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The fixed point theorem for simplicial nonpositive curvature

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Chapter I

Introduction

1 Simplicial nonpositive curvature

Systolic complexes and systolic groups were introduced by T. Januszkiewicz and J. Świątkowski in [6] and independently by F. Haglund in [4]. Systolic complexes are simply–connected simplicial complexes satisfying certain link conditions (see Definition 3.2). Some of their properties are very similar to the properties of CAT(0) (i.e. metrically nonpositively curved or satisfying the Alexandrov condition) metric spaces, therefore one calls them complexes of *simplicial nonpositive curvature*. On the other hand, all systolic groups (which are groups acting on systolic complexes properly and cocompactly) satisfy some exotic properties that make them in dimension ≥ 3 different from the previously studied classes of groups. There is a variety of examples, since systolic groups can have arbitrarily large virtual cohomological dimension [6].

Let us first list the similarities between systolic complexes and groups and CAT(0) metric spaces and groups. First we want to point out that a simplicial complex of dimension 2 is systolic iff it is CAT(0) with respect to the piecewise Euclidean metric, for which edges have length 1 (this makes the triangles equilateral). In higher dimension, however, there are examples of a group G acting by simplicial automorphisms on systolic complex X, for which there does not exist a G–invariant piecewise Euclidean metric on X, which is CAT(0). It is not known if all systolic groups act properly and cocompactly on some CAT(0) spaces.

One of the most important similarities is that systolic complexes are contractible (Theorem 4.1(1) in [6]). This can be thought of as an analogue of the Cartan–Hadamard theorem in the systolic setting. Moreover, any simplicially nonpositively curved complex of groups is developable [6]. This should be compared with the theorem that all metrically nonpositively curved complexes of groups are developable (Theorem 4.17, Chapter III.$\mathcal{C}$ in [1]). Januszkiewicz and Świątkowski established the previously mentioned properties of systolic complexes by introducing and exploiting the notion of convexity. They have also found a particularly nice system of geodesics (so called bicombing), which turns out to satisfy the so called fellow traveler property. In particular they proved that systolic groups are biautomatic [6]. This implies, for example, that free abelian subgroups (which in fact must have rank ≤ 2, see below) of systolic groups are undistorted.

This direction was pursued by T. Elsner in [2] and [3]. In [2] Elsner proved that for any group $H \cong \mathbb{Z}^2$ acting properly on a systolic complex X there is an associated so called flat systolic plane in X, which is H–invariant. In CAT(0) setting this is known as the Flat Torus Theorem (Theorem 7.1, Chapter II.7 of [1]). Elsner also proved that for any $H \cong \mathbb{Z}$ there exists a 1–skeleton geodesic, invariant under some finite index subgroup of H (see [3]).

Our dissertation, which concerns the fixed point theorem for finite automorphism groups, is a part of the program of seeking analogies between systolic complexes and CAT(0) metric spaces. See Section 2 for the overview of the results in the dissertation.

It should be mentioned that systolic complexes are not far from being Gromov hyperbolic. In fact any systolic complex with no flat systolic planes is Gromov hyperbolic, as shown in [10], and independently in [2]. For example, if one puts a slightly stronger condition on links of a systolic complex (7-largeness, c.f. Definition 3.2), it turns out to be Gromov hyperbolic [6]. Moreover, for each natural number n there exists a number $k(n)$ such that if all links of a systolic complex of dimension $\leq n$ are $k(n)$–large, then this complex is CAT(0) with respect to the piecewise Euclidean metric, for which the lengths of edges equal 1, see [6]. Although many of the known constructions of systolic groups (see [6], for example) yield word–hyperbolic groups, in general systolic groups are not word–hyperbolic.

Now let us list some exotic properties of systolic groups. Januszkiewicz and Świątkowski observed in [7] that all full subcomplexes of systolic complexes are aspherical. This has the following consequences. First of all, for 7–systolic groups (which are word–hyperbolic) one can study their Gromov boundary. It was established by D. Osajda [9] that this boundary is hereditary aspherical. Very roughly, this means that all closed subspaces of the boundary are aspherical. Recently Świątkowski has obtained another result of this kind. Namely, he proved that for a closed subspace F of the Gromov boundary ∂G of a 7–systolic group G, the morphism induced by the inclusion $F \subset \partial G$ on the associated pro–fundamental groups is a monomorphism. See [16] for details. Both of the above mentioned properties of the boundaries are exotic for topological spaces of dimension ≥ 2, i.e. for groups of virtual cohomological dimension ≥ 3.

Another application of the asphericity of full subcomplexes of a systolic complex is establishing the so called asymptotic hereditary asphericity for systolic groups [7]. One should view it as a coarse version of the notion of hereditary asphericity described above. Groups that are asymptotically hereditary aspherical do not allow subgroups isomorphic to fundamental groups of nonpositively curved closed riemannian manifolds of dimension ≥ 3. In particular they do not allow free abelian subgroups of rank ≥ 3.

2 Overview of the results

Now let us give an overview of the results, which constitute the dissertation, although for a more detailed introduction we refer to the introductory sections of Chapter II and Chapter III. The content of the dissertation, up to some rearrangements, agrees with papers [11] and [12].

The main objective of the dissertation is to consider systolic analogues of the following CAT(0) fixed point theorem. Namely, if X is a complete CAT(0) space and G is a finite group of isometries of X then the fixed–point set of G is non–empty and convex, hence contractible (Corollary 2.8, Chapter II.2 in [1]). In systolic setting one asks if for any finite group G of simplicial automorphisms of a systolic complex X the fixed point set of G is non–empty. While the proof of the CAT(0) fixed point theorem is quite easy, in systolic setting it is highly nontrivial. In fact, we are only able to establish a coarse analogue of the fixed point theorem for general systolic complexes, which suffices, however, for applications. Namely, we prove that for a finite group G acting by simplicial automorphisms on a systolic complex X there exists a G–invariant subcomplex of X of diameter ≤ 5. As a consequence, we prove that systolic groups have only finitely many conjugacy classes of finite subgroups.

We provide a proof of the honest fixed point theorem for locally finite 7–systolic complexes, though. We deduce from this that the family of k–systolic groups, for $k \geq 7$, is closed under amalgamating over finite subgroups and taking HNN extensions over finite subgroups. However, the honest fixed point theorem for general systolic complexes is out of reach via our current techniques, though we have some partial results, which could turn out to be useful in future attempts.

One can further analyze the structure of the fixed point set. In fact, we prove that for any group G acting by simplicial automorphisms on a systolic complex X, the fixed point set of G is either empty or contractible. Moreover, this statement remains valid if we substitute the complex X with its Rips complex X_n for any $n \geq 1$, where X_n is obtained from X by adding simplices spanned on sets of vertices of X which have diameter $\leq n$.

Combining the coarse fixed point theorem for systolic complexes with the above alternative for the fixed point set in the Rips complex of a systolic complex, we obtain the following. Let G be a group acting properly on a systolic complex X, this action extends to the proper action of G on X_n. Then for $n \geq 5$ the fixed point set in X_n for any finite subgroup of G is contractible. CW–complexes equipped with the proper action of G satisfying this last property are called models for $\underline{E}G$, see [8]. Thus X_n is a finite dimensional model for $\underline{E}G$. Moreover, if the action of G on X is cocompact (i.e. when G is systolic), then X_n is a so called finite model for $\underline{E}G$.

$\underline{E}G$ is also called a classifying space for finite subgroups (or for proper actions). The geometry of $\underline{E}G$ reflects many algebraic properties of the group G, see Section 5 in [8]. In particular $\underline{E}G$ appears in the formulation of the famous Baum–Connes Conjecture. Moreover, among the geometrical methods of proving the Novikov Conjecture, there is a method via constructing the boundary of a model for $\underline{E}G$, see [13]. The boundaries that can be used in this approach need to satisfy properties similar to the properties of Gromov boundaries for word–hyperbolic groups. This is the current area of research of D. Osajda and the author.

Let us state very briefly how the dissertation is organized. In Section 3 of Chapter I we recall the basic notions and properties of systolic complexes and groups which will be used throughout the dissertation.

In Chapter II we prove that for an action of a finite group G on a systolic complex X there exists a G–invariant subcomplex of X of diameter ≤ 5. We deduce from this that systolic groups have only finitely many conjugacy classes of finite subgroups. For 7–systolic locally finite complexes we prove that there is a fixed point for an action of any finite G. From this we deduce that free products with amalgamation (and HNN extensions) of 7–systolic groups over finite subgroups are also 7–systolic.

In Chapter III we prove that the fixed point set of an action of any group on a systolic complex or its Rips complex is either empty or contractible. We deduce from this that if a group G acts properly by simplicial automorphisms on a systolic complex X, then X_n is a finite dimensional model for $\underline{E}G$, for $n \geq 5$.

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3 Systolic complexes

Let us recall (from [6]) the definition of a systolic complex and a systolic group.

Definition 3.1. A subcomplex K of a simplicial complex X is called *full* in X if any simplex of X spanned by vertices of K is a simplex of K. The *span* of a subcomplex $K \subset X$ is the smallest full subcomplex of X containing K. We will denote it by $\mathrm{span}(K)$. A simplicial complex X is called *flag* if any set of vertices, which are pairwise connected by edges of X, spans a simplex in X. A simplicial complex X is called *k–large*, $k \geq 4$, if X is flag and there are no embedded cycles of length $< k$, which are full subcomplexes of X (i.e. X is flag and every simplicial loop of length $< k$ and ≥ 4 "has a diagonal").

Definition 3.2. A simplicial complex X is called *systolic* if it is connected, simply connected and links of all simplices in X are 6–large. A group Γ is called *systolic* if it acts cocompactly and properly by simplicial automorphisms on a systolic complex X. (*Properly* means X is locally finite and for each compact subcomplex $K \subset X$ the set of $\gamma \in \Gamma$ such that $\gamma(K) \cap K \neq \emptyset$ is finite.) If the links of all simplices of X are additionally k–large with $k \geq 6$ we call it (and the group) *k–systolic*.

Recall [6], Proposition 1.4, that systolic complexes are themselves 6–large. In particular they are flag. Moreover, connected and simply connected full subcomplexes of systolic (respectively k–systolic) complexes are themselves systolic (resp. k–systolic). It turns out a simplicial complex is k–systolic with $k \geq 6$ iff it is connected, simply connected and k–large.

Now we briefly treat the definitions and facts concerning convexity.

Definition 3.3. For every pair of subcomplexes (usually vertices) A, B in a simplicial complex X denote by $|A, B|$ ($|ab|$ for vertices $a, b \in X$) the combinatorial distance between $A^{(0)}, B^{(0)}$ in $X^{(1)}$, the 1–skeleton of X. The *diameter* $\mathrm{diam}(A)$ is the maximum of $|a_1a_2|$ over vertices a_1, a_2 in A.

A subcomplex K of a simplicial complex X is called *3–convex* if it is a full subcomplex of X and for every pair of edges ab, bc such that $a, c \in K$, $|ac| = 2$, we have $b \in K$. A nonempty subcomplex K of a systolic complex X is called *convex* if it is connected and links of all simplices in K are 3–convex subcomplexes of links of those simplices in X.

In Lemma 7.2 of [6] authors conclude that convex subcomplexes of a systolic complex X are contractible, full and 3–convex in X. For a subcomplex $Y \subset X$, $n \geq 0$, *the combinatorial ball* $B_n(Y)$ *of radius* n *around* Y is the span of $\{p \in X^{(0)} \colon |p, Y| \leq n\}$. (Similarly $S_n(Y) = \mathrm{span}\{p \in X^{(0)} \colon |p, Y| = n\}$.) If Y is convex (in particular, if Y is a simplex) then $B_n(Y)$ is also convex, as proved in [6], Corollary 7.5. The intersection of a family of convex subcomplexes is convex and we can define the *convex hull* of any subcomplex $Y \subset X$ as the intersection of all convex subcomplexes of X containing Y. We denote the convex hull of Y by $\mathrm{conv}(Y)$.

We include the proof of the following easy but useful lemma, since it does not appear elsewhere.

Lemma 3.4. $\mathrm{diam}(\mathrm{conv}(Y)) = \mathrm{diam}(Y)$.

Proof. If Y is unbounded then there is nothing to prove. Otherwise, denote $d = \mathrm{diam}(Y)$. The inequality $\mathrm{diam}(\mathrm{conv}(Y)) \geq d$ is obvious. For the other direction, let y_1, y_2 be any two vertices in $\mathrm{conv}(Y)$. We want to prove that $|y_1y_2| \leq d$. Observe that for any vertex $y \in Y$ the ball $B_d(y)$ is convex and contains Y, hence by the definition of the convex hull we have $\mathrm{conv}(Y) \subset B_d(y)$. This means that $|yy_1| \leq d$. Thus Y is contained in $B_d(y_1)$ and by convexity of balls we have $\mathrm{conv}(Y) \subset B_d(y_1)$. We get $|y_1y_2| \leq d$, as desired. $\square$

The paper [5] of F. Haglund and J. Świątkowski contains a proof of the following proposition, which will be used throughout the dissertation.

Proposition 3.5 ([5], Proposition 4.9)**.** *A nonempty full subcomplex Y of a systolic complex X is convex if and only if $Y^{(1)}$ is geodesically convex in $X^{(1)}$ (i.e. if all geodesics in $X^{(1)}$ joining vertices of Y lie in $Y^{(1)}$).*

We will need a crucial projection lemma. The *residue* of a simplex σ in X is the union of all simplices in X, which contain σ.

Lemma 3.6 ([6], Lemma 7.7)**.** *Let Y be a convex subcomplex of a systolic complex X and let σ be a simplex in $B_1(Y)$ disjoint with Y. Then the intersection of the residue of σ and of the complex Y is a simplex (in particular it is nonempty).*

Definition 3.7. The simplex as in Lemma 3.6 is called the *projection* of σ onto Y.

Now let us recall some definitions and facts concerning flat minimal surfaces in systolic complexes proved by T. Elsner in [2].

Definition 3.8. The *flat systolic plane* is a systolic 2–complex obtained by equilaterally triangulating Euclidean plane. We denote it by $\mathbb{E}^2_\Delta$. A *systolic disc* is a systolic triangulation of a 2–disc and a *flat disc* is any systolic disc Δ, which can be embedded into $\mathbb{E}^2_\Delta$, such that $\Delta^{(1)}$ is embedded isometrically into 1–skeleton of $\mathbb{E}^2_\Delta$. A systolic disc Δ is called *wide* if $\partial\Delta$ is a full subcomplex of Δ. For any vertex $v \in \Delta$ the *defect* of v (denoted by $\mathrm{def}(v)$) is defined as $6 - t(v)$ for $v \notin \partial\Delta$, and as $3 - t(v)$ for $v \in \partial\Delta$, where $t(v)$ is the number of triangles in Δ containing v. It is clear that internal vertices of a systolic disc have nonpositive defects.

We will need the following easy and well known fact.

Lemma 3.9 (Gauss-Bonnet Lemma)**.** *If Δ is any triangulation of a 2–disc, then*

$$\sum_{v \in \Delta^{(0)}} \mathrm{def}(v) = 6$$

Definition 3.10. Let X be a systolic complex. Any simplicial map $S\colon \Delta \to X$, where Δ is a triangulation of a 2–disc, is called a *surface*. A surface S is *systolic, flat* or *wide* if the disc Δ satisfies the corresponding property. If S is injective on $\partial\Delta$ and minimal (the smallest number of triangles in Δ) among surfaces with the given image of $\partial\Delta$, then S is called *minimal*. A geodesic in $\Delta^{(1)}$ is called *neat* if it stays out of $\partial\Delta$ except possibly at its ends. A surface S is called *almost geodesic* if it maps neat geodesics in $\Delta^{(1)}$ isometrically into $X^{(1)}$.

Lemma 3.11 ([2], Lemma 2.5)**.** *A systolic disc D is flat if and only if it satisfies the following three conditions:*

(i) D has no internal vertices of defect < 0

(ii) D has no boundary vertices of defect < -1

(iii) any segment in ∂D connecting vertices with defect < 0 contains a vertex of defect > 0.

Theorem 3.12 ([2], Theorem 3.1)**.** *Let X be a systolic complex. If S is a wide flat minimal surface in X then S is almost geodesic.*

Finally, recall the following powerful observation.

Lemma 3.13 ([7], Lemma 4.4)**.** *Every full subcomplex of a systolic complex is aspherical.*

Chapter II

The fixed point theorem

4 Introduction

For CAT(0) spaces we have the following fixed point theorem.

Theorem 4.1 ([1], Chapter II.2, Corollary 2.8)**.** *If X is a complete CAT(0) space and G is a finite group of isometries then the fixed–point set of G is non–empty.*

Theorem 4.1 follows from an observation that for every bounded subset Y of a CAT(0) space we can define a special point y, the *circumcenter* of Y, which is the center of the (unique) minimal ball containing Y. The circumcenter y is invariant under isometries which leave Y invariant.

For simplicial nonpositive curvature a minimal combinatorial ball containing finite set is not unique and the distance between centers of various minimal balls can be arbitrarily large. Thus there is no immediate way to define circumcenter.

We have found an analogue of Theorem 4.1 for actions of finite groups on systolic complexes. This chapter is devoted to the proof of the following theorem.

Theorem 4.2. *Let G be a finite group acting by simplicial automorphisms on a systolic complex X. Then there exists a bounded subcomplex $Y \subset X$ which is invariant under the action of G and whose diameter is ≤ 5.*

This is a coarse version of the fixed point theorem. We use it to obtain in systolic setting the property, which for CAT(0) groups is implied by Theorem 4.1.

Corollary 4.3. *Any systolic group contains only finitely many conjugacy classes of finite subgroups.*

The class of 7–systolic complexes is a special subclass of the class of systolic complexes. 7–systolic complexes are hyperbolic metric spaces ([6], Theorem 2.1) thus 7–systolic groups (defined as the groups which act properly and cocompactly on 7–systolic complexes) are hyperbolic. All hyperbolic groups have finitely many conjugacy classes of finite subgroups, so Corollary 4.3 is nothing new for 7–systolic groups. However, for locally finite 7–systolic complexes we can go further than Theorem 4.2 and obtain an honest fixed point theorem.

Theorem 4.4. *Let G be a finite group acting by simplicial automorphisms on a locally finite 7–systolic complex X . Then there exists a simplex $\sigma \in X$, which is invariant under the action of G. (The barycenter of σ is a fixed point for G).*

We apply Theorem 4.4 to show that the class of 7–systolic groups is closed under certain algebraic operations.

Theorem 4.5. *Free products of 7–systolic groups amalgamated over finite subgroups are 7–systolic. HNN extensions of 7–systolic groups over finite subgroups are 7–systolic.*

The chapter is organized in the following way. In Section 5 we introduce the notion of round complexes (which are an obstruction to finding the circumcenter) and study their properties. In Section 6 we obtain a weaker version of Theorem 4.2, which still implies Corollary 4.3. This weaker version is more elegant in proof and it serves as a demonstration of our method, which is fully applied in Section 7. In Section 8 we prove Theorem 4.4 and in Section 9 we apply it to prove Theorem 4.5.

5 Round complexes

Let us make some remarks to motivate the forthcoming definition.

Remark 5.1.

(1) If a finite group G acts by simplicial automorphisms on a systolic complex, then there exists a bounded convex subcomplex $Y \subset X$ that is invariant under the action of G. To see this, take any vertex $x \in X$ and take the convex hull Y of the set $Gx = \{g(x) \colon g \in G\}$. Since the set Gx is finite,

the points of Gx are at distance $< d$ from x for some finite d. Convexity of combinatorial balls implies that the points of Y are also at distance $< d$ from x. The fact that Y is invariant under the action of G is immediate.

(2) Let Y be a bounded systolic complex of diameter d and let G be a group acting on Y by simplicial automorphisms. Then the convex subcomplex $\bigcap_{y \in Y^{(0)}} B_{d-1}(y)$ is invariant under the action of G and its diameter is $\leq d-1$.

(3) This looks like a plan for getting invariant subcomplexes of arbitrarily small diameter. However, this plan is difficult to execute, since it is unclear how to exclude the possibility that $\bigcap_{y \in Y^{(0)}} B_{d-1}(y)$ is empty.

Definition 5.2. A bounded systolic complex Y of diameter d is called *round* if

$$\bigcap_{y \in Y^{(0)}} B_{d-1}(y) = \emptyset.$$

Note that this is equivalent to the property that for each vertex $v \in Y^{(0)}$ there is a vertex $w \in Y^{(0)}$ such that $|vw| = d$.

We now start developing properties of round complexes, which will result in establishing a bound for their diameter.

Lemma 5.3. *Let Y be a round complex of diameter d. Then there exists an edge $ab \in Y$ and vertices $v, w \in Y$ such that $|vb| = |wa| = d$ and $|va| = |wb| = d - 1$.*

Proof. Fix any maximal simplex $\sigma \subset Y$. Let k be the maximal number such that for each simplex $\sigma' \subset \sigma$ of dimension k there exists a vertex $v \in Y$ with $|v, \sigma'| = d$. Note that since Y is round we have $k \geq 0$. On the other hand if we denote by n the dimension of σ we have $k < n$, since if σ would be at distance d from some vertex v, then the projection of σ onto $B_{d-1}(v)$ together with σ would form a strictly greater simplex (c.f. Lemma 3.6 and Definition 3.7).

Take a simplex $\tau \subset \sigma$ of dimension $k+1$ for which there does not exist a vertex $v \in Y$ with $|v, \tau| = d$. Denote by a, b any two vertices of τ. Denote by τ_a, τ_b faces of codimension 1 in τ not containing a, b respectively. By definition of k there exist vertices v, w such that $|v, \tau_a| = |w, \tau_b| = d$. This implies $|vb| = |wa| = d$. The choice of τ implies $|va| = |wb| = d - 1$. $\square$

Definition 5.4. A *k–hexagon* is a subcomplex of $\mathbb{E}^2_\Delta$ obtained by taking $6k^2$ 2–dimensional simplices forming a regular hexagon of edge length k. A *k–triangle* is a subcomplex of $\mathbb{E}^2_\Delta$ obtained by taking k^2 2–dimensional simplices forming a triangle of edge length k.

Lemma 5.5. *Let $k \geq 1$ be an integer and let Y be a round complex of diameter $d \geq 3k$. Then there exists a k–hexagon $H \subset Y$ whose 1–skeleton is isometrically emdedded in the 1–skeleton $Y^{(1)}$.*

Proof. Lemma 5.3 guarantees the existence of the vertices $a, b, v, w \in Y$ at appropriate distances.

Let $v' \in Y$ be the furthermost vertex from v which is common for some geodesics connecting v to a and v to w. Then let $w' \in Y$ be the furthermost vertex from w which is common for some geodesics connecting w to b and w to v'. Take any loop obtained by concatenating some 1–skeleton geodesics connecting a to v', v' to w' then w' to b and then the edge ba. We claim that this loop does not have self–intersections. Indeed, the segments av' and bw' do not intersect by the choice of a, b, v, w (any geodesics av and bw must be disjoint, since vertices on av are further from b than from a and vertices on bw are nearer to b that to a). The segment $v'w'$ does not intersect av' (outside of v') by the choice of v'. Finally, $v'w'$ does not intersect bw' (outside of w') by the choice of w'.

Now among all surfaces, whose boundary is any such piecewise geodesic loop $av'w'ba$ choose a surface $S\colon \Delta \to X$ of minimal area. Clearly, S is a minimal surface. Moreover, since our loop is piecewise geodesic and these geodesics are chosen arbitrarily, the defect at every boundary vertex of Δ different from a, b, v', w' is at most 0. Since the segments ab and bw' form a geodesic, the defect at b is at most 1, and similar argument shows the same for a. Gauss–Bonnet Lemma 3.9 implies now that the defects at v' and w' are equal to 2, at a, b equal to 1, at other boundary and interior vertices equal to 0. Lemma 3.11 implies that Δ is flat. Analyzing possible subcomplexes of $\mathbb{E}^2_\Delta$ one easily sees that Δ has to be a trapezoid with sides $ab, v'w'$ parallel.

Denote $d' = |v'w'| \leq d$, so that the other edges of the trapezoid Δ have lengths $|av'| = |bw'| = d' - 1, |ab| = 1$. Then $|vv'| = |av| - |av'| = (d-1) - (d'-1)$ and similarly $|ww'| = |bw| - |bw'| = (d-1) - (d'-1)$. Then the geodesic vw has length $|vv'| + |v'w'| + |w'w| = 2d - d' = d + (d - d') \geq d$. Since the diameter of Y is d we must have $d' = d$, which implies $v = v'$ and $w = w'$.

Since $3k \leq d$, there exists a k–hexagon $H \subset \Delta$. Let T be a subcomplex obtained from Δ by deleting the two triangles containing $S^{-1}(v)$ and $S^{-1}(w)$. Then our surface S restricted to T is flat, wide and minimal. Therefore, by Elsner's Theorem 3.12, all neat geodesics in $T^{(1)}$ are mapped by S onto geodesics in $Y^{(1)}$. This implies that the 1–skeleton of the k–hexagon $H \subset T$ is mapped isometrically into $Y^{(1)}$. Identification of H with $S(H)$ finishes the proof. □

Lemma 5.6. *Let Y be a bounded systolic complex of diameter d, and let $H \subset Y$ be a k–hexagon, whose 1–skeleton is isometrically embedded into $Y^{(1)}$. Let $v \in H$ be the vertex, which is the Euclidean center of H. Take m such that $2m \leq k$. Suppose that $w \in Y$ is a vertex at distance d from v. Then there exists an m–hexagon $H' \subset H$, such that $H' \subset S_d(w)$.*

Proof. Since $H^{(1)}$ is isometrically embedded in $Y^{(1)}$, the intersection $C = H \cap B_{d-1}(w)$ is convex in H (Proposition 3.5). Denote by $S_0, \ldots, S_5$ the six k–triangles in H whose vertices are v and two consecutive vertices (in the Euclidean sense) of the boundary of H. Notice that since v is not in C, then for each $i = 0, 1, 2$ at least one of the opposite S_i, S_{i+3} must have empty intersection with C. Moreover, if $C \cap S_i \neq \emptyset$ and $C \cap S_{i+2} \neq \emptyset$ (we treat indices $i = 0, \ldots, 5$ modulo 6) then $C \cap S_{i+1} \neq \emptyset$. Together this implies that we have an i such that $C \cap \big(S_i \cup S_{i+1} \cup S_{i+2}\big) = \emptyset$. We find the m–hexagon H' inside the union of these k–triangles. □

Corollary 5.7. *Let Y be a round complex of diameter $d \geq 12$. Then there exists a vertex w and a 2–hexagon $H' \subset Y$, whose 1–skeleton is isometrically embedded into $Y^{(1)}$ and such that $H' \subset S_d(w)$.*

Proof. We apply Lemma 5.5 to Y and obtain a 4–hexagon H, whose 1–skeleton is isometrically embedded into $Y^{(1)}$. Let $v \in H$ be the vertex, which is the Euclidean center of H. Since Y is round, there exists a vertex $w \in Y$ such that $|vw| = d$. Now applying Lemma 5.6 gives us the desired 2–hexagon H'. □

6 The main argument

Lemma 6.1. *Let $w \in Y$ be a vertex in a systolic complex. Then for any sphere $S_n(w) \subset Y$ with $n \geq 1$ there is no 2–hexagon $H \subset S_n(w)$, whose 1–skeleton is isometrically embedded into $Y^{(1)}$.*

Before we present the proof, we note that Corollary 5.7 and Lemma 6.1 immediately imply the following.

Corollary 6.2. *The diameter of any round complex is ≤ 11.*

Proof of Lemma 6.1. We will prove the lemma by contradiction. Suppose there is a 2–hexagon $H \subset S_n(w)$, whose 1–skeleton is isometrically embedded into $Y^{(1)}$. Then $n \geq 2$ (because the diameter of H is 4). Denote by

$a_0, a_1 \ldots, a_{11}$ the vertices of the boundary of H in their natural cyclic order. Now from the projections of the edges $a_i a_{i+1}$ (we treat i's modulo 12) onto $B_{n-1}(w)$ choose single vertices b_i. Denote $B = \operatorname{span}(\bigcup_i b_i)$. Note that for any i the vertices b_i, b_{i+1} are either equal or connected by an edge (this follows from projection Lemma 3.6).

Let us denote $D_1 = \operatorname{span}(B \cup H), D_2 = \operatorname{span}(B \cup B_{n-2}(w))$. Observe first that $D_1 \cup D_2 \subset Y$ is full because there are no edges between the complexes $H \subset S_n(w)$ and $B_{n-2}(w)$. We will use Meyer–Vietoris sequence for the pair D_1, D_2. Namely consider the loop $b_0 b_1 \ldots b_{11} b_0$, which is contractible both in D_1 and D_2 (which is clear). These contractions form a 2–sphere in $D_1 \cup D_2$ which is contractible, as $D_1 \cup D_2$ is aspherical (Lemma 3.13). Thus the existence (and the form) of the homomorphism $H_2(D_1 \cup D_2) \to H_1(D_1 \cap D_2)$ in the Meyer–Vietoris sequence implies $b_0 b_1 \ldots b_{11} b_0$ is homological to zero in $D_1 \cap D_2 = B$.

We will show this is not possible. Namely, we will construct a continuous mapping from B into $\mathbb{R}^2 \setminus \{0\}$ such that the loop $b_0 b_1 \ldots b_{11} b_0$ is mapped to a nontrivial loop. First we construct a map $f \colon B^{(0)} \to \mathbb{R}^2$. Denote $e_k = e^{2\pi i \frac{k}{12}} \in \mathbb{C} = \mathbb{R}^2$. For each vertex of B we choose any i such that this vertex is equal to b_i and we map it to e_i (this mapping is not unique only if for some $i \neq j$ we have $b_i = b_j$). Then we extend linearly to all simplices of B. Let $\langle \cdot, \cdot \rangle$ denote the standard scalar product in $\mathbb{R}^2$. Fix a simplex $\sigma \subset B$ and suppose that $b_i, b_j \in \sigma$, where i, j are the indices chosen when we defined f on $B^{(0)}$. Since $|b_i b_j| \leq 1$ we have $|a_i a_j| \leq 3$, so $|i - j| \leq 3$ and thus $\langle f(b_i), f(b_j) \rangle \geq 0$. If we fix i, j and consider some other $b_k \in \sigma$ (if k is the index chosen for b_k) then the same observation yields that $f(b_k)$ belongs to $\{v \colon \|v\|^2 = 1, \langle v, f(b_i) \rangle \geq 0, \langle v, f(b_j) \rangle \geq 0\}$. The convex hull of this set omits 0. This proves that the image of f lies in $\mathbb{R}^2 \setminus \{0\}$. Now observe that if $b_i = b_j$ for $i \neq j$ (now the indices are arbitrary), then the distances $|a_i a_j|, |a_i a_{j+1}|, |a_{i+1} a_j|, |a_{i+1} a_{j+1}|$ are all ≤ 2, which implies $|i - j| = 1$. From this we see that the image under f of each b_i is e_{i-1}, e_i or e_{i+1}. It follows that the image of each edge $b_i b_{i+1}$ lies in the sector between e_{i-1} and e_{i+2}. This implies that the loop $b_0 b_1 \ldots b_{11} b_0$ is mapped to a nontrivial loop. Thus we have reached a contradiction. □

Proof of Theorem 4.2 with bound ≤ 11 instead of ≤ 5. Let $Y \subset X$ be a bounded convex subcomplex invariant under the action of G, with the minimal possible (nonzero) diameter d. Such subcomplexes exist, as it follows from Remark 5.1(1). Now let $Y' = Y \cap \left(\bigcap_{y \in Y^{(0)}} B_{d-1}(y) \right)$. As it was noticed in Remark 5.1(2), the diameter of Y' is $< d$, so by minimality of the diameter of Y, Y' must be empty. Thus Y is a round complex and by Corollary 6.2

its diameter is ≤ 11. $\square$

Remark 6.3. For any bounded convex subcomplex $Y \subset X$ of a systolic complex X we can define a sequence $Y = Y_0, Y_1, Y_2 \dots$ by the formula $Y_{i+1} = Y_i \cap \left(\bigcap_{y \in Y_i^{(0)}} B_{d(Y_i)-1}(y) \right)$, where $d(Y_i)$ is the diameter of Y_i. The round complex on which this sequence terminates can be treated as the *circumcenter* of Y.

Proof of Corollary 4.3. We argue by contradiction. Suppose we have infinitely many conjugacy classes of finite subgroups represented by $H_1, H_2, \dots \subset G$. Denote by $K \subset X$ the compact subset such that $\bigcup_{g \in G} g(K) = X$. For all $i \geq 1$ let K_i be bounded subcomplexes of X with diameter ≤ 11 invariant under H_i. Find $g_i \in G$ such that $g_i(K_i) \cap K \neq \emptyset$. Then the subgroups $g_i H_i g_i^{-1}$, which leave $g_i(K_i)$ invariant, still represent different conjugacy classes. In particular, the union $\bigcup_i g_i H_i g_i^{-1}$ is infinite. But for all elements g of this union we have $g(B_{11}(K)) \cap B_{11}(K) \neq \emptyset$, which contradicts the properness of the action of G. $\square$

7 Round complexes have diameter ≤ 5

In this section we give the proof of even sharper bound for diameter of round complexes. It involves the same techniques, but takes considerably more case checking. First observe that as a special case of Lemma 5.5 we get the following.

Corollary 7.1. *Let Y be a round complex of diameter $d \geq 6$. Then there exists a 2–hexagon $H \subset Y$ whose 1–skeleton is isometrically emdedded in the 1–skeleton $Y^{(1)}$.*

Lemma 7.2. *Let Y be a bounded systolic complex of diameter d, and let $H \subset Y$ be a 2–hexagon, whose 1–skeleton is isometrically embedded into $Y^{(1)}$. Let $v \in H$ be the vertex, which is the Euclidean center of H. Suppose $w \in Y$ is a vertex at distance d from v. Denote by $a_0 a_1 \dots a_{11} a_0$ the boundary of H. Then there exists a simplicial loop $\gamma = b_0 b_1 \dots b_{11} b_0$ in $S_{d-1}(w)$, such that each b_i is connected by an edge with a_i and a_{i+1} (or is equal to one of them) and γ is contractible in* $\mathrm{span}(\gamma \cup (H \cap S_d(w)))$.

Proof. As in the proof of Lemma 5.6 we obtain three consecutive 2–triangles in H, say S_0, S_1, S_2, such that $S_0 \cup S_1 \cup S_2 \subset S_d(w)$. Denote the boundary vertices of H by $a_0, \dots, a_{11}$ so that $a_5, a_6, \dots, a_{11}$ lie in $S_0 \cup S_1 \cup S_2$. Pick vertices b_i with $5 \leq i \leq 10$ from the projections of edges $a_i a_{i+1}$ onto $B_{d-1}(w)$.

Now we will pick b_4 and b_{11}. If $a_0 \in S_d(w)$, then pick b_{11} in the projection of $a_{11}a_0$ onto $B_{d-1}(w)$. In the other case pick $b_{11} = a_0$. Similarly pick b_4 from the projection of a_4a_5 if $a_4 \in S_d(w)$ or pick $b_4 = a_4$ in the other case.

Now we will pick b_0 and b_1. Denote by u the vertex in H which is the common neighbor of a_0, a_2 and v. Note that since u is a neighbor of v, we have either $u \in S_{d-1}(w)$ or $u \in S_d(w)$. In the first case, pick $b_0 = b_1 = u$. In the second case proceed as follows. First we will pick b_0. Consider the edge a_0a_1. If it lies in $S_d(w)$, pick b_0 from the projection of a_0a_1 onto $B_{d-1}(w)$, as usual. If not, choose $b_0 = a_1$ if $a_1 \in S_{d-1}(w)$ and $b_0 = a_0$ in the other case. Now we will pick b_1. If a_1a_2 lies in $S_d(w)$ then pick b_1 from the projection of a_1a_2, as usual. If not, choose $b_1 = a_2$ if a_2 lies in $S_{d-1}(w)$ and $b_1 = a_1$ in the other case.

Define b_3, b_2 exactly as b_0, b_1 substituting b_0, b_1, a_0, a_1 with b_3, b_2, a_4, a_3 respectively in the previous construction.

Note that our choice guarantees that for each $0 \leq i \leq 11$ the vertex b_i is connected by an edge with a_i and a_{i+1} (or is equal to one of them).

First we will check that $b_0b_{11} \dots b_{11}b_0$ is a simplicial loop, i.e. that $|b_ib_{i+1}| \leq 1$ for all $0 \leq i \leq 11$. For $4 \leq i \leq 10$ this follows from the projection Lemma 3.6. Now consider $i = 11$. If $a_0 \in S_d(w)$ then $|b_{11}b_0| \leq 1$ follows again from the projection Lemma 3.6. In the other case $b_{11} = a_0$, so it is also a neighbor of b_0. The analogous argument works for $i = 3$.

Now focus on $i = 0$. If $u \in S_{d-1}(w)$ (where u is defined as in the construction of b_0, b_1), then $b_0 = u = b_1$ and we are done. If not, then if $a_1 \in S_d(w)$ then both b_0, b_1 are neighbors of a_1 in $S_{d-1}(w)$, so they are neighbors by Lemma 3.6. If $a_1 \notin S_d(w)$, then $b_0 = a_1$, so b_1 is its neighbor and we are also done. The analogous argument works for $i = 1$.

Finally consider $i = 2$. Define u' to be the common neighbor in H of a_2, a_4, v. If both u and u' are in $S_{d-1}(w)$, then $b_1 = u, b_2 = u'$ and we are done. If any of u, u' lies in $S_{d-1}(w)$, then a_2 lies either in $S_d(w)$ or in $S_{d-1}(w)$. In the first case, both b_1, b_2 are neighbors of a_2 in $S_{d-1}(w)$, so they are connected by Lemma 3.6. In the second case $b_1 = a_2 = b_2$ and we are also done.

We have thus proved that $b_0b_1 \dots b_{11}b_0$ is a simplicial loop, which we now denote by γ. We will prove that γ is contractible in $\operatorname{span}(\gamma \cup (H \cap S_d(w)))$. First observe that if the vertex u (defined as before) is not in $H \cap S_d(w)$ then, by the construction of b_0, $u = b_0$ lies in γ. The same holds for u'. So $S_0 \cup S_1 \cup S_2 \cup u \cup u' \subset \gamma \cup (H \cap S_d(w)$. This is enough to guarantee that

$\mathrm{span}\big(H \cap \big(\gamma \cup (H \cap S_d(w))\big)\big) \subset H$ is contractible. Denote this subcomplex of H by H_0.

Thus it is enough to prove that γ can be homotoped into H_0 in $\mathrm{span}(\gamma \cup (H \cap S_d(w)))$. If γ is disjoint with H_0, then γ is disjoint with H and, by the construction of γ, we have $H_0 = H \subset S_d(w)$. Hence, in this case, γ is homotopic to ∂H in $\mathrm{span}(\gamma \cup (H \cap S_d(w)))$ (as in Lemma 6.1) and we are done. If γ intersects H_0, then let $b_i b_{i+1} \dots b_j$ denote any segment of γ such that $b_i, b_j \in H_0$ and $b_l \notin H_0$ for all $i < l < j$ (with respect to the cyclic order). Then, by the construction of b_l, we have $a_l \in S_d$ for $i < l < j+1$, which implies $a_l \in H_0$. Thus the segment $b_i b_{i+1} \dots b_j$ is homotopic (relative its endpoints) to the segment $b_i a_{i+1} a_{i+2} \dots a_j b_j \subset H_0$ in $\mathrm{span}(\gamma \cup (H \cap S_d(w)))$. Thus all segments of γ lying outside H_0 can be homotoped into H_0 in $\mathrm{span}(\gamma \cup (H \cap S_d(w)))$ and we are done. □

Lemma 7.3. *Every round complex has diameter* ≤ 5.

Proof. Suppose, on the contrary, that Y is a round complex of diameter $d \geq 6$. Then, by Corollary 7.1, we get a 2–hexagon $H \subset Y$ whose 1–skeleton is isometrically embedded in $Y^{(1)}$. Let $v \in H$ be the Euclidean center of H. Since Y is round, there exists a vertex w at distance d from v. Denote by $a_0 a_1 \dots a_{11} a_0$ the boundary of H. Then, by Lemma 7.2, there exists a loop $\gamma = b_0 b_1 \dots b_{11} b_0$ in $S_{d-1}(w)$, such that b_i is connected by an edge with a_i and a_{i+1} and γ is contractible in $\mathrm{span}\,(\gamma \cup (H \cap S_d(w)))$. Let us denote $D_1 = \mathrm{span}(\gamma \cup (H \cap S_d(w))), D_2 = \mathrm{span}(\gamma \cup B_{n-2}(w))$. Observe that $D_1 \cup D_2 \subset Y$ is full because there are no edges between the vertices in $S_d(w)$ and $B_{n-2}(w)$. Now we can proceed word–by–word following the proof of Lemma 6.1 and get a contradiction. □

Proof of Theorem 4.2. As in the previous version of the proof in Section 6, we obtain an invariant convex subcomplex, which is round, so by Lemma 7.3 its diameter is ≤ 5. □

8 Fixed point for 7–systolic complexes

In this section we prove Theorem 4.4, the fixed point theorem for 7–systolic complexes (c.f. Definition 3.2). Notice that we already know that round 7–systolic complexes have diameter ≤ 2, since for diameter ≥ 3 Lemma 5.5 would produce a 1-hexagon with 1–skeleton isometrically embedded, which is not allowed in a 7–systolic complex. We can however skip this argument using a lemma by D. Osajda.

Lemma 8.1 ([9], Lemma 3.1)**.** *Suppose $|vq| = |wq| = n > 0$ for some vertices v, w, q of a 7–systolic complex X. Denote by $P(v), P(w)$ the projections onto $B_{n-1}(q)$ of v, w respectively. Then either $P(v) \subset P(w)$ or $P(w) \subset P(v)$ (or both).*

Proof of Theorem 4.4. Let Y be a minimal (nonempty) connected and simply connected full subcomplex of X invariant under G. By Remark 5.1(1) such subcomplexes exist. Y is round by Remark 5.1(2). Moreover, Y is 7–systolic, since it is a connected and simply connected full subcomplex of a 7–systolic complex X. We will show that Y must be a simplex. Suppose on the contrary that $d = \mathrm{diam}(Y) \geq 2$.

We will show there exists a vertex $v \in Y$ such that for some vertex w with $|vw| = 1$ we have $B_1(v) \subsetneq B_1(w)$. In other words, (excluding v and w) the set of neighbors of w is strictly greater than the set of neighbors of v. One may then view v as "more exposed" in Y than w.

To prove this, pick any vertex $q \in Y$ and consider the family $\{P(v)\}_{v \in S_d(q)}$ of all projections $P(v)$ onto $B_{d-1}(q)$ of vertices in $v \in S_d(q)$. Since Y is round this family is nonempty. By projection Lemma 3.6 the elements of this family are simplices. Now consider a vertex v such that $P(v)$ is minimal (for inclusion) simplex of the family. Take any vertex $w \in P(v)$. We will prove that v, w have the desired property. Consider any neighbor u of v. If $u \in S_{d-1}(q)$, then $u \in P(v)$, so u is a neighbor of w. If $u \in S_d(v)$, then by Lemma 8.1 and by minimality of $P(v)$ we get that $P(v) \subset P(u)$, hence $w \in P(u)$ and in this case also u is a neighbor of w. Note that if we project w onto $B_{d-2}(q)$ (recall that $d \geq 2$) we obtain some neighbor of w, which is not a neighbor of v. This ends the proof of $B_1(v) \subsetneq B_1(w)$.

Notice that since the strict inclusion of 1–balls is a transitive relation and since Y is finite, there exists a pair of vertices $|vw| = 1, B_1(v) \subsetneq B_1(w)$, such that no neighbor u of v satisfies $B_1(u) \subsetneq B_1(v)$ (i.e. v is a minimal element of this relation). Now consider the set $\mathcal{V}$ of all vertices $v \in Y$, which have the above minimality property. For each such vertex denote by v' its corresponding vertex w (this choice may be not unique). Notice that for all $v \in \mathcal{V}$ we have $v' \notin \mathcal{V}$, since v' cannot be minimal. We will now show that the subcomplex $Y' \subset Y$ spanned on the vertices $Y^{(0)} \setminus \mathcal{V}$ is connected and simply connected. Since it is nonempty, invariant under G and a strict subset of Y this will contradict the minimality of Y and will finish the proof.

To prove that Y' is connected and simply connected, we will construct a retraction $r \colon Y \to Y'$. First we define r on $Y^{(0)}$. For $v \in Y'^{(0)}$ put $r(v) = v$. For $v \in \mathcal{V}$ put $r(v) = v'$. We will prove that r can be extended to a simplicial mapping. Since Y' is flag all we have to show is that for any adjacent vertices

$v_1, v_2 \in Y^{(1)}$ we have $|r(v_1)r(v_2)| \leq 1$. If $v_1, v_2 \in Y'$ then this is obvious. If $v_1 \in Y', v_2 \in \mathcal{V}$, then since $r(v_1) = v_1$ is a neighbor of v_2, it is also a neighbor of $r(v_2)$ and we are done. If $v_1, v_2 \in \mathcal{V}$ then since v_1 is neighbor of v_2, it is also a neighbor of $r(v_2)$ and now since $r(v_2)$ is a neighbor of v_1, it is also a neighbor of $r(v_1)$ or equals $r(v_1)$ and we are done. Thus we can extend r to a simplicial mapping $r\colon Y \to Y'$ fixing Y' and thus, since Y is connected and simply connected, so is Y'. As observed earlier, this contradicts the minimality of Y. Thus $d \leq 1$ and Y is a simplex. □

Remark 8.2. We do not know if the assumption of Theorem 4.4 that X is locally finite may be omitted.

9 Amalgamated free products of 7–systolic groups

We will prove Theorem 4.5 by constructing 7–systolic complexes on which the amalgamated products and HNN extensions act. These complexes will have a form of trees of 7–systolic complexes, as defined below, related to the Bass-Serre trees of the corresponding products.

Definition 9.1. A *tree of k–systolic complexes* ($k \geq 6$) is a simplicial complex E together with a simplicial mapping $p\colon E \to T$ onto a simplicial tree T satisfying the following properties. For a vertex $v \in T$ the preimage $p^{-1}(v) \subset E$ is a k–systolic complex. For an open edge $e \in T$ the closure of the preimage $p^{-1}(e) \subset E$ is a simplex.

Lemma 9.2. *If $p\colon E \to T$ is a tree of k–systolic complexes, $k \geq 6$, then E is itself k–systolic.*

Proof. To prove E is k–systolic we need to prove E is connected, simply connected and k–large (c.f. remarks after Definition 2.2). Obviously, E is connected, simply connected and flag, since the preimage of each vertex and the closure of the preimage of each open edge in T is flag and contractible and the same holds for their intersections. Let γ be any loop of length l with $4 \leq l < k$ in E. Then $p(\gamma)$ is a loop in the tree T. If $p(\gamma)$ is a single vertex, then γ lies in a k–systolic subcomplex of E and thus has a diagonal. If $p(\gamma)$ is not a vertex, then there exists two different edges $ab, a'b' \in \gamma$ such that $p(ab) = p(a'b')$. This implies that a, b, a', b' lie in a common simplex. Since at least three of those vertices are different vertices of γ, we obtain a diagonal in the loop γ. □

Construction 9.3. Let G, H be 7–systolic groups acting properly and cocompactly on 7–systolic complexes X, Y, respectively. Let $F \subset G, F \subset H$ be some finite common subgroup. Let $\sigma \subset X, \tau \subset Y$ be some simplices fixed under F, as guaranteed by Theorem 1.4. We define the *amalgamated complex* $X * Y$ *for* $G *_F H$ as follows. Take the product space $G *_F H \times X \sqcup Y$ and identify (ag, x) with (a, gx) and (ah, y) with (a, hy) for all $a \in G *_F H, g \in G, h \in H, x \in X, y \in Y$. Note that this is an equivalence relation. As for now this is just a disjoint union of copies of X and Y corresponding to right cosets of G and H in $G *_F H$ respectively. Let α be an abstract simplex spanned on σ and τ (the join of σ and τ). Extend the action of F on σ and τ to an affine (i.e. simplicial) action on α. Now add extra simplicies (a, α) spanned on the pairs $(a, \sigma), (a, \tau)$ for $a \in G *_F H$ and identify (af, z) with (a, fz) for $a \in G *_F H, f \in F, z \in \alpha$. Hence we added a copy of α for each coset aF of F in $G *_F H$. This copy of α is glued to the copies of X and Y corresponding to aG and aH respectively. Note that what we get is a simplicial complex, i.e. there are no multiple edges. The only multiple edges could occur as a result of gluing two copies of α, say $(a, \alpha), (b, \alpha)$, where $a, b \in G *_F H$, to the same pair of copies of X and Y. This would imply $(a, X) = (b, X)$ and $(a, Y) = (b, Y)$. Thus $b^{-1}a \in G \cap H = F$, hence $(a, \alpha) = (b, \alpha)$. Also note that since the action of G on X and H on Y is proper, the complex $X * Y$ is locally finite.

Now we define the action of $G *_F H$ on $X * Y$. Take $a, b \in G *_F H$ and $z \in X \sqcup Y$ or $z \in \alpha$. Define $a(b, z) = (ab, z)$. This is a simplicial automorphism of $X * Y$.

Construction 9.4. Let G be a 7–systolic group acting properly and cocompactly on a 7–systolic complex X. Let F_1, F_2 be some finite subgroups of G isomorphic through a fixed isomorphism $i\colon F_1 \to F_2$. Let $\sigma, \tau \subset X$ be some simplices fixed under F_1, F_2 respectively, as guaranteed by Theorem 1.4. We define the *HNN extended complex* $X*$ *for* $G*_i$ as follows. Denote by t the element of $G*_i$ given in the presentation $t^{-1}ft = i(f), f \in F_1$. Take the product space $G *_i \times X$ and identify (ag, x) with (a, gx) for all $a \in G*_i, g \in G, x \in X$. Let α be an abstract simplex spanned on σ and τ (treated as disjoint abstract simplices). Extend the action of F_1 on σ and τ (on which F_1 acts as $F_2 = i(F_1)$) to an affine (i.e. simplicial) action on α. Now add extra simplicies (a, α) spanned on the pairs $(a, \sigma), (at, \tau)$ for $a \in G*_i$ and identify (af, z) with (a, fz) for $a \in G*_i, f \in F_1, z \in \alpha$. Again what we get is a simplicial complex. Loops cannot occur since $t \notin G \subset G*_i$ in the HNN extension and thus the copies of X corresponding to cosets aG and atG are different. The only multiple edges could occur as a result of gluing two copies of α, say

$(a, \alpha), (b, \alpha)$, where $a, b \in G*_i$, to the same pair of copies of X. This would imply $(a, X) = (b, X)$ and $(at, Y) = (bt, Y)$. Thus $b^{-1}a \in G \cap tGt^{-1} = F_1$, hence $(a, \alpha) = (b, \alpha)$. Since the action of G on X is proper, the complex $X*$ we obtained is locally finite.

Now we define the action of $G*_i$ on $X*$. Take $a, b \in G*_i$ and $z \in X$ or $z \in \alpha$. Define $a(b, z) = (ab, z)$. This is a simplicial automorphism of $X*$.

Lemma 9.5. *Consider the complexes and groups acting on them from the Construction 9.3 and the Construction 9.4. The action of $G *_F H$ on $X * Y$ is proper and cocompact. The action of $G*_i$ on $X*$ is proper and cocompact.*

Proof. We prove the first part of the lemma. Let $K_X \subset X$ and $K_Y \subset Y$ be compact sets, such that their translates through the elements of G, H respectively fill in the corresponding complexes. Take $K \subset X * Y$ defined as $K = (1, K_X) \sqcup (1, K_Y) \sqcup (1, \alpha)$. The translates of K through $G *_F H$ fill in $X * Y$, so the action is cocompact.

Now to prove the properness, since $X * Y$ is locally finite, it is enough to show that vertex stabilizers are finite. To do this, fix $b \in G *_F H, x \in X$, and suppose that $a(b, x) = (b, x)$ for some $a \in G *_F H$. Then there exists $g \in G$ such that $gx = x, a = bgb^{-1}$. Since g determines a and since the set of such g is finite by the properness of the action of G on X, the stabilizer of (b, x) is finite. For $x \in Y$ the argument is the same.

The second part of the lemma can be proved in the same fashion. □

Lemma 9.6. *Consider the complexes from the Construction 9.3 and the Construction 9.4. Then $X * Y$ and $X*$ are both trees of 7–systolic complexes.*

Proof. We prove the first part of the lemma. Define a graph T as follows. Let V_G, V_H be right cosets of the subgroups G, H in $G *_F H$. Let $V = V_G \sqcup V_H$ be the set of vertices of T. Let edges of T be right cosets aF of F in $G *_F H$ joining the vertices aG and aH. This graph is a tree, in fact it is the Bass–Serre tree of this amalgamated free product [14] (special case of Theorem 3.14). Now define the simplicial mapping p from $X * Y$ onto T. Define $p(a, x) = aG$ for $x \in X$ and $p(a, y) = aH$ for $y \in Y$. We can extend p to a simplicial mapping. Then $p(a, \alpha) = aF$, where by aF we mean the corresponding edge of T. From this construction it follows immediately that the closures of preimages of open edges in T are simplices (the copies of α) and that the preimages of vertices in T are 7–systolic (these are copies of X or Y).

For the second part, let V be the set of the right cosets of G in $G*_i$ and let edges be cosets aF_1 of F_1 in $G*_i$ joining the vertices aG and atG in V. This graph T is again a tree and we can define the simplicial mapping $p\colon X* \to T$ by $p(a,x) = aG$ for $a \in G*_i, x \in X*$ and $p(a,\alpha) = aF_1$. As before, the preimages of vertices are 7–systolic copies of X and the closures of preimages of open edges are simplices. □

Proof of Theorem 4.5. Groups we consider act properly and cocompactly (Lemma 9.5) on trees of 7–systolic complexes (Lemma 9.6), which are 7–systolic by Lemma 9.2. □

Remark 9.7. Using the same argument one can prove the following extension of Theorem 4.5. Let $k \geq 7$. Free products of k–systolic groups amalgamated over finite subgroups are k–systolic. HNN extensions of k–systolic groups over finite subgroups are k–systolic.

Remark 9.8. Note that the Constructions 9.3 and 9.4 work also for general (6–)systolic complexes whenever we amalgamate over groups which fix some simplices (for example if we amalgamate over the trivial group). We do not know if in general amalgamated products (and HNN extensions) of systolic groups over finite subgroups are systolic.

10 Final remarks on the general systolic case

Remark 10.1. It seems that with current techniques we cannot get a sharper bound for the diameter of round complexes. We suspect, however, that round complexes have diameter ≤ 2, because all round complexes we know have diameter ≤ 2. An example of a round complex of diameter 2 is the 2–triangle.

If it was true that round complexes have diameter ≤ 2, we claim we could prove there is a fixed point for any simplicial action of a finite group on a locally finite systolic complex (and this would imply that amalgamates of systolic groups over finite subgroups are systolic). As a first step, we would find, like in the proof of Theorem 4.2, an invariant round complex, whose diameter would be ≤ 2. Then we would use the following lemma that we proved together with P. Zawiślak.

Lemma 10.2. *In every finite systolic complex Y of diameter ≤ 2 there is a simplex, which is invariant under simplicial automorphisms of Y.*

Note that we could have used Lemma 10.2 to prove Theorem 4.4. However, the proof of Theorem 4.4 we presented is simpler than the proof of Lemma 10.2. Since we treat Lemma 10.2 only as a digression, we do not enclose the proof.

Chapter III

$\underline{E}G$ for systolic groups

11 Introduction

Recall that it was shown in [6], Theorem 4.1(1), that systolic complexes are contractible. Thus if a group G is systolic and torsion free, then X is a finite model for EG.

Similarly, if G acts properly on a CAT(0) space X and if G is torsion free, then X is a model for EG. If we do not assume that G is torsion free, then the stabilizer of any point in X is finite and the fixed point set of any finite subgroup of G is contractible (in particular nonempty). This means that X is the so called *model for* $\underline{E}G$ — the classifying space for finite subgroups [8].

There are other families of groups G, which admit nice models for $\underline{E}G$. For example, if G is word–hyperbolic, and if S is a finite generating set for G, then for sufficiently large real number d the Rips complex $P_d(G, S)$ is a model for $\underline{E}G$. What makes this model attractive for applications is that it is a *finite* model, i.e. the action of G on it is cocompact. See [8] for details.

In this chapter we give an explicit finite model for $\underline{E}G$ for a systolic group G. We prove that an appropriate Rips complex of any systolic complex X on which G acts properly is a model for $\underline{E}G$. We define the Rips complex in our context as follows.

Definition 11.1. Let X be any simplicial complex. For any $n \geq 1$, the *Rips complex* X_n is a simplicial complex with the same set of vertices as X and with a simplex spanned on any subset $S \subset X^{(0)}$ such that $\mathrm{diam}(S) \leq n$ in $X^{(1)}$. If G acts on X properly (and cocompactly), then the natural extension of this action to X_n is also proper (and cocompact).

Our main result is the following.

Theorem 11.2. *Let X be a systolic complex on which a group G acts properly. Then for $n \geq 5$ the Rips complex X_n is a finite dimensional model for $\underline{E}G$. If additionally G acts cocompactly on X then X_n is a finite model for $\underline{E}G$.*

Theorem 11.2 extends and its proof is based on Theorem 4.2 (which also explains the appearance of the constant 5 in the above formulation). To apply Theorem 4.2, let H be a group acting by automorphisms on a simplicial complex X. Then the fixed point set of the action of H on X is a subcomplex of the barycentric subdivision X' of X. Denote this subcomplex by $\mathrm{Fix}_H X'$. Similarly denote the fixed point set of the action of H on the Rips complex X_n by $\mathrm{Fix}_H X'_n$. It is a subcomplex of X'_n. By Theorem 4.2, if X is systolic, H is finite and $n \geq 5$, then $\mathrm{Fix}_H X'_n$ is nonempty.

Now the proof of Theorem 11.2 reduces to the following.

Proposition 11.3. *Let H be any group acting by automorphisms on a systolic complex X. Then for any $n \geq 1$ the complex $Fix_H X'_n$ is either empty or contractible.*

The remaining part of this chapter is devoted to the proof of Proposition 11.3. This will be done without using the contractibility of systolic complexes [6]. In fact, by applying Proposition 11.3 to the case of H trivial and $n = 1$, we reprove that systolic complexes are contractible (since $X_1 = X$ by flagness of systolic complexes).

Our proof may seem more sophisticated than the original proof [6], but the reason for this is that we deal at the same time with contractibility of the systolic complex X and with contractibility of its Rips complexes.

In fact, our proof is simpler than the original proof. By using the methods of Section 11 (not present in [6]), we are able to avoid writing down explicit homotopies.

Note that if Theorem 4.2 could be strengthened to guarantee a true fixed point instead of an invariant subcomplex (which is possible for example for 7–systolic complexes, by Theorem 4.4), then under the hypothesis of Theorem 11.2 we would get a stronger assertion: Proposition 11.3 would imply that the original complex X is a model for $\underline{E}G$.

This chapter is organized as follows. In Section 12 we introduce the key notion of the chapter, the expansion by projection, and establish its basic properties. In Section 13 we present two abstract ways of producing homotopies in geometric realizations of posets, which will be needed later. The proof of Proposition 11.3 occupies Section 14.

12 Expansion by projection

The proof of contractibility of systolic complexes given by T. Januszkiewicz and J. Świątkowski in [6] uses Lemma 3.6 and the notion of projection (Definition 3.7). To be able to deal with the Rips complex we need to extend this notion: we need to be able to project not only simplices, but all convex subcomplexes. In this section we introduce the necessary definitions for this and establish the basic properties of the corresponding notions.

Definition 12.1. Let Y be a convex subcomplex of a systolic complex X and let σ be a simplex in $B_1(Y)$. The *expansion by projection* of σ (denoted by $e_Y(\sigma)$) is a simplex in $B_1(Y)$ defined in the following way. If $\sigma \subset Y$ then $e_Y(\sigma) = \sigma$. Otherwise $e_Y(\sigma)$ is the join of $\sigma \cap S_1(Y)$ (which is nonempty) and its projection (c.f. Definition 3.7) onto Y.

Remark 12.2. Observe that $\sigma \subset e_Y(\sigma)$. Moreover, by Lemma 3.6, $e_Y(\sigma) \cap Y$ is nonempty.

Definition 12.3. Let Y be a convex subcomplex of a systolic complex X and let Z be a convex subcomplex in $B_1(Y)$. The *expansion by projection* of Z (denoted by $e_Y(Z)$) is a subcomplex of $B_1(Y)$ defined in the following way. Let $e_Y(Z)$ be the span of the union of $e_Y(\sigma)$ over all maximal (with respect to inclusion) simplices $\sigma \subset Z$. Clearly this definition extends Definition 12.1.

Remark 12.4. Observe that $Z \subset e_Y(Z)$. Moreover $e_Y(Z) \cap Y$ is nonempty. Note that $e_Y(Z)$ does not have to be convex.

Remark 12.5. Let g be an automorphism of X which leaves Y and Z invariant. Then g leaves also $e_Y(Z)$ invariant.

The following property of the expansion by projection is not at all obvious.

Lemma 12.6. $\mathrm{diam}(e_Y(Z)) \leq \max\{\mathrm{diam}(Z), 1\}$.

In fact, since by Remark 12.4 we have $Z \subset e_Y(Z)$, this is an equality unless Z is a single vertex of Y.

Before giving the proof we need to establish some facts about the distance between maximal simplices in convex subcomplexes.

Lemma 12.7. *Let Z be a convex subcomplex in a systolic complex X. Let d be the diameter of Z. Assume $d \geq 2$. Let σ, τ be any maximal simplices of Z and let v be any vertex of Z. Then*

(1) $|\sigma, v| \leq d - 1$,

(2) $|\sigma, \tau| \leq d - 2$.

Proof. First we prove assertion (1). We do this by contradiction. Assume $|\sigma, v| = d$. This means that $\sigma \subset S_d(v)$, so $e_{B_{d-1}(v)}(\sigma)$ (the expansion by projection onto $B_{d-1}(v)$, c.f. Definition 12.1 and Remark 12.2) is a simplex strictly greater than σ. All vertices in $e_{B_{d-1}(v)}(\sigma)$ lie on some 1–skeleton geodesics from v to vertices in σ. Hence by Proposition 3.5 and by convexity of Z we have $e_{B_{d-1}(v)}(\sigma) \subset Z$. Thus σ is not maximal in Z, contradiction.

Now we prove assertion (2). We do this again by contradiction. Assume $|\sigma, \tau| > d - 2$. By (1) this implies that $|\sigma, v| = d - 1$ for all $v \in \tau$. Thus $\tau \subset S_{d-1}(\sigma)$. As before, by Proposition 3.5 and by convexity of Z we get $e_{B_{d-2}(\sigma)}(\tau) \subset Z$. Since $e_{B_{d-2}(\sigma)}(\tau)$ is strictly greater than τ, we obtain contradiction with maximality of τ. □

Proof of Lemma 12.6 Denote by d the diameter of Z. Suppose $d \geq 2$ (otherwise the lemma is obvious). Take any $v, w \in e_Y(Z)$. We must prove that $|vw| \leq d$. If $v, w \in Z$ then there is nothing to prove. Now assume that $v \in Z, w \notin Z$. Thus there exists a maximal simplex $\sigma \subset Z$ such that $w \in e_Y(\sigma)$. By Lemma 12.7(1) we have $|\sigma, v| \leq d - 1$, hence there exists a vertex $s \in \sigma$ such that $|vs| \leq d - 1$. Since $|sw| \leq 1$, we are done.

Now assume that both $v, w \notin Z$. Thus there exist maximal simplices $\sigma, \tau \subset Z$ such that $v \in e_Y(\sigma), w \in e_Y(\tau)$. By Lemma 12.7(2) there exist vertices $s \in \sigma, t \in \tau$ such that $|st| \leq d - 2$. Since $|vs| \leq 1$ and $|wt| \leq 1$, we are done. □

We end this section with a lemma which though seems technical, nevertheless lies at the heart of the proof of Proposition 11.3, which will be presented in Section 12. This lemma states, roughly speaking, that expanding by projection has not too bad monotonicity properties (although usually it is not true that $Z \subset Z'$ implies $e_Y(Z) \subset e_Y(Z')$ or $e_Y(Z) \supset e_Y(Z')$).

Lemma 12.8. *Let $Z_1 \subset Z_2 \subset \ldots \subset Z_n \subset B_1(Y)$ be an increasing sequence of convex subcomplexes of $B_1(Y)$. Then the intersection*

$$\Big(\bigcap_{i=1}^{n} e_Y(Z_i)\Big) \cap Y$$

is nonempty.

Proof. If $Z_1 \cap Y$ is nonempty then any vertex $v \in Z_1 \cap Y$ belongs to the required intersection. Otherwise take any maximal (in Z_1) simplex $\sigma_1 \subset Z_1$. We define inductively an increasing sequence of simplices $\sigma_i \subset Z_i$ for $i = 2, \ldots, n$. Namely choose σ_i to be any maximal simplex in Z_i containing σ_{i-1}. Take any vertex $v \in e_Y(\sigma_n) \cap Y$. Since σ_i do not lie entirely in Y, we have by definition of $e_Y(\sigma_i)$ that $v \in e_Y(\sigma_i)$ for all i. Since each σ_i is maximal in the corresponding Z_i, this implies that $v \in e_Y(Z_i)$ for all i, hence v belongs to the required intersection. $\square$

13 Homotopies

We will use the following well known results. The proof of the first proposition can be found, for example, in the paper of G. Segal [15]. However, for completeness, we give an indication of an argument.

Proposition 13.1 ([15], Proposition 1.2)**.** *If $\mathcal{C}, \mathcal{D}$ are posets and $F_0, F_1 \colon \mathcal{C} \to \mathcal{D}$ are functors (i.e. they respect the order) such that for each $c \in \mathcal{C}$ we have $F_0(c) \leq F_1(c)$, then the maps induced by F_0, F_1 on geometric realizations of $\mathcal{C}, \mathcal{D}$ are homotopic. Moreover this homotopy is constant on the geometric realization of the subposet of $\mathcal{C}$ of objects on which F_0 and F_1 agree.*

Proof. We need to extend the natural homotopy on vertices of geometric realizations to higher skeleta. This is done by performing the so called prism subdivision of the cells of the homotopy. Then the homotopy can be realized simplicially, it can be explicitly written down. $\square$

In the next proposition we will consider a functor $F \colon \mathcal{C}' \to \mathcal{C}$ from the flag poset $\mathcal{C}'$ of a poset $\mathcal{C}$ into the poset $\mathcal{C}$, assigning to each object in $\mathcal{C}'$, which is a chain of objects of $\mathcal{C}$, its minimal element. F is covariant if we take on $\mathcal{C}'$ the partial order inverse to the inclusion. Geometric realizations of $\mathcal{C}, \mathcal{C}'$ are homeomorphic in a canonical way (one is the barycentric subdivision of the other), which allows us to identify them.

Proposition 13.2. *The map induced by F on geometric realizations of $\mathcal{C}', \mathcal{C}$ is homotopic to identity.*

Proof. We give only a sketch. Take any simplex in the geometric realization of $\mathcal{C}'$, suppose it corresponds to a chain $c'_1 \subset \ldots \subset c'_n$ (c'_i are chains of objects in $\mathcal{C}$). This simplex and its image under the map induced by F both lie in the simplex of the image, which corresponds to the chain c'_n. Thus the homotopy can be realized affinely on each simplex. $\square$

14 Nonempty fixed point sets are contractible

As observed in Section 11, Theorem 11.2 is implied by Theorem 4.2 and Proposition 11.3. Thus to prove Theorem 11.2 it is enough to prove Proposition 11.3, which we do in this section.

Let us give an outline of the proof. Suppose the fixed point set we are considering is nonempty. We define an increasing sequence of subcomplexes exhausting the Rips complex, with an invariant simplex as the first subcomplex. We then prove that the intersection of the fixed point set with a subcomplex from our family is homotopy equivalent to the intersection of the fixed point set with the subsequent subcomplex. Since we know that the first of those intersections is contractible, it follows by induction that any of the intersections is contractible. Since we choose an exhausting family, this means that the whole fixed point set is contractible.

We define now this exhausting family.

Definition 14.1. Let X be any simplicial complex. Let $\sigma \subset X_n$ be any simplex in the Rips complex of X for some $n \geq 1$. Let $A \subset X_n^{(0)} = X^{(0)}$ be the set of vertices of σ. Recall that $B_i(A)$ is the combinatorial ball of radius i around A in X. Now define an increasing sequence of full subcomplexes $D_i(\sigma) \subset X'_n$, where $i \geq 0$, in the following way. Let $D_{2i}(\sigma)$ be the span of all vertices in X'_n corresponding to simplices in X_n, which have all their vertices in $B_i(A)$ (i.e. $D_{2i}(\sigma)$ is equal to the barycentric subdivision of the span in X_n of vertices in $B_i(A) \subset X$). Let $D_{2i+1}(\sigma)$ be the span of those vertices in X'_n, which correspond to those simplices in X_n that have all their vertices in $B_{i+1}(A)$ and at least one vertex in $B_i(A)$ (where the balls are taken in X).

In case of a flag complex X for $n = 1$ we have $X_1 = X$ and the subcomplexes $D_i(\sigma)$ are combinatorial balls in X' around the barycentric subdivision of σ.

Remark 14.2. Notice that $\bigcup_{i=0}^{\infty} D_i(\sigma) = X'_n$. Moreover, any compact subcomplex of X'_n is contained in some $D_i(\sigma)$.

Proof of Proposition 11.3 Assume that $\mathrm{Fix}_H X'_n$ is nonempty. Let $\sigma \subset X_n$ be a maximal H–invariant simplex in X_n. Denote the set of vertices of σ in $X_n^{(0)} = X^{(0)}$ by A. We claim that the span of A in X is convex. Otherwise, by Lemma 3.4, the vertices of $\mathrm{conv}(A)$ in X span a simplex in X_n, which is also H–invariant and strictly greater than σ, contradiction. Let $D_i(\sigma) \subset X'_n$ be as in Definition 14.1. In the further discussion we will use an abbreviated notation $D_i = D_i(\sigma)$.

We will prove the following three assertions.

(i) $D_0 \cap \mathrm{Fix}_H X'_n$ is contractible,

(ii) the inclusion $D_{2i} \cap \mathrm{Fix}_H X'_n \subset D_{2i+1} \cap \mathrm{Fix}_H X'_n$ is a homotopy equivalence,

(iii) the identity on $D_{2i+2} \cap \mathrm{Fix}_H X'_n$ is homotopic to a mapping with image in $D_{2i+1} \cap \mathrm{Fix}_H X'_n \subset D_{2i+2} \cap \mathrm{Fix}_H X'_n$.

Suppose for a moment that (i)–(iii) hold. We will show how this implies the theorem. We will prove by induction on k the following.

Claim. $D_k \cap \mathrm{Fix}_H X'_n$ is contractible.

For $k = 0$ this is stated in assertion (i). Suppose we have proved the claim for some $k \geq 0$. If k is even, $k = 2i \geq 0$, then assertion (ii) implies the claim for $k = 2i + 1$. If k is odd, $k = 2i + 1$, then the identity mapping from assertion (iii) is homotopic to the mapping with image in a contractible subspace, hence the identity mapping is homotopically trivial. This proves the claim for $k = 2i + 2$. We have thus completed the induction step.

By Remark 14.2, the image of any sphere mapped into $\mathrm{Fix}_H X'_n$ is contained in some $D_i \cap \mathrm{Fix}_H X'_n$, which is contractible. Thus all homotopy groups of $\mathrm{Fix}_H X'_n$ are trivial and since $\mathrm{Fix}_H X'_n$ is a simplicial complex, it is contractible, by Whitehead's Theorem, as desired. To complete the proof we must now prove assertions (i)–(iii).

Assertion (i). Since D_0 is the barycentric subdivision of the simplex $\sigma \subset X_n$ and the barycenter of σ belongs to $\mathrm{Fix}_H X'_n$, we have that $D_0 \cap \mathrm{Fix}_H X'_n$ is a cone over the barycenter of σ, hence it is contractible.

Assertion (ii). Let $\mathcal{C}$ be the poset of H–invariant simplices in X_n with vertices in $B_{i+1}(A)$ (ball in X) and at least one vertex in $B_i(A)$. Its geometric realization is $D_{2i+1} \cap \mathrm{Fix}_H X'_n$. Consider a functor $F\colon \mathcal{C} \to \mathcal{C}$ assigning to each object of $\mathcal{C}$ i.e. a simplex in X_n its subsimplex spanned by vertices in $B_i(A)$. Notice that this subsimplex is H–invariant (i.e. it is an object of $\mathcal{C}$) since A and hence $B_i(A)$ are H–invariant. By Proposition 13.1 the geometric realization of F is homotopic to identity (which is the geometric realization of the identity functor). Moreover this homotopy is constant on $D_{2i} \cap \mathrm{Fix}_H X'_n$. The image of the geometric realization of F is contained in $D_{2i} \cap \mathrm{Fix}_H X'_n$. Hence $D_{2i} \cap \mathrm{Fix}_H X'_n$ is a deformation retract of $D_{2i+1} \cap \mathrm{Fix}_H X'_n$, as desired.

Assertion (iii). Let $\mathcal{C}$ be the poset of H–invariant simplices in X'_n with vertices in $B_{i+1}(A)$ and let $\mathcal{C}'$ be its flag poset, with the partial order inverse to the inclusion. Let $F_0\colon \mathcal{C}' \to \mathcal{C}$ be the functor (from Proposition 13.2) assigning to each object in $\mathcal{C}'$, which is a chain of objects of $\mathcal{C}$, its minimal element. The geometric realization of both $\mathcal{C}$ and $\mathcal{C}'$ is equal to $D_{2i+2} \cap \mathrm{Fix}_H X'_n$ and by Proposition 13.2 the geometric realization of F_0 is homotopic to identity.

Now we define another functor $F_1 \colon \mathcal{C}' \to \mathcal{C}$. This is the heart of the proof. First notice that since $\mathrm{span}(A)$ is convex in X, we have that the ball $B_i(A)$ is also convex. Hence for any convex subcomplex $Z \subset B_{i+1}(A)$ there exists its expansion by projection (c.f. Definition 12.3) $e_{B_i(A)}(Z)$. Now we define F_1. For any object $c' \in \mathcal{C}'$, which is a chain of objects $c_1 < c_2 < \ldots < c_k$ of $\mathcal{C}$, recall that c_j (where $1 \leq j \leq k$) are some H–invariant simplices in X_n with vertices in $B_{i+1}(A)$. Denote the set of vertices of c_j by S_j and treat it as a subset of $X^{(0)}$. Notice that the subcomplexes $\mathrm{conv}(S_j) \subset X$ are of diameter $\leq n$ (by Lemma 3.4), they form an increasing sequence and they are all contained in $B_{i+1}(A)$ by monotonicity of taking the convex hull and by convexity of balls. Thus if we define S'_j to be the set of vertices in $e_{B_i(A)}(\mathrm{conv}(S_j))$, then by Lemma 12.8 the intersection $\bigcap_{j=1}^{k} S'_j$ contains at least one vertex in $B_i(A)$. Also note that this intersection is contained in $B_{i+1}(A)$. Moreover, by Lemma 12.6, all the sets S'_j, and hence their intersection, have diameter $\leq n$. Thus we can treat the set $\bigcap_{j=1}^{k} S'_j$ as a simplex in X_n with vertices in $B_{i+1}(A)$. By Remark 12.5 this simplex is H–invariant, hence it is an object in $\mathcal{C}$. We define $F_1(c')$ to be this object. In geometric realization of $\mathcal{C}$, which is $D_{2i+2} \cap \mathrm{Fix}_H X'_n$ the object $F_1(c')$ corresponds to a vertex in $D_{2i+1} \cap \mathrm{Fix}_H X'_n$, by our previous remarks. It is obvious that F_1 preserves the partial order (inverse to the inclusion on $\mathcal{C}'$), since the greater the chain, the more sets S'_j we have to intersect.

Now notice that by Remark 12.4 for any $c' \in \mathcal{C}'$ we have $F_0(c') \subset F_1(c')$, hence by Proposition 13.1 the geometric realizations of F_0 and F_1 are homotopic. But as observed at the beginning, F_0 is homotopic to the identity. On the other hand, F_1 has image in $D_{2i+1} \cap \mathrm{Fix}_H X'_n$. Thus we are done. $\square$

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