DIHEDRAL TWISTS IN THE TWIST CONJECTURE

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Abstract. Under the assumption that a defining graph of a Coxeter group admits only subsequent elementary twists in $\mathbb{Z}_2$ or dihedral groups and is of type FC, we prove Bernhard Mühlherr's Twist Conjecture.

1. Introduction

We make progress towards verifying Bernhard Mühlherr's Twist Conjecture. This conjecture predicts that angle-compatible Coxeter generating sets differ by a sequence of elementary twists. By [7] and [10] the Twist Conjecture solves the Isomorphism Problem for Coxeter groups.

Main Theorem. *Let S be a Coxeter generating set of type* FC *angle-compatible with a Coxeter generating set S'. Suppose that any Coxeter generating set twist equivalent to S admits only elementary twists in $\mathbb{Z}_2$ or the dihedral groups. Then S is twist equivalent to S'.*

Note that in the case where S does not admit any elementary twist, we proved the Twist Conjecture in [3]. The bookkeeping in the proof was much simpler assuming S is of FC type, but we managed to remove that assumption in the last section of [3]. In [9] we kept that assumption and we confirmed the Twist Conjecture in the case where we allow elementary twists but require they are all in $\mathbb{Z}_2$. In the current article we follow this strategy amounting to allowing gradually elementary twists in larger groups. We believe that eventually we will understand the necessary bookkeeping to resolve the entire Twist Conjecture both under FC assumption and without it. For more historical background, see our previous paper [9]. Note that we will be invoking some basic lemmas from [9], but not its Main Theorem.

Definitions. A *Coxeter generating set S* of a group W is a set such that (W, S) is a Coxeter system. This means that S generates W subject only to relations of the form $s^2 = 1$ for $s \in S$ and $(st)^{m_{st}} = 1$,

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where $m_{st} = m_{ts} \geq 2$ for $s \neq t \in S$ (possibly there is no relation between s and t, and then we put by convention $m_{st} = \infty$). An S-*reflection* (or a *reflection*, if the dependence on S does not need to be emphasised) is an element of W conjugate to some element of S. We say that S is *reflection-compatible* with another Coxeter generating set S' if every S-reflection is an S'-reflection. Furthermore, S is *angle-compatible* with S' if for every $s, t \in S$ with $\langle s, t\rangle$ finite, the set $\{s, t\}$ is conjugate to some $\{s', t'\} \subset S'$.

We call a subset $J \subseteq S$ *spherical* if $\langle J\rangle$ is finite. If J is spherical, let w_J denote the longest element of $\langle J\rangle$. We say that two elements $s \neq t \in S$ are *adjacent* if $\{s, t\}$ is spherical. This gives rise to a graph whose vertices are the elements of S and whose edges (labelled by m_{st}) correspond to adjacent pairs in S. This graph is called the *defining graph* of S. Occasionally, when all m_{st} are finite, we will use another graph, whose vertices are still the elements of S, but (labelled) edges correspond to pairs of non-commuting elements of S. This graph is called the *Coxeter–Dynkin diagram* of S. Whenever we talk about adjacency of elements of S, we always mean adjacency in the defining graph unless otherwise specified.

Given a subset $J \subseteq S$, we denote by $J^\perp$ the set of those elements of $S \setminus J$ that commute with J. A subset $J \subseteq S$ is *irreducible* if it is not contained in $K \cup K^\perp$ for some non-empty proper subset $K \subset J$. We say that S is of *type* FC if each $J \subseteq S$ consisting of pairwise adjacent elements is spherical.

Let $J \subseteq S$ be an irreducible spherical subset. We say that $C \subseteq S \setminus (J \cup J^\perp)$ is a *component*, if the subgraph induced on C in the defining graph of S is a connected component of the subgraph induced on $S\setminus(J\cup J^\perp)$. Assume that we have a nontrivial partition $S\setminus(J\cup J^\perp) = A \sqcup B$, where each component C is contained entirely in A or in B. In other words, for all $a \in A$ and $b \in B$, we have that a and b are non-adjacent. We then say that J *weakly separates* S. The map $\tau\colon S \to W$ defined by

$$\tau(s) = \begin{cases} s & \text{for } s \in A \cup J \cup J^\perp, \\ w_J s w_J^{-1} & \text{for } s \in B, \end{cases}$$

is called an *elementary twist in* $\langle J\rangle$ (see [1, Def 4.4]). Coxeter generating sets S and S' of W are *twist equivalent* if S' can be obtained from S by a finite sequence of elementary twists and a conjugation. We say that S is *k-rigid* if for each weakly separating $J \subset S$ we have $|J| < k$. Thus the assumption in the Main Theorem amounts to all

Coxeter generating sets twist equivalent to S being 3-rigid. Note that being of type FC is invariant under elementary twists.

Proof outline. Let $\mathbb{A}_{\mathrm{amb}}$ be the Davis complex for (W, S'), and for each reflection $r \in W$, let $\mathcal{W}_r$ be its wall in $\mathbb{A}_{\mathrm{amb}}$. In Section 2, following [2], we explain that to prove that S (which might differ from the original S by elementary twists) is conjugate to S', we must find a 'geometric' set of halfspaces for s with $s \in S$. To this end, we will use 'markings', introduced in [3] and discussed in Section 3. These are triples $\mu = ((s,w),m)$ with $w = j_1 \cdots j_n$ where $j_i, s, m \in S$ satisfy certain conditions guaranteeing in particular $\mathcal{W}_s \cap w\mathcal{W}_m = \emptyset$. This determines a halfspace Φ^μ_s for s containing $w\mathcal{W}_m$. As in [3], to prove that the set of these halfspaces is geometric, it suffices to prove that Φ^μ_s depends only on s.

Until the last section, our goal becomes to prove the following 'consistency' of irreducible spherical $\{s,t\} \subset S$. Consistency means that all Φ^μ_s with $j_1 = t$ are equal, all Φ^μ_t with $j_1 = s$ are equal, and these two halfspaces form a geometric pair. To this end, we introduce the 'complexity' $(\mathcal{K}_1(S), \mathcal{K}_2(S))$ of S with respect to S'. The first entry $\mathcal{K}_1(S)$ is the sum of the distances in $\mathbb{A}^{(1)}_{\mathrm{amb}}$ between all the pairs of residues C_L fixed by maximal spherical $L \subset S$. The second entry $\mathcal{K}_2(S)$ is the sum of the distances between more subtle objects. Namely, for maximal spherical $L \subset S$ let $D_L \subseteq C_L$ consist of chambers adjacent to each $\mathcal{W}_l$ with $l \in L$. The contribution to $\mathcal{K}_2(S)$ of a pair L, I of maximal spherical subsets of S is the distance between particular $E_{L,I} \subseteq D_L$ and $E_{I,L} \subseteq D_I$. Let us explain in detail what $E_{L,I}$ is for L irreducible.

First notice that then D_L consists of exactly two opposite chambers. We say that L is 'exposed' if $|L| \leq 2$ or $|L| = 3$ and there are at least two elements of L not adjacent to any element of $S \setminus (L \cup L^\perp)$. For L exposed we set $E_{L,I} = D_L$. Otherwise, we can predict which of the two chambers is better positioned with respect to I, and we set $E_{L,I}$ to be that chamber. Namely, we choose $E_{L,I}$ inside Φ^μ_s for $m \in I$ and 'good' s and $\{s, j_1\}$. The notion of 'good' is discussed in Section 4. For example if m adjacent to both j_1 and j_2, then s and $\{s, j_1\}$ are good. We designed this notion to make $E_{L,I}$ independent of the choice of s, j_1, which is proved in Sections 5 and 6. This allows us to define the complexity in Section 7. From now on we assume that the complexity of S is minimal among all Coxeter generating sets twist equivalent to S.

Going back to the goal of proving the consistency of $\{s,t\}$, we consider the components of $S \setminus (\{s,t\} \cup \{s,t\}^\perp)$. Using the 3-rigidity of S and 'moves', markings $\mu = ((s, tp\cdots), m)$ and $((s,t),p)$ with various

p in a fixed component A give rise to the same $\Phi_{s,A} := \Phi^{\mu}_s$. Thus to prove the consistency of $\{s,t\}$ one needs to prove that the pair $\Phi_{s,A}, \Phi_{t,A}$ is geometric (which we call the 'self-compatibility' of A), and that $\Phi_{s,A} = \Phi_{s,B}$ and $\Phi_{t,A} = \Phi_{t,B}$ for every other component B (which we call the 'compatibility' of A and B). We gradually show that in the subsequent sections. There we call A 'small' if all the elements of A are adjacent to both s,t; we call A 'big' otherwise. We call (small or big) A 'exposed' if there is an exposed $L \supset \{s,t\}$ intersecting A.

In Section 8 we prove that small components are self-compatible, and that each exposed component is self-compatible and compatible with any other component. This is done using various elementary twists provided by an exposed L, which allow to turn $E_{L,I} = D_L$ 'towards' C_I and decrease $\mathcal{K}_2$ in the case of incompatibility. In Section 9, we prove the compatibility of big components. A crucial concept there is that of 'peripherality', which picks out the 'least' inconsistent $\{s,t\}$ and allows to decrease $\mathcal{K}_1$. Finally, in Section 10 we prove the self-compatibility of big components, and their compatibility with small ones.

Having established the consistency of doubles, it is not hard to prove that Φ^{μ}_s depends only on s (which as we explained implies the Main Theorem), following a simplified version of the main argument of [9], which we present in Section 11.

Reading the article. Upon a first reading, we recommend to ignore $\mathcal{K}_2$. This means skipping Sections 4–8 except for the definition of $\mathcal{K}_1$ and the ones in Section 8, and focusing on understanding the details of Section 9. After that, it should become clear that to treat small components it is not enough to use only $\mathcal{K}_1$, which motivates the introduction of $\mathcal{K}_2$ with all its technical aspects.

Let us also mention that our construction of a 'folding' in Section 9 for $m_{st} = 4$ agrees with the construction in the article of Weigel [12, Fig 1]. His assumptions on the defining graph do not allow for irreducible spherical subsets L with $|L| > 2$, so he does not need to discuss small components or $\mathcal{K}_2$. However, our Main Theorem does not immediately imply the Main Theorem of [12] since Weigel allows for some subsets of S that violate FC.

While we could modify our article to also allow for these subsets, we refrain from that in order not to complicate the notation. In the future work, we plan to divide the subsets violating FC into two types. One type will be treated similarly to subsets of S containing a non-adjacent pair. The second type will have to be treated similarly to weakly separating spherical subsets of S of cardinality ≥ 3. The methods to prove the consistency of such subsets still need to be developped.

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2. Preliminaries

2.1. **Davis complex.** Let $\mathbb{A}$ be the *Davis complex* of a Coxeter system (W, S) (see [4, §7.3] for a precise definition). The 1-skeleton of $\mathbb{A}$ is the Cayley graph of (W, S) with vertex set W and a single edge spanned on $\{w, ws\}$ for each $w \in W, s \in S$. Higher dimensional cells of $\mathbb{A}$ are spanned on left cosets in W of remaining finite $\langle J \rangle$. The left action of W on itself extends to the action on $\mathbb{A}$.

A *chamber* is a vertex of $\mathbb{A}$. Collections of chambers corresponding to cosets $w\langle J \rangle$ are called *J-residues* of $\mathbb{A}$. A *gallery* is an edge-path in $\mathbb{A}$. For two chambers $c_1, c_2 \in \mathbb{A}$, we define their *gallery distance*, denoted by $d(c_1, c_2)$, to be the length of a shortest gallery from c_1 to c_2.

Let $r \in W$ be an S-reflection. The fixed point set of the action of r on $\mathbb{A}$ is called its *wall* $\mathcal{Y}_r$. The wall $\mathcal{Y}_r$ determines r uniquely. Moreover, $\mathcal{Y}_r$ separates $\mathbb{A}$ into two connected components, which are called *halfspaces (for r)*. If a non-empty subset $K \subset \mathbb{A}$ is contained in a single halfspace, then $\Phi(\mathcal{Y}_r, K)$ denotes this halfspace. An edge of $\mathbb{A}$ crossed by $\mathcal{Y}_r$ is *dual* to $\mathcal{Y}_r$. A chamber is *incident* to $\mathcal{Y}_r$ if it is an endpoint of an edge dual to $\mathcal{Y}_r$. The *distance* of a chamber c to $\mathcal{Y}_r$, denoted by $d(c, \mathcal{Y}_r)$, is the minimal gallery distance from c to a chamber incident to $\mathcal{Y}_r$.

2.2. **Geometric set of reflections.** Let (W, S) be a Coxeter system. Let $\mathbb{A}_{\mathrm{ref}}$ be the Davis complex for (W, S) ('ref' stands for 'reference complex'). For each reflection r, let $\mathcal{Y}_r$ be its wall in $\mathbb{A}_{\mathrm{ref}}$. Suppose that S is **angle-compatible** with another Coxeter generating set S'. Let $\mathbb{A}_{\mathrm{amb}}$ be the Davis complex for (W, S') ('amb' stands for 'ambient complex'). For each reflection r, let $\mathcal{W}_r$ be its wall in $\mathbb{A}_{\mathrm{amb}}$. Let $P \subseteq S$.

Definition 2.1. Let $\{\Phi_p\}_{p \in P}$ be a collection of halfspaces of $\mathbb{A}_{\mathrm{amb}}$ for $p \in P$. The collection $\{\Phi_p\}_{p \in P}$ is 2-*geometric* if for any pair $p, r \in P$, the set $\Phi_p \cap \Phi_r \cap \mathbb{A}^{(0)}_{\mathrm{amb}}$ is a fundamental domain for the action of $\langle p, r \rangle$ on $\mathbb{A}^{(0)}_{\mathrm{amb}}$. The collection $\{\Phi_p\}_{p \in P}$ is *geometric* if additionally $F = \bigcap_{p \in P} \Phi_p \cap \mathbb{A}^{(0)}_{\mathrm{amb}}$ is non-empty. The set P is 2-*geometric* if there exists a 2-geometric collection of halfspaces $\{\Phi_p\}_{p \in P}$.

Theorem 2.2 ([2, Thm 4.2]). *If $\{\Phi_p\}_{p\in P}$ is 2-geometric, then after possibly replacing each Φ_p by opposite halfspace, the collection $\{\Phi_p\}_{p\in P}$ is geometric.*

Theorem 2.2 justifies calling 2-geometric P *geometric* for simplicity. We call F as above a *geometric fundamental domain for* P, since by [6] (see also [8, Thm 1.2] and [2, Fact 1.6]), we have:

Proposition 2.3. *If P is geometric, then F is a fundamental domain for the action of $\langle P\rangle$ on $\mathbb{A}^{(0)}_{\mathrm{amb}}$, and for each $p \in P$ there is a chamber in F incident to $\mathcal{W}_p$. In particular, if $P = S$, then S is conjugate to S'.*

Corollary 2.4 ([9, Cor 2.6]). *Let $J \subseteq S$ be spherical. Then J is conjugate to a spherical $J' \subseteq S'$. In particular, J is geometric, and if it is irreducible, there exist exactly two geometric fundamental domains for J.*

We will need the following compatibility result.

Lemma 2.5. *Let $J \subset S$ be irreducible spherical, and let $r_1, r_2 \in S \setminus J$ with $J \cup \{r_1, r_2\}$ geometric. Let $\mathcal{W}_1$ and $\mathcal{W}_2$ be walls of $\mathbb{A}_{\mathrm{amb}}$ fixed by some reflections in $\langle J\rangle$ and satisfying $\mathcal{W}_i \cap \mathcal{W}_{r_i} = \emptyset$ for $i = 1, 2$. Let Δ_1, Δ_2 be the geometric fundamental domains for J satisfying $\Phi(\mathcal{W}_i, \Delta_i) = \Phi(\mathcal{W}_i, \mathcal{W}_{r_i})$ for $i = 1, 2$. Then $\Delta_1 = \Delta_2$.*

Proof. Let $F \subset \mathbb{A}^{(0)}_{\mathrm{amb}}$ be the geometric fundamental domain for $J \cup \{r_1, r_2\}$. By Proposition 2.3, for $i = 1, 2$, there is chamber $x_i \in F$ incident to $\mathcal{W}_{r_i}$. Let Δ be the geometric fundamental domain for J containing F. Then for $i = 1, 2$, we have $\Phi(\mathcal{W}_i, \mathcal{W}_{r_i}) = \Phi(\mathcal{W}_i, x_i) = \Phi(\mathcal{W}_i, F) = \Phi(\mathcal{W}_i, \Delta)$, and so $\Delta_i = \Delta$. □

We close with the following result, which is [9, Lem 5.4]. Note that we assumed there that $\mathcal{W} = \mathcal{W}_r$ for some $r \in S$, but the proof works word for word without that assumption.

Lemma 2.6. *Let $\{j_1, j_2\} \subset S$ be irreducible spherical. Suppose that a wall $\mathcal{W}$ in $\mathbb{A}_{\mathrm{amb}}$ is disjoint from both $\mathcal{W}_{j_2}$ and $j_1\mathcal{W}_{j_2}$, and we have $\Phi(\mathcal{W}_{j_2}, \mathcal{W}) = \Phi(\mathcal{W}_{j_2}, j_1\mathcal{W})$. Let F be a geometric fundamental domain for $\{j_1, j_2\}$. Then $\mathcal{W}$ is disjoint from $j_2\mathcal{W}_{j_1}$ and we have $\Phi(\mathcal{W}_{j_2}, \mathcal{W}) = \Phi(\mathcal{W}_{j_2}, F)$ if and only if $\Phi(\mathcal{W}_{j_1}, j_2\mathcal{W}) = \Phi(\mathcal{W}_{j_1}, F)$.*

We have the following immediate consequence, which is a variant of [3, Lem 5.1].

Corollary 2.7. *Let $\{j_1, j_2\} \subset S$ be irreducible spherical. Suppose that a wall $\mathcal{W}$ in $\mathbb{A}_{\mathrm{amb}}$ is disjoint from both $\mathcal{W}_{j_2}$ and $j_1\mathcal{W}_{j_2}$, and intersects $\mathcal{W}_{j_1}$. Let F be a geometric fundamental domain for $\{j_1, j_2\}$. Then $\mathcal{W}$ is disjoint from $j_2\mathcal{W}_{j_1}$ and we have $\Phi(\mathcal{W}_{j_2}, \mathcal{W}) = \Phi(\mathcal{W}_{j_2}, F)$ if and only if $\Phi(\mathcal{W}_{j_1}, j_2\mathcal{W}) = \Phi(\mathcal{W}_{j_1}, F)$.*

3. Bases and markings

Henceforth, in the entire article we assume that S **is irreducible, not spherical, and of type** FC. (The reducible case easily follows from the irreducible.)

In this section we recall several central notions from [3]. Let $W, S, \mathbb{A}_{\mathrm{ref}}, \mathcal{Y}_r$ (and later $S', \mathbb{A}_{\mathrm{amb}}, \mathcal{W}_r$) be as in Section 2.2. Let c_0 be the identity chamber of $\mathbb{A}_{\mathrm{ref}}$.

3.1. **Bases.**

Definition 3.1. A *base* is a pair (s, w) with *core* $s \in S$ and $w \in W$ satisfying

(i) $w = j_1 \cdots j_n$, where $n \geq 0$, and $j_i \in S$,

(ii) $d(w.c_0, \mathcal{Y}_s) = n$,

(iii) the *support* $J = \{s, j_1, \ldots, j_n\}$ is spherical.

Note that this agrees with [3, Def 3.1]. Indeed, Condition (ii) from [3, Def 3.1] saying that every wall that separates $w.c_0$ from c_0 intersects $\mathcal{Y}_s$ follows immediately from our Condition (iii). On the other hand, our Condition (iii) follows from [3, Lem 3.5] since S is of type FC. Note also that our Condition (ii) implies that J is irreducible. A base is *simple* if s and all j_i are distinct. In [3, Lem 3.7] and the paragraph preceding it, we established the following.

Remark 3.2. If $J \subset S$ is irreducible spherical and $s \in J$, then there is a unique simple base (s, w) with support J and core s. We have $w = j_1 \cdots j_n$ for any ordering of the elements of $J \setminus \{s\}$ into a sequence (j_i) with each $\{s, j_1, \ldots, j_i\}$ irreducible. We often denote that base (s, w) by (s, J).

The following result is a straightforward generalisation of [9, Lem 3.3], where a base was assumed to be simple.

Lemma 3.3. *Let $J \subset S$ be irreducible spherical, and let F be a geometric fundamental domain for J. Then for any base (s, w) with support J we have $\Phi(\mathcal{W}_s, F) = \Phi(\mathcal{W}_s, wF)$.*

3.2. **Markings.**

Definition 3.4. A *marking* is a pair $\mu = ((s,w),m)$, where (s,w) is a base with support J and where the *marker* $m \in S$ is such that $J \cup \{m\}$ is not spherical. The *core* and the *support* of the marking μ are the core and the support of its base. We say that μ is *simple*, if its base is simple.

Our definition of a marking agrees with the notion of a *complete marking* from [3, Def 3.8]. To see that, note that since S if of type FC, m is not adjacent to some element of J and hence by [3, Rem 3.2(ii)] we have that $w\mathcal{Y}_m$ is disjoint from $\mathcal{Y}_s$. We decided to drop the term 'complete' since we will not be discussing any other markings in this article. Similarly, our definition of a simple marking agrees with the notion of a *good marking* from [3, Def 3.13], since by FC there are no semicomplete markings described in [3, Def 3.11].

Remark 3.5 ([9, Rem 3.5])**.** For each $s \in I \subset S$ with I irreducible spherical, there exists a simple marking with support containing I and core s.

Definition 3.6. Let $\mu = ((s,w),m)$ be a marking. Since $w\mathcal{Y}_m$ is disjoint from $\mathcal{Y}_s$, the element $wmw^{-1}s$ is of infinite order, and hence also $w\mathcal{W}_m$ is disjoint from $\mathcal{W}_s$. We define $\Phi^\mu_s = \Phi(\mathcal{W}_s, w\mathcal{W}_m)$.

Proposition 3.7 ([3, Prop 5.2])**.** *Let $s_1, s_2 \in S$. Suppose that for each $i = 1, 2$, any simple marking μ with core s_i gives rise to the same $\Phi_{s_i} = \Phi^\mu_{s_i}$. Then the pair Φ_{s_1}, Φ_{s_2} is geometric.*

We summarise Proposition 3.7, Theorem 2.2, and Proposition 2.3 in the following.

Corollary 3.8 ([9, Cor 3.8])**.** *If for each $s \in S$ any simple marking μ with core s gives rise to the same Φ^μ_s, then S is conjugate to S'.*

3.3. **Moves.**

Definition 3.9. Let $((s,w),m), ((s,w'),m')$ be markings with common core. We say that they are related by *move*

(M1) if $w = w'$, and the markers m and m' are adjacent;

(M2) if there is $j \in S$ such that $w = w'j$ and moreover m equals m' and is adjacent to j.

We will write $((s,w),m) \sim ((s,w'),m')$ if there is a finite sequence of moves M1 or M2 that brings $((s,w),m)$ to $((s,w'),m')$.

The following is a special case of [3, Lem 4.2].

Lemma 3.10. *If markings μ and μ' with common core s are related by move* M1 *or* M2*, then $\Phi_s^{\mu} = \Phi_s^{\mu'}$.*

We have a straightforward generalisation of [9, Prop 4.3].

Proposition 3.11. *Let (s, w) be a base with support I. Suppose that no irreducible spherical $I' \supsetneq I$ weakly separates S. Let $\mu_1 = ((s, ww_1), m_1)$ and $\mu_2 = ((s, ww_2), m_2)$ be markings with supports J_1, J_2, where each of w_i is a product of distinct elements of $J_i \setminus I$. Moreover, for $i = 1, 2$ define $K_i = J_i \setminus (I \cup I^{\perp})$ when $I \subsetneq J_i$, and $K_i = \{m_i\}$ when $J_i = I$. Suppose that K_1 and K_2 are in the same component of $S \setminus (I \cup I^{\perp})$. Then $\mu_1 \sim \mu_2$. Consequently $\Phi_s^{\mu_1} = \Phi_s^{\mu_2}$.*

3.4. **Applications to 3-rigid S.** We start with choosing the notation for the K_i above in the case where μ_i is simple.

Definition 3.12. Let $\mu = ((s, J), m)$ be a simple marking with the base (s, J) defined in Remark 3.2. Let $I \subseteq J$ with $s \in I$. Then we denote by K_I^{μ} the set $J \setminus (I \cup I^{\perp})$ if $J \neq I$, and the set $\{m\}$ otherwise. We simplify the notation $K_{\{s\}}^{\mu}$ to K_s^{μ} etc.

Let $\{s, t\} \subset S$ be irreducible spherical. By Remark 3.5, for each component A of $S \setminus (\{s, t\} \cup \{s, t\}^{\perp})$, there exists a simple marking μ with support containing t, core s, and $K_{s,t}^{\mu} \subseteq A$. If S is 3-rigid, then by Proposition 3.11 if we have μ' with $K_{s,t}^{\mu'} \subseteq A$, then $\Phi_s^{\mu} = \Phi_s^{\mu'}$. Thus each component A of $S \setminus (\{s, t\} \cup \{s, t\}^{\perp})$ determines a halfspace $\Phi_{A,s} := \Phi_s^{\mu}$ for s.

The following is another variation on [9, Prop 4.3].

Proposition 3.13. *Suppose that S is 3-rigid. Let μ_1, μ_2 be simple markings with common core s. Suppose that $K_s^{\mu_1} \cap K_s^{\mu_2} = \emptyset$ and that there is an embedded path ω in the defining graph of S outside $s \cup s^{\perp}$ starting in $k_1 \in K_s^{\mu_1}$ and ending in $k_2 \in K_s^{\mu_2}$ such that*

(i) for any vertex $k \neq k_1, k_2$ of ω adjacent to s and any simple markings ν_1, ν_2 with supports containing k and core s we have $\Phi_s^{\nu_1} = \Phi_s^{\nu_2}$.

(ii) if k_1 is adjacent to s, then $K_{s,k_1}^{\mu_1}$ lies in the same component of $S \setminus (\{s, k_1\} \cup \{s \cup k_1\}^{\perp})$ as k_2, and

(iii) condition (ii) holds with indices 1 and 2 interchanged.

Then $\Phi_s^{\mu_1} = \Phi_s^{\mu_2}$.

Proof. We proceed by induction of the length of ω. Consider first the case where k_1 and k_2 are adjacent. If neither k_1 nor k_2 is adjacent to s, then $\mu_1 \sim \mu_2$ by move M1. If exactly one of k_1, k_2, say k_1, is adjacent to s, then let $\mu = ((s, k_1), k_2)$. We have $\mu \sim \mu_2$ by move M2. Moreover, $\mu_1 \sim \mu$ by condition (ii) and Proposition 3.11. If both k_1, k_2 are adjacent to s, then let μ be a simple marking with support containing k_1, k_2, and core s, which exists by Remark 3.5. By conditions (ii) and (iii) and Proposition 3.11 we have $\mu_1 \sim \mu \sim \mu_2$. Thus $\Phi_s^{\mu_1} = \Phi_s^{\mu_2}$ by Lemma 3.10.

Now consider the case where k_1 and k_2 are not adjacent. Let k be a vertex of ω distinct from k_1, k_2. Note that if k_1 is adjacent to s, then all the vertices of ω except for k_1 are contained in $S \setminus (\{s, k_1\} \cup \{s \cup k_1\}^{\perp})$, and thus condition (ii) holds with k_2 replaced with k.

First suppose that k is not adjacent to s. Then by the previous paragraph the pair of markings $\mu_1, \nu = ((s, \emptyset), k)$ satisfies the hypotheses of the proposition and thus by induction we have $\Phi_s^{\mu_1} = \Phi_s^{\nu}$. Analogously, $\Phi_s^{\nu} = \Phi_s^{\mu_2}$. Finally, suppose that k is adjacent to s. For $i = 1, 2$ let ν_i be a simple marking with support containing k and core s such that $K_{s,k}^{\nu_i}$ lies in the same component of $S \setminus (\{s, k\} \cup \{s \cup k\}^{\perp})$ as k_i, which exists by Remark 3.5. As before, by induction we have $\Phi_s^{\mu_i} = \Phi_s^{\nu_i}$. Furthermore, $\Phi_s^{\nu_1} = \Phi_s^{\nu_2}$ by condition (i). □

4. Good pairs

Let (W, S) be a Coxeter system. **Throughout the remaining part of the article, we will assume that all Coxeter generating sets twist equivalent to S are 3-rigid**.

The following notion of a good element t varies slightly from the one in [9], where we allowed r to be adjacent to t.

Definition 4.1. Let $L \subset S$ be irreducible spherical and let $r \in S$. An element $t \in L$ *is good with respect to* r, if

- $r \neq t$ and r is not adjacent to t, and
- $L \setminus (t \cup t^{\perp})$ is non-empty and in the same component of $S \setminus (t \cup t^{\perp})$ as r.

Note that being good depends on L. However, we often write shortly 't is good with respect to r' (or even just 't is good'), if L (and r) are fixed.

A non-commuting pair $\{s, t\} \subset L$ *is good with respect to* r, if

- $\{s, t, r\}$ is not spherical, and
- $L \setminus (\{s,t\} \cup \{s,t\}^{\perp})$ is non-empty and in the same component of $S \setminus (\{s,t\} \cup \{s,t\}^{\perp})$ as r.

The following lemma and its corollary exceptionally do not require the 3-rigidity assumption on S.

Lemma 4.2. *Let $\{s,t\} \subset S$ be spherical irreducible, and let $r \in S$ with $\{s,t,r\}$ not spherical. Suppose that $s \in L = \{s,t\}$ is not good with respect to r and $t \in L$ is not good with respect to r. Then r lies in a component of $S \setminus (\{s,t\} \cup \{s,t\}^{\perp})$ that has no element adjacent to s or t.*

Proof. If r is not adjacent to, say, s, then r is not adjacent to t (since s is not good). For contradiction, suppose that r lies in a component of $S \setminus (\{s,t\} \cup \{s,t\}^{\perp})$ that has an element adjacent to s or t. Let $\omega = r \cdots k$ be a minimal length path in the defining graph of S outside $\{s,t\} \cup \{s,t\}^{\perp}$ ending with a vertex k adjacent to s or t, say t. Since s is not good, there is a vertex k' of ω that lies in $s^{\perp}$. By the minimality of ω, we have $k' = k$, and hence k is also adjacent to s. Analogously, since t is not good, we have $k \in t^{\perp}$. Consequently, $k \in \{s,t\}^{\perp}$, which is a contradiction. □

Lemma 4.2 immediately implies the following.

Corollary 4.3. *Let $L \subset S$ be irreducible spherical and let $\{s,t\} \subset L$ be a non-commuting pair. Let $r \in S \setminus L$. If $\{s,t\}$ is good with respect to r, then s or t is good with respect to r.*

Lemma 4.4. *Let $L \subset S$ be irreducible spherical and let $r \in S$. Let s, t, p be consecutive vertices in the Coxeter–Dynkin diagram of L with $\{s,t,p,r\}$ not spherical. If $\{s,t\}$ is not good with respect to r and $\{t,p\}$ is not good with respect to r, then none of the elements in $S \setminus (\{s,t,p\} \cup \{s,t,p\}^{\perp})$ are adjacent to s or p.*

Proof. If r is not adjacent to, say, s or t, then since $\{s,t\}$ is not good with respect to r, we have that r is also not adjacent to p. For contradiction, suppose that an element in $S \setminus (\{s,t,p\} \cup \{s,t,p\}^{\perp})$ is adjacent to s or p. Since S is 3-rigid, we have a minimal length path $\omega = r \cdots k$ in the defining graph of S outside $\{s,t,p\} \cup \{s,t,p\}^{\perp}$ with k adjacent to s or p, say p. Since $\{s,t\}$ is not good, a vertex k' of ω lies in $\{s,t\}^{\perp}$. Then $k' = k$ by the minimality of ω. Thus $k \in \{s,t\}^{\perp}$. Analogously, since $\{t,p\}$ is not good, a vertex of ω lies in $\{t,p\}^{\perp}$ giving $k \in \{t,p\}^{\perp}$. Thus $k \in \{s,t,p\}^{\perp}$, which is a contradiction. □

We have the following immediate consequence of Lemma 4.4.

Corollary 4.5. *Let $L \subset S$ be irreducible spherical with $|L| \geq 4$ and let $r \in S$. Let s, t, p be consecutive vertices in the Coxeter–Dynkin diagram of L with $\{s, t, p, r\}$ not spherical. Then at least one of $\{s, t\}$ or $\{t, p\}$ is good with respect to r.*

We have also the following variant of Lemma 4.4.

Lemma 4.6. *Let $L \subset S$ be irreducible spherical and let $r \in S$. Let s, t, p be consecutive vertices in the Coxeter–Dynkin diagram of L with $\{s, t, p, r\}$ not spherical. If $\{s, t\}$ is not good with respect to r and p is not good with respect to r, then none of the elements in $S \setminus (\{s, t, p\} \cup \{s, t, p\}^{\perp})$ are adjacent to t or p.*

Proof. Again, if r is not adjacent to s or t, then since $\{s, t\}$ is not good, we have that r is also not adjacent to p. On the other hand, if r is not adjacent to p, then since p is not good, we have that r is also not adjacent to t. For contradiction, suppose that an element in $S \setminus (\{s, t, p\} \cup \{s, t, p\}^{\perp})$ is adjacent to t or p. Since S is 3-rigid, we have a minimal length path $\omega = r \cdots k$ in the defining graph of S outside $\{s, t, p\} \cup \{s, t, p\}^{\perp}$ with k adjacent to t or p, say t (the other case is similar). Since p is not good, a vertex k' of ω lies in $p^{\perp}$. Then $k' = k$ by the minimality of ω. Thus $k \in p^{\perp}$. Analogously, since $\{s, t\}$ is not good, a vertex of ω lies in $\{s, t\}^{\perp}$ giving $k \in \{s, t\}^{\perp}$. Thus $k \in \{s, t, p\}^{\perp}$, which is a contradiction as before. □

5. Fundamental domains for good pairs

Let $S, S', W, \mathbb{A}_{\mathrm{ref}}$ and $\mathbb{A}_{\mathrm{amb}}$ be as in Section 3. In this and the following section, we fix $L \subset S$ irreducible spherical.

Definition 5.1. Let $\mu = ((s, w), m)$ be a marking with support contained in L. By Δ^{μ} (or $\Delta^{(s,w),m}$) we denote the geometric fundamental domain for L that is contained in $\Phi^{\mu}_s = \Phi(\mathcal{W}_s, w\mathcal{W}_m)$. Equivalently, by Lemma 3.3, it is the geometric fundamental domain for L that is contained in $\Phi(w^{-1}\mathcal{W}_s, \mathcal{W}_m)$.

Note that Δ^{μ} depends on L but we suppress this in the notation.

Remark 5.2. By Lemma 3.10, for $\mu \sim \nu$ we have $\Delta^{\mu} = \Delta^{\nu}$.

In the remaining part of the section, let $\{s, t\} \subset L$ be a non-commuting pair, and let $r \in S$ with $\{s, t, r\}$ not spherical. Here is the main result of the section.

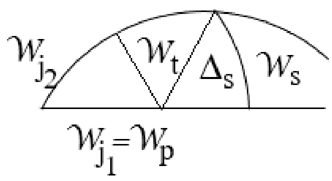

FIGURE 1

Proposition 5.3. *Suppose that $\{s, t\}$ is good with respect to r. Suppose that both s and t are good with respect to r. Then $\Delta^{(t,s),r} = \Delta^{(s,t),r}$.*

Proposition 5.3 makes the following notion well-defined.

Definition 5.4. Suppose that $\{s, t\}$ is good with respect to r. By Corollary 4.3, at least one of s, t, say s, is good with respect to r. Then we define $\Delta^{\{s,t\},r}$ to be $\Delta^{(s,t),r}$.

Proposition 5.3 follows immediately from the following.

Proposition 5.5. *Suppose that both $\{s, t\}$ and s are good with respect to r and that there are consecutive vertices s, t, p in the Coxeter–Dynkin diagram of L. Then $\Delta^{(t,s),r} = \Delta^{(s,t),r}$.*

In preparation for the proof of Proposition 5.5 we discuss several lemmas. We will denote shortly $\Delta_s = \Delta^{(s,t),r}$, $\Delta_t = \Delta^{(t,s),r}$.

Lemma 5.6. *Suppose that s, t, p are consecutive vertices in the Coxeter–Dynkin diagram of L, and $m_{st} \neq 3$. Suppose also $((s, t), r) \sim ((s, tp), r)$ and $((t, s), r) \sim ((t, spt), r)$. Then $\Delta_t = \Delta_s$.*

Note that it is easy to check that for $m_{st} \neq 3$ the pair (t, spt) is indeed a base. Making use of this base and its extensions is exactly the reason for which we need to discuss in this article bases that are not simple.

Proof. By the classification of finite Coxeter groups, we have $m_{tp} = 3$. We want to apply Lemma 2.6 to the conjugate $tpSpt$, to j_1, j_2 the conjugates of t, s, so that $j_2 = tpspt = tst, j_1 = tptpt = p$, and to $\mathcal{W} = \mathcal{W}_r$. Since $((s, t), r) \sim ((s, tp), r)$, by Lemma 3.10 we have $\Phi(\mathcal{W}_{j_2}, \mathcal{W}) = \Phi(t\mathcal{W}_s, \mathcal{W}_r) = \Phi(t\mathcal{W}_s, p\mathcal{W}_r) = \Phi(\mathcal{W}_{j_2}, j_1\mathcal{W})$, so the assumption of Lemma 2.6 is satisfied. It is easy to see (Figure 1) that Δ_s lies in a geometric fundamental domain F for $\{j_1, j_2\}$. By the definition of Δ_s we have $\Phi(\mathcal{W}_{j_2}, \mathcal{W}) = \Phi(\mathcal{W}_{j_2}, F)$, so by Lemma 2.6 (and Lemma 3.3), we have $\Phi(j_2\mathcal{W}_{j_1}, \mathcal{W}) = \Phi(j_2\mathcal{W}_{j_1}, F)$. This implies $\Phi(tst\mathcal{W}_p, \mathcal{W}_r) = \Phi(tst\mathcal{W}_p, \Delta_s)$. Since $tst\mathcal{W}_p = tps\mathcal{W}_t$, and $((t,s),r) \sim ((t,spt),r)$, Remark 5.2 implies $\Delta_t = \Delta_s$. □

Corollary 5.7. *Suppose that $L = \{u,s,t,p\} \subset S$ is of type* F_4*, and that u,s,t,p are consecutive vertices in the Coxeter–Dynkin diagram of L. Suppose also that r is not adjacent to s and that both $\{s,t\}$ and $\{u,s\}$ are good with respect to r. Then $\Delta_t = \Delta_s$.*

Proof. Since $\{s,t\}$ is good, by Proposition 3.11 and the 3-rigidity of S we have $((s,t),r) \sim ((s,tp),r)$ and $((t,s),r) \sim ((t,spu),r)$. Since $\{u,s\}$ is good, there is a minimal length path $rr_1\cdots r_n t$ in the defining graph of S outside $\{u,s\} \cup \{u,s\}^\perp \supset \{u,s,p\} \cup \{u,s,t,p\}^\perp$. By the classification of finite Coxeter groups $\{u,s,t,p\}$ is maximal irreducible spherical. Thus using moves M1 and M2 we obtain

$$\begin{aligned}((t,spu),r) \sim & ((t,spu),r_1) \sim \cdots \sim ((t,spu),r_n) \sim \\ & ((t,sput),r_n) \sim \cdots \sim ((t,sput),r_1) \sim ((t,sput),r).\end{aligned}$$

By the 3-rigidity of S and Proposition 3.11 we have $((t,sput),r) = ((t,sptu),r) \sim ((t,spt),r)$. Thus Lemma 5.6 applies. □

Lemma 5.8. *Suppose that $L = \{u,s,t,p\}$ and that u,s,t,p are consecutive vertices in the Coxeter–Dynkin diagram of L. Suppose that $\{s,t,r\}$ is not spherical. Then $\{u,s\}$ or $\{t,p\}$ is good with respect to r.*

Proof. If r is adjacent to s or t, then the lemma follows, so suppose otherwise. By the classification of finite Coxeter groups we can assume without loss of generality $m_{us} = 3$. Suppose that $\{u,s\}$ is not good. Let Γ_τ be the defining graph of the Coxeter generating set S_τ obtained by the elementary twist in $\langle u,s\rangle$ that conjugates by the longest word w_{us} in $\langle u,s\rangle$ all the elements of the component B of $S \setminus (\{u,s\} \cup \{u,s\}^\perp)$ containing r. Note that $t \notin B$ as $\{u,s\}$ is not good. Since S_τ is 3-rigid, $\{s,t,p\}$ does not weakly separate S_τ. Consider then a minimal length path ω_τ in Γ_τ from $w_{us}rw_{us}^{-1}$ to u outside $\{s,p,t\} \cup \{s,p,t\}^\perp$. Note that by the minimality of ω_τ all the vertices of ω_τ are conjugates by w_{us} of the elements in B, except for u and possibly the vertex preceding u, which might be in $\{u,s\}^\perp$. Thus conjugating ω_τ back, we obtain a path ω from r to s in the defining graph of S, contained in $B \cup \{u,s\}^\perp \cup \{s\}$ and outside $\{u,p,t\} \cup \{u,s,p,t\}^\perp$. We claim that ω lies outside $\{t,p\}^\perp$ justifying that $\{t,p\}$ is good. Otherwise, let k be the first vertex of

ω in $\{t,p\}^\perp$, and let ω_k be the subpath $r \cdots k$ of ω. Since $t \notin B$, the path ω_k must have a vertex in $\{u,s\}^\perp$. By the minimality of ω_τ, this must be the vertex k. Thus k is a vertex of ω lying in $\{u,s,p,t\}^\perp$, which is a contradiction. $\square$

Proof of Proposition 5.5. Assume $m_{st} \neq 3$, since otherwise $s\mathcal{W}_t = t\mathcal{W}_s$ and so $\Delta_t = \Delta_s$ is immediate.

Suppose first that r is adjacent to one of s,t, say t. By definition we have $\Phi(\mathcal{W}_t, s\mathcal{W}_r) = \Phi(\mathcal{W}_t, \Delta_t)$. Thus applying Corollary 2.7 with $j_1 = t, j_2 = s, F \supset \Delta_t$ we obtain $\Phi(\mathcal{W}_s, \mathcal{W}_r) = \Phi(\mathcal{W}_s, \Delta_t)$. As $((s,\emptyset),r) \sim ((s,t),r)$ (move M2), by Remark 5.2 we have $\Delta_t = \Delta_s$, as required.

It remains to consider the case where r is adjacent neither to s nor t. Since $\{s,t\} \subset L$ is good with respect to r, by Proposition 3.11 and the 3-rigidity of S we have $((s,t),r) \sim ((s,tp),r)$ and $((t,s),r) \sim ((t,sp),r)$. Since $s \in L$ is good with respect to r, there is a path in the defining graph of S from r to t outside $s \cup s^\perp$. If for each vertex $u \neq t$ on this path the set $\{s,t,p,u\}$ is not spherical, then using moves M1 and M2 we have $((t,sp),r) \sim ((t,spt),r)$ and Lemma 5.6 applies. Otherwise, if u is a vertex of that path adjacent to all s,t,p, we are in the setup of Lemma 5.8, with L replaced by $L' = \{u,s,t,p\}$. Thus one of $\{u,s\},\{t,p\} \subset L'$ is good with respect to r. Note that L' and L both contain s,t,p, so we have that $\{s,t\} \subset L'$ is still good with respect to r. Then, possibly after interchanging s with t and u with p, Corollary 5.7 applies, with L' in place of L (which is of type F_4 by $m_{st} \neq 3$ and the classification of finite Coxeter groups). Thus for Δ'_s, Δ'_t defined as Δ_s, Δ_t, with L' in place of L, we have $\Delta'_t = \Delta'_s$. Since by definition Δ'_t and Δ_t (and analogously Δ'_s and Δ_s) are contained in the same geometric fundamental domain for $\{s,t\}$, we have $\Delta_t = \Delta_s$, as desired. $\square$

6. Independence of fundamental domains

This section is devoted to the proof of the following.

Proposition 6.1. *Let $L \subset S$ be irreducible spherical and $I \subset S$ be spherical. Suppose that $\{s,t\}$ and $\{p,q\}$ are non-commuting pairs in L. Let $r, r' \in I$ be such that $\{s,t\} \subset L$ is good with respect to r, and $\{p,q\} \subset L$ is good with respect to r'. Then $\Delta^{\{s,t\},r} = \Delta^{\{p,q\},r'}$.*

The key to the proof is:

Lemma 6.2. *Let $L \subset S$ be irreducible spherical with $|L| \geq 3$ and let $r \in S$ with $L \cup \{r\}$ not spherical. Consider $s \in L$ that is not a leaf of* *the Coxeter–Dynkin diagram of L. Let $\mu = ((s, L), r)$. Then Δ^μ does not depend on s.*

Here (s, L) denotes the unique simple base (s, w) with support L and core s from Remark 3.2. Before we give the proof of Lemma 6.2, we record the following.

Remark 6.3. Let $L \subset S$ be irreducible spherical and let $r \in S$. Suppose that $\{s, t\} \subset L$ is a non-commuting pair that is good with respect to r.

- (i) Let $\nu = ((s, t), r)$. Since we can assume that w above starts with t, by the 3-rigidity of S, Proposition 3.11, and Remark 5.2 we have $\Delta^\nu = \Delta^\mu$.
- (ii) Since $s \in L$ is not a leaf of the Coxeter–Dynkin diagram of L, by Proposition 5.5 we have $\Delta^{\{s,t\},r} = \Delta^\nu$.

Note that Proposition 6.1 follows immediately from Remark 6.3 and Lemma 6.2 since $\mathcal{W}_r \cap \mathcal{W}_{r'} \neq \emptyset$ and hence $\Delta^{(s,L),r} = \Delta^{(s,L),r'}$.

Proof of Lemma 6.2. Suppose that $t \in L$ is also not a leaf of the Coxeter–Dynkin diagram of L. If we have $m_{st} = 3$, then $s\mathcal{W}_t = t\mathcal{W}_s$ and $\Delta^{(s,L),r} = \Delta^{(t,L),r}$ follows. It remains to analyse the situation where u, s, t, p are consecutive vertices in the Coxeter–Dynkin diagram of L of type F_4. If $\{s, t\}$ is good, then we have $\Delta^{(s,L),r} = \Delta^{(t,L),r}$ by Proposition 5.5 and Remark 6.3(i).

Suppose now that $\{s, t\}$ is not good. We claim that at least one of $\{u, s\}, \{t, p\}$ is good. We first establish that r is not adjacent to at least one of u, p. Indeed, if $\{s, t, r\}$ is not spherical, then since $\{s, t\}$ is not good we have that r is neither adjacent to u nor to p. If $\{s, t, r\}$ is spherical, then since $L \cup \{r\}$ is not spherical, r is not adjacent to at least one of u, p, say u. Then by Corollary 4.5 the pair $\{u, s\}$ is good, justifying the claim. In particular, we have

$$((s, u), r) \sim ((s, ut), r) \sim ((s, utp), r), \tag{6.1}$$

where $\sim$ follow from the assumptions that $\{u, s\}$ is good, that S is 3-rigid and from Proposition 3.11.

Let $\nu = ((s, u), r)$. Let $H_1 = \Phi(\mathcal{W}_t, \Delta^\nu)$ and $H_2 = \Phi(\mathcal{W}_t, psu\mathcal{W}_r)$. By Remark 6.3(i), to prove $\Delta^{(s,L),r} = \Delta^{(t,L),r}$ it suffices to show $H_1 = H_2$.

Let $H = \Phi(\mathcal{W}_s, u\mathcal{W}_r)$. By Equation (6.1) and Lemma 3.10, we have $\mathcal{W}_r \subset uH \cap ptuH =: U$. Thus $\mathcal{W}_r \subset U \cap uspH_2$. On the other hand, by

Lemma 3.3, we have $\Delta^\nu \subset U \cap uspH_1$. Hence $H_1 = H_2$ by Corollary 2.4 and Lemma 6.4 below. □

Lemma 6.4. *Suppose W is a Coxeter group of type F_4 with u, s, t, p consecutive vertices in its Coxeter–Dynkin diagram. Consider the Tits representation $W \curvearrowright \mathbb{E}^4$. Let H_j^+ and H_j^- be the two open halfspaces in $\mathbb{E}^4$ bounded by the hyperplane fixed by a generator j. Let $U = uH_s^+ \cap ptuH_s^+$. Then one of $U \cap uspH_t^+$ and $U \cap uspH_t^-$ is empty.*

Proof. The simple roots associated to u, s, t, p are $\alpha_u = (1, -1, 0, 0), \alpha_s = (0, 1, -1, 0), \alpha_t = (0, 0, 1, 0)$ and $\alpha_p = (-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2})$. One computes directly $u\alpha_s = \alpha_u + \alpha_s, tu\alpha_s = \alpha_u + \alpha_s + 2\alpha_t, ptu\alpha_s = \alpha_u + \alpha_s + 2\alpha_t + 2\alpha_p$. Moreover, $p\alpha_t = \alpha_t + \alpha_p, sp\alpha_t = \alpha_s + \alpha_t + \alpha_p$, and $usp\alpha_t = \alpha_u + \alpha_s + \alpha_t + \alpha_p$. Note that $u\alpha_s + ptu\alpha_s = 2usp\alpha_t$. Thus for any vector $v \in \mathbb{E}^4$, if $\langle v, u\alpha_s\rangle > 0$ and $\langle v, ptu\alpha_s\rangle > 0$, then $\langle v, usp\alpha_t\rangle > 0$, as desired. □

7. Complexity

In this section, we introduce the complexity of the Coxeter generating set S with respect to S'. This extends the ideas of [9, §6]. We keep the setup from Section 5. To start with, we need to distinguish particular spherical subsets.

Definition 7.1. Let $L \subset S$ be irreducible spherical. L is *exposed* if $|L| \leq 2$ or $|L| = 3$ and there are at least two elements of L not adjacent to any element of $S \setminus (L \cup L^\perp)$.

Here are several criteria for identifying exposed L.

Lemma 7.2. *Let $L \subset S$ be irreducible spherical, and let $r \in S$ with $L \cup \{r\}$ not spherical. Suppose that each non-commuting pair $\{s, t\} \subset L$ is not good with respect to r. Then L is exposed.*

Proof. Suppose $|L| \geq 3$. If for some non-commuting pair $\{s, t\} \subset L$ we have that r is adjacent to both s, t, then let $p \in L \setminus \{s, t\}$ be non-commuting with one of s, t, say t. Since r is adjacent to s and $\{t, p\}$ is not good, we have that r is adjacent to p. Proceeding in this way we get that r is adjacent to all the elements of L, which contradicts our hypothesis. Thus by Corollary 4.5 we have $|L| = 3$, and by Lemma 4.4 there are at least two elements of L not adjacent to any element of $S \setminus (L \cup L^\perp)$. □

Lemmas 4.4 and 4.6 give also immediately the following.

Lemma 7.3. *Let $L \subset S$ be irreducible spherical, and s, t, p be consecutive vertices in the Coxeter–Dynkin diagram of L. Let $r \in S$ be distinct from and not adjacent to p. Suppose that both of the following hold:*

- *$\{s, t\} \subset L$ is not good with respect to r,*
- *$\{t, p\} \subset L$ is not good with respect to r or $p \in L$ is not good with respect to r.*

Then L is exposed.

We now describe particular subsets of pairs of maximal spherical residues.

Definition 7.4. Let $L \subset S$ be a maximal spherical subset. By Corollary 2.4, $\langle L\rangle$ stabilises a unique maximal cell $\sigma_L \subset \mathbb{A}_{\mathrm{amb}}$. Let C_L be the collection of vertices in σ_L and let D_L be the elements of C_L incident to each $\mathcal{W}_l$ for $l \in L$.

When L is irreducible, then by Corollary 2.4 it is easy to see that D_L consists of two antipodal vertices. In general, let $L = L_1 \sqcup \cdots \sqcup L_k$ be the decomposition of L into maximal irreducible subsets. Let $\sigma_L = \sigma_1 \times \cdots \times \sigma_k$ be the induced product decomposition of the associated cell. Then D_L is a product of pairs of antipodal vertices $\{u_i, v_i\}$ for each σ_i. Let $\pi_i \colon D_L \to \{u_i, v_i\}$ be the coordinate projections.

Definition 7.5. For each ordered pair (L, I) of maximal spherical subsets of S, we define the following subset $E_{L,I} \subseteq D_L$. First, for each $i = 1, \ldots, k$, consider the following $E^i_{L,I} \subseteq D_L$. If L_i is exposed or $L_i \subset I$, then we take $E^i_{L,I} = D_L$. Otherwise, since I is maximal spherical, there is $r \in I$ with $L_i \cup \{r\}$ not spherical. Moreover, by Lemma 7.2, there is $\{s, t\} \subset L_i$ that is good with respect to r. Then we take $E^i_{L,I} = C_L \cap \Delta^{\{s,t\},r}$, where in Definition 5.1 we substitute L with L_i. Note that such $E^i_{L,I}$ is contained in D_L and equal $\pi_i^{-1}(u_i)$ or $\pi_i^{-1}(v_i)$. Furthermore, $E^i_{L,I}$ does not depend on $\{s, t\}$ and r by Proposition 6.1. We define $E_{L,I} = E^1_{L,I} \cap \cdots \cap E^k_{L,I}$.

Remark 7.6. In Definition 7.5, in the case where L_i is neither exposed nor a subset of I, the set $E^i_{L,I}$ can be characterised in the following alternate way that does not involve the notion of a good pair. Namely, by Remark 6.3 and Lemma 6.2 we have that $\Delta^{\{s,t\},r}$ is the fundamental domain for L_i that is contained in $\Phi(\mathcal{W}_{s'}, wC_I)$, for any $s' \in S$ that is not a leaf of the Coxeter–Dynkin diagram of L_i and (s', w) the unique simple base with support L_i and core s'.

Definition 7.7. We define the *complexity* of S, denoted $\mathcal{K}(S)$, to be the ordered pair of numbers

$$\big(\mathcal{K}_1(S), \mathcal{K}_2(S)\big) = \Big(\textstyle\sum_{L\neq I} d(C_L, C_I), \sum_{L\neq I} d(E_{L,I}, E_{I,L})\Big),$$

where L and I range over all maximal spherical subsets of S. For two Coxeter generating sets S and S_τ, we define $\mathcal{K}(S_\tau) < \mathcal{K}(S)$ if $\mathcal{K}_1(S_\tau) < \mathcal{K}_1(S)$, or $\mathcal{K}_1(S_\tau) = \mathcal{K}_1(S)$ and $\mathcal{K}_2(S_\tau) < \mathcal{K}_2(S)$.

In the following lemma we prove that elementary twists preserve exposed L. This will enable us later to trace the change of $\mathcal{K}_2(S)$.

Definition 7.8. Let $L \subset S$ be maximal spherical and let τ be an elementary twist with $S \setminus (J \cup J^\perp) = A \sqcup B$ as in the definition of an elementary twist in the Introduction. We define the following spherical subset $L_\tau \subset \tau(S)$. If $L \subseteq A \cup J \cup J^\perp$, then we set $L_\tau = L$. If $L \subseteq B \cup J \cup J^\perp$, then we set $L_\tau = w_J L w_J^{-1}$. Note that this definition is not ambiguous if $L \subseteq J \cup J^\perp$, since then by the maximality of L we have $J \subseteq L$ and hence $L = w_J L w_J^{-1}$. If L' is an irreducible subset of L, then similarly L'_τ denotes L' or $w_J L' w_J^{-1}$ depending on whether $L \subseteq A \cup J \cup J^\perp$ or $L \subseteq B \cup J \cup J^\perp$ as before. Note that L'_τ might depend on L, but only if $L' \subsetneq J$. In particular this cannot happen for $|L'| \geq 3$ and $|J| = 2$.

Note that $L_\tau \subset \tau(S)$ is still maximal spherical and the assignment $L \to L_\tau$ is a bijection between the maximal spherical subsets of S and $\tau(S)$.

Lemma 7.9. *Let τ be an elementary twist of S. Let L be a maximal irreducible subset of a maximal spherical subset of S. If $|L| = 3$ and L is exposed in S, then L_τ is exposed in $\tau(S)$.*

Proof. We can assume that τ is an elementary twist with $J = \{s, t\}$ and m_{st} odd, since otherwise the defining graph of S is invariant under τ. Let $S \setminus (\{s,t\} \cup \{s,t\}^\perp)$ decompose into $A \sqcup B$ as in the definition of an elementary twist. Without loss of generality assume $L \subset A \cup \{s,t\} \cup \{s,t\}^\perp$. Then $L_\tau = L$.

First consider the case where $\{s, t\}$ is disjoint from L. Then $l \in L$ is adjacent to $r \in S$ if and only if $\tau(l) = l$ is adjacent to $\tau(r)$ in the defining graph of $\tau(S)$ and $m_{lr} = m_{\tau(l)\tau(r)}$. In particular $(L_\tau)^\perp = \tau(L^\perp)$. Then L_τ is exposed.

Secondly, consider the case where $\{s, t\} \subset L$. Then $(L_\tau)^\perp = \tau(L^\perp)$. Moreover, since $k = 1$ or 2 elements among s, t are not adjacent to any element of $S \setminus (L \cup L^\perp)$, we have that $S \setminus (L \cup L^\perp)$ is contained entirely in A or B. Consequently, there are k elements among $\tau(s)=s, \tau(t)=t$ that are not adjacent to any elements of $\tau(S\setminus(L\cup L^{\perp}))$, and thus L_τ is exposed.

Thirdly, consider the case where $\{s,t\}\cap L=\{t\}$. Then t is the only element of L adjacent to some element of $S\setminus(L\cup L^{\perp})$. Note also that since $t\in L, s\notin L$, we have $L^{\perp}\subseteq A\cup\{s,t\}^{\perp}$. Thus $(L_\tau)^{\perp}=\tau(L^{\perp})$. Moreover, none of the elements of $\tau(S\setminus(L\cup L^{\perp}))$ is adjacent to an element of $L_\tau\setminus\{t\}$, and so L_τ is exposed. □

Remark 7.10. (i) Suppose that L_i in Definition 7.5 is not exposed and that we have $L_i\subset I$. Then we also have $(L_i)_\tau\subset I_\tau$ in $\tau(S)$. Thus by Lemma 7.9 $E^i_{L,I}=D_L$ if and only if $E^i_{L_\tau,I_\tau}=D_{L_\tau}$.

(ii) Consequently, by Remark 7.6, if $L\cup I\subseteq A\cup J\cup J^{\perp}$, then $E_{(L_i)_\tau,I_\tau}=E_{L_i,I}$ and so in particular we have $d(E_{L,I},E_{I,L})=d(E_{L_\tau,I_\tau},E_{I_\tau,L_\tau})$. We have the same conclusion for $L\cup I\subseteq B\cup J\cup J^{\perp}$, since then $E_{(L_i)_\tau,I_\tau}=w_JE_{L_i,I}$.

(iii) Suppose that $L\subseteq A\cup J\cup J^{\perp}$ and $I\subseteq B\cup J\cup J^{\perp}$ with $J\subseteq I$. Then $C_I=w_JC_I=C_{I_\tau}$. Consequently, by Remark 7.6 we have $E_{(L_i)_\tau,I_\tau}=E_{L_i,I}$. Analogously, if $L\subseteq B\cup J\cup J^{\perp}$ and $J\subseteq I$, then $E_{(L_i)_\tau,I_\tau}=w_JE_{L_i,I}$.

8. Proof of the main theorem : exposed components

We keep the setup from Section 5. The Main Theorem reduces to the following.

Theorem 8.1. *Let S be a Coxeter generating set of type* FC *angle-compatible with a Coxeter generating set S'. Suppose that any Coxeter generating set twist-equivalent to S is* 3*-rigid. Assume moreover that S has minimal complexity among all Coxeter generating sets twist-equivalent to S. Then S is conjugate to S'.*

The main step in the proof of Theorem 8.1 will be to establish the consistency of doubles.

Definition 8.2. Let S be a Coxeter generating set and let $I\subset S$ be irreducible spherical with $|I|=2$. We say that I is *consistent* if for any simple markings μ_1,μ_2 with supports containing I and cores $s_1,s_2\in I$ the pair $\Phi^{\mu_1}_{s_1},\Phi^{\mu_2}_{s_2}$ is geometric (which means $\Phi^{\mu_1}_{s_1}=\Phi^{\mu_2}_{s_2}$ for $s_1=s_2$). Otherwise we say that I is *inconsistent*. We say that S has *consistent doubles*, if any such I is consistent.

In the following we use the notation from Definition 3.12.

Definition 8.3. Let $\{s,t\} \subset S$ be irreducible spherical. We say that components A_1, A_2 of $S \setminus (\{s,t\} \cup \{s,t\}^\perp)$ are *compatible* if $\Phi_{A_1,s} = \Phi_{A_2,s}$ and $\Phi_{A_1,t} = \Phi_{A_2,t}$. We say that a component A of $S \setminus (\{s,t\} \cup \{s,t\}^\perp)$ is *self-compatible* if the pair $\Phi_{A,s}, \Phi_{A,t}$ is geometric.

Note that if all components of $\{s,t\}$ are compatible and self-compatible, then $\{s,t\}$ is consistent. We will prove the compatibility in different ways depending on the type of the components.

Definition 8.4. Let $\{s,t\} \subset S$ be irreducible spherical. A component A of $S \setminus (\{s,t\} \cup \{s,t\}^\perp)$ is *big* if there is $r \in A$ with $\{s,t,r\}$ not spherical. Otherwise A is *small*. We say that a component A of $S \setminus (\{s,t\} \cup \{s,t\}^\perp)$ is *exposed* if there is $p \in A$ such that $\{s,t,p\}$ is exposed.

The goal of this section is the following.

Proposition 8.5. *Under the assumptions of Theorem 8.1, if there is an exposed component of $S \setminus (\{s,t\} \cup \{s,t\}^\perp)$, then $\{s,t\}$ is consistent.*

In the proof we will need the following terminology and lemmas.

Definition 8.6. Let $J \subset S$ be irreducible spherical. By $\mathcal{W}_J$ we denote the union of $\mathcal{W}_j$ over all reflections $j \in \langle J \rangle$. The components of $\mathbb{A}_{\mathrm{amb}} \setminus \mathcal{W}_J$ are called *sectors for J*. The two sectors containing the geometric fundamental domains for J are called *geometric*.

Lemma 8.7. *Under the hypotheses of Theorem 8.1, let $J \subset S$ be exposed with $|J| = 3$. Suppose that we have simple markings μ_1, μ_2 with supports contained in J, and cores s_1, s_2. Then the pair $\Phi^{\mu_1}_{s_1}, \Phi^{\mu_2}_{s_2}$ is geometric.*

Proof. **Case 1. The unique component of $S \setminus (J \cup J^\perp)$ has no element adjacent to an element of J.** Since $S \setminus (J \cup J^\perp)$ is a single component, all the walls $\mathcal{W}_r$ for $r \in S \setminus (J \cup J^\perp)$ lie in $\mathbb{A}_{\mathrm{amb}}$ in a single sector Λ for J. If Λ is a geometric sector, then the pair $\Phi^{\mu_1}_{s_1}, \Phi^{\mu_2}_{s_2}$ is geometric, since by Lemma 3.3 each $\Phi^{\mu_i}_{s_i}$ is the halfspace for s_i containing Λ.

If Λ is not geometric, suppose that it is of form $w\Lambda_0$ for Λ_0 a geometric sector for J and $w \in \langle J \rangle$. Let $w = t_0 \cdots t_{n-1}$ with $t_i \in J$ and minimal n. Consider the following Coxeter generating sets S_i with $S_0 = S$ and elementary twists τ_i with $S_{i+1} = \tau_i(S_i)$. Namely, we set $J_i = \{t_i\}, A_i = J \setminus (t_i \cup t_i^\perp)$ (which is a component of $S \setminus (t_i \cup t_i^\perp)$), and $B_i = S \setminus (J_i \cup J_i^\perp \cup A_i)$. The elementary twist τ_i conjugates B_i by

t_i and fixes the other elements of S_i. Let $\tau = \tau_{n-1} \circ \cdots \circ \tau_0$, so that $S_n = \tau(S)$.

We now argue, similarly as in [9, §7.2], that $\mathcal{K}_1(\tau(S)) = \mathcal{K}_1(S)$ and $\mathcal{K}_2(\tau(S)) < \mathcal{K}_2(S)$. A maximal spherical subset L of S either contains J, and is then called *idle* or intersects $S \setminus (J \cup J^\perp)$. Thus all D_I with $I \subset S$ maximal spherical that are not idle, are contained in Λ. For L idle we have $D_{L_\tau} = D_L$. In particular, for all maximal spherical $I \subset S$ we have $C_{I_\tau} = w^{-1}C_I$, implying $\mathcal{K}_1(\tau(S)) = \mathcal{K}_1(S)$.

To compare $\mathcal{K}_2(\tau(S))$ and $\mathcal{K}_2(S)$, first note that if both L and I are maximal spherical and idle (resp. not idle), then by Remark 7.10(ii) we have $d(E_{L,I}, E_{I,L}) = d(E_{L_\tau,I_\tau}, E_{I_\tau,L_\tau})$. Now suppose that L is idle and I is not idle. Then by Remark 7.10(iii) we have $E_{I_\tau,L_\tau} = w^{-1}E_{I,L}$. Furthermore, $J \subset L$ is maximal irreducible, and so the decomposition of L into maximal irreducible subsets has the form $L = L_1 \sqcup \cdots \sqcup L_k$ with $L_1 = J$. Since J is exposed, by Lemma 7.9 we have $E^1_{L_\tau,I_\tau} = D(L_\tau) = D(L) = E^1_{L,I}$. For $i \neq 1$ we have $L_i \subseteq J^\perp$ and so by Remark 7.10(ii) we have $E^i_{L_\tau,I_\tau} = w^{-1}E^i_{L,I}$, which equals $E^i_{L,I}$ since w commutes with L_i. Thus $E_{L_\tau,I_\tau} = E_{L,I}$. Let $\beta = \beta'\beta''$ be a minimal gallery from a chamber in $E_{I,L}$ to a chamber $x \in E_{L,I}$, where $\beta' \subset \Lambda$ and β'' is contained in the J-residue containing x. (Such a gallery exists by [11, Thm 2.9].) Then $w^{-1}\beta'$ connects a chamber in E_{I_τ,L_τ} to a chamber in E_{L_τ,I_τ}, proving $\mathcal{K}_2(\tau(S)) < \mathcal{K}_2(S)$.

Case 2. The unique component of $S \setminus (J \cup J^\perp)$ has an element r' adjacent to an element $t \in J$. Let Λ be a sector for J with $\Lambda \cup t\Lambda$ containing $\mathcal{W}_{r'}$. If Λ or $t\Lambda$ is a geometric sector, then the pair $\Phi^{\mu_1}_{s_1}, \Phi^{\mu_2}_{s_2}$ is geometric as in Case 1. Suppose now that neither Λ nor $t\Lambda$ is geometric. Let $w \in \langle J \rangle$ be of minimal word length with $w\Lambda_0 = \Lambda$ or $t\Lambda$ and Λ_0 a geometric sector for J. Say we have $w\Lambda_0 = \Lambda$.

Since $\mathcal{W}_{r'}$ intersects $\mathcal{W}_t$, there is $t' \in J$ satisfying $wt'w^{-1} = t$. By [5, Prop 5.5] there is $n \geq 0$, elements $t_0 = t, \ldots, t_n = t' \in J$ and $s_0, \ldots s_{n-1} \in J$ such that for each $i = 0, \ldots, n-1$ we have $s_i \neq t_i$, and for

$$w_i = \begin{cases} s_i, \text{ if } s_i \text{ and } t_i \text{ commute} \\ \text{the longest word in } \langle s_i, t_i \rangle, \text{ otherwise} \end{cases}$$

we have

- $w_i t_{i+1} w_i^{-1} = t_i$, and
- $w = w_0 \cdots w_{n-1}$ or $w = tw_0 \cdots w_{n-1}$.

We focus on the case where $w = w_0 \cdots w_{n-1}$. Construct the following Coxeter generating sets $S_i \supset J$ with $S_0 = S$ and elementary twists τ_i with $S_{i+1} = \tau_i(S_i)$. We will also get inductively that the unique component of $S_i \setminus (J \cup J^\perp)$ does not have an element adjacent to an element of J distinct from t_i.

If s_i and t_i commute, we set $J_i = \{s_i\}, A_i = J \setminus (s_i \cup s_i^\perp), B_i = S \setminus (J_i \cup J_i^\perp \cup A_i)$. The elementary twist τ_i conjugates B_i by $s_i = w_i$ and fixes the other elements of S_i. Note that A_i is a component of $S \setminus (J_i \cup J_i^\perp)$ since $t_i \in s_i^\perp$. If s_i and t_i do not commute, we set $J_i = \{s_i, t_i\}$ and keep the same formulas for A_i, B_i. Then the elementary twist τ_i conjugates B_i by w_i and fixes the other elements of S_i.

We argue analogously as in Case 1 to obtain $\mathcal{K}_1(S_n) = \mathcal{K}_1(S_0)$. For L idle and I not idle we also obtain analogously $E_{I_\tau, L_\tau} = w^{-1} E_{I,L}, E_{L_\tau, I_\tau} = E_{L,I}$. Let $\beta = \beta' \beta''$ be a minimal gallery from a chamber in $E_{I,L}$ to a chamber $x \in E_{L,I}$, where $\beta' \subset \Lambda$ or $t\Lambda$ and β'' is contained in the J-residue containing x. Then $w^{-1}\beta'$ connects a chamber in E_{I_τ, L_τ} to $x \in E_{L_\tau, I_\tau}$ for $\beta' \subset \Lambda$ or to a chamber adjacent to x for $\beta' \subset t\Lambda$. Moreover, in the latter case β'' has length at least 2 by the minimality assumption on w. This shows $d(E_{L_\tau, I_\tau}, E_{I_\tau, L_\tau}) < d(E_{L,I}, E_{I,L})$ and hence $\mathcal{K}_2(S_n) < \mathcal{K}_2(S_0)$.

If $w = tw_0 \cdots w_{n-1}$, then we start with an additional elementary twist in $\langle t \rangle$ and we continue analogously. □

Lemma 8.8. *Under the assumptions of Theorem 8.1, if $S \setminus (\{s,t\} \cup \{s,t\}^\perp)$ is a single component that is small, then it is self-compatible.*

Proof. Let $A = S \setminus (\{s,t\} \cup \{s,t\}^\perp)$. Let μ be a simple marking with support J containing s, t guaranteed by Remark 3.5. Without loss of generality, discarding part of J, we can assume that the Coxeter–Dynkin diagram of J is a path starting with s. Thus $\mu = ((s, tpw), r))$ where $p \in A$ and s commutes with pw. Since A is the unique component of $S \setminus (\{s,t\} \cup \{s,t\}^\perp)$, we have that r is adjacent to s. Thus $\mathcal{W} = pw\mathcal{W}_r$ intersects $\mathcal{W}_s$ and so by Corollary 2.7 there is a geometric fundamental domain F for $\{s,t\}$ that is contained in both $\Phi(t\mathcal{W}_s, \mathcal{W})$, and $\Phi(\mathcal{W}_t, \mathcal{W}) = \Phi(\mathcal{W}_t, s\mathcal{W})$, which is $\Phi_t^{\mu'}$ for $\mu' = ((t, spw), r)$. Thus by Lemma 3.3 the pair $\Phi_s^\mu, \Phi_t^{\mu'}$ is geometric, as desired. □

Proof of Proposition 8.5. Let $J = \{s, t, p\}$ be exposed and let A be the component of $S \setminus (\{s,t\} \cup \{s,t\}^\perp)$ containing p. Consider first the case where $A = S \setminus (\{s,t\} \cup \{s,t\}^\perp)$. If A is small, then it suffices to apply Lemma 8.8. If A is big, then let $r \in A$ with $\{s,t,r\}$ not spherical. By Lemma 8.7, the halfspaces for s,t determined by the markings $((s,t),r),((t,s),r)$ are geometric and hence A is self-compatible.

It remains to consider the case where $A \subsetneq S \setminus (\{s,t\} \cup \{s,t\}^{\perp})$. Let B be a component of $S \setminus (\{s,t\} \cup \{s,t\}^{\perp})$ distinct from A and let $r \in B$. Since the unique component of $S \setminus (J \cup J^{\perp})$ has no element adjacent to one of s,t, we have that $\{s,t,r\}$ is not spherical. By Lemma 8.7, the halfspaces for s,t determined by the markings $((s,t),r),((s,tp),r),((t,s),r),((t,sp),r)$ are geometric and hence B is compatible with A and they are both self-compatible. □

Corollary 8.9. *Under the assumptions of Theorem 8.1, each small component of $S \setminus (\{s,t\} \cup \{s,t\}^{\perp})$ is self-compatible.*

Proof. Let A be a small component of $S \setminus (\{s,t\} \cup \{s,t\}^{\perp})$. If $A = S \setminus (\{s,t\} \cup \{s,t\}^{\perp})$, then it suffices to apply Lemma 8.8. Otherwise, let r be an element of a component B of $S \setminus (\{s,t\} \cup \{s,t\}^{\perp})$ distinct from A. Let $p \in A$, suppose without loss of generality $m_{sp} = 2$, and set $\mu = ((s,tp),r)$. If A is exposed, then we can apply Proposition 8.5. Otherwise, by Lemma 7.3 we have that $\{t,p\} \subset \{s,t,p\}$ is good with respect to r. Thus by the 3-rigidity of S and Proposition 3.11, we have $\Phi(\mathcal{W}_t, p\mathcal{W}_r) = \Phi(\mathcal{W}_t, ps\mathcal{W}_r)$. By Lemma 2.6 there is a geometric fundamental domain F for $\{s,t\}$ that is contained in both $\Phi(t\mathcal{W}_s, p\mathcal{W}_r)$ and $\Phi(\mathcal{W}_t, p\mathcal{W}_r)$, which is $\Phi_t^{\mu'}$ for $\mu' = ((t,sp),r)$. Thus by Lemma 3.3 the pair $\Phi_s^{\mu}, \Phi_t^{\mu'}$ is geometric, and so A is self-compatible. □

9. Big components

The content of this section was designed together with Pierre-Emmanuel Caprace.

Lemma 9.1. *Under the assumptions of Theorem 8.1, if $m_{st} = 3$, then $\{s,t\}$ is consistent.*

Proof. By Proposition 8.5 we can assume that no component of $S \setminus (\{s,t\} \cup \{s,t\}^{\perp})$ is exposed.

For a big component B of $S \setminus (\{s,t\} \cup \{s,t\}^{\perp})$ and $r \in B$ with $\{s,t,r\}$ not spherical, let F be the geometric fundamental domain for $\{s,t\}$ lying in $\Phi(s\mathcal{W}_t, \mathcal{W}_r) = \Phi(t\mathcal{W}_s, \mathcal{W}_r)$. By Lemma 3.3 we have that F lies in $\Phi_{B,s} \cap \Phi_{B,t}$, which thus form a geometric pair. Hence B is self-compatible and by Corollary 8.9 it remains to prove that all $\Phi_{B,s}$ coincide (including small B).

Otherwise, let A be the union of all components A_i with one $\Phi_{A_i,s}$, and B the union of components B_i with the other $\Phi_{B_i,s}$. Let τ be the elementary twist that sends each element $b \in B$ to $w_{st} b w_{st}^{-1}$, where $w_{st} = tst$, and fixes the other elements of S. For a contradiction, we will first prove that if there are incompatible big components, then $\mathcal{K}_1(\tau(S)) < \mathcal{K}_1(S)$. For maximal spherical $L \subset S$ we say that L is *twisted* if it contains an element of B. We then have $C_{L_\tau} = w_{st} C_L$. If I is maximal spherical and not twisted, then we have $C_{I_\tau} = C_I$. Consequently $d(C_{L_\tau}, C_{I_\tau})$ might vary from $d(C_L, C_I)$ only if, say, L is twisted and I is not twisted, and $\{s,t\} \not\subseteq L, I$. Such L, I exist exactly if there are incompatible big components. Then C_L, C_I lie in the opposite halfspaces of $t\mathcal{W}_s = \mathcal{W}_{w_{st}}$, and consequently $d(C_{L_\tau}, C_{I_\tau}) < d(C_L, C_I)$, as desired.

If all big components are compatible, we have $\mathcal{K}_1(\tau(S)) = \mathcal{K}_1(S)$, and we need to analyse the effect of τ on $\mathcal{K}_2$. Consider maximal spherical subsets $L, I \subset S$. If both L, I are twisted, or both are not twisted, then by Remark 7.10(ii) we have $d(E_{L_\tau, I_\tau}, E_{I_\tau, L_\tau}) = d(E_{L,I}, E_{I,L})$. Suppose now that L is twisted and intersects $B_i \subseteq B$ and I is not twisted. If $I \subseteq \{s,t\} \cup \{s,t\}^\perp$, the same equality holds, so we can assume $I \not\subseteq \{s,t\} \cup \{s,t\}^\perp$.

We claim $E_{L,I} \subset \Phi_{B,s}$. Indeed, let $L_1 \subseteq L$ be maximal irreducible containing $\{s,t\}$, and let $u \in L_1$ with $\{s,t,u\}$ irreducible, so that $u \in B_i$. Let $r \in I \setminus (\{s,t\} \cup \{s,t\}^\perp)$. Since B is self-compatible, after possibly interchanging s with t, we can assume that u, s, t are consecutive in the Coxeter–Dynkin diagram of L_1. Then s is not a leaf in the Coxeter–Dynkin diagram of L_1 and by Remark 7.6 we have $E_{L,I} \subset \Phi_s^\mu$ for $\mu = ((s, L_1), r)$. Since $K_{s,t}^\mu \subset B_i$, the claim follows. The proof of the lemma splits now into two cases.

Case 1. ***I*** **contains** $\{s,t\}$**.** Interchanging the roles of L and I, from the claim we have $E_{I,L} \subset \Phi_{A,s}$. Consequently, $E_{L,I}$ and $E_{I,L}$ lie in the opposite geometric fundamental domains for $\{s,t\}$. In particular, they lie in the opposite halfspaces of $t\mathcal{W}_s = \mathcal{W}_{w_{st}}$. By Remark 7.10(iii), we have $E_{L_\tau, I_\tau} = w_{st} E_{L,I}$ and $E_{I_\tau, L_\tau} = E_{I,L}$. Thus $d(E_{L_\tau, I_\tau}, E_{I_\tau, L_\tau}) < d(E_{L,I}, E_{I,L})$.

Case 2. ***I*** **contains an element** r **not adjacent to** s **or** t**.** By the claim and Lemma 3.3 we have $E_{L,I} \subset t\Phi_{B,s}$. Consider the marking $\mu = ((s,t), r)$. Since $K_{s,t}^\mu \subseteq A$, we have $\mathcal{W}_r \subset t\Phi_{A,s}$, and so $E_{I,L} \subset t\Phi_{A,s}$. Furthermore, we have $E_{I_\tau, L_\tau} = E_{I,L}$ as in Case 1. To finish as in Case 1, it remains to prove $E_{L_\tau, I_\tau} = w_{st} E_{L,I}$.

To this end, let $u \in L_1$ as in the proof of the claim. Note that in the Coxeter–Dynkin diagram of $(L_1)_\tau$ we have consecutive vertices s, t and $\tau(u)$. We have that $(L_1)_\tau$ is not exposed by Lemma 7.9. By Lemma 7.3, $u \in L_1, \{u, s\} \subset L_1$ are good with respect to r and $\tau(u) \in (L_1)_\tau, \{\tau(u), t\} \subset (L_1)_\tau$ are good with respect to $\tau(r) = r$. Consequently it suffices to prove $\Delta^{(\tau(u),t),r} = w_{st}\Delta^{(u,s),r}$. This follows from the fact that the reflections sts and sus commute, hence each of the halfspaces of $s\mathcal{W}_u$ is preserved by w_{st}, and thus $\Phi(tw_{st}\mathcal{W}_u, \mathcal{W}_r) = \Phi(w_{st}s\mathcal{W}_u, \mathcal{W}_r) = w_{st}\Phi(s\mathcal{W}_u, \mathcal{W}_r)$. □

For $m_{st} \neq 3$, we will need the following measure of consistency.

Definition 9.2. Let $\{s, t\} \subset S$ be irreducible spherical and let V be one of the two geometric fundamental domains for $\{s, t\}$. We define the *consistency* $\mathcal{C}_V(s, t) = \mathcal{C}_V(t, s)$ as the number of maximal spherical $L \subset S$ with C_L intersecting $sV \cup V \cup tV$. We say that inconsistent $\{s, t\}$ is *peripheral* if $\mathcal{C}_V(s, t)$ is maximal among all inconsistent $\{s, t\} \subset S$ and both V.

Obviously, if S does not have consistent doubles, then there is peripheral $\{s, t\}$. The following remark describes the role of the union $sV \cup V \cup tV$.

Remark 9.3. Let $\{s, t\} \subset S$ be irreducible spherical and let V be a geometric fundamental domain for $\{s, t\}$. Suppose that we have $r \in S$ with $\mathcal{W}_r \subset sV \cup V \cup tV$. Then $\mu = ((s, t), r), \mu' = ((t, s), r)$ are markings, and we have $V \subset \Phi^\mu_s, \Phi^{\mu'}_t$. Consequently, the component of $S \setminus (\{s, t\} \cup \{s, t\}^\perp)$ containing r is self-compatible. Conversely, if $\{s, t, r\}$ is not spherical, and the component B of $S \setminus (\{s, t\} \cup \{s, t\}^\perp)$ containing r is self-compatible, then $\mathcal{W}_r \subset sV \cup V \cup tV$ for a geometric fundamental domain V for $\{s, t\}$. Furthermore, V depends only on B, not on r.

Proposition 9.4. *Under the assumptions of Theorem 8.1, if $\{s, t\}$ is peripheral, then big components of $S \setminus (\{s, t\} \cup \{s, t\}^\perp)$ are compatible. Moreover, if there is a big component that is not self-compatible, then all $\mathcal{W}_r$ with $\{s, t, r\}$ not spherical are contained in a single sector for $\{s, t\}$.*

In the proof we will need the following key notion.

Definition 9.5. Let $\{s, t\} \subset S$ be irreducible spherical. A *folding* is a map $f\colon \langle s, t\rangle \to \{s, \mathrm{Id}, t\}$ such that for each $w \in \langle s, t\rangle$ we have

- $f(ws) = f(w)$ or $f(ws) = f(w)s$, and

- $f(wt) = f(w)$ or $f(wt) = f(w)t$.

In other words, f is a simplicial type-preserving map on the Cayley graph of $\langle s,t\rangle$.

Example 9.6. Let $m_{st} = 3$ and $w_{st} = tst$. Let $f\colon \langle s,t\rangle \to \{s, \mathrm{Id}, t\}$ be the map whose restriction to $\{s, \mathrm{Id}, t\}$ is the identity map and whose restriction to $\{w_{st}s, w_{st}, w_{st}t\}$ is the reflection w_{st}. It is easy to see that f is a folding.

Lemma 9.7. *Let $f\colon \langle s,t\rangle \to \{s, \mathrm{Id}, t\}$ be a folding. Let V be a geometric fundamental domain for $\{s,t\}$. Let $\tilde{f}\colon \mathbb{A}^{(0)}_{\mathrm{amb}} \to sV \cup V \cup tV$ be the map sending each wV to $f(w)V$ via $f(w)w^{-1}$, where $w \in \langle s,t\rangle$. Then $\tilde{f}$ induces a simplicial map on $\mathbb{A}^{(1)}_{\mathrm{amb}}$. Moreover, for $x \in wV, y \in w'V$ we have $d(\tilde{f}(x), \tilde{f}(y)) = d(x,y)$ if and only if the restriction of f to the vertices of some path π from w to w' in the Cayley graph of $\langle s,t\rangle$ is injective.*

Proof. To prove the first assertion, consider adjacent chambers g, gp of $\mathbb{A}^{(1)}_{\mathrm{amb}}$, where $g \in W, p \in S'$. If g, gp belong to the same wV, then $\tilde{f}(g) = f(w)w^{-1}g$ and $\tilde{f}(gp) = f(w)w^{-1}gp$ are obviously adjacent. If g, gp belong to distinct translates of V, then we have, say, $g \in wV, gp \in wsV$. In that case we also have $wsw^{-1}g = gp$ and so g and gp are in the same orbit of the action of $\langle s,t\rangle$ on $\mathbb{A}^{(0)}_{\mathrm{amb}}$. Consequently, if $f(ws) = f(w)$, then since $f(w)V$ intersects each $\langle s,t\rangle$-orbit in one chamber, we have $\tilde{f}(g) = \tilde{f}(gp)$. On the other hand, if $f(ws) = f(w)s$, then $f(ws)(ws)^{-1} = f(w)w^{-1}$, and hence $\tilde{f}(g) = f(w)w^{-1}g$ and $\tilde{f}(gp) = f(ws)(ws)^{-1}gp$ are adjacent.

For the second assertion, let γ be a minimal gallery from x to y, let $wV, \ldots, w'V$ be the distinct consecutive translates of V traversed by γ and let $\pi = w \cdots w'$ be the corresponding path in the Cayley graph of $\langle s,t\rangle$. If $d(\tilde{f}(x), \tilde{f}(y)) = d(x,y)$, then in view of the previous paragraph the consecutive vertices of the path $f(\pi)$ are distinct, as desired. Conversely, if $d(\tilde{f}(x), \tilde{f}(y)) < d(x,y)$, then a pair of consecutive vertices of $f(\pi)$ coincides. Since γ was minimal, the length of of π is at most m_{st}, and consequently the length of the second path π' from w to w' in the Cayley graph of $\langle s,t\rangle$ is $\geq m_{st} > 2$. Since f takes only values s, Id, t, the restriction of f to π' is also not injective. $\square$

Proof of Proposition 9.4. Let Λ_0 be the geometric sector containing V from Definition 9.2. First consider the case where m_{st} is odd, so the longest word w_{st} in $\langle s,t\rangle$ is a reflection. This case will not require the peripherality hypothesis.

We begin with focusing entirely on the case where a component B of $S\setminus(\{s,t\}\cup\{s,t\}^\perp)$ is not self-compatible. Observe that if $p\in B$ is adjacent to s, then it is also adjacent to t (and vice versa): indeed, otherwise the pair of halfspaces determined by markings $((s,t),p),((t,s),p)$ would be geometric by Corollary 2.7 (and Lemma 3.3).

We now claim that if $r\in B$ is not adjacent to s, then $s\in\{s,t\}$ is not good with respect to r. Indeed, otherwise let $\mu_1=((s,\emptyset),r),\mu_2=((s,t),r)$. If $\Phi_s^{\mu_1}=\Phi_s^{\mu_2}$, then Lemma 2.6 (and Lemma 3.3) contradict the assumption that B is not self-compatible. Thus by Proposition 3.13, there is a vertex $p\neq t$ on a minimal length path from r to t in the defining graph of S outside $s\cup s^\perp$, with p adjacent to s and $\{s,p\}$ inconsistent. By Lemma 9.1, we have $m_{st},m_{sp}>3$, so from FC it follows that p is not adjacent to t. This contradicts the observation above, and justifies the claim.

Analogously $t\in\{s,t\}$ is not good with respect to r. Consequently, by Lemma 4.2, the elements of B are adjacent neither to s nor t. Thus B is also a component of $S\setminus(s\cup s^\perp)$ and a component of $S\setminus(t\cup t^\perp)$. Furthermore, all $\mathcal{W}_r$ for $r\in B$ lie in a single sector $w_B\Lambda_0$ for some $w_B\in\langle s,t\rangle$ and by Remark 9.3 we have $w_B\neq s,\mathrm{Id},t,w_{st}s,w_{st},w_{st}t$. Consequently, for L maximal spherical intersecting B, we have $C_L\subset w_BV$.

Let j be the first letter in the minimal length word representing w_B. We set τ_B to be the composition of elementary twists conjugating B by the letter s or t in the order in which they appear as consecutive letters in w_Bj. As a result, for L maximal spherical intersecting B, we have $C_{L_{\tau_B}}=jw_B^{-1}C_L\subset jV$. (Here by L_{τ_B} with τ_B a composition $\tau_n\circ\cdots\circ\tau_1$ of elementary twists and $\sigma=\tau_{n-1}\circ\cdots\circ\tau_1$ we mean, inductively, $(L_\sigma)_{\tau_n}$.)

Consider now a self-compatible component B of $S\setminus(\{s,t\}\cup\{s,t\}^\perp)$. By Remark 9.3 either

(i) for each L maximal spherical intersecting B, we have that C_L intersects $sV\cup V\cup tV$, or

(ii) each such C_L intersects $w_{st}sV\cup w_{st}V\cup w_{st}tV$.

If B is big, then there is L for which we can replace the word 'intersects' by 'is contained in' in the preceding statement. In case (ii), we perform an elementary twist τ_B with $J=\{s,t\}$, which sends each $p\in B$ to $w_{st}pw_{st}^{-1}$. As a result, for L maximal spherical intersecting B, we have $C_{L_{\tau_B}}=w_{st}C_L$, which intersects $sV\cup V\cup tV$. Let τ be the composition of all τ_B above.

To summarise, consider the folding $f\colon \langle s,t\rangle \to \{s,\mathrm{Id},t\}$ defined by:

- $f(w) = w$ for $w = s, \mathrm{Id}, t$,
- $f(w) = w_{st}w$ for $w = w_{st}s, w_{st}, w_{st}t$,
- $f(w) = j$ for other w, where j is the first letter in the minimal length word representing w.

By Lemma 9.7, for $x \in wV, y \in w'V$ we have $d(\tilde{f}(x), \tilde{f}(y)) \leq d(x,y)$ with equality if and only if $w = w'$ or both w, w' lie in $\{s, \mathrm{Id}, t\}$ or they both lie in $\{w_{st}s, w_{st}, w_{st}t\}$. Furthermore, for each L maximal spherical we have $C_{L_\tau} \supseteq \tilde{f}(C_L)$ (where the inclusion is strict exactly when $L \supseteq \{s,t\}$). Thus we get $\mathcal{K}_1(\tau(S)) \leq \mathcal{K}_1(S)$. Moreover, we have strict inequality as soon as there are two incompatible big components or a big component B that is not self-compatible, and $\mathcal{W}_r \not\subset w_B\Lambda_0$ with $\{s,t,r\}$ not spherical.

Secondly, consider the case where m_{st} is even. We treat components B that are not self-compatible exactly as before. Suppose now that B is a self-compatible component of $S \setminus (\{s,t\} \cup \{s,t\}^\perp)$ as in case (ii). A *refined component* of B is a component of $B \setminus (s^\perp \cup t^\perp)$.

Let $L \subset S$ be maximal spherical intersecting B. Suppose that C_L does not intersect $w_{st}V$. Then it is contained in one of $sw_{st}V, tw_{st}V$, say $sw_{st}V$. By the maximality of L, there is $r \in L$ that is not adjacent to s and so $\mathcal{W}_r \subset sw_{st}\Lambda_0$. In particular r is not adjacent to t and so r lies in a refined component B' of B. We claim that $s \in \{s,t\}$ is not good with respect to r. Indeed, otherwise as before let $\mu_1 = ((s,\emptyset),r), \mu_2 = ((s,t),r)$ so that $\Phi_s^{\mu_1} \neq \Phi_s^{\mu_2}$. Thus by Proposition 3.13, there is a vertex $p \neq t$ on a minimal length path ω from r to t in the defining graph of S outside $s \cup s^\perp$, with p adjacent to s and $\{s,p\}$ inconsistent. Note that all the vertices of ω distinct from t lie in B' except for possibly the vertex preceding t that might lie in $B \cap t^\perp$, which is excluded below.

By Lemma 9.1, we have again $m_{st}, m_{sp} > 3$, so from FC it follows that p is not adjacent to t. Then $p\mathcal{W}_s, s\mathcal{W}_p$ are disjoint from $s\mathcal{W}_t, t\mathcal{W}_s$. Moreover, since $p \in B'$, we have $\mathcal{W}_p \subset w_{st}s\Lambda_0 \cup w_{st}\Lambda_0 \cup w_{st}t\Lambda_0$, and so $p\mathcal{W}_s, s\mathcal{W}_p \subset w_{st}s\Lambda_0 \cup w_{st}\Lambda_0 \cup w_{st}t\Lambda_0$. Consequently, there is a geometric fundamental domain V' for $\{s,p\}$ that contains V and intersects C_L for some L maximal spherical containing s,p. Since C_L is disjoint from $sV \cup V \cup tV$, we have $\mathcal{C}_{V'}(s,p) > \mathcal{C}_V(s,t)$, contradicting the hypothesis that $\{s,t\}$ is peripheral. This justifies the claim.

By the claim, there is no element in B' adjacent to t or to $B \cap t^\perp$. Thus B' is a component of $S \setminus (s \cup s^\perp)$.

Furthermore, we will prove that for each $L' \subset S$ maximal spherical intersecting B' we have that $C_{L'}$ intersects $sw_{st}V$. Otherwise, for $r' \in L'$ that is not adjacent to s we have $r' \in B'$ and $\mathcal{W}_{r'} \subset w_{st}\Lambda_0$. Consequently, for $\mu_1 = ((s, \emptyset), r), \mu_2 = ((s, \emptyset), r')$ we have $\Phi_s^{\mu_1} \neq \Phi_s^{\mu_2}$. Then by Proposition 3.13, there is an element $p \in B'$ with p adjacent to s and $\{s, p\}$ inconsistent. As before, this contradicts the hypothesis that $\{s, t\}$ is peripheral.

Consequently, for each self-compatible component B of $S \setminus (\{s, t\} \cup \{s, t\}^\perp)$ as in case (ii), and its refined component B', there is at least one element $w_{B'}$ among sw_{st}, w_{st}, tw_{st} with $C_L \subset w_{B'}V$ for all L maximal spherical intersecting B'.

We perform now a sequence of elementary twists as follows. We treat the components that are not self-compatible as before. For a self-compatible component B of $S\setminus(\{s, t\}\cup\{s, t\}^\perp)$ as in case (ii) we do the following. First, for each refined component $B' \subset B$ satisfying $w_{B'} = sw_{st}$ (resp. $w_{B'} = tw_{st}$) we apply the elementary twist with $J = \{s\}$ (resp. $J = \{t\}$) that conjugates all the elements of B' by s (resp. t) and fixes all the other elements of S. Afterwards, we apply the elementary twist with $J = \{s, t\}$ that conjugates the entire image of B under the preceding elementary twists by w_{st}. Let τ be the composition of all these elementary twists. Then $C_{L_\tau} \supseteq \tilde{f}(C_L)$ with $f \colon \langle s, t\rangle \to \{s, \mathrm{Id}, t\}$ the folding defined by:

- $f(w) = w$ for $w = s, \mathrm{Id}, t$,
- $f(w) = \mathrm{Id}$ for $w = w_{st}s, w_{st}, w_{st}t$,
- $f(w) = j$ for other w, where j is the first letter in the minimal length word representing w.

We can thus apply Lemma 9.7 as before. □

10. Small components

Proposition 10.1. *Under the assumptions of Theorem 8.1, doubles are consistent.*

Proof. Otherwise, let $\{s, t\} \subset S$ be peripheral and let V be as in Definition 9.2. By Proposition 8.5, Corollary 8.9, and Proposition 9.4, we can assume that none of the components of $S \setminus (\{s, t\} \cup \{s, t\}^\perp)$ are exposed, that all small components are self-compatible, and that big components are compatible. Thus it remains to prove that each small component is compatible with any other component and that all big components are self-compatible. Divide the components into two families $\{A_i\}$ and $\{B_i\}$ such that all $\Phi_{A_i,s}$ coincide and are distinct from all $\Phi_{B_i,s}$, which also coincide. Let A (resp. B) be the union of all A_i (resp. B_i) and suppose that all big components are in B. If there are self-compatible big components, then this implies $V \subset \Phi_{B_i,s}$. If there are no self-compatible big components, then, after possibly switching V, we can also assume $V \subset \Phi_{B_i,s}$.

Let w_{st} be the longest word in $\langle s,t\rangle$ and for each A_i let τ_{A_i} be the elementary twist that sends each element $a \in A_i$ to $w_{st}aw_{st}^{-1}$, and fixes the other elements of S. For a big component B_i that is not self-compatible, we define w_{B_i}, τ_{B_i} as in the proof of Proposition 9.4. Let τ be the composition of all these τ_{A_i} and τ_{B_i}. Let $L \subset S$ be a maximal spherical subset. L is *twisted* if it contains an element of A. In that case $s,t \in L$. L is *rotated* if it contains an element of B_i that is not self-compatible. If L is neither twisted nor rotated, it is *idle*.

Note that if we have rotated subsets, then by Proposition 9.4 we have no idle subsets not containing $\{s,t\}$, and that all w_{B_i} coincide. Consequently $\mathcal{K}_1(S) = \mathcal{K}_1(\tau(S))$. We will now prove $\mathcal{K}_2(S) < \mathcal{K}_2(\tau(S))$.

Consider maximal spherical subsets $L, I \subset S$. If both L, I are twisted, both are rotated, or both are idle, by Remark 7.10(ii) we have $d(E_{L_\tau,I_\tau}, E_{I_\tau,L_\tau}) = d(E_{L,I}, E_{I,L})$. Suppose for a moment that L is twisted and I is rotated or idle. If $I \subseteq \{s,t\} \cup \{s,t\}^\perp$, the same equality holds, so we can assume $I \nsubseteq \{s,t\} \cup \{s,t\}^\perp$. We then have $E_{L,I} \subset \Phi_{A,s}$, word for word as in the proof of the claim in Lemma 9.1, and so $E_{L,I} \subset w_{st}V$. Analogously, if L is idle and contains $\{s,t\}$, and I is rotated or twisted, we have $E_{L,I} \subset V$, except in the 'special' case where $L \subseteq \{s,t\} \cup \{s,t\}^\perp$ and so, say, $L_1 = \{s,t\}$ is exposed and $E^1_{L,I} = D_L$. Furthermore, for L idle not containing $\{s,t\}$ we have $C_L \subset sV \cup V \cup tV$, and for $L \subset B_i$ rotated we have $C_L \subset w_{B_i}V$. This accounts for all possible positions of $E_{L,I}$. We now need to analyse the effect of τ on all $E_{L,I}$. Let f be the folding from the proof of Proposition 9.4. We will prove that except in the 'special' case where $L_1 = \{s,t\}$, we have

$$E_{L_\tau,I_\tau} = \tilde{f}(E_{L,I}). \qquad (*)$$

Case 1. ***I* is twisted or idle containing $\{s,t\}$.** Then $(*)$ follows from Remark 7.10.(iii).

Case 2. ***I* is rotated or idle not containing $\{s,t\}$.** In that case L contains $\{s,t\}$. Suppose first that I is idle not containing $\{s,t\}$, and so L is twisted. Then $(*)$ amounts to $E_{L_\tau,I_\tau} = w_{st}E_{L,I}$. Let $r \in I \backslash (\{s,t\} \cup \{s,t\}^\perp)$. We have $m_{st} = 4$ or 5. If $m_{st} = 5$, we choose $u \in L_1$ as in

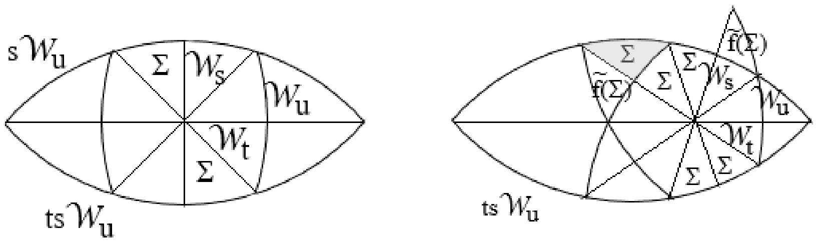

FIGURE 2. On the right both possible positions of $\tilde{f}(\Sigma)$ for the shaded sector of Σ

the proof of the claim in Lemma 9.1 (possibly interchanging s with t). Each of the halfspaces of $ts\mathcal{W}_u$ is preserved by w_{st}, since the reflections $tstst$ and $tsust$ commute, and so $w_{st}\Phi(ts\mathcal{W}_u, \mathcal{W}_r) = \Phi(w_{st}ts\mathcal{W}_u, \mathcal{W}_r)$. This implies $E_{L_\tau, I_\tau} = w_{st}E_{L,I}$ as in Case 2 of the proof of Lemma 9.1. If $m_{st} = 4$, we have $(stst)ts\mathcal{W}_u = st\mathcal{W}_u = s\mathcal{W}_u$. Thus w_{st} exchanges the halfspaces of $ts\mathcal{W}_u$ and $s\mathcal{W}_u$, and in fact acts on them as t does, so in particular $\Phi(w_{st}s\mathcal{W}_u, \mathcal{W}_r) = \Phi(ts\mathcal{W}_u, \mathcal{W}_r)$. Since $\{u, s\} \subset L_1$ is good with respect to r, and S is 3-rigid, by Proposition 3.11 we have $\Phi(ts\mathcal{W}_u, \mathcal{W}_r) = w_{st}\Phi(s\mathcal{W}_u, \mathcal{W}_r)$, and $(*)$ follows.

It remains to consider the case where I is rotated. Let $r \in I$ and suppose first $m_{st} = 4$. Let $K = \Phi(s\mathcal{W}_u, \mathcal{W}_r) \cap t\Phi(s\mathcal{W}_u, \mathcal{W}_r)$, which contains $\mathcal{W}_r$ as in the preceding paragraph. Since the pair $s\Phi(s\mathcal{W}_t, \mathcal{W}_r), t\Phi(t\mathcal{W}_s, \mathcal{W}_r)$ is not geometric, $\mathcal{W}_r$ may lie only in two sectors for $\{u, s, t\}$, indicated in Figure 2, left. Denoting by Σ the union of the interiors of these two sectors, we have that $\langle s, t\rangle\Sigma$ lies entirely in K. This implies $(*)$ for L idle since Σ and its image $\tilde{f}(\Sigma)$ under the folding lie in the same halfspace of $s\mathcal{W}_u$. It also implies $(*)$ for L twisted, since $\Phi(ts\mathcal{W}_u, \tilde{f}(\Sigma)) = w_{st}\Phi(s\mathcal{W}_u, \Sigma)$. The case $m_{st} = 5$ is similar: though the union Σ of possible sectors containing $\mathcal{W}_r$ is larger (see Figure 2, right), its image $\tilde{f}(\Sigma)$ under the folding, in both possible cases for V, still lies entirely in one halfspace of $ts\mathcal{W}_u$, which is w_{st} invariant.

This ends the proof of $(*)$ as long as $L_1 \neq \{s, t\}$. Then by Lemma 9.7, as long as $L_1, I_1 \neq \{s, t\}$, we have $d(E_{L_\tau, I_\tau}, E_{I_\tau, L_\tau}) \leq d(E_{L,I}, E_{I,L})$, with strict inequality if L, I are not both idle, both twisted or both rotated. It remains to consider the case where I is idle with $I_1 = \{s, t\}$. Recall that then $E^1_{I,L} = D_I$ and so $E_{I,L} = w_{st}E_{I,L}$. Consequently, if L is twisted, by $(*)$ we have $d(E_{L_\tau, I_\tau}, E_{I_\tau, L_\tau}) = d(E_{L,I}, E_{I,L})$. If L

is rotated, and chambers $x \in E_{L,I}, y \in E_{I,L}$ realise the distance $d(E_{L,I}, E_{I,L})$, then $\tilde{f}(y) \in E_{I_\tau,L_\tau}$, and so by $(*)$ and Lemma 9.7 we have $d(E_{L_\tau,I_\tau}, E_{I_\tau,L_\tau}) \leq d(\tilde{f}(x), \tilde{f}(y)) < d(E_{L,I}, E_{I,L})$.

To summarise, if there is a big component that is not self-compatible, then there is maximal spherical L that is rotated and maximal spherical I that contains $\{s,t\}$, hence not rotated. If all big components are self-compatible, and there is a small component incompatible with another component, then one of them is twisted and another is idle, so there is maximal spherical L that is twisted and maximal spherical I that is idle with $I_1 \neq \{s,t\}$. In both situations we obtain $\mathcal{K}_2(S) < \mathcal{K}_2(\tau(S))$, which is a contradiction. □

11. Making use of consistent doubles

In this section we prove Theorem 8.1, which as pointed out in Section 8 implies the Main Theorem.

Lemma 11.1. *Suppose that S has consistent doubles. Let $L \subset S$ be irreducible spherical and let $r \in S$ with $L \cup \{r\}$ not spherical. Consider non-commuting $s,t \in L$ with $\{s,t,r\}$ not spherical. Then $\Delta^{(s,t),r}$ does not depend on s,t, and we can denote it $\Delta^{L,r}$*

Moreover, for L exposed, for C_L the vertex set of any cell of $\mathbb{A}_{\mathrm{amb}}$ fixed by L, and for any chamber x incident to $\mathcal{W}_r$, we have $d(C_L \cap \Delta^{L,r}, x) < d(C_L \cap w_L\Delta^{L,r}, x)$.

Proof. To start we focus on the first assertion. Since doubles are consistent, by Remark 5.2 we have $\Delta^{(s,t),r} = \Delta^{(s,L),r}$. Consider first the case where $|L| \geq 3$. Then by Lemma 6.2 we have that $\Delta^{(s,L),r}$ does not depend on s as long as s is not a leaf of the Coxeter–Dynkin diagram of L. However, if s is a leaf and t is not a leaf, since the doubles are consistent observe that the pair $\Phi_s^{(s,t),r}, \Phi_t^{(t,s),r}$ is geometric. This implies $\Delta^{(s,t),r} = \Delta^{(t,s),r}$ and the assertion follows. In the case where $|L| = 2$ it is enough to invoke that last observation.

For the second assertion, assume first that we have $|L| = 2$ and that V is the sector for L containing $\Delta := \Delta^{L,r}$. Then by Remark 9.3 we have $\mathcal{W}_r \subset sV \cup V \cup tV$. The required inequality follows then from e.g. Lemma 9.7 applied to one of the two foldings from the proof of Proposition 9.4. If $|L| = 3$, suppose that s,t,p are consecutive vertices in the Coxeter–Dynkin diagram of L. Since

$$\mathcal{W}_r \subset \Phi(t\mathcal{W}_s, \Delta) \cap \Phi(t\mathcal{W}_p, \Delta) \cap \Phi(s\mathcal{W}_t, \Delta) \cap \Phi(p\mathcal{W}_t, \Delta) \cap \Phi(ps\mathcal{W}_t, \Delta),$$

we have that $\mathcal{W}_r$ is contained in a sector for L separated by at most two walls in $\mathcal{W}_L$ from V. Since $\mathcal{W}_L$ consists of at least 6 walls separating Δ from $w_L\Delta$, the inequality follows. □

Proof of Theorem 8.1. By Proposition 10.1, S has consistent doubles. By Corollary 3.8, to prove Theorem 8.1 it suffices to show that for any simple markings μ and μ' with common core $s \in S$, we have $\Phi_s^\mu = \Phi_s^{\mu'}$. For each component A of $S \setminus (s \cup s^\perp)$, by Remark 3.5 there exists a simple marking μ with core s such that $K_s^\mu \subseteq A$ (where K_s^μ is as in Definition 3.12). We now repeat the construction of halfspaces associated to components from Definition 3.12, with $\{s, t\}$ replaced by s. Namely, since S has consistent doubles, by Proposition 3.13, if $K_s^{\mu'} \subseteq A$, then $\Phi_s^\mu = \Phi_s^{\mu'}$. Thus each component A of $S \setminus (s \cup s^\perp)$ determines a halfspace $\Phi_A := \Phi_s^\mu$ for s. Two components A_1, A_2 are *compatible* if $\Phi_{A_1} = \Phi_{A_2}$. We will show that all components of $S \setminus (s \cup s^\perp)$ are compatible.

Let A be a component of $S \setminus (s \cup s^\perp)$ and let $L \subset S$ be maximal spherical intersecting A. Note that if $s \notin L$, then there is $m \in L$ not adjacent to s and hence using the marking $((s, \emptyset), m)$ we observe that $C_L \subset \Phi_A$. Suppose now that L contains s, and let $L_1 \subset L$ be maximal irreducible containing s and hence also containing some $t \in A$. Let $I \subset S$ be maximal spherical with some $r \in I \cap B$ for another component B of $S \setminus (s \cup s^\perp)$. If L_1 is not exposed, then using the marking $\mu = ((s, t), r)$, by the first assertion in Lemma 11.1, we have $E_{L,I} \subseteq \Phi_A$. If L is exposed, then by the second assertion in Lemma 11.1, for each chamber y in $E_{L,I}$ realising the distance to any fixed chamber of $E_{I,L}$, we have $y \in \Phi_A$ as well.

If some components of $S \setminus (s \cup s^\perp)$ are not compatible, let A be the union of all components A_i with one $\Phi_{A_i,s}$, and B the union of components B_i with the other $\Phi_{B_i,s}$. Let τ be the elementary twist that sends each element $b \in B$ to sbs, and fixes the other elements of S. Let $L, I \subset S$ be maximal spherical. By the observation above on C_L, we have $d(C_{L_\tau}, C_{I_\tau}) \leq d(C_L, C_I)$ with strict inequality if and only if $s \notin L, I$ and $L \cap A_i, I \cap B_j \neq \emptyset$ or vice versa. Thus we can assume that such L, I do not exist, and hence $\mathcal{K}_1(\tau(S)) = \mathcal{K}_1(S)$ so that we can focus on $\mathcal{K}_2$. Then again from the above paragraph if $L \cap A_i, I \cap B_j \neq \emptyset$, then the chambers realising the distance between $E_{L,I}$ and $E_{I,L}$ lie in the opposite halfspaces for s. Thus $d(E_{L_\tau,I_\tau}, E_{I_\tau,L_\tau}) < d(E_{L,I}, E_{I,L})$ and consequently, $\mathcal{K}_2(\tau(S)) < \mathcal{K}_2(S)$, which is a contradiction. □

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