

A PAIR OF GARSIDE SHADOWS

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ABSTRACT. We prove that the smallest elements of Shi parts and cone type parts exist and form Garside shadows. The latter resolves a conjecture of Parkinson and the second author as well as a conjecture of Hohlweg, Nadeau and Williams.

1. INTRODUCTION

A *Coxeter group* W is a group generated by a finite set S subject only to relations $s^2 = 1$ for $s \in S$ and $(st)^{m_{st}} = 1$ for $s \neq t \in S$, where $m_{st} = m_{ts} \in \{2, 3, \dots, \infty\}$. Here the convention is that $m_{st} = \infty$ means that we do not impose a relation between s and t . By X^1 we denote the *Cayley graph* of W , that is, the graph with vertex set $X^0 = W$ and with edges (of length 1) joining each $g \in W$ with gs , for $s \in S$. For $g \in W$, let $\ell(g)$ denote the *word length* of g , that is, the distance in X^1 from g to id . We consider the action of W on $X^0 = W$ by left multiplication. This induces an action of W on X^1 .

For $r \in W$ a conjugate of an element of S , the *wall* \mathcal{W}_r of r is the fixed point set of r in X^1 . We call r the *reflection* in \mathcal{W}_r (for fixed \mathcal{W}_r such r is unique). Each wall \mathcal{W} separates X^1 into two components, called *half-spaces*, and a geodesic edge-path in X^1 intersects \mathcal{W} at most once [Ron09, Lem 2.5]. Consequently, the distance in X^1 between $g, h \in W$ is the number of walls separating g and h .

We consider the partial order \preceq on W (called the ‘weak order’ in algebraic combinatorics), where $p \preceq g$ if p lies on a geodesic in X^1 from id to g . Equivalently, there is no wall separating p from both id and g .

Shi parts. Let \mathcal{E} be the set of walls \mathcal{W} such that there is no wall separating \mathcal{W} from id (these walls correspond to so-called ‘elementary roots’). The components of $X^1 \setminus \bigcup \mathcal{E}$ are *Shi components*. For a Shi component Y , we call $P = Y \cap X^0$ the corresponding *Shi part*.

Our first result is the following.

Theorem 1.1. *Let P be a Shi part. Then P has a smallest element with respect to \preceq .*

Theorem 1.1 was proved independently in a more general form by Dyer, Fishel, Hohlweg and Mark in [DFHM23, Theorem 1.1(1)]. Here we give a short proof following the lines of the proof of a related result of the first author and Osajda [OP22, Thm 2.1].

In [Shi87], Shi proved Theorem 1.1 for affine W . The family \mathcal{E} , which is finite by [BH93], has been extensively studied ever since and has become an important object in algebraic combinatorics, geometric group theory and representation theory. See for example see the survey article [Fis20].

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By [BH93], Shi parts are in correspondence with the states of an automaton recognising the language of reduced words of the Coxeter group. This partition of a Coxeter group is thus one of the primary examples of ‘regular’ partitions, see [PY22].

For $g \in W$, let $m(g)$ be the smallest element in the Shi part containing g , guaranteed by Theorem 1.1. Let $M \subset W$ be the set of elements of the form $m(g)$ for $g \in W$.

The *join* of $g, g' \in W$ is the smallest element h (if it exists) satisfying $g \preceq h$ and $g' \preceq h$. A subset $B \subseteq W$ is a *Garside shadow* if it contains S , contains $g^{-1}h$ for every $h \in B$ and $g \preceq h$, and contains the join, if it exists, of every $g, g' \in B$.

Theorem 1.2. *M is a Garside shadow.*

Theorem 1.2 was also obtained in [DFHM23, Thm 1.1(2)], where the authors showed that M is the set of so-called ‘low elements’ introduced in [DH16]. We give an alternative proof using ‘bipodality’, a notion introduced in [DH16] and rediscovered in [OP22].

Cone type parts. For each $g \in W$, let $T(g) = \{h \in W \mid \ell(gh) = \ell(g) + \ell(h)\}$. For $T \subset W$, the *cone type part* $Q(T) \subset W$ is the set of all g^{-1} with $T(g) = T$. In other words, $Q(T)$ consists of g such that T is the set of vertices on geodesic edge-paths starting at g and passing through id that appear after id , including id .

We obtain a new proof of the following.

Theorem 1.3. [PY22, Thm 1] *Let Q be a cone type part. Then Q has a smallest element with respect to \preceq .*

For $g \in W$, let $\mu(g)$ be the smallest element in the cone type part containing g . Let $\Gamma \subset W$ be the set of elements of form $\mu(g)$ for $g \in W$. These elements are called the *gates* of the cone type partition in [PY22].

We also obtain the following new result, confirming in part [PY22, Conj 1].

Theorem 1.4. *For any $g, g' \in \Gamma$, if the join of g and g' exists, then it belongs to Γ .*

By [PY22, Prop 4.27(i)], this implies that Γ is a Garside shadow. Furthermore, Γ is the set of states of a the minimal automaton (in terms of the number of states) recognising the language of reduced words of a Coxeter group. This verifies [HNW16, Conj 1].

The paper is organised as follows. In Section 2 we discuss ‘bipodality’ and use it to prove Theorem 1.1 and Theorem 1.2. In Section 3 we focus on the cone type parts and give the proofs of Theorem 1.3 and Theorem 1.4.

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2. SHI PARTS

The following property was called *bipodality* in [DH16]. It was rediscovered in [OP22].

Definition 2.1. Let $r, q \in W$ be reflections. Distinct walls $\mathcal{W}_r, \mathcal{W}_q$ *intersect*, if \mathcal{W}_r is not contained in a half-space for \mathcal{W}_q (this relation is symmetric). Equivalently, $\langle r, q \rangle$ is a finite group. We say that such r, q are *sharp-angled*, if r and q do not commute and $\{r, q\}$ is conjugate into S . In particular, there is a component of

$X^1 \setminus (\mathcal{W}_r \cup \mathcal{W}_q)$ whose intersection F with X^0 is a fundamental domain for the action of $\langle r, q \rangle$ on X^0 . We call such F a *geometric fundamental domain* for $\langle r, q \rangle$.

Lemma 2.2 ([OP22, Lem 3.2], special case of [DH16, Thm 4.18]). *Suppose that reflections $r, q \in W$ are sharp-angled, and that $g \in W$ lies in a geometric fundamental domain for $\langle r, q \rangle$. Assume that there is a wall \mathcal{U} separating g from \mathcal{W}_r or from \mathcal{W}_q . Let \mathcal{W}' be a wall distinct from $\mathcal{W}_r, \mathcal{W}_q$ that is the translate of \mathcal{W}_r or \mathcal{W}_q under an element of $\langle r, q \rangle$. Then there is a wall \mathcal{U}' separating g from \mathcal{W}' .*

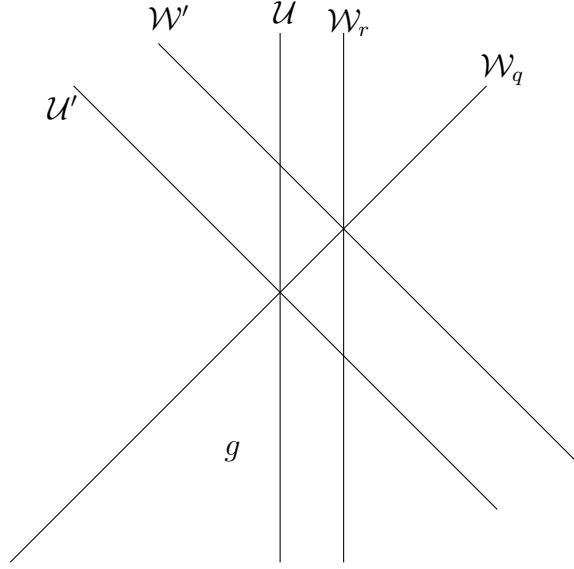


FIGURE 1. Lemma 2.2 for the case $m_{rq} = 4$

The following proof is surprisingly the same as that for a different result [OP22, Thm 2.1].

Proof of Theorem 1.1. Let $P = Y \cap X^0$, where Y is a Shi component. It suffices to show that for each $p_0, p_n \in P$ there is $p \in P$ satisfying $p_0 \succeq p \preceq p_n$. Let (p_0, p_1, \dots, p_n) be the vertices of a geodesic edge-path π in X^1 from p_0 to p_n , which lies in Y . Let $L = \max_{i=0}^n \ell(p_i)$.

We will now modify π and replace it by another embedded edge-path from p_0 to p_n with vertices in P , so that there is no p_i with $p_{i-1} \prec p_i \succ p_{i+1}$. Then we will be able to choose p to be the smallest p_i with respect to \preceq .

If $p_{i-1} \prec p_i \succ p_{i+1}$, then let $\mathcal{W}_r, \mathcal{W}_q$ be the (intersecting) walls separating p_i from p_{i-1}, p_{i+1} , respectively. Moreover, if r and q do not commute, then r, q are sharp-angled, with id in a geometric fundamental domain for $\langle r, q \rangle$. We claim that all the elements of the *residue* $R = \langle r, q \rangle(p_i)$ lie in P .

Indeed, since p_{i-1}, p_{i+1} are both in P , we have that $\mathcal{W}_r, \mathcal{W}_q \notin \mathcal{E}$. It remains to justify that each wall $\mathcal{W}' \neq \mathcal{W}_r, \mathcal{W}_q$ that is the translate of \mathcal{W}_r or \mathcal{W}_q under an element of $\langle r, q \rangle$ does not belong to \mathcal{E} . We can thus assume that r and q do not commute, since otherwise there is no such \mathcal{W}' . Since $\mathcal{W}_r \notin \mathcal{E}$, there is a wall \mathcal{U} separating id from \mathcal{W}_r . By Lemma 2.2, there is a wall \mathcal{U}' separating id from \mathcal{W}' , justifying the claim.

We now replace the subpath (p_{i-1}, p_i, p_{i+1}) of π by the second embedded edge-path with vertices in the residue R from p_{i-1} to p_{i+1} . Since all the elements of R

are $\prec p_i$ [Ron09, Thm 2.9], this decreases the complexity of π defined as the tuple (n_L, \dots, n_2, n_1) , where n_j is the number of p_i in π with $\ell(p_i) = j$, with lexicographic order. After possibly removing a subpath, we can assume that the new edge-path is embedded. After finitely many such modifications, we obtain the desired path. \square

Lemma 2.3. *For $g \preceq h$, we have $m(g) \preceq m(h)$.*

Proof. Let k be the minimal number of distinct Shi components traversed by a geodesic edge-path γ from h to g . We proceed by induction on k , where for $k = 1$ we have $m(g) = m(h)$. Suppose now $k > 1$. If a neighbour f of h on γ lies in the same Shi component as h , then we can replace h by f . Thus we can assume that f lies in a different Shi component than h . Consequently, the wall \mathcal{W}_r separating h from f belongs to \mathcal{E} . Since $g \preceq f$, by the inductive assumption we have $m(g) \preceq m(f)$. Thus it suffices to prove $m(f) \preceq m(h)$.

In the first case, where for every neighbour h' of h on a geodesic edge-path from h to id , the wall separating h from h' belongs to \mathcal{E} , we have $h = m(h)$ and we are done. Otherwise, let \mathcal{W}_q be such a wall separating h from h' outside \mathcal{E} . If r and q do not commute, then r, q are sharp-angled, with id in a geometric fundamental domain for $\langle r, q \rangle$. By Lemma 2.2, among the walls in $\langle r, q \rangle \{ \mathcal{W}_r, \mathcal{W}_q \}$ only \mathcal{W}_r belongs to \mathcal{E} . Let \bar{h}, \bar{f} be the vertices opposite to f, h in the residue $\langle r, q \rangle h$. We have $m(\bar{h}) = m(h), m(\bar{f}) = m(f)$. Replacing h, f by \bar{h}, \bar{f} , and possibly repeating this procedure finitely many times, we arrive at the first case. \square

Lemma 2.3 has the following immediate consequence.

Corollary 2.4. *For any $g, g' \in M$, if the join of g and g' exists, then it belongs to M .*

For completeness, we include the proof of the following.

Lemma 2.5 ([DH16, Prop 4.16]). *For any $h \in M$ and $g \preceq h$, we have $g^{-1}h \in M$.*

Proof. For any neighbour h' of h on a geodesic edge-path from h to g , the wall \mathcal{W} separating h from h' belongs to \mathcal{E} . Consequently, we also have $g^{-1}\mathcal{W} \in \mathcal{E}$, and so $g^{-1}h \in M$. \square

Also note that for each $s \in S$, we have $\mathcal{W}_s \in \mathcal{E}$ and so $m(s) = s$ implying $S \subset M$. Thus Corollary 2.4 and Lemma 2.5 imply Theorem 1.2.

3. CONE TYPE PARTS

Let $T = T(g)$ for some $g \in W$. We denote by ∂T the set of walls separating adjacent vertices $h \in T$ and $h' \notin T$. In particular, the walls in ∂T separate id from g^{-1} .

We note that one of the primary differences between the cone type parts and the Shi parts is that the cone type parts do not correspond to a ‘hyperplane arrangement’. See for example Figure 2.

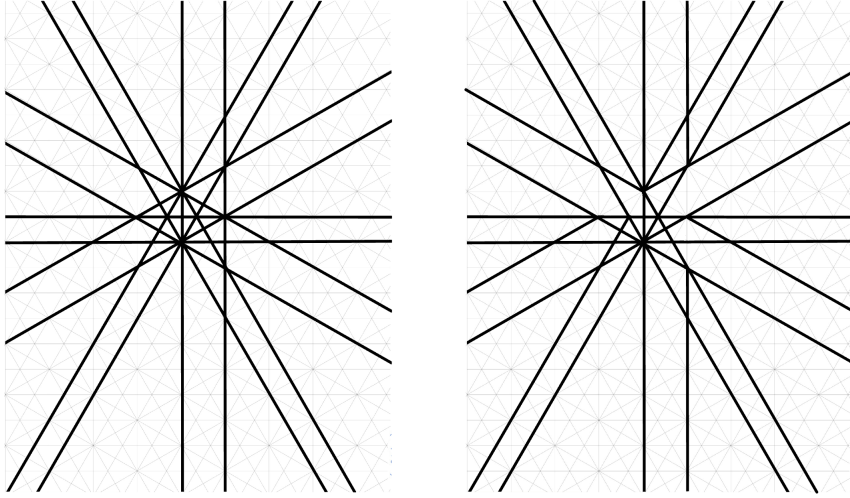


FIGURE 2. Shi parts and cone type parts for the Coxeter group of type \tilde{G}_2

Remark 3.1. Note that for $g, g' \in Q(T)$ any geodesic edge-path from g to g' has all vertices f in $Q(T)$. Indeed, for $h \in T$, any wall separating id from f separates id from g or g' and so it does not separate id from h . Thus $h \in T(f^{-1})$ and so $T \subseteq T(f^{-1})$. Conversely, if we had $T \subsetneq T(f^{-1})$ then there would be a vertex $h \in T$ with a neighbour $h' \in T(f^{-1}) \setminus T$ separated from h by a wall \mathcal{W} (in ∂T) that does not separate h from f . The wall \mathcal{W} would not separate h' from g or g' , contradicting $h' \notin T(g^{-1})$ or $h' \notin T(g'^{-1})$. See also [PY22, Thm 2.14] for a more general statement.

Proof of Theorem 1.3. The proof is identical to that of Theorem 1.1, with P replaced by Q . The vertices of a geodesic edge-path π in X^1 from p_0 to p_n belong to Q by Remark 3.1. We also make the following change in the proof of the claim that all the elements of $R = \langle r, q \rangle(p_i)$ lie in Q . Namely, since $T = T(p_i^{-1})$ equals $T(p_{i-1}^{-1})$, we have $\mathcal{W}_r \notin \partial T$. Analogously we obtain $\mathcal{W}_q \notin \partial T$. If r and q do not commute, we have that T is contained in a geometric fundamental domain for $\langle r, q \rangle$, and so we also have $\mathcal{W}' \notin \partial T$ for any \mathcal{W}' that is a translate of \mathcal{W}_r or \mathcal{W}_q under an element of $\langle r, q \rangle$. This justifies the claim. \square

Proof of Theorem 1.4. The proof structure is similar to that of Lemma 2.3. We need to justify that for $g \preceq h$, we have $\mu(g) \preceq \mu(h)$, where we induct on the minimal number k of distinct cone type components traversed by a geodesic edge-path γ from h to g . Suppose $k > 1$, and let $Q = Q(T)$ be the cone type component containing h . If a neighbour f of h on γ lies in Q , then we can replace h by f . Thus we can assume $f \notin Q$. Consequently, the wall \mathcal{W}_r separating h from f belongs to ∂T . Since $g \preceq f$, by the inductive assumption we have $\mu(g) \preceq \mu(f)$. Thus it suffices to prove $\mu(f) \preceq \mu(h)$.

If for every neighbour h' of h on a geodesic edge-path from h to id , the wall separating h from h' belongs to ∂T , we have $h = \mu(h)$ and we are done. Otherwise, let \mathcal{W}_q be such a wall separating h from h' outside ∂T . Let \bar{h}, \bar{f} be the vertices opposite to f, h in the residue $\langle r, q \rangle h$, and let $f' = rqh$. It suffices to prove $\mu(\bar{h}) = \mu(h), \mu(\bar{f}) = \mu(f)$. To justify $\mu(\bar{h}) = \mu(h)$, or, equivalently, $\bar{h} \in Q$, it suffices to

observe that among the walls in $\langle r, q \rangle \{ \mathcal{W}_r, \mathcal{W}_q \}$ only \mathcal{W}_r belongs to ∂T : Indeed, if r and q do not commute, then r, q are sharp-angled, with T in the geometric fundamental domain F for $\langle r, q \rangle$ containing id .

It remains to justify $\mu(\bar{f}) = \mu(f)$, or, equivalently, $T(\bar{f}^{-1}) = \tilde{T}$ for $\tilde{T} = T(f^{-1})$. Since $\tilde{T} \cap F = T$, to show, for example, $T(f'^{-1}) = \tilde{T}$, it suffices to show that the wall $\mathcal{W} = r\mathcal{W}_q$ does not belong to $\partial\tilde{T}$.

Otherwise, let $b \in \tilde{T}$ be adjacent to \mathcal{W} . Then $rb \in F$ is adjacent to \mathcal{W}_q , which is outside ∂T . Consequently, $rb \notin T$. Thus there is a wall \mathcal{W}' separating id from h and rb . Note that $\mathcal{W}' \neq \mathcal{W}_r$ and so \mathcal{W}' separates id from f . Since id lies on a geodesic edge-path from f to b , we have that \mathcal{W}' does not separate id from b . Thus $r\mathcal{W}'$ separates r and rb from f, h, b , and id , since, again, id lies on a geodesic edge-path from f to b .

Consider the distinct connected components $\Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4$ of $X^1 \setminus (\mathcal{W}_r \cup r\mathcal{W}')$ with $\text{id} \in \Lambda_1, b \in \Lambda_2, r \in \Lambda_3, rb \in \Lambda_4$. Since id and r are interchanged by the reflection r and they lie in the opposite connected components, we have $r\Lambda_2 \subsetneq \Lambda_1$. On the other hand, since b and rb lie in the opposite connected components, we have $r\Lambda_1 \subsetneq \Lambda_2$, which is a contradiction.

This proves that the wall \mathcal{W} does not belong to $\partial\tilde{T}$, and hence neither does any other wall in $\langle r, q \rangle \{ \mathcal{W}_r, \mathcal{W}_q \}$. Consequently $T(\bar{f}^{-1}) = \tilde{T}$, as desired. \square

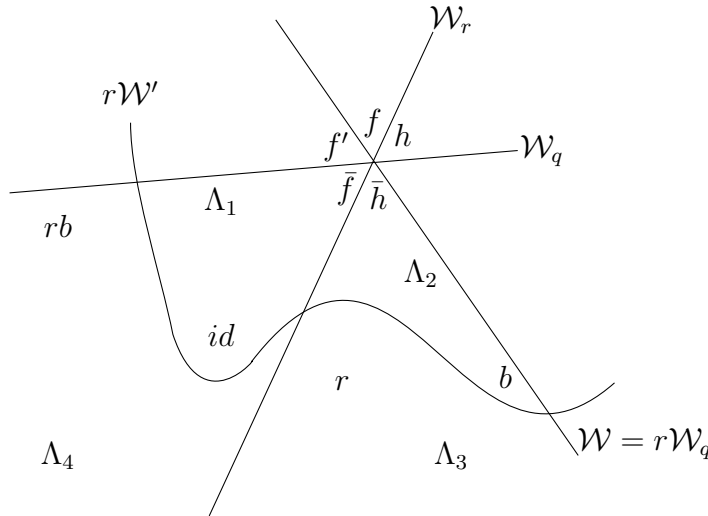


FIGURE 3. Proof of Theorem 1.4, the case of $m_{rq} = 3$

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