j} = s_{k-j}$.

Lemma 4.2. *Let $w = s_0 \cdots s_{2k-1} \in W$. Then for any $n \in \mathbb{N}$, we have that $s_0 w^n = s_1 s_2 \cdots s_{2k-1}(s_0 s_1 \cdots s_{2k-1})^{n-1}$ is a geodesic word in W (by convention, s_0 cancels the first letter s_0 of w^n).*

Proof. Let $s, r \in W$ be such that $\Omega = M_r, \Omega' = M_s$. Let γ be the above minimal length geodesic gallery between panels in the walls Ω and Ω' having type $s_0 w = s_1 \cdots s_{k-1}$. Then $s_0 w^n = s_1 s_2 \cdots s_{2k-1}(s_0 s_1 \cdots s_{2k-1})^{n-1}$ is the type of the gallery $\gamma_n := \bigcup_{i=0}^{n-1}(sr)^i(\gamma \cup s\gamma)$ (see Figure 2).

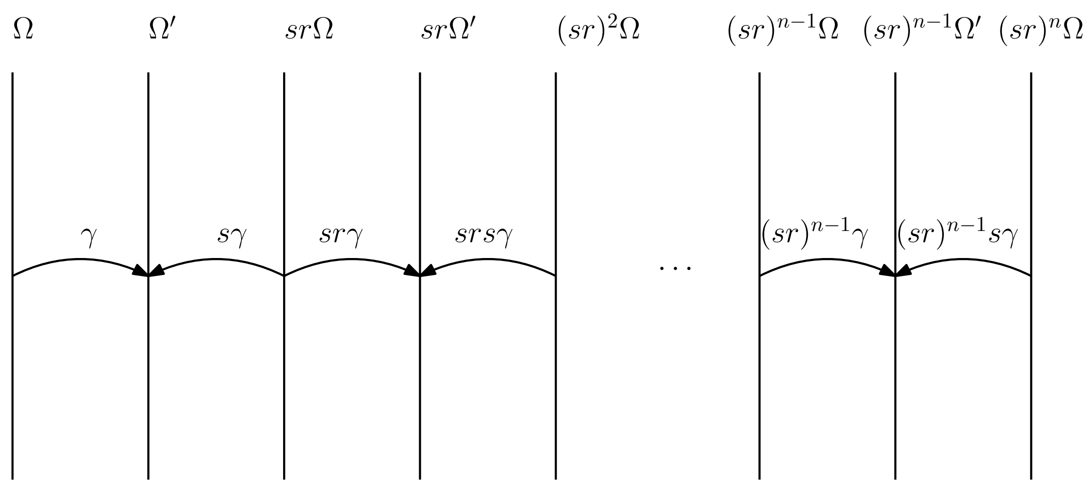

Figure 2: The concatenation of geodesics in the definition of γ_n.

We claim that γ_n is a geodesic gallery. Indeed, if α were a gallery with the same starting and ending chambers as γ_n, then by [13, Lemma 2.5(ii)], we must have that α crosses each such wall $(sr)^i\Omega$. Let α_i denote the segment of α between the successive parallel walls $(sr)^i\Omega$ and $(sr)^{i+1}\Omega$. Then $(sr)^{-i}\alpha_i$ is a gallery between the walls Ω and $(sr)\Omega = s\Omega$. Since the chambers contained in Ω and $s\Omega$ are in different roots of s, by [13, Lemma 2.5(ii)] we have that $(sr)^{-i}\alpha_i$ crosses Ω'. Since γ is a minimum length geodesic gallery between panels in the walls Ω and Ω', we have that $s\gamma$ is a minimum length gallery between panels in Ω' and $s\Omega$. Therefore, the length of the subgallery of $(sr)^{-i}\alpha_i$ between Ω and Ω' is at least $\ell(\gamma)$ and similarly the length of the subgallery of $(sr)^{-i}\alpha_i$ between Ω' and $s\Omega$ is at least $\ell(s\gamma)$. Therefore, $\ell((sr)^{-i}\alpha_i) \geq \ell(\gamma \cup s\gamma)$. Translating by $(sr)^i$, we obtain $\ell(\alpha_i) \geq \ell((sr)^i(\gamma \cup s\gamma))$. As this holds for all $i = 0, 1, \ldots n-1$, we conclude that $\ell(\alpha) \geq \ell(\gamma_n)$, and therefore that γ_n is a geodesic gallery. Therefore, since $s_0 w^n$ is the type of $\gamma_1 \cdots \gamma_{2n}$, it follows that $s_0 w^n$ is a geodesic word. $\square$

Using ideas from the work of Ballmann and Brin in [2], we construct the following Markov chain. The set A of states will consist of pairs (c, j), where c is a chamber of Δ and $j \in \mathbb{Z}$ is an index taken modulo $2k$ (recall that k is the length of the minimum length gallery γ between the walls Ω and Ω'). We define the **transition probability** $p(a \to a')$ from $a = (c, j) \in A$ to $a' = (c', i) \in A$ to be positive if $i = j + 1$ and c and c' share a panel σ with type s_j, in which case we set $p(a \to a') = \frac{1}{\deg \sigma - 1}$, otherwise we put $p(a \to a') = 0$. We have an action of G on A via $g(c, j) = (gc, j)$ for each chamber c of Δ and $j = 0, 1 \ldots, 2k - 1$.

Given a sequence of states $a_0, \ldots, a_n$, we put $p_n(a_0, \ldots, a_n) = \prod_{i=0}^{n-1} p(a_i \to a_{i+1})$. Given a finite sequence $(a_0, \ldots, a_n)$ of states, denote the **cylinder set** $[a_0 \cdots a_n]_{(N,N+n)} := \{(b_i)_{i \in \mathbb{Z}} : b_i = a_{i-N}$ for all $i = N, \ldots, N + n\} \subseteq A^{\mathbb{Z}}$.

Lemma 4.3. *There exists a shift-invariant measure μ on $A^{\mathbb{Z}}$ such that $\mu([a_0 \cdots a_n]_{(N,N+n)}) = p_n(a_0, \ldots, a_n)$ for each $a_0, \ldots, a_n \in A$ $(n \geq 0)$.*

Proof. By [16, Example (8)], we need to check that the following properties of p are satisfied:

(i) For any $a \in A$, $\sum_{a' \in A} p(a \to a') = 1$

(ii) For any $a \in A$, $\sum_{a' \in A} p(a' \to a) = 1$

For (i), given $a = (c, j) \in A$, we have $p(a \to a') \neq 0$ only if the chambers of a' and a share a panel σ of type s_j. In this case, we then have $p(a \to a') = \frac{1}{\deg \sigma - 1}$. Since there are exactly $\deg \sigma - 1$ chambers other than the chamber of a having σ as a panel, we obtain:

$$\sum_{a' \in A} p(a \to a') = (\deg \sigma - 1) \cdot \frac{1}{\deg \sigma - 1} = 1$$

For (ii), given $a = (c, j) \in A$, we have $p(a' \to a) \neq 0$ only if the chambers of a' and a share a panel σ of type s_{j-1}, and in this case we have $p(a' \to a) = \frac{1}{\deg \sigma - 1}$. We then have:

$$\sum_{a' \in A} p(a' \to a) = (\deg \sigma - 1) \cdot \frac{1}{\deg \sigma - 1} = 1$$

Therefore, p induces a shift invariant measure μ on $A^{\mathbb{Z}}$ with the desired value on cylinder sets. $\square$

The measure μ is G-invariant, since the action of G on Δ is type-preserving and hence preserves adjacency. Therefore, μ descends to a measure $\bar{\mu}$ on $A^{\mathbb{Z}}/G$ by putting $\bar{\mu}(\bar{S}) = \mu(S)$ where $\bar{S} \subseteq A^{\mathbb{Z}}/G$ is (Borel) measurable and $S \subseteq A^{\mathbb{Z}}$ is a (Borel) measurable set of lifts to $A^{\mathbb{Z}}$ of each element of $\bar{S}$ (where the Borel structure on $A^{\mathbb{Z}}$ comes from putting the discrete topology on A and equipping $A^{\mathbb{Z}}$ with the product topology). Note that every measurable $\bar{S} \subseteq A^{\mathbb{Z}}/G$ admits a measurable set $S \subseteq A^{\mathbb{Z}}$ of lifts. Indeed, by [9, Theorem 6.4.4] and the fact that G is countable (since $G \curvearrowright (\Delta, d)$ is proper and Δ is countable), it suffices to show that the orbit equivalence relation E_G of $G \curvearrowright A^{\mathbb{Z}}$ is a closed subset of $A^{\mathbb{Z}} \times A^{\mathbb{Z}}$. Let $((a^n)_n, (g_n a^n)_n)$ be a sequence in E_G converging to $(a^\infty, b^\infty) \in A^{\mathbb{Z}} \times A^{\mathbb{Z}}$. Then the sequence $(a_0^n)_n$ is eventually equal to a_0^∞ and the sequence $(g_n a_0^n)_n$ is eventually equal to b_0^∞. Thus, we have that $g_n a_0^\infty = b_0^\infty$ for all sufficiently large n. By properness of the action of G on Δ, there are only finitely many $g \in G$ with $g a_0^\infty = b_0^\infty$, so passing to a subsequence, we may assume that the sequence $(g_n)_n$ is constant, with all g_n equal to some $g \in G$. We then have that $b^\infty = \lim_{n\to\infty}(ga^n)_n = ga^\infty$, so we conclude that $(a^\infty, b^\infty) \in E_G$ and hence that E_G is closed.

Since μ is invariant under the forward shift map $T : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ given by $T((a_i)_{i\in\mathbb{Z}}) = (a_{i+1})_{i\in\mathbb{Z}}$, so is $\bar{\mu}$.

Since the action of G on Δ is cocompact, we have that A/G is finite, and so the measure $\bar{\mu}$ is finite (and non-zero).

To produce an F_2 subgroup inside G, we will construct a tree of chambers in Δ and a subgroup of G acting freely on this tree using the Poincaré recurrence lemma. The next proposition is the key ingredient involved in this construction.

Proposition 4.4. *Given a pair of states $(d, 0)$ and $(d', 1)$ with $p((d, 0) \to (d', 1)) > 0$, there exists a sequence $(a_1, \dots, a_n)$ of elements of A with the following properties:*

- $a_1 = (d', 1)$ *and* $a_n = g(d, 0)$ *for some* $g \in G$.
- $p(a_i \to a_{i+1}) > 0$ *for each* $i = 0, 1, \dots, n-1$.

Proof. Denoting $a = (d, 0)$ and $a' = (d', 1)$, consider the cylinder set $[aa']_{(0,1)} = \{(a_i)_{i\in\mathbb{Z}} : a_0 = a, a_1 = a'\} \subset A^{\mathbb{Z}}$ and consider the image of this cylinder set under the quotient map to $A^{\mathbb{Z}}/G$: $\overline{[aa']_{(0,1)}} = \{(hb_i)_{i\in\mathbb{Z}} : b_0 = a, b_1 = a', h \in G\} \subset A^{\mathbb{Z}}/G$. We have $\bar{\mu}(\overline{[aa']_{(0,1)}}) > 0$. Indeed, let $G' = \mathrm{Stab}(a) \cap \mathrm{Stab}(a')$. Then $G' \curvearrowright [aa']_{(0,1)}$ and G' is finite by properness of the action of G on Δ. Since G' is finite, we have a measurable fundamental domain F for the action of G' on $\overline{[aa']_{(0,1)}}$ ([9, Exercise 7.1.1] and [9, Exercise 7.1.6]). We have that F is a set of lifts of elements of $\overline{[aa']_{(0,1)}}$ and $\mu(F) = \frac{1}{|G'|}\mu([aa']_{(0,1)}) > 0$. Thus, $\bar{\mu}(\overline{[aa']_{(0,1)}}) = \mu(F) > 0$.

Let $Y = \overline{[aa']_{(0,1)}} \setminus \{(hb_i)_{i\in\mathbb{Z}} : h \in G \text{ and } p(b_j \to b_{j+1}) = 0 \text{ for some } j \in G\}$. Since A is countable (since Δ is countable), we have that $\bar{\mu}(Y) = \bar{\mu}(\overline{[aa']_{(0,1)}})$. Note that all elements of Y are then of the form $(hb_i)_{i\in\mathbb{Z}}$, where for each j, we have that $p(b_j \to b_{j+1}) > 0$, so that $b_j = (c, \ell)$ and $b_{j+1} = (c', \ell + 1)$, and c, c' share a panel of type s_ℓ.

By Poincaré recurrence (see, e.g. [16, Thm 1.4]) applied to the set Y and the shift map $T \curvearrowright A^{\mathbb{Z}}/G$, we have that there exist $n > 0$ and some $(hb_i)_{i\in\mathbb{Z}} \in Y$ such that $T^n((hb_i)_{i\in\mathbb{Z}}) \in Y$. Lifting back up to $A^{\mathbb{Z}}$, we obtain a sequence $(a_i)_{i\in\mathbb{Z}} \in [aa']_{(0,1)}$ such that $p(a_j \to a_{j+1}) > 0$ for all j and such that $a_n = ga$ for some $g \in G$. □

Note that in the proof of Proposition 4.4, we have that $n = 0$ modulo $2k$ since $p(a_j \to a_{j+1}) > 0$ for all j.

Conclusion of the proof of Proposition 3.2:

Let c_1 be the first chamber of the minimal length geodesic gallery γ between Ω and Ω' and let σ be the panel of type s_0 in Ω containing c_1. By Lemma 4.1, since Ω has a branching panel, every panel in Ω branches, so σ branches. Let c_2, c_3 be two other distinct chambers in σ.

Apply Proposition 4.4 to produce a sequence of states $((d_1, 1), \dots, (d_n, 0))$ whose chambers d_i form a gallery ω from c_2 to gc_1 for some $g \in G$. Similarly, produce a sequence of states whose chambers form a gallery ω'' from c_3 to $g''c_1$ for some $g'' \in G$ and a sequence of states whose chambers form a gallery ω' from $g''c_3$ to $g'g''c_2$ for some $g' \in G$; see Figure 3. Note that in the sequences produced by Proposition 4.4, adjacent states have positive transition probability, hence each of $\omega, \omega', \omega''$ has type of the form $s_0(s_0 \cdots s_{2k-1})^m$ for some $m \in \mathbb{N}$, and hence is a geodesic gallery by Lemma 4.2.

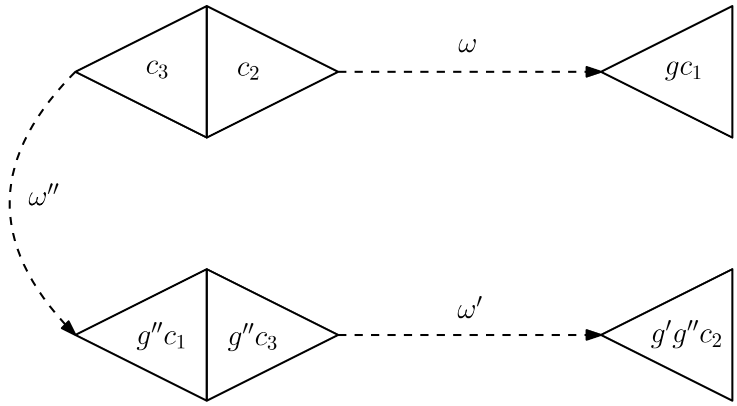

Figure 3: The "dumbbell graph" produced from the branching panel σ.

Claim: Let σ be the initial branching panel above in the wall Ω. Then for any non-trivial freely reduced word u in g, g', we have $u\sigma \neq \sigma$ in Δ.

Proof of claim. Write u as a word $u = u_1 u_2 \cdots u_m$, where the u_i are alternating powers of g and g'. Denote $u(i) = u_1 u_2 \cdots u_i$ for each $1 \leq i \leq m$ and let $u(0) = 1$. We show that for each i, we can connect $u(i-1)\sigma$ to $u(i)\sigma$ with a gallery ω_i satisfying the following:

(a) ω_i is a concatenation of $\langle g, g' \rangle$-translates of ω, ω' and ω'',

(b) ω_i has type of the form $s_0 w^{n_i}$ for some $n_i \in \mathbb{N}$, (recall that $w = s_0 \cdots s_{2k-1}$),

(c) the ending chamber of ω_i is different from the starting chamber of ω_{i+1}.

We first show that each ω_i satisfies (a) and (b). We consider the following cases:

(i) $u_i = g^n$ for some $n \in \mathbb{Z} \setminus \{0\}$. Then $u(i)\sigma = u(i-1)g^n\sigma$. We can connect σ to $g^n\sigma$ by $\bigcup_{j=0}^{n-1} g^j\omega$ if $n > 0$ or $\bigcup_{j=1}^{-n} g^{-j}\omega^{-1}$ if $n < 0$. Therefore, we set $\omega_i = u(i-1)\bigcup_{j=0}^{n-1} g^j\omega$ if $n > 0$ and $\omega_i = u(i-1)\bigcup_{j=1}^{-n} g^{-j}\omega^{-1}$ if $n < 0$. Thus, in this case we have that the type of ω_i is of the form $s_0 w^{n_i}$, since the type of ω is of this form and the starting and ending chambers of ω are different. See Figure 4 for an illustration of an example. Note that the type of σ (and hence all of its translates) is s_0.

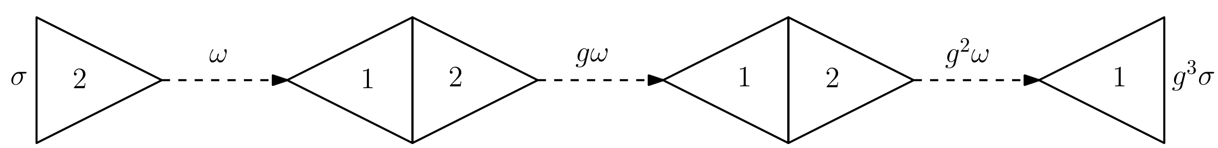

Figure 4: An example of a gallery joining σ and $g^3\sigma$. The numbers on the chambers indicate of which c_i they are translates.

(ii) $u_i = (g')^n$ for some $n \in \mathbb{Z} \setminus \{0\}$. Then $u(i)\sigma = u(i-1)(g')^n\sigma$. For $n > 0$, we can connect σ to $(g')^n\sigma$ by the concatenation $\omega'' \cup (\bigcup_{j=0}^{n-1}(g')^j\omega') \cup (g')^n(\omega'')^{-1}$ and if $n < 0$, we can connect σ to $(g')^n\sigma$ by the concatenation $\omega'' \cup (\bigcup_{j=1}^{-n}(g')^{-j}(\omega')^{-1}) \cup (g')^n(\omega'')^{-1}$, which has type of the form $s_0 w^{n_i}$ for some $n_i \in \mathbb{N}$ since each of ω'', ω' has type of this form and the starting and ending chambers of ω' and ω'' are distinct. Thus, $u(i-1)\sigma$ and $u(i)\sigma$ are connected by

$$\omega_i = u(i-1)(\omega'' \cup (\bigcup_{j=0}^{n-1}(g')^j\omega') \cup (g')^n(\omega'')^{-1}) \text{ if } n > 0, \text{ or}$$

$$\omega_i = u(i-1)(\omega'' \cup (\bigcup_{j=1}^{-n} (g')^{-j}(\omega')^{-1}) \cup (g')^{n}(\omega'')^{-1}) \text{ if } n < 0,$$

which therefore have labels of the desired form $s_0 w^{n_i}$ for some $n_i \in \mathbb{N}$. See Figure 5 for an illustration of an example.

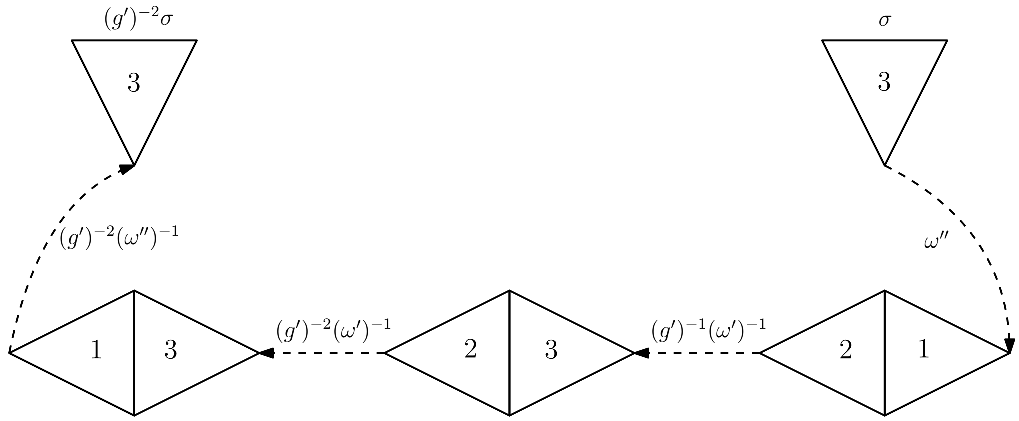

Figure 5: An example of a gallery joining σ and $(g')^{-2}\sigma$. The numbers on the chambers indicate of which c_i they are translates.

Now we show that the ending chamber of ω_i is different from the starting chamber of ω_{i+1}.

By the cases (i) and (ii) above, either for some $h \in G$, the ending chamber of ω_i is of the form hc_1 or hc_2 and the starting chamber of ω_{i+1} is of the form hc_3 (when u_i is a power of g and u_{i+1} is a power of g'), or for some $h \in G$, the ending chamber of ω_i is of the form hc_3 and the starting chamber of ω_{i+1} is of the form hc_1 or hc_2 (when u_i is a power of g' and u_{i+1} is a power of g). Therefore, the ω_i satisfy (c).

Thus, the type of each ω_i is of the form $s_0 w^n$ and the starting and ending chamber of ω_i and ω_{i+1} are distinct. Therefore, letting $\gamma = \bigcup_{i=1}^{n} \omega_i$ be the concatenation of the ω_i galleries, we have that γ has type of the form $s_0 w^{n_1+n_2+\cdots+n_m}$. By Lemma 4.2, $s_0 w^{n_1+n_2+\cdots+n_m}$ is a geodesic word, and so we have that γ is a geodesic gallery in Δ. Therefore, γ has distinct endpoints, and so $\sigma \neq u\sigma$. In Figure 6, see an example of the gallery γ for $u = (g')^{-2}g^{-1}g'$. $\square$

By the above claim, we obtain that $\langle g, g' \rangle \cong F_2$. Therefore, we have that G contains F_2 as a subgroup, concluding the proof of Proposition 3.2. $\square$

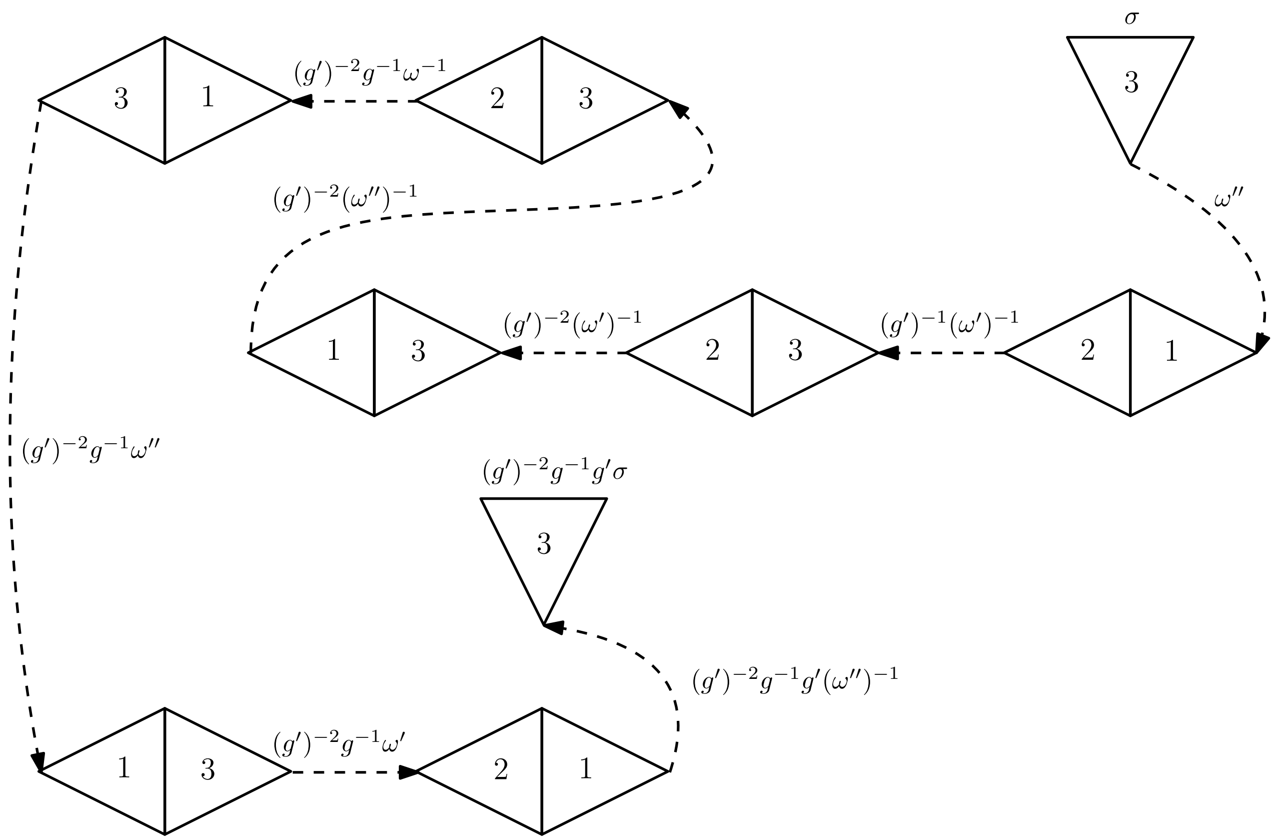

Figure 6: An example of a gallery joining σ and $(g')^{-2}g^{-1}g'\sigma$. The numbers on the chambers indicate of which c_i they are translates.

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