353-COMBINATORIAL CURVATURE AND THE 3-DIMENSIONAL $K(\pi,1)$ CONJECTURE

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Abstract. We prove the $K(\pi,1)$ conjecture for Artin groups of dimension 3. As an ingredient, we introduce a new form of combinatorial non-positive curvature.

1. Introduction

The $K(\pi,1)$ conjecture for Artin groups, due to Arnold, Brieskorn, Pham, and Thom, predicts that each Artin group has a $K(\pi,1)$ space that is a complex manifold described in the terms of the canonical linear representation of the associated Coxeter group. See [Par14, GP12a, Del72, CD95a], for background and a summary of progress on this conjecture before the 2010s, and [MS17, Juh18, PS21, Pao21, DPS22, Juh23, Gol24, Hae24, Hae22a, HH23, Hua24b, Hua24a, GH25] for more recent developments. In this article we prove the following.

Theorem 1.1. *Let A be an Artin group of dimension ≤ 3. Then A satisfies the $K(\pi,1)$ conjecture. In particular, A is torsion free.*

The *dimension* of an Artin group A is the maximal cardinality of a subset S' of the standard generating set of A such that the subgroup of A generated by S' is spherical. It is conjectured that this quantity is equal to the cohomological dimension of A, and by Theorem 1.1 this is true if one of these two quantities is ≤ 3. The dimension ≤ 2 case of Theorem 1.1 was established in 1995 [CD95b].

Using [JS23], we also deduce the centre conjecture.

Theorem 1.2. *Let A be an Artin group of dimension ≤ 3. If A has no nontrivial spherical factor, then it has trivial centre.*

Theorem 1.1 is a special case of Theorem 12.20, where we establish the $K(\pi,1)$ conjecture for many new Artin groups in each dimension, since the class of Artin groups that we treat contains arbitrarily large irreducible spherical parabolic subgroups.

1.1. **Combinatorial non-positive curvature.** A key ingredient of the proof is a contractibility criterion for a class of complexes satisfying a new form of combinatorial non-positive curvature, called 353-square complexes.

Theorem 1.3. *The thickening of a wide stable 353-square complex is contractible.*

Let us define 353-square complexes and their thickenings (for the notions of *wide* and *stable*, see Section 8). A *square complex* is a 2-dimensional combinatorial complex X, where X^1 is a bipartite simplicial graph, with vertex set partitioned into sets $\mathcal{A}$ and $\mathcal{D}$, and with attaching maps of the 2-cells distinct embedded cycles of length 4 (a *cycle* in a graph is a closed edge-path, or, shortly, an edge-loop). We often identify a 2-cell with its attaching map, and we call it a *square*. Not all embedded cycles of length 4 are assumed to be squares. We refer to Section 2.3 for the background on (minimal) disc diagrams in X.

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Vertices $a, a' \in \mathcal{A}$ (or $d, d' \in \mathcal{D}$) are *close* if they belong to a common square. The *thickening* $X^{\boxtimes}$ of a square complex X is the flag simplicial complex whose 1-skeleton is obtained from X^1 by adding edges between close vertices. We adopt the convention that if we label a vertex of X by a, a', a_1 etc, then it belongs to $\mathcal{A}$.

Definition 1.4. A *cube corner* C is a square complex isomorphic to the subcomplex of the boundary of a 3-dimensional cube formed of three squares containing a common vertex, called the *centre* of C. A *cube corner* in X is a disc diagram $C \to X$. A cube corner in X is *minimal* if its boundary 6-cycle does not bound a disc diagram in X with < 3 squares.

Definition 1.5. A 353-*square complex* is a simply connected square complex satisfying the following properties.

(1) If dad_1a' and dad_2a' are squares, then d_1ad_2a' is a square (see Figure 1(1)).
(2) Let d be a vertex of a minimal cube corner C lying in exactly two squares ada_1d_1, ada_2d_2 of C. Then there is no square $d'a_1da_2$ (see Figure 1(2)).
(3) Let d be a vertex of a minimal cube corner C lying in exactly two squares ada_1d_1, ada_2d_2 of C. Suppose that there is $a' \neq a$ with a_1da', a_2da' also lying in squares (see Figure 1(3)). Then a' is a neighbour of d_1 and d_2 and $ada'd_1, ada'd_2$ are squares.
(4) Let E, E' be as in Figure 1(4a), (4b). For any disc diagram $f\colon E \to X$ whose restriction to each cube corner of E is minimal, there is a diagram $f'\colon E' \to X$ with the same boundary as f, such that $f'(a')$ is a neighbour of $f(d)$.
(5) Previous properties hold if we interchange $\mathcal{A}$ and $\mathcal{D}$.

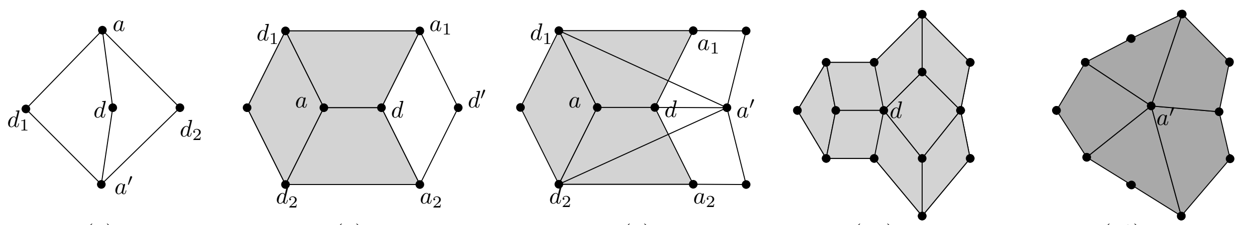


FIGURE 1.

Definition 1.5 is motivated by the structure of the *icosahedral honeycomb* of the hyperbolic 3-space $\mathbb{H}^3$, with Schläfli symbol $\{3, 5, 3\}$. It is one of the four compact, regular, space-filling honeycombs in $\mathbb{H}^3$, and it was the least understood one from the perspective of combinatorial non-positive curvature (while the other three honeycombs, viewed as cell complexes, are cell-Helly [HO21, Def 3.5]).

Given the icosahedral honeycomb of $\mathbb{H}^3$, viewed as a combinatorial complex Z, we define the associated square complex X. Vertices in $\mathcal{A}$ correspond to the icosahedra of Z, and vertices in $\mathcal{D}$ correspond to the vertices of Z. Vertices $x \in \mathcal{A}$ and $y \in \mathcal{D}$ are neighbours if the icosahedron corresponding to x contains the vertex corresponding to y. We span squares on all embedded 4-cycles of X^1. Definition 1.5 is conceived by listing local combinatorial features of X of non-positive curvature flavour. The list of local properties in Definition 1.5 leads to a collection of global properties in Lemma 8.5, which is an analogue of the Cartan–Hadamard theorem. These complexes have quadratic Dehn function (Lemma 8.5(ii)), and their balls satisfy a weak form of convexity [Can87] (Lemma 8.5(iv))). Finally, we have contractibility as in Theorem 1.3.

Let Λ be the Coxeter diagram that is the linear graph formed of three edges with consecutive labels $3, 5, 3$, and let Δ be the Artin complex (see Definition 1.6) of A_Λ, which is a 3-dimensional simplicial complex. A major step towards Theorem 1.1 is showing that Δ is contractible. This is done via Theorem 1.3, see Definition 9.1 and Corollary 9.11.

In [CD95b], Charney and Davis proposed to equip Δ with a piecewise Euclidean metric (the Moussong metric) and to show that Δ is CAT(0), hence contractible. Proving CAT(0) amounts to studying loops of length $< 2\pi$ in the links of the vertices of Δ. Each such loop gives an equation of the form $w_1 w_2 \cdots w_n = 1$ in the Artin group A_{H_3} of type H_3, subject to the constraint that each w_i lies in an appropriate parabolic subgroup of A_{H_3}. Thus proving that Δ is CAT(0) relies on understanding the 'varieties' of solutions to a finite (but large) set of such equations over A_{H_3}. There are no established theories in algebraic geometry to understand the solution varieties of such equations, and the ambient group A_{H_3} being exceptional further obscures the picture. This is the main difficulty of the CAT(0) approach.

The CAT(0) approach inspired us to look for a 'softer' notion of non-positive curvature, leading to Definition 1.5 and its simplicial companion, Definition 9.1. Like in the CAT(0) approach, proving the contractility of Δ reduces to studying a collection of short loops in the link of each vertex. However, it is a much smaller collection of loops, hence the number of the associated equations over G that we need to solve is significantly reduced. Miraculously, we avoid solving some of the most sophisticated equations needed in the CAT(0) approach. However, we do not completely avoid the task of analysing the solution varieties of some of these equations, which takes a substantial portion of the article.

1.2. **Reading guide.** Section 2 consists of preliminaries. The article is divided into Part I ranging from Section 3 to Section 9, and Part II ranging from Section 10 to Section 12. Part I takes up most of the article, and it concerns the Artin complex of a **single** Artin group.

Definition 1.6. Let A_Λ be an Artin group with Coxeter graph Λ and generating set S. Its *Artin complex* Δ_Λ [CD95b, GP12b, CMV20] is a simplicial complex defined as follows. For each $s \in S$, let $A_{\hat{s}}$ be the standard parabolic subgroup generated by $\hat{s} = S \setminus \{s\}$. The vertices of Δ_Λ correspond to the left cosets of $\{A_{\hat{s}}\}_{s\in S}$. Moreover, vertices span a simplex if the corresponding cosets have non-empty common intersection. A vertex of Δ_Λ corresponding to a left coset of $A_{\hat{s}}$ has *type* $\hat{s}$.

Let Λ be the 353 Coxeter diagram from the previous subsection. The main goal of Part I is to establish two properties of Δ_Λ. First, Δ_Λ is contractible, which implies the $K(\pi,1)$ conjecture for A_Λ. Second, each induced embedded 4-cycle in Δ_Λ of type $\hat{s}\hat{t}\hat{s}\hat{t}$ has a common neighbour in Δ_Λ of type $\hat{r}$, where r is a vertex of Λ separating s and t. This second property is useful for proving the $K(\pi,1)$ conjecture for other Artin groups.

Part I is performed in two steps. In Step 1, we show that the link of each vertex of Δ_Λ satisfies a list of properties (Sections 3-7). In Step 2, we introduce a more general family of complexes, called 353-*simplicial complexes*, that are simply connected and whose vertex links satisfy the same list of properties (Definition 9.1). We prove, under two minor assumptions, that such a complex is contractible and has the desired 4-cycle property mentioned in the previous paragraph (Sections 8 and 9).

Let us discuss Step 2 in more detail. In Section 8, we study 353-square complexes, establishing properties of minimal disc diagrams bounded by certain cycles in these complexes and proving the contractility of their thickenings (Theorem 1.3). In Section 9, we introduce the notion of a 353-simplicial complex. For each 353-simplicial complex Δ, we can construct an associated 353-square complex X whose thickening is homotopy equivalent to Δ (under mild assumptions), implying the contractibility of Δ and the desired 4-cycle property.

Coming back to Step 1, it remains to show that Δ_Λ is a 353-simplicial complex. By definition, this reduces to proving that two kinds of *critical cycles* in Δ_{H_3} are *admissible.* Here Δ_{H_3} denotes the Artin complex of the spherical Artin group A_{H_3}, which is isomorphic to the vertex link of Δ_Λ. Critical cycles in Δ_{H_3} and the notion of their *admissibility* are introduced at the beginning of Sections 6 and 7, and most of Sections 3-7 is the proof of the admissibility of critical cycles.

In Section 3, we give the background on hyperplane arrangements and associated complexes needed later. For each collection $\mathcal{A}$ of affine hyperplanes in $\mathbb{R}^n$ passing through the origin, we consider the complex manifold $M(\mathcal{A}\otimes\mathbb{C}^n) = \mathbb{C}^n - \bigcup_{H\in\mathcal{A}}(H\otimes\mathbb{C})$. Let $\mathcal{A}$ be the collection of reflection hyperplanes for the canonical linear representation of the Coxeter group of type H_3 acting on $\mathbb{R}^3$. Then $\pi_1 M(\mathcal{A}\otimes\mathbb{C}^n)$ is the pure Artin group PA_{H_3} of A_{H_3}. It is difficult to analyse a cycle ω in Δ_{H_3} directly. Instead, we consider a subset $\mathcal{A}'\subset\mathcal{A}$, which gives an inclusion $M(\mathcal{A}\otimes\mathbb{C}^n)\to M(\mathcal{A}'\otimes\mathbb{C}^n)$. This induces a quotient map between groups $PA_{H_3}\to\pi_1 M(\mathcal{A}'\otimes\mathbb{C}^n)$, and a surjective simplicial map from Δ_{H_3} to another complex $\Delta_{\mathcal{A}'}$ (we use this notation only in the Introduction). The cycle $\omega\subset\Delta_{H_3}$ is sent to a cycle $\omega'\subset\Delta_{\mathcal{A}'}$. It turns out that for suitable choices of $\mathcal{A}'$, the complex $\Delta_{\mathcal{A}'}$ contains large subcomplexes that are 'non-positively curved'. If the subcomplex is large enough to contain ω', then we can use the non-positive curvature to analyse ω', and then lift the information back to ω. This last step is nontrivial, since we are losing information in the quotient map $PA_{H_3}\to\pi_1 M(\mathcal{A}'\otimes\mathbb{C}^n)$.

In Section 4, we discuss the possible subset $\mathcal{A}'$. Actually, we choose two subsets $\mathcal{A}_1$ and $\mathcal{A}_2$, so certain information that is lost as a consequence of one choice survives for the other choice. For each $\mathcal{A}_i$, we indicate what is the non-positively curved subcomplex of $\Delta_{\mathcal{A}_i}$ that we have found. This section is mostly a review of [Hua24a].

The material in Section 5 is new. The non-positively curved subcomplex of $\Delta_{\mathcal{A}_2}$ found in [Hua24a] is not large enough for our purpose. We show in Section 5 that there is a larger subcomplex of $\Delta_{\mathcal{A}_2}$ that satisfies a new form of non-positive curvature, governed by what we have called a *splitting system* (Definition 5.13). We use it to understand minimal disc diagrams in the subcomplex. Given these ingredients, we treat critical 8-cycles in Section 6 and critical 10-cycles in Section 7.

We prove Theorem 1.1 in Part II of the paper (Sections 10-12). Our point of departure is the following criterion by Godelle and Paris.

Theorem 1.7 ([Hua24b, Thm 2.2], which is a reformulation of [GP12b, Thm 3.1])**.** *Let Λ be non-spherical with Δ_Λ contractible. If $A_{\Lambda'}$ satisfies the $K(\pi,1)$ conjecture for all subdiagrams Λ' induced on all but one vertex of Λ, then A_Λ satisfies the $K(\pi,1)$ conjecture.*

To show the contractibility of Δ_Λ, we adopt the strategy from [Hua24b]. Roughly speaking, we first show that Δ_Λ deformation retracts to a suitable subcomplex, which is the *relative Artin complex* (Definition 10.1). We then show that this subcomplex is non-positively curved in an appropriate sense, and so it is contractible. Section 10 summarises the properties of relative Artin complexes from [Hua24b] and some additional contracitibility criteria from [Bes06, Hae24].

In Section 11, we introduce a geometric tool needed for executing our strategy, the notion of convexity for a class of simplicial complexes that are closely related to Garside categories [Bes99, CMW04, Bes06, Hae22b]. A convex subcomplex, as defined here, can be detected locally—specifically, by examining the links of vertices—using a purely combinatorial criterion (see Definition 11.7). This notion is inspired by the Bestvina normal form [Bes99]. Intriguingly, even for the tessellation of $\mathbb{E}^2$ by equilateral triangles, our notion differs from the more classical ones.

In Section 12, we prove Theorem 1.1 by induction on the number of generators of the Artin group. In the process, we obtain the following byproducts or enhancements of Theorem 1.1, some of which (items 3 and 4) might have applications outside the $K(\pi,1)$ conjecture.

(1) We show that the $K(\pi,1)$ conjecture holds not only for all the 3-dimensional Artin groups, but also for many higher dimensional ones (Theorem 12.20).
(2) We derive a general result that reduces the $K(\pi,1)$ conjecture for arbitrary Artin groups to properties of Artin groups whose Coxeter diagrams do not contain triangles (Corollary 12.19).
(3) We show that the 'girth condition' holds for each 3-dimensional Artin group (Theorem 12.2)
(4) For each 3-dimensional Artin group that is not spherical, we construct a 'non-positively curved' relative Artin complex on which the group acts.

2. Preliminaries

2.1. **Artin complex.** A *Coxeter diagram* Λ is a finite simplicial graph with vertex set $S=\{s_i\}_i$ and labels $m_{ij}=3,4,\ldots,\infty$ for each edge s_is_j. If s_is_j is not an edge, we define $m_{ij}=2$. The Artin group A_Λ is the group with generator set S and relations $s_is_js_i\cdots=s_js_is_j\cdots$ with both sides alternating words of length m_{ij}, whenever $m_{ij}<\infty$. The Coxeter group W_Λ is obtained from A_Λ by adding relations $s_i^2=1$.

The *pure Artin group* PA_Λ is the kernel of the obvious homomorphism $A_\Lambda\to W_\Lambda$. We say that A_Λ (or Λ) is *spherical*, if W_Λ is finite. Recall that any $S'\subset S$ generates a subgroup of A_Λ isomorphic to $A_{\Lambda'}$, where Λ' is the subdiagram of Λ induced on S'. Such a subgroup is called a *standard parabolic subgroup*.

We refer to Definition 1.6 for the notion of the Artin complex Δ_Λ of the Artin group A_Λ. It follows from [GP12b, Prop 4.5] that Δ_Λ is a flag complex. Note that given $g\in A_\Lambda$, the vertices corresponding to the collection of the left cosets $\{gA_{\hat{s}}\}_{s\in S}$ span a top-dimensional simplex of Δ_Λ. This gives a bijective correspondence between the elements of A_Λ and the top-dimensional simplices of Δ_Λ. The *Coxeter complex* $\mathfrak{C}_\Lambda$ is defined analogously, where we replace $A_{\hat{s}}$ by $W_{\hat{s}}<W_\Lambda$ generated by $\hat{s}$. A vertex of $\mathfrak{C}_\Lambda$ corresponding to a left coset of $W_{\hat{s}}$ has *type* $\hat{s}$. We have that $\mathfrak{C}_\Lambda$ is the quotient of Δ_Λ under the action of PA_Λ.

Remark 2.1 ([Hua24b, Cor 6.5])**.** For $i=1,2,3$, let $x_i\in\Delta^0_\Lambda$ be of type $\hat{s}_i$. Suppose that s_1 and s_3 belong to distinct components of $\Lambda\setminus\{s_2\}$. If x_2 is a neighbour of both x_1 and x_3, then x_1 is a neighbour of x_3.

We need the following generalisation of Remark 2.1. The *type* of a face of Δ_Λ is the intersection of the types of its vertices. Again, faces of type $\widehat{T}=S\setminus T$ are in bijective correspondence with the left cosets of $A_{\Lambda\setminus T}$, where $\Lambda\setminus T\subset\Lambda$ is the subdiagram induced on $\widehat{T}$. The *type* of a vertex v of the barycentric subdivision Δ'_Λ of Δ_Λ is the type of the face of Δ_Λ with barycentre v. Given two vertices x,y of Δ'_Λ, we write $x\sim y$ if they are contained a common simplex of Δ_Λ. Then $x\sim y$ if and only if the corresponding two left cosets intersect.

Lemma 2.2. *Let x_1,x_2,x_3 be vertices of Δ'_Λ of type $\widehat{S}_1,\widehat{S}_2,\widehat{S}_3$, respectively. Suppose that any $s_1\in S_1\setminus S_2$ and $s_3\in S_3\setminus S_2$ belong to distinct components of $\Lambda\setminus S_2$. If $x_1\sim x_2$ and $x_2\sim x_3$, then $x_1\sim x_3$.*

Proof. The proof is identical to that of [Hua24b, Lem 10.4]. We include it for the convenience of the reader. We can assume that S_2 does not contain S_1 (or S_3), since otherwise the left coset corresponding to x_2 would be contained in the left coset corresponding to x_1, and so $x_2 \sim x_3$ would imply $x_1 \sim x_3$.

Up to the left translation, we can assume that x_2 corresponds to the identity coset $A_{\Lambda \setminus S_2}$. For $i = 1, 3$, let Λ_i be the union of the components of $\Lambda \setminus S_2$ that are disjoint from S_i. By our hypotheses, we have $\Lambda_i \neq \emptyset$ and $\Lambda_1 \cup \Lambda_3$ contains all the vertices of $\Lambda \setminus S_2$. Since $A_{\Lambda \setminus S_2}$ is the direct sum of the Artin groups with Coxeter diagrams the components of $\Lambda \setminus S_2$, any left coset of A_{Λ_1} in $A_{\Lambda \setminus S_2}$ and any left coset of A_{Λ_3} in $A_{\Lambda \setminus S_2}$ intersect. For $i = 1, 3$, let H_i be the left coset of $A_{\Lambda \setminus S_i}$ in A_Λ corresponding to x_i. For $i = 1, 3$, since $\Lambda_i \subset \Lambda \setminus S_i$, we have that $A_{\Lambda \setminus S_2} \cap H_i$ contains a left coset of A_{Λ_i} in $A_{\Lambda \setminus S_2}$. Thus we have $A_{\Lambda \setminus S_2} \cap H_1 \cap H_3 \neq \emptyset$ and so $x_1 \sim x_3$. □

2.2. **Posets.** Let S be a set of size n. A simplicial complex X is of *type* S if all the maximal simplices of X have dimension $n-1$ and there is a function $\mathrm{Type}\colon X^0 \to S$ such that $\mathrm{Type}(x) \neq \mathrm{Type}(y)$ if x and y are neighbours. Note that the restriction of Type to the vertex set of each maximal simplex is a bijection.

As an example, if A_Λ is an Artin group, and S is the vertex set of Λ, then the Artin complex Δ_Λ is a simplicial complex of type S (or, rather, $\widehat{S}$).

Definition 2.3. Let X be a simplicial complex of type S. Any total order $<$ on S induces the following relation $<$ on X^0. We declare $x < y$ if x and y are neighbours, and $\mathrm{Type}(x) < \mathrm{Type}(y)$.

Let P be a poset, i.e. a partially ordered set. Let $Q \subset P$. An *upper bound* (resp. *lower bound*) for Q is an element $x \in P$ such that $q \leq x$ (resp. $x \leq q$) for any $q \in Q$. An upper bound x of Q is the *join* of Q if $x \leq y$ for any upper bound y of Q. A lower bound x of Q is the *meet* of Q if $y \leq x$ for any lower bound y of Q. We write $x \vee y$ and $x \wedge y$ for the join and meet of $\{x, y\}$ (if they exist). We say that P is a *lattice* if P is a poset and any two elements of P have the join and the meet. For $a, b \in P$ with $a \leq b$, the *interval* $[a, b]$ *between* a *and* b is the collection of all the elements x of P satisfying $a \leq x$ and $x \leq b$. A poset P is *weakly graded* if there is a *poset map* $r\colon P \to \mathbb{Z}$, i.e. for every $x < y$ in P, we have $r(x) < r(y)$. Such r is called a *rank* function. A *bowtie* $x_1y_1x_2y_2$ consists of distinct elements of P satisfying $x_i < y_j$ for all $i, j = 1, 2$. The name comes from the fact that if we draw y_1, y_2 above x_1, x_2 in the Hasse diagram, then we obtain a bowtie shaped configuration.

Definition 2.4. A poset P is *bowtie free* if for any bowtie $x_1y_1x_2y_2$ there exists $z \in P$ satisfying $x_i \leq z \leq y_j$ for all $i, j = 1, 2$.

Lemma 2.5 ([BM10, Prop 1.5] and [HH23, Prop 2.4])**.** *If P is a bowtie free weakly graded poset, then any subset $Q \subset P$ with an upper bound has the join, and any $Q \subset P$ with a lower bound has the meet.*

Proof. The case of $|Q| = 2$ is [HH23, Prop 2.4]. This easily implies the case of finite Q. Thus for infinite Q with an upper bound u, we have that each finite subset of Q has the join, which is $\leq u$. Let T be a finite subset of Q such that the join u_T of T has largest rank among all the joins of the finite subsets of Q. We claim that u_T is the join of Q. To justify the claim, it is enough to show that for any $q \in Q$, we have $q \leq u_T$. Let u be the join of $T \cup \{q\}$. Then we have $u_T \leq u$. On the other hand, we have $r(u) \leq r(u_T)$ by our choice of T. Thus $u_T = u$, and so $q \leq u_T$, as desired. The assertion on the meet is proved analogously. □

Definition 2.6. A poset is *upward flag* if any three pairwise upper bounded elements have an upper bound. A poset is *downward flag* if any three pairwise lower bounded elements have a lower bound. A poset is *flag* if it is upward flag and downward flag.

Definition 2.7. A poset is *weakly upward flag* if any three elements pairwise upper bounded by non-maximal elements have an upper bound. Analogously, we define *weakly downward flag* and *weakly flag* posets.

We will be often discussing Coxeter diagrams Λ that are linear graphs with consecutive vertices $s_1,\dots,s_n$. In that case, we write shortly $\Lambda = s_1\cdots s_n$.

Theorem 2.8 ([Hae24, Prop 6.6]). *Let $\Lambda = s_1\cdots s_n$ be the Coxeter diagram of type B_n with $m_{s_{n-1}s_n} = 4$, and total order $\hat{s}_1 < \cdots < \hat{s}_n$. Then the induced relation $<$ on Δ^0_Λ from Definition 2.3 is a partial order that is weakly graded, bowtie free and upward flag*

Theorem 2.9 ([Hua24a, Thm 7.1]). *Let $\Lambda = s_1s_2s_3$ be the Coxeter diagram of type H_3 with $m_{s_2s_3} = 5$, and $\hat{s}_1 < \hat{s}_2 < \hat{s}_3$. Then the induced relation $<$ on Δ^0_Λ from Definition 2.3 is a partial order that is weakly graded, bowtie free and upward flag.*

2.3. **Disc diagrams.** A map from a CW complex Y to a CW complex X is *combinatorial* if its restriction to each open cell of Y is a homeomorphism onto an open cell of X. A CW complex X is *combinatorial*, if the attaching map of each open cell of X is combinatorial for some subdivision of the sphere.

A *disc diagram* D is a finite contractible combinatorial complex with a fixed embedding in the plane $\mathbb{R}^2$. We can then view $\mathbb{R}^2 \cup \{\infty\}$ as the combinatorial complex that is a union of D and a 2-cell *at infinity*. The *boundary cycle* of D is the edge-loop in D that is the attaching map of the cell at infinity. A *disc diagram in a combinatorial complex* X is a combinatorial map $f\colon D \to X$, where D is a disc diagram. The *boundary cycle* of f is the composition of the boundary cycle of D with f. A disc diagram $f\colon D \to X$ is *minimal* if it has minimal area (i.e. number of 2-cells in D) among all diagrams in X with the same boundary cycle. We say that f is *reduced* if it is locally injective at $D \setminus D^0$. The following is a well-known variation of a result by Van Kampen.

Lemma 2.10 ([MW02, Lem 2.16 and 2.17]). *Any homotopically trivial cycle ω in X is the boundary cycle of a disc diagram $f\colon D \to X$. Any minimal disc diagram is reduced.*

Note that if ω is not embedded, then D might not be homeomorphic to a disc.

Suppose that the corners of each p-gon of a disc diagram D are assigned real numbers, called *angles*, with sum $(p-2)\pi$. Let v be a vertex of D whose link in D has n_v components. We define the *curvature at v of D* to be $(2-n_v)\pi$ minus the sum of all the angles at v. We will use the following 'Gauss–Bonnet theorem'.

Theorem 2.11 ([MW02, Thm 4.6]). *The sum of the curvatures at all the vertices v of D equals 2π.*

Here is an example of an application of Theorem 2.11. However, we will be using without reference various similar results on 2-dimensional CAT(0) simplicial complexes, especially in Sections 6–7.

Lemma 2.12. *Let Y be a 2-dimensional* CAT(0) *simplicial complex of type $\{\hat{s},\hat{t},\hat{p}\}$, all of whose triangles have type $\hat{s}\hat{t}\hat{p}$ with angles $\frac{\pi}{4},\frac{\pi}{2},\frac{\pi}{4}$ or $\frac{\pi}{6},\frac{\pi}{2},\frac{\pi}{3}$.*

(i) Then any induced 4-cycle ω in Y^1 has type $\hat{s}\hat{p}\hat{s}\hat{p}$ and has a common neighbour of type $\hat{t}$.

(ii) In the $\frac{\pi}{6},\frac{\pi}{2},\frac{\pi}{3}$ case, for $n \le 6$, any embedded $2n$-cycle ω in Y^1 of type $(\hat{t}\hat{p})^n$ has a common neighbour of type $\hat{s}$ and satisfies $n = 6$.

(iii) In the $\frac{\pi}{4},\frac{\pi}{2},\frac{\pi}{4}$ case, for $n \le 4$, any embedded $2n$-cycle ω in Y^1 of type $(\hat{t}\hat{p})^n$ has a common neighbour of type $\hat{s}$ and satisfies $n = 4$.

Proof. For part (i), let $D \to Y$ be a minimal disc diagram bounded by the 4-cycle ω. Let $T_1, \dots, T_4 \subset D$ be the triangles containing the boundary edges. Note that since ω is induced, the T_i are distinct. Furthermore, the sum of the angles of T_i incident to ∂D is at least $4 \cdot \frac{\pi}{2}$. Since D is minimal, it is locally CAT(0), and by Theorem 2.11 the sum of the angles at ∂D is at most 2π. Consequently, we have equality, and there are no other triangles in D incident to ∂D. As a result, there are no other triangles in D, as desired.

For part (ii), we consider $2n$ triangles $T_i \subset D$ containing the boundary edges. The sum of the angles of T_i incident to ∂D is at least $2n\left(\frac{\pi}{2}+\frac{\pi}{3}\right) = \frac{5n\pi}{3}$. By Theorem 2.11, the sum of the angles at ∂D is at most $2n\pi - 2\pi$. Consequently $\frac{5n\pi}{3} \leq 2n\pi - 2\pi$, and so $n \geq 6$ and we conclude as before. This part also follows from [Hua24b, Lem 9.8]. The proof of part (iii) is analogous to that of part (ii). $\square$

3. Complexes for hyperplane arrangements

3.1. **Hyperplane arrangements and their dual polyhedra.** A *hyperplane arrangement* in the vector space $\mathbb{R}^n$ is a locally finite family $\mathcal{A}$ of affine hyperplanes. Let $\mathcal{Q}(\mathcal{A})$ be the set of nonempty affine subspaces that are intersections of subfamilies of $\mathcal{A}$ (here $\mathbb{R}^n \in \mathcal{Q}(\mathcal{A})$ as the intersection of an empty family). Each point $x \in \mathbb{R}^n$ belongs to a unique element of $\mathcal{Q}(\mathcal{A})$ that is minimal with respect to inclusion, called the *support* of x. A *fan* of $\mathcal{A}$ is a maximal connected subset of $\mathbb{R}^n$ consisting of points with the same support. Denote the collection of all fans of $\mathcal{A}$ by $\mathrm{Fan}(\mathcal{A})$. Note that $\mathbb{R}^n$ is the (disjoint) union of $\mathrm{Fan}(\mathcal{A})$. We define a partial order on $\mathrm{Fan}(\mathcal{A})$ so that $U_1 < U_2$ if U_1 is contained in the closure of U_2. Let $b\Sigma_{\mathcal{A}}$ be the simplicial complex that is the geometric realisation of this poset. For each $U \in \mathrm{Fan}(\mathcal{A})$, we choose a point $x_U \in U$. This gives a piecewise linear embedding $b\Sigma_{\mathcal{A}} \subset \mathbb{R}^n$ sending the vertex of $b\Sigma_{\mathcal{A}}$ corresponding to U to x_U.

By [Sal87, pp. 606-607], the simplicial complex $b\Sigma_{\mathcal{A}}$ is the barycentric subdivision of a combinatorial complex $\Sigma_{\mathcal{A}}$ whose vertices correspond to the top-dimensional fans. Namely, for each vertex of $b\Sigma_{\mathcal{A}}$ corresponding to $U \in \mathrm{Fan}(\mathcal{A})$, the union of all the simplices of $b\Sigma_{\mathcal{A}}$ corresponding to chains with smallest element U is homeomorphic to a closed disc [Sal87, Lem 6], which becomes the face of $\Sigma_{\mathcal{A}}$ corresponding to U. We will sometimes view $b\Sigma_{\mathcal{A}}$ and $\Sigma_{\mathcal{A}}$ as subspaces of $\mathbb{R}^n$. For $B \in \mathcal{Q}(\mathcal{A})$, a face F of $\Sigma_{\mathcal{A}}$ is *dual* to B, if B contains the fan U corresponding to F and $\dim(B) = \dim(U)$. We equip the 1-skeleton of $\Sigma_{\mathcal{A}}$ with the path metric d such that each edge has length 1. Given vertices $x, y \in \Sigma^0_{\mathcal{A}}$, it turns out that $d(x,y)$ is the number of hyperplanes separating x and y [Del72, Lem 1.3].

Lemma 3.1 ([Sal87, Lem 3])**.** *Let $x \in \Sigma^0_{\mathcal{A}}$ and let F be a face of $\Sigma_{\mathcal{A}}$. Then there exists unique $\Pi_F(x) \in F^0$ such that $d(x, \Pi_F(x)) \leq d(x,y)$ for any $y \in F^0$.*

The vertex $\Pi_F(x)$ is called the *projection* of x to F. A hyperplane $H \in \mathcal{A}$ *crosses* a face F of $\Sigma_{\mathcal{A}}$ if H is dual to an edge of F. For an edge xy of $\Sigma_{\mathcal{A}}$, if the hyperplane dual to xy crosses F, then $\Pi_F(x)\Pi_F(y)$ is an edge dual to the same hyperplane, otherwise we have $\Pi_F(x) = \Pi_F(y)$. Thus Π_F extends naturally to a map $\Sigma^1_{\mathcal{A}} \to F^1$.

Lemma 3.2. *Let E and F be faces of $\Sigma_{\mathcal{A}}$. Then $\Pi_F(E^0) = F'^0$ for some face $F' \subset F$.*

Proof. Let $\mathcal{A}'$ be the collection of all the hyperplanes that cross both E and F. Note that for any edge-path P in E, any edge of $\Pi_F(P)$ is dual to an element of $\mathcal{A}'$.

Let $B \in \mathcal{Q}(\mathcal{A})$ be the intersection of all the elements of $\mathcal{A}'$. If $\mathcal{A}' = \emptyset$, then we set $B = \mathbb{R}^n$. Let E' be any face of E dual to B, which is a vertex for $B = \mathbb{R}^n$, and let w be any vertex of E'. By the above discussion, $\Pi_F(w)$ is contained in a face

F' of Σ dual to B. Moreover, we have $F' \subset F$. Furthermore, $\Pi_F(E'^0) = F'^0$ and $\Pi_F(E^0) \subset F'^0$, as desired. □

In the situation of Lemma 3.2, we write $F' = \Pi_F(E)$. The assignment $E \to \Pi_F(E)$ gives rise to a piecewise linear map $\Pi_F \colon \Sigma_{\mathcal{A}} \cong b\Sigma_{\mathcal{A}} \to bF \cong F$.

3.2. **Salvetti complex.** Let $V = \Sigma^0_{\mathcal{A}}$. Consider the set of pairs (F, v), where F is a face of $\Sigma_{\mathcal{A}}$ and $v \in V$. We define an equivalence relation $\sim$ on this set by $(F, v) \sim (F', v')$ whenever $F = F'$ and $\Pi_F(v') = \Pi_F(v)$. Note that each equivalence class $[F, v']$ contains a unique representative of form (F, v) with $v \in F^0$. The *Salvetti complex* $\widehat{\Sigma}_{\mathcal{A}}$ is obtained from $\Sigma_{\mathcal{A}} \times V$ (a disjoint union of copies of $\Sigma_{\mathcal{A}}$) by identifying faces $F \times v$ and $F \times v'$ whenever $[F, v] = [F, v']$ [Sal87, pp. 608]. For example, for each edge $F = v_0v_1$ of $\Sigma_{\mathcal{A}}$, we obtain two edges $F \times v_0$ and $F \times v_1$ of $\widehat{\Sigma}_{\mathcal{A}}$, glued along their endpoints $v_0 \times v_0$ and $v_1 \times v_1$. We orient the edge $F \times v_0$ from $v_0 \times v_0$ to $v_1 \times v_0 = v_1 \times v_1$. Then $\widehat{\Sigma}^0_{\mathcal{A}} = V$, while $\widehat{\Sigma}^1_{\mathcal{A}}$ is obtained from $\Sigma^1_{\mathcal{A}}$ by doubling each edge. Thus each edge of the form $F \times v$ is oriented so that its endpoint is farther from v in F^1 than its starting point.

There is a natural map $p \colon \widehat{\Sigma}_{\mathcal{A}} \to \Sigma_{\mathcal{A}}$ forgetting the second coordinate. For each subcomplex Y of $\Sigma_{\mathcal{A}}$, we write $\widehat{Y} = p^{-1}(Y)$. If F is a face of $\Sigma_{\mathcal{A}}$, then $\widehat{F}$ is a *standard subcomplex* of $\widehat{\Sigma}_{\mathcal{A}}$.

Lemma 3.3 ([Hua24a, Lem 4.5])**.** *Let E and F be faces of $\Sigma_{\mathcal{A}}$. If $[E, v_1] = [E, v_2]$, then $[\Pi_F(E), v_1] = [\Pi_F(E), v_2]$.*

Definition 3.4. Let F be a face of $\Sigma_{\mathcal{A}}$. Consider the disjoint union of V copies of the map Π_F, where $\Pi_F \times v \colon \Sigma_{\mathcal{A}} \times v \to F \times v$. It follows from Lemma 3.3 that this map factors to a map $\Pi_{\widehat{F}} \colon \widehat{\Sigma}_{\mathcal{A}} \to \widehat{F}$, which is a retraction (see [GP12b, Thm 2.2]).

The following key property of $\Pi_{\widehat{F}}$ follows directly from Definition 3.4.

Lemma 3.5. *Let E and F be faces of $\Sigma_{\mathcal{A}}$. Then $\Pi_{\widehat{F}}(\widehat{E}) = \widehat{\Pi_F(E)}$.*

Let $\mathcal{A} \otimes \mathbb{C}$ be the complexification of $\mathcal{A}$, which is a collection of affine complex hyperplanes in $\mathbb{C}^n$. Define

$$M(\mathcal{A} \otimes \mathbb{C}) = \mathbb{C}^n - \bigcup_{H \in \mathcal{A}} (H \otimes \mathbb{C}).$$

It follows from [Sal87, Thm 1] that $\widehat{\Sigma}_{\mathcal{A}}$ is homotopy equivalent to $M(\mathcal{A} \otimes \mathbb{C})$, and so they have isomorphic fundamental groups.

In the remaining part of this subsection, we assume that W_Λ is a finite Coxeter group with its canonical representation $\rho \colon W_\Lambda \to \mathbf{GL}(n, \mathbb{R})$ [Dav08, Chap 6.12]. A *reflection* of W_Λ is a conjugate of $s \in S$. Each reflection fixes a hyperplane in $\mathbb{R}^n$, which we call a *reflection hyperplane*. Let $\mathcal{A}$ be the family of all reflection hyperplanes. The hyperplane arrangement $\mathcal{A}$ is the *reflection arrangement* associated with W_Λ. We denote $\Sigma_\Lambda = \Sigma_{\mathcal{A}}$ and $\widehat{\Sigma}_\Lambda = \widehat{\Sigma}_{\mathcal{A}}$. Since W_Λ permutes the elements of $\mathcal{A}$, there is an induced action $W_\Lambda \curvearrowright M(\mathcal{A} \otimes \mathbb{C})$ and an induced action $W_\Lambda \curvearrowright \widehat{\Sigma}_{\mathcal{A}}$, which are free. The union of $\mathcal{A}$ cuts the unit sphere of $\mathbb{R}^n$ into a simplicial complex, which is isomorphic to the Coxeter complex $\mathfrak{C}_\Lambda$ and dual to Σ_Λ. The following are standard [Par14, §3.2 and 3.3].

- $\pi_1 M(\mathcal{A} \otimes \mathbb{C}) = PA_\Lambda$ [vdL83],
- $\pi_1(M(\mathcal{A} \otimes \mathbb{C})/W_\Lambda) = \pi_1(\widehat{\Sigma}_\Lambda/W_\Lambda) = A_\Lambda$,
- $\widehat{\Sigma}^2_\Lambda/W_\Lambda$ is isomorphic to the presentation complex of A_Λ.

Definition 3.6. Note that $\widehat{\Sigma}^1_\Lambda$ is isomorphic to the Cayley graph of W_Λ (with edges appropriately oriented), and Σ^1_Λ is isomorphic to the unoriented Cayley graph of W_Λ (obtained by collapsing each double edge of the usual Cayley graph to a single edge). Thus the edges of $\widehat{\Sigma}_\Lambda$ and Σ_Λ are labelled by the elements of S. The *type* of a face of Σ_Λ or a standard subcomplex of $\widehat{\Sigma}_\Lambda$ is defined to be the collection of the labels of edges of this subcomplex.

Remark 3.7 (Alternative description of the Artin complex Δ_Λ)**.** Let $\widetilde{\Sigma}_\Lambda$ be the universal cover of $\widehat{\Sigma}_\Lambda$. Then by the last bullet point above, $\widetilde{\Sigma}^1_\Lambda$ can be identified with the Cayley graph of A_Λ. An *elevation* of a subcomplex of $\widehat{\Sigma}_\Lambda$ to $\widetilde{\Sigma}_\Lambda$ is a connected component of the preimage of this subcomplex under the covering map. Vertices of Δ_Λ are in bijective correspondence with the elevations of the standard subcomplexes of type $\hat{s}$ for $s \in S$, since the vertex set of such an elevation is a left coset $gA_{\hat{s}} \subset A_\Lambda = \widetilde{\Sigma}^0_\Lambda$. Vertices of Δ_Λ span a simplex if their corresponding elevations have non-empty common intersection. We will call these elevations *standard subcomplexes of* $\widetilde{\Sigma}_\Lambda$. By [vdL83], the intersection of a collection of standard subcomplexes of $\widetilde{\Sigma}_\Lambda$ of types S_i is empty or is a standard subcomplex of type $\bigcap_i S_i$.

3.3. **Collapsing hyperplanes.** Let $\mathcal{A}$ be a hyperplane arrangement and let $\mathcal{A}' \subset \mathcal{A}$. Note that each fan of $\mathcal{A}$ is contained in a unique fan of $\mathcal{A}'$. Since the vertices of $b\Sigma_\mathcal{A}$ (the barycentric subdivision of $\Sigma_\mathcal{A}$) correspond to the fans of $\mathcal{A}$, this gives a map from the vertex set of $b\Sigma_\mathcal{A}$ to the vertex set of $b\Sigma_{\mathcal{A}'}$. This map extends to a simplicial map $\kappa = \kappa_{\mathcal{A}'} \colon b\Sigma_\mathcal{A} \to b\Sigma_{\mathcal{A}'}$, which can also be viewed as a piecewise linear map from $\Sigma_\mathcal{A}$ to $\Sigma_{\mathcal{A}'}$.

By the description of the faces of $\Sigma_\mathcal{A}$ in the terms of the simplices of $b\Sigma_\mathcal{A}$ at the beginning of Section 3.1, κ maps each face F of $\Sigma_\mathcal{A}$ onto a face of $\Sigma_{\mathcal{A}'}$ that we denote $\kappa(F)$. Note that if an edge e of $\Sigma_\mathcal{A}$ is dual to a hyperplane outside $\mathcal{A}'$, then $\kappa_{\mathcal{A}'}(e)$ is a vertex, otherwise $\kappa_{\mathcal{A}'}(e)$ is an edge.

Furthermore, for $v, v' \in \Sigma^0_\mathcal{A}$ satisfying $\Pi_F(v') = \Pi_F(v)$, we have $\Pi_{\kappa(F)}(\kappa(v')) = \Pi_{\kappa(F)}(\kappa(v))$. Thus κ induces a piecewise linear map $\widehat{\kappa} \colon \widehat{\Sigma}_\mathcal{A} \to \widehat{\Sigma}_{\mathcal{A}'}$.

3.4. **Central arrangements.** Let $\mathcal{A}$ be a hyperplane arrangement in $\mathbb{R}^n$ that is *central*, that is, all its hyperplanes pass through the origin. Let $H \in \mathcal{A}$ and let $\mathbb{R}^{n-1} \subset \mathbb{R}^n$ be parallel to and distinct from H. The *deconing* $\mathcal{A}_H$ of $\mathcal{A}$ with respect to H is the hyperplane arrangement in $\mathbb{R}^{n-1}$ consisting of the intersections of the elements of $\mathcal{A}$ with $\mathbb{R}^{n-1}$. Note that $\mathcal{A}_H$ is well-defined, since choosing a different parallel hyperplane $\mathbb{R}^{n-1}$ gives rise to a hyperplane arrangement differing from the first one by an affine isomorphism. It is well-known that $M(\mathcal{A}\otimes\mathbb{C})$ is homeomorphic to $M(\mathcal{A}_H \otimes \mathbb{C}) \times \mathbb{C}^*$, where $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$, see e.g. [OT13, Prop 5.1]. Thus $\pi_1 M(\mathcal{A} \otimes \mathbb{C}) \cong \pi_1 M(\mathcal{A}_H \otimes \mathbb{C}) \oplus \mathbb{Z}$. It is also possible to see this isomorphism on the level of the Salvetti complex, where we identify $\Sigma_{\mathcal{A}_H}$ with the subcomplex of $\Sigma_\mathcal{A}$ on one side of H.

Lemma 3.8. *Let $\mathcal{A}$ be a central arrangement and $H \in \mathcal{A}$. Then the inclusion $i \colon \widehat{\Sigma}_{\mathcal{A}_H} \to \widehat{\Sigma}_\mathcal{A}$ is π_1-injective. Moreover, $\pi_1 \widehat{\Sigma}_\mathcal{A} = i_*(\pi_1 \widehat{\Sigma}_{\mathcal{A}_H}) \oplus \mathbb{Z}$.*

Proof. Let $b\widehat{\Sigma}_\mathcal{A}$ denote the barycentric subdivision of $\widehat{\Sigma}_\mathcal{A}$. Recall that, in [Sal87, pp. 608], Salvetti constructed a piecewise linear embedding $\phi : b\widehat{\Sigma}_\mathcal{A} \to M(\mathcal{A} \otimes \mathbb{C})$. He proved in [Sal87, pp. 611] that ϕ is a homotopy equivalence. Let $\mathbb{R}^{n-1} \subset \mathbb{R}^n$ be as above. Then $M(\mathcal{A}_H \otimes \mathbb{C})$ is a subspace of $\mathbb{R}^{n-1} \otimes \mathbb{C}$. We can homotopy ϕ so that $\phi(\widehat{\Sigma}_{\mathcal{A}_H}) \subset M(\mathcal{A}_H \otimes \mathbb{C})$ and $\phi|_{\widehat{\Sigma}_{\mathcal{A}_H}} : \widehat{\Sigma}_{\mathcal{A}_H} \to M(\mathcal{A}_H \otimes \mathbb{C})$ is a homotopy equivalence. Thus the lemma follows from the paragraph preceding its statement. $\square$

3.5. **Line arrangements.** Let $\mathcal{A}$ be a central arrangement of lines in $\mathbb{R}^2$. Let ω be a locally embedded edge-path in $\widehat{\Sigma}_{\mathcal{A}}$, and $\ell \in \mathcal{A}$. An *ℓ-segment* of ω is a maximal subpath mapped to an edge dual to a fan in ℓ under $\widehat{\Sigma}_{\mathcal{A}} \to \Sigma_{\mathcal{A}}$.

Lemma 3.9 ([Fal95, Lem 3.6])**.** *Suppose that P is a locally embedded homotopically trivial edge-loop in $\widehat{\Sigma}_{\mathcal{A}}$, and $\ell \in \mathcal{A}$. Then P contains at least two ℓ-segments.*

A *minimal positive path* in $\widehat{\Sigma}_{\mathcal{A}}$ is a minimal length path between its endpoints that is positively oriented. (The orientation of the edges was introduced at the beginning of Section 3.2 and discussed in Definition 3.6.) Note that the boundary of each 2-cell of $\widehat{\Sigma}_{\mathcal{A}}$ is a union of two minimal positive paths from some vertex x to its antipodal vertex y. We call x the *source* of this 2-cell, and y the *sink*. Let Δ_x be the concatenation of a minimal positive path from x to y and a minimal positive path from y to x. The element represented by Δ_x in $\pi_1(\widehat{\Sigma}_{\mathcal{A}}, x)$ is independent of the choice of the paths.

Below, we denote the edges of the two minimal length paths from x to y in $\Sigma^1_{\mathcal{A}}$ by $e_1 \cdots e_n$ and $d_n \cdots d_1$. For an edge e_i of $\Sigma^1_{\mathcal{A}}$, we label both edges of $\widehat{e}_i$ by e_i. Note that they are oriented in opposite directions. Let z be the common vertex of e_1 and e_2.

Lemma 3.10. *(1) Let P be an edge-path in $\widehat{\Sigma}_{\mathcal{A}}$ from x_1 to x_2. Then $\Delta_{x_1}P$ is homotopic, relative to the endpoints, to $P\Delta_{x_2}$. In particular, Δ_{x_1} is central in $\pi_1(\widehat{\Sigma}_{\mathcal{A}}, x_1)$.*
(2) Paths $\Delta_z e_2^{-1}e_3^{-1}\cdots e_n^{-1}e_n^{-1}\cdots e_3^{-1}e_2^{-1}$ and e_1e_1 represent the same element of $\pi_1(\widehat{\Sigma}_{\mathcal{A}}, z)$.

Proof. Assertion 1 is a consequence of [Del72, Lem 1.26 and Prop 1.27]. For Assertion 2, note that $\Delta_z \sim e_2 \cdots e_n d_1 d_1 e_n \cdots e_2$, where $\sim$ stands for a homotopy relative to the endpoints. The union of the two 2-cells of $\widehat{\Sigma}_{\mathcal{A}}$ with sources the two endpoints of e_1 form a cylinder in $\widehat{\Sigma}_{\mathcal{A}}$ with boundary paths e_1^2 and d_1^2. More precisely, $e_1^2 \sim e_2e_3\cdots e_n d_1^2 e_n^{-1}\cdots e_3^{-1}e_2^{-1}$. Thus

$$\Delta_z \sim e_2e_3\cdots e_n d_1^2 e_n^{-1}\cdots e_3^{-1}e_2^{-1}e_2e_3\cdots e_ne_n\cdots e_2 \sim e_1^2e_2e_3\cdots e_ne_n\cdots e_2.$$

□

Let e_1, d_1 be dual to the fans in $\ell \in \mathcal{A}$. Let $i\colon \widehat{\Sigma}_{\mathcal{A}_l} \to \widehat{\Sigma}_{\mathcal{A}}$ be as in Lemma 3.8.

Lemma 3.11. *We have a short exact sequence*

$$\pi_1(\widehat{\Sigma}_{\mathcal{A}_l}, z) \xrightarrow{i_*} \pi_1(\widehat{\Sigma}_{\mathcal{A}}, z) \xrightarrow{p_*} \pi_1(\widehat{e}_1, z),$$

where $p = \Pi_{\widehat{e}_1}$. In particular, if for a representative P of $\alpha \in \pi_1(z, \widehat{\Sigma}_{\mathcal{A}})$ the path $\Pi_{\widehat{e}_1}(P)$ is homotopically trivial, then α can be represented by a loop in $\widehat{\Sigma}_{\mathcal{A}_l}$.

Proof. Since $\pi_1(\widehat{e}_1, z)$ is isomorphic to $\mathbb{Z}$, this follows from Lemma 3.8, from $\operatorname{im} i_* <$ $\ker p_*$, and from the surjectivity of p_*. □

Lemma 3.12. *Let $\mathcal{A}' = \mathcal{A} \setminus \{l\}$, where l is dual to e_1. Let P be an edge-path in $\widehat{e}_3 \cup \widehat{e}_4 \cup \cdots \cup \widehat{e}_n$. If $\widehat{\kappa}_{\mathcal{A}'}(P)$ is homotopic, relative to the endpoints, into $\widehat{\kappa}_{\mathcal{A}'}(\widehat{e}_j)$ in $\widehat{\Sigma}_{\mathcal{A}'}$, for some $j \neq 1$, then P is homotopic, relative to the endpoints, into $\widehat{e}_j$ in $\widehat{\Sigma}_{\mathcal{A}}$.*

For $j = 2$ this means that P is a homotopically trivial edge-loop by Lemma 3.8.

Proof. Let $f_i = \kappa_{\mathcal{A}'}(e_i)$. By Lemma 3.8, $\widehat{f}_3 \cup \widehat{f}_4 \cup \cdots \cup \widehat{f}_n \subset \widehat{\Sigma}_{\mathcal{A}'}$ is π_1-injective. Thus for $j \neq 2$ the edge-path $\widehat{\kappa}_{\mathcal{A}'}(P)$ is homotopic in $\widehat{f}_3 \cup \widehat{f}_4 \cup \cdots \cup \widehat{f}_n \subset \widehat{\Sigma}_{\mathcal{A}'}$ into $\widehat{f}_j$. Consequently, P is homotopic in $\widehat{e}_3 \cup \widehat{e}_4 \cup \cdots \cup \widehat{e}_n$ into $\widehat{e}_j$.

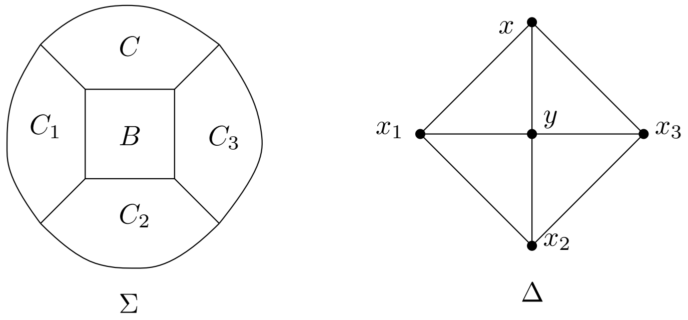


FIGURE 2. Denting

For $j = 2$, since P is contained in $\widehat{e}_3\cup\widehat{e}_4\cup\cdots\cup\widehat{e}_n$ and $\widehat{\kappa}_{\mathcal{A}'}(P)$ is homotopic, relative to the endpoints, into $\widehat{f}_2$, we have that P is an edge-loop. If P is homotopically nontrivial in $\widehat{e}_3\cup\widehat{e}_4\cup\cdots\cup\widehat{e}_n$, then by Lemma 3.8 $\widehat{\kappa}_{\mathcal{A}'}(P)$ is homotopically nontrivial in $\widehat{\Sigma}_{\mathcal{A}'}$. However, by considering the retraction $\Pi_{\widehat{f}_2}\colon \widehat{\Sigma}_{\mathcal{A}'} \to \widehat{f}_2$, we obtain that a loop in $\widehat{f}_2$ cannot be homotopic in $\widehat{\Sigma}_{\mathcal{A}'}$ to a homotopically nontrivial loop in $\widehat{f}_3\cup\widehat{f}_4\cup\cdots\cup\widehat{f}_n$, which is a contradiction. □

4. SOME SUB-ARRANGEMENTS OF THE H_3-ARRANGEMENT

Let Λ be the Coxeter diagram of type H_3, which is the linear graph with consecutive vertices abc and $m_{ab} = 3, m_{bc} = 5$. Let $\mathcal{A}$ be the reflection arrangement in $\mathbb{R}^3$ associated with W_Λ. Denoting the quotient map from the Artin complex $\Delta = \Delta_\Lambda$ to the Coxeter complex $\mathfrak{C} = \mathfrak{C}_\Lambda$ by π, we say that a vertex x of Δ has *face type* C, where C is the face of Σ dual to $\pi(x)$.

We start with describing a procedure of converting an n-cycle in Δ to a concatenation of n words in A_Λ (cf. [Hua24b, Def 6.14]). These n words are well-defined up to an appropriate notion of equivalence.

Construction 4.1. Let $\omega = x_1\cdots x_n$ be a cycle in Δ of type $\hat{s}_1\cdots\hat{s}_n$. For each $i \in \mathbb{Z}/n\mathbb{Z}$, consider a triangle containing x_ix_{i+1} and corresponding $g_i \in A_\Lambda$. Then $g_i = g_{i-1}w_i$ for $w_i \in A_{\hat{s}_i}$ and $w_1\cdots w_n = 1$. A different choice of such triangles would lead to a word $u_1\cdots u_n$ with $u_i = q_{i-1}^{-1}w_iq_i$ for some $q_i \in A_{S\setminus\{s_i,s_{i+1}\}}$. In this case we say that the words $u_1\cdots u_n$ and $w_1\cdots w_n$ are *equivalent.*

Given ω, we construct a homotopically trivial edge-loop $P = P_1\cdots P_n$ in $\widehat{\Sigma}$ as follows. Let $\widetilde{\Sigma}$ be the universal cover of $\widehat{\Sigma} = \widehat{\Sigma}_\Lambda$, with standard subcomplexes $\mathcal{T}_i$ corresponding to x_i. Let $\widetilde{P}_i$ be edge-paths in $\mathcal{T}_i$ from g_{i-1} to g_i representing w_i. We define P_i to be the image of $\widetilde{P}_i$ in $\widehat{\Sigma}$. We have $P_i \subset \widehat{C}_i$, where C_i is the face type of x_i.

Conversely, consider a homotopically trivial edge-loop $P = P_1\cdots P_n$ in $\widehat{\Sigma}$, with the P_i contained in *hosts* $\widehat{C}_i$. Then we can construct a cycle in Δ as follows. For any lift $\widetilde{P}$ of P to $\widetilde{\Sigma}$, each $\widetilde{P}_i$ is contained in a standard subcomplex that is an elevation of $\widehat{C}_i$ corresponding to a vertex x_i of Δ. Then $\omega = x_1\cdots x_n$ is a cycle of Δ.

Definition 4.2. We refer to Figure 2 for the following discussion. Let ω and P be as in Construction 4.1. Suppose that $x_1 \neq x_2 \neq x_3$ are of types $\hat{a}, \hat{c}, \hat{a}$ or $\hat{c}, \hat{a}, \hat{c}$, and P_2 is also contained in $\widehat{B}$ with $B \subset \Sigma$ a square. Then there is a vertex $y \in \Delta^0$ of face type B that is a neighbour of x_1, x_2, x_3. Let C be the face of Σ intersecting B along the edge opposite to $B \cap C_2$. Then there is a vertex $x \in \Delta^0$ of face type C that is a neighbour of y and consequently of x_1, x_3 by Remark 2.1. Replacing x_2 by x in ω is called *denting* x_2 *to* C.

4.1. **Sub-arrangement of type I.**

Definition 4.3. Consider consecutive vertices $\theta_1, \theta_2, \theta_3$ of $\mathfrak{C}$ of types $\hat{a}, \hat{b}, \hat{a}$ in a hyperplane of $\mathcal{A}$. Let $\mathcal{H} \subset \mathcal{A}$ be the collection of hyperplanes passing through at least one of the θ_i, see Figure 3, left. The central arrangement $\mathcal{H}$ in $\mathbb{R}^3$ is called the *sub-arrangement of type I.* Let $H \in \mathcal{H}$ be the hyperplane passing through θ_1 represented as the boundary circle in Figure 3, left. Consider the deconing $\mathcal{H}' = \mathcal{H}_H$, which is a hyperplane arrangement in $\mathbb{R}^2$, see Figure 3, right.

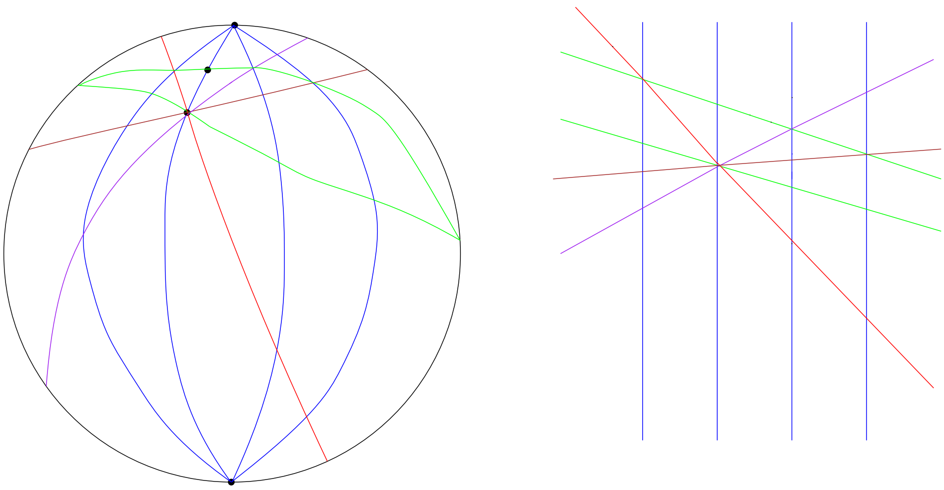


FIGURE 3. Sub-arrangement of type I

Let $X = \Sigma_{\mathcal{H}'}$ and $\widehat{X} = \widehat{\Sigma}_{\mathcal{H}'}$. Denote the four vertical hyperplanes of $\mathcal{H}'$ by h_1, h_2, h_3, h_4, from left to right. Let X_i be the union of all the closed faces of X that intersect h_i. For $i = 1, 2, 3$, let $\widehat{Y}_i = \widehat{X}_i \cap \widehat{X}_{i+1}$. We define subcomplexes X_{ij} of X_i for $j = 1, 2$ as follows. For $i = 1, 3$, let X_{i1} be the subcomplex of X_i coloured white in Figure 4 (the hexagon), and let X_{i2} be the subcomplex of X_i coloured gray (the union of three squares). For $i = 2, 4$, let X_{i1} be the subcomplex of X_i coloured gray (the square on the top), and let X_{i2} be the subcomplex of X_i coloured white.

We now define a simple complex of groups $\mathcal{U}_{12}$ (see [BH99, II.12]) with fundamental group $\pi_1(\widehat{X}_1 \cup \widehat{X}_2)$ as in Figure 5, whose underlying complex U_{12} is the union of two triangles. The vertex groups and the edge groups are the fundamental groups of the subcomplexes of $\widehat{X}$ as labelled in Figure 5, and the remaining local groups are trivial. The morphisms between the local groups are induced by the inclusions of the associated subcomplexes, which are injective by Lemma 3.8. By [Hua24a, §6.1], $\mathcal{U}_{12}$ is developable with $\pi_1\mathcal{U}_{12} = \pi_1(\widehat{X}_1 \cup \widehat{X}_2)$.

Definition 4.4. Let $\mathbb{U}_{12}$ be the development of $\mathcal{U}_{12}$ (cf. [BH99, II.12]). Equivalently, the vertices of $\mathbb{U}_{12}$ (of *face type* $\widehat{X}_{ij}$) correspond to the elevations of $\widehat{X}_{ij}$ to the universal cover of $\widehat{X}_1 \cup \widehat{X}_2$, which we also call *standard subcomplexes of face type* $\widehat{X}_{ij}$. By [Hua24a, Lem 6.3], the intersection of a pair of standard subcomplexes and in fact of any collection of standard subcomplexes) is either empty or connected. Vertices of $\mathbb{U}_{12}$ are neighbours if their corresponding subcomplexes intersect. Vertices of $\mathbb{U}_{12}$ form a triangle, if their corresponding subcomplexes have non-empty common intersection (which is then a single vertex).

Below, we consider the inclusion $\widehat{X} = \widehat{\Sigma}_{\mathcal{H}'} \subset \widehat{\Sigma}_{\mathcal{H}}$ introduced in Section 3.4.

Lemma 4.5 ([Hua24a, Lem 6.6])**.** *The inclusion* $\widehat{X}_1 \cup \widehat{X}_2 \subset \widehat{\Sigma}_{\mathcal{H}}$ *is* π_1*-injective.*

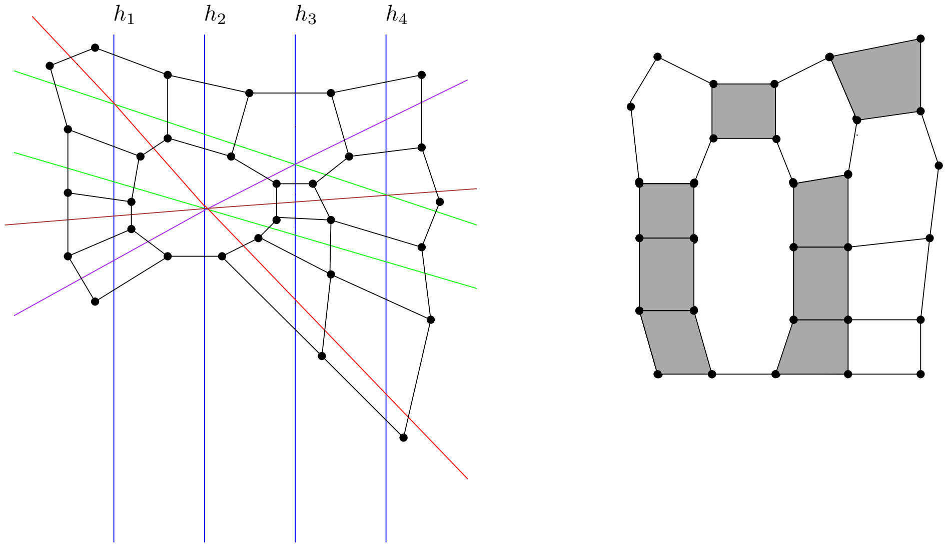


FIGURE 4. Dual complex

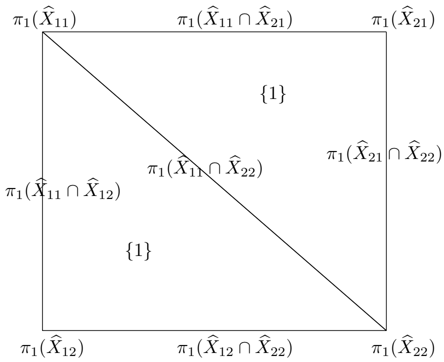


FIGURE 5.

There is a natural action $\pi_1(\widehat{X}_1 \cup \widehat{X}_2) \curvearrowright \mathbb{U}_{12}$, with quotient U_{12}. We equip U_{12} with the piecewise Euclidean metric of the unit square. It pulls back to a piecewise Euclidean metric on $\mathbb{U}_{12}$.

Lemma 4.6 ([Hua24a, Lem 6.4])**.** $\mathbb{U}_{12}$ *is* CAT(0).

4.2. **Converting paths in $\widehat{X}_1 \cup \widehat{X}_2$ to paths in $\mathbb{U}_{12}$.**

Definition 4.7. Let $\mathcal{T} = \{\widehat{X}_{11}, \widehat{X}_{12}, \widehat{X}_{21}, \widehat{X}_{22}\}$. Let P be a homotopically trivial edge-loop in $\widehat{X}_1 \cup \widehat{X}_2$ that decomposes as a concatenation of edge-paths P_i contained in *hosts* $\mathcal{T}_i \in \mathcal{T}$. Let $\widetilde{P}$ be a lift of P to the universal cover of $\widehat{X}_1 \cup \widehat{X}_2$. Then each $\widetilde{P}_i$ is contained in a standard subcomplex of face type $\mathcal{T}_i$ corresponding to a vertex x_i of $\mathbb{U}_{12}$. For each i, we have that x_{i+1} is equal to, or a neighbour of x_i. Thus $\omega = x_1 x_2 \cdots$ is a cycle in $\mathbb{U}_{12}$ *corresponding* to P (or $\widetilde{P}$). Note that ω depends on the decomposition of P, the choice of the hosts and (least importantly) the choice of the lift of P. In practice, we will be looking for a minimal decomposition.

Definition 4.8. Let $\widetilde{P}$ be an edge-path in the universal cover of $\widehat{X}_1 \cup \widehat{X}_2$. We say that $\widetilde{P}$ *starts* (resp. *ends*) with a triangle $\sigma \subset \mathbb{U}_{12}$ if $\widetilde{P}$ starts (resp. ends) with the vertex corresponding to σ.

Under the same notation as in Definition 4.7, since non-empty intersections of standard subcomplexes were connected, we have the following.

Lemma 4.9. *Suppose that $y_1\cdots y_k$ is a locally embedded edge-path of face type $\mathcal{T}_{i1}\cdots\mathcal{T}_{ik}$ in $\mathrm{lk}(x_i,\mathbb{U}_{12})$ from x_{i-1} to x_{i+1}. Then $\widetilde{P}_i$ is homotopic in its elevation of $\mathcal{T}_i$, relative to its endpoints, to a concatenation of locally embedded edge-paths $\widetilde{P}_{i1}\cdots\widetilde{P}_{ik}$, with $\widetilde{P}_{ij}$ projecting into $\mathcal{T}_i\cap\mathcal{T}_{ij}$. Moreover,*

(1) if $2\le j\le k-1$, then $\widetilde{P}_{ij}$ is nontrivial in the sense that its endpoints are distinct,

(2) if $\widetilde{P}_i$ starts with the triangle $x_ix_{i-1}y_2$ and ends with the triangle $x_iy_{k-1}x_{i+1}$, then $\widetilde{P}_{i1}$ and $\widetilde{P}_{ik}$ are trivial.

Note that an analogous result holds for $\mathbb{U}_{12},\widehat{X}_1\cup\widehat{X}_2$ replaced by $\Delta,\widehat{\Sigma}$.

Definition 4.10. Let $\widehat{\kappa}\colon\widehat{\Sigma}\to\widehat{\Sigma}_{\mathcal{H}}$ be as in Section 3.3. Let $\widetilde{\kappa}$ be the induced map between the universal covers. We view $\widehat{X}$ as a subcomplex $\widehat{\Sigma}_{\mathcal{H}}$ as in Section 3.4. Let E be an elevation of $\widehat{X}_1\cup\widehat{X}_2$ to the universal cover of $\widehat{\Sigma}_{\mathcal{H}}$, which is the universal cover of $\widehat{X}_1\cup\widehat{X}_2$ by Lemma 4.5. Consider the subcomplex $\Sigma^*\subset\Sigma$ (depending on $\mathcal{H}$) that is the union of 2-cells C with $\kappa(C)$ a 2-cell of $X_1\cup X_2$. Let Δ^* be the subcomplex of Δ spanned by the vertices of face type in Σ^* whose corresponding standard subcomplexes map under $\widetilde{\kappa}$ into E. Then $\widetilde{\kappa}$ induces a simplicial map $\kappa^*\colon\Delta^*\to\mathbb{U}_{12}$. For a vertex x of Δ^*, we denote $x^{\mathcal{H}}=\kappa^*(x)$. Whenever the dependence on $\mathcal{H}$ is relevant, we write $\Delta^*_{\mathcal{H}},\kappa_{\mathcal{H}}$ instead of Δ^*,κ.

We say a that a 2-cell C of Σ^* and its image $\kappa(C)$ in $X_1\cup X_2$ are *non-collapsed* if $\kappa(C)=X_{11},X_{12}$, or X_{22}. In particular, $\widehat{\kappa}_{|\widehat{C}}$ is a homeomorphism. A vertex of Δ^* (resp. $\mathbb{U}_{12}$) is *non-collapsed* if its face type is non-collapsed. Let Δ^{nc} (resp. $\mathbb{U}_{12}^{\mathrm{nc}}$) be the subcomplex of Δ^* (resp. $\mathbb{U}_{12}$) spanned on non-collapsed vertices.

Lemma 4.11. *Let $x\in\Delta^*$ be non-collapsed of face type C. If $\kappa(C)=X_{11}$ or X_{21}, then the map $\mathrm{lk}(x,\Delta^*)\to\mathrm{lk}(x^{\mathcal{H}},\mathbb{U}_{12})$ induced by κ^* is an isomorphism. Furthermore, if $\kappa(C)=X_{22}$, then the map $\mathrm{lk}(x,\Delta^{\mathrm{nc}})\to\mathrm{lk}(x^{\mathcal{H}},\mathbb{U}_{12}^{\mathrm{nc}})$ induced by κ^* is an isomorphism.*

Proof. By the the description of edges and triangles in Δ and $\mathbb{U}_{12}$ in Remark 3.7 and Definition 4.4, all the relevant neighbours of $x,x^{\mathcal{H}}$ correspond to lines in the isomorphic standard subcomplexes corresponding to $x,x^{\mathcal{H}}$. Two such neighbours span an edge of the link exactly when these lines intersect, which is invariant under the isomorphism. □

4.3. **Sub-arrangement of type II.**

Definition 4.12. Let $\mathcal{A}$ be as at the beginning of the Section 4. Consider consecutive vertices $\theta_1,\dots,\theta_4$ of $\mathfrak{C}$ of types $\hat{a},\hat{c},\hat{b},\hat{c}$ in a hyperplane of $\mathcal{A}$. Let $\mathcal{K}\subset\mathcal{A}$ be the collection of hyperplanes passing through at least one of the θ_i. See Figure 6, left. The central arrangement $\mathcal{K}$ in $\mathbb{R}^3$ is called the *sub-arrangement of type II.* Let $H\in\mathcal{K}$ be the hyperplane passing through θ_1 represented as the boundary circle in Figure 6, left. We consider the deconing $\mathcal{K}'=\mathcal{K}_H$, which is a hyperplane arrangement in $\mathbb{R}^2$, see Figure 6, right.

Let $\mathsf{X}=\Sigma_{\mathcal{K}'}$ and $\widehat{\mathsf{X}}=\widehat{\Sigma}_{\mathcal{K}'}$. We view $\widehat{\Sigma}_{\mathcal{K}'}$ as a subcomplex of $\widehat{\Sigma}_{\mathcal{K}}$, as in Section 3.4. Denote the four vertical hyperplanes of $\mathcal{K}'$ by h_1,h_2,h_3,h_4 (from left to right in Figure 7(I)). Let X_i be the union of all the closed faces of X that intersect h_i. For $2\le i\le 4$, X_i is formed of three 2-cells, denoted from top to bottom by $\mathsf{X}_{i1},\mathsf{X}_{i2}$ and X_{i3}. The following lemma follows from Lemma 3.8 and [Hua24a, Lem 6.8].

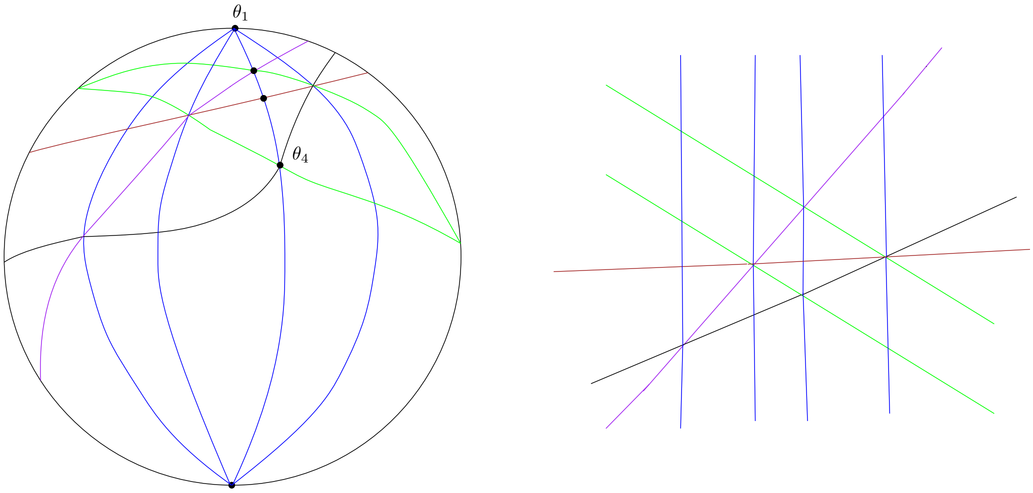


FIGURE 6. Sub-arrangement of type II

Lemma 4.13. *Inclusions* $\widehat{\mathsf{X}}_2 \cup \widehat{\mathsf{X}}_3 \subset \widehat{\Sigma}_{\mathcal{K}}$*,* $\widehat{\mathsf{X}}_3 \cup \widehat{\mathsf{X}}_4 \subset \widehat{\Sigma}_{\mathcal{K}}$*, and* $\widehat{\mathsf{X}}_2 \cup \widehat{\mathsf{X}}_3 \cup \widehat{\mathsf{X}}_4 \subset \widehat{\Sigma}_{\mathcal{K}}$ *are* π_1*-injective.*

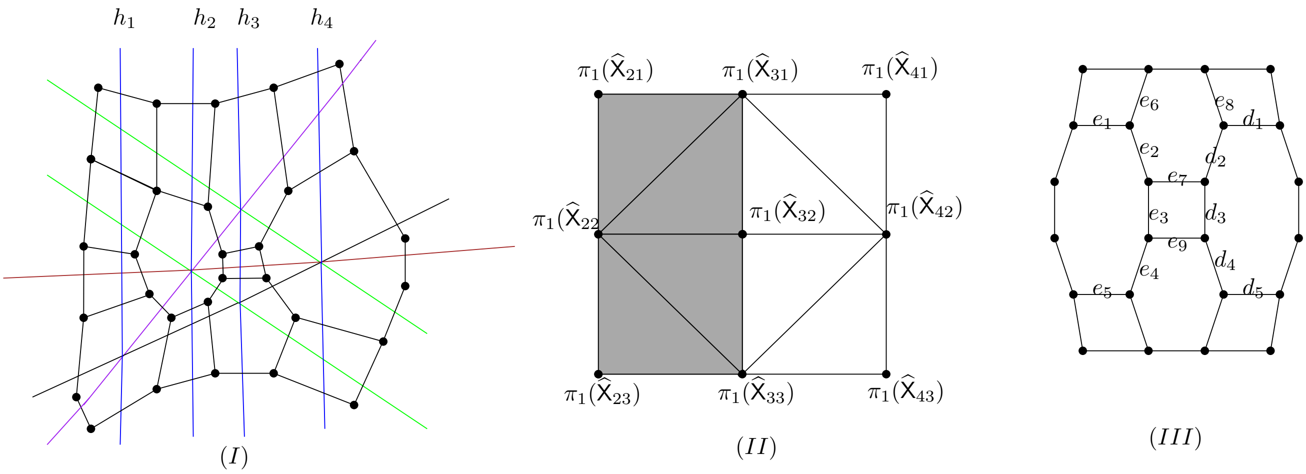


FIGURE 7. Dual complex

Let W_{234} be the Coxeter group of type B_3, and let $\mathcal{A}_{234}$ be its reflection arrangement. Namely, $\mathcal{A}_{234}$ has the following hyperplanes: $x_i = 0$ for $1 \leq i \leq 3$, and $x_i \pm x_j = 0$ for $1 \leq i \neq j \leq 3$. Let Σ_{234} and $\widehat{\Sigma}_{234}$ be the associated dual polyhedron and the Salvetti complex. Let $\mathcal{A}'_{234}$ be the deconing of $\mathcal{A}_{234}$ with respect to $x_1 = 0$. Then we have isomorphisms of combinatorial complexes

$$\Sigma_{\mathcal{A}'_{234}} \cong \mathsf{X}_2 \cup \mathsf{X}_3 \cup \mathsf{X}_4 \text{ and } \widehat{\Sigma}_{\mathcal{A}'_{234}} \cong \widehat{\mathsf{X}}_2 \cup \widehat{\mathsf{X}}_3 \cup \widehat{\mathsf{X}}_4.$$

Let $\mathcal{V}_{234}$ be the simple complex of groups with the underlying complex V_{234} described in Figure 7(II). Note that $\pi_1\mathcal{V}_{234} = \pi_1\widehat{\Sigma}_{\mathcal{A}'_{234}}$. Hence the local groups embed in $\pi_1\mathcal{V}_{234}$, and so $\mathcal{V}_{234}$ is developable. Its development is called the *Falk complex* $\mathbb{V}_{234}$. Let V_{23} be the gray subcomplex of V_{234} in Figure 7(II). Let $\mathcal{V}_{23}$ be the simple complex of groups with the underlying complex V_{23} induced from $\mathcal{V}_{234}$. Then $\pi_1(\mathcal{V}_{23}) = \pi_1(\widehat{\mathsf{X}}_2 \cup \widehat{\mathsf{X}}_3)$. Let $\mathbb{V}_{23}$ be the development of $\mathcal{V}_{23}$. We analogously define $\mathbb{V}_{34}$.

Let $\widetilde{K}_{234}$ (resp. $\widetilde{K}_{23}$) be the universal cover of $\widehat{\mathsf{X}}_2 \cup \widehat{\mathsf{X}}_3 \cup \widehat{\mathsf{X}}_4$ (resp. $\widehat{\mathsf{X}}_2 \cup \widehat{\mathsf{X}}_3$). Vertices of $\mathbb{V}_{234}$ are in bijective correspondence with the elevations of $\widehat{\mathsf{X}}_{ij}$ in $\widetilde{K}_{234}$, for $2 \leq$

$i \leq 4, 1 \leq j \leq 3$, called *standard subcomplexes of face type* $\widehat{\mathsf{X}}_{ij}$. The *face type* of the corresponding vertex of $\mathbb{V}_{234}$ is also $\widehat{\mathsf{X}}_{ij}$. We can describe edges and triangles of $\mathbb{V}_{234}$ (and of $\mathbb{V}_{23}$ and $\mathbb{V}_{34}$) using these standard subcomplexes in the same way as in Definition 4.4. We equip V_{23} with the piecewise Euclidean metric of a rectangle with sides of lengths 1 and 2. It pulls back to a piecewise Euclidean metric on $\mathbb{V}_{23}$. We define an analogous piecewise Euclidean metric on $\mathbb{V}_{34}$. We will use the vocabulary from Definitions 4.7 and 4.8 in the context of $\mathbb{V}_{234}$ as well.

Definition 4.14. Consider the subcomplex $\Sigma^* \subset \Sigma$ (depending on $\mathcal{K}$) that is the union of 2-cells C with $\kappa(C)$ a 2-cell of $\mathsf{X}_2 \cup \mathsf{X}_3 \cup \mathsf{X}_4$. Using $\widehat{\kappa}\colon \widehat{\Sigma} \to \widehat{\Sigma}_{\mathcal{K}}$, we can define a subcomplex $\Delta^* \subset \Delta$ arising from Σ^*, and a simplicial map $\kappa^*\colon \Delta^* \to \mathbb{V}_{234}$ in a similar way as in Definition 4.10. We say a that a 2-cell C of Σ^* and its image $\kappa(C)$ in $\mathsf{X}_2 \cup \mathsf{X}_3 \cup \mathsf{X}_4$ are *non-collapsed* if $\kappa(C) = \mathsf{X}_{21}, \mathsf{X}_{31}, \mathsf{X}_{41}, \mathsf{X}_{32}$, or X_{33} — these are the faces for which $\kappa|_C$ is a homeomorphism. A vertex of Δ^* (resp. $\mathbb{V}_{234}$) is *non-collapsed* if its face type is non-collapsed.

The following has the same proof as Lemma 4.11.

Lemma 4.15. *Let $x \in \Delta^*$ be non-collapsed of face type C. If $\kappa(C) \neq \mathsf{X}_{33}$, then the map $\mathrm{lk}(x, \Delta^*) \to \mathrm{lk}(x^{\mathcal{K}}, \mathbb{V}_{234})$ induced by κ^* is an isomorphism. If $\kappa(C) = \mathsf{X}_{33}$, then this map is an isomorphism onto the subcomplex spanned by the vertices of face types $\widehat{\mathsf{X}}_{22}, \widehat{\mathsf{X}}_{32}$, and $\widehat{\mathsf{X}}_{42}$.*

5. Filling cycles in $\mathbb{V}_{234}$

Let Λ be the Coxeter diagram of type B_3, which is the linear graph with consecutive vertices $s_1s_2s_3$ and $m_{s_1s_2} = 3, m_{s_2s_3} = 4$, and total order $s_1 < s_2 < s_3$. We shortly write $\Lambda = 234$. Let $\mathbb{V}_{234}, V_{234}, \mathsf{X}_i, \widehat{\mathsf{X}}_i$ and $\widehat{\mathsf{X}}_{ij}$ be as in Section 4.3.

The goal of this section is to establish the properties of certain 8-cycles and 10-cycles in $\mathbb{V}_{234}$, namely Propositions 5.10, 5.11, and 5.12. We will start with lemmas on vertex links in Section 5.1, which will be used to study the cycles in Section 5.2.

5.1. Vertex links in $\mathbb{V}_{234}$.

Let $\mathsf{Y} = \mathsf{X}_2 \cup \mathsf{X}_3 \cup \mathsf{X}_4$. We label the edges of Y (and $\widehat{\mathsf{Y}}$) as in Figure 7(III). Identifying $\widehat{\Sigma}^1_{234}$ with the Cayley graph of W_{234}, for each edge e of Y, the two edges of $\widehat{e} \subset \widehat{\mathsf{Y}} \subset \widehat{\Sigma}_{234}$ are oriented in opposite directions.

Lemma 5.1. *Let ω be a locally embedded cycle in the link of a vertex of type $\hat{s}_3$ (resp. $\hat{s}_1$) in $\mathbb{V}_{234}$. Then ω contains at least two vertices of face type $\widehat{\mathsf{X}}_{32}$ (resp. of type $\hat{s}_2$ but not of face type $\widehat{\mathsf{X}}_{32}$).*

Proof. Suppose that ω lies in the link of a vertex of face type $\widehat{X}_{22}$. We apply Lemma 4.9 to ω to produce a locally embedded edge-loop P in $\widehat{\mathsf{X}}_{22}$. By Lemma 3.9 applied to P, the cycle ω has at least two vertices of face type in $\{\widehat{\mathsf{X}}_{21}, \widehat{\mathsf{X}}_{23}\}$. Other cases are analogous. □

Lemma 5.2. *Let x be a vertex of $\mathbb{V}_{234}$ of face type $\widehat{\mathsf{X}}_{22}$. Let ω be a locally embedded n-cycle in $\mathrm{lk}(x, \mathbb{V}_{234})$. Then $n \geq 8$. Moreover, the equality holds if and only if ω corresponds, in the sense of Lemma 4.9, up to a cyclic permutation of vertices, to an edge-loop in $\widehat{\mathsf{X}}_{22}$ of form $e_1^{2k}e_2e_3e_4e_5^{-2k}e_4^{-1}e_3^{-1}e_2^{-1}$ or $e_1^{2k}e_2^{-1}e_3^{-1}e_4^{-1}e_5^{-2k}e_4e_3e_2$. An analogous statement holds for x of face type $\widehat{\mathsf{X}}_{42}$, with e_i replaced by d_i.*

Proof. As before, we apply Lemma 4.9 to ω to produce a locally embedded edge-loop in $\widehat{\mathsf{X}}_{22}$, and then we apply Lemma 3.9 to this edge-loop to deduce that ω has at least two vertices whose face types belong to $\{\widehat{\mathsf{X}}_{21}, \widehat{\mathsf{X}}_{23}\}$, at least two vertices with face type $\widehat{\mathsf{X}}_{31}$, at least two vertices with face type $\widehat{\mathsf{X}}_{32}$, and at least two vertices with face type $\widehat{\mathsf{X}}_{33}$. Hence $n \geq 8$. When $n = 8$, up to a cyclic permutation, the only possible face type of ω is $\widehat{\mathsf{X}}_{21}\widehat{\mathsf{X}}_{31}\widehat{\mathsf{X}}_{32}\widehat{\mathsf{X}}_{33}\widehat{\mathsf{X}}_{23}\widehat{\mathsf{X}}_{33}\widehat{\mathsf{X}}_{32}\widehat{\mathsf{X}}_{31}$. Thus the corresponding edge-loop in $\widehat{\mathsf{X}}_{22}$ is of form $e_1^{2k_1}e_2^*e_3^*e_4^*e_5^{2k_2}e_4^*e_3^*e_2^*$, where k_i and $*$ are non-zero integers. Since $\widehat{\mathsf{X}}_{22}$ is the cover of the presentation complex of a dihedral Artin group corresponding to the pure Artin group, it remains to apply [Cri05, Lem 39]. ☐

We record the following corollary, which will be used in the later sections.

Corollary 5.3. $\mathbb{V}_{23}$ *(and* $\mathbb{V}_{34}$*) are* CAT(0)*.*

Proof. Since $\mathbb{V}_{23}$ is the development of a complex of groups, it is simply connected. It remains to that show that each $\mathrm{lk}(x, \mathbb{V}_{23})$ is CAT(1), i.e. each embedded cycle in $\mathrm{lk}(x, \mathbb{V}_{23})$ has length $\geq 2\pi$. This is clear if x has face type $\widehat{\mathsf{X}}_{21}$, $\widehat{\mathsf{X}}_{32}$, or $\widehat{\mathsf{X}}_{23}$, as its link is a bipartite graph with edge length $\frac{\pi}{2}$. The case where x has face type $\widehat{\mathsf{X}}_{22}$ follows from Lemma 5.2. It remains to consider x of face type $\widehat{\mathsf{X}}_{31}$ (or $\widehat{\mathsf{X}}_{33}$). By Lemma 4.9 and Lemma 3.9, any embedded cycle in $\mathrm{lk}(x, \mathbb{V}_{31})$ has at least two vertices of face type $\widehat{\mathsf{X}}_{21}$, and at least two vertices of face type $\widehat{\mathsf{X}}_{32}$. Any such cycle has ≥ 8 edges (of length $\frac{\pi}{4}$), as desired. ☐

The following lemma has the proof analogous to Lemma 5.2.

Lemma 5.4. *Let x be a vertex of $\mathbb{V}_{234}$ of face type $\widehat{\mathsf{X}}_{31}$. Let ω be a locally embedded n-cycle in $\mathrm{lk}(x, \mathbb{V}_{234})$. Then $n \geq 6$. If $n = 6$, then ω corresponds, up to a cyclic permutation of vertices, to an edge-loop in $\widehat{\mathsf{X}}_{31}$ of form $e_6^{2k}e_2e_7d_2^{-2k}e_7^{-1}e_2^{-1}$, $e_6^{2k}e_2^{-1}e_7^{-1}d_2^{-2k}e_7e_2$, $e_2^{2k}e_7d_2e_8^{-2k}d_2^{-1}e_7^{-1}$, or $e_2^{2k}e_7^{-1}d_2^{-1}e_8^{-2k}d_2e_7$. An analogous statement holds for x of face type $\widehat{\mathsf{X}}_{33}$.*

Lemma 5.5. *Let ω be a locally embedded cycle in the link of a vertex of type $\hat{s}_1$. If ω contains a subpath of type $\hat{s}_2\hat{s}_3\hat{s}_2$, where none of the type $\hat{s}_2$ vertices are of face type $\widehat{\mathsf{X}}_{32}$, then $|\omega| \geq 12$.*

Proof. We can assume that ω lies in the link of a vertex of face type $\widehat{X}_{22}$, and contains a subpath of face type $\widehat{X}_{21}\widehat{X}_{31}\widehat{X}_{21}$. By Lemma 4.9 and Lemma 3.9, ω has at least two vertices of face type $\widehat{X}_{33}$. So if $|\omega| < 12$, then it is a 10-cycle of face type $\widehat{\mathsf{X}}_{31}\widehat{\mathsf{X}}_{21}\widehat{\mathsf{X}}_{31}\widehat{\mathsf{X}}_{21}\widehat{\mathsf{X}}_{31}\widehat{\mathsf{X}}_{32}\widehat{\mathsf{X}}_{33}\widehat{\mathsf{X}}_{32}\widehat{\mathsf{X}}_{33}\widehat{\mathsf{X}}_{32}$ or $\widehat{\mathsf{X}}_{31}\widehat{\mathsf{X}}_{21}\widehat{\mathsf{X}}_{31}\widehat{\mathsf{X}}_{21}\widehat{\mathsf{X}}_{31}\widehat{\mathsf{X}}_{32}\widehat{\mathsf{X}}_{33}\widehat{\mathsf{X}}_{23}\widehat{\mathsf{X}}_{33}\widehat{\mathsf{X}}_{32}$.

In the first case, by Lemma 4.9 we obtain a locally embedded edge-loop in $\widehat{\mathsf{X}}_{22}$ of form $P_\omega = e_1^{2k_1}e_2^{2m}e_1^{2k_2}e_2^*e_3^*e_4^*e_3^*e_4^*e_3^*e_2^*$, where k_1, m, k_2 and $*$ are non-zero integers. By considering $\Pi_{\widehat{e}_1}\colon \widehat{\mathsf{X}}_{22} \to \widehat{e}_1$ (see Definition 3.4), we obtain $k_1 + k_2 = 0$. By Lemma 3.10, $e_1^{2k_1}e_2^{2m}e_1^{2k_2}$ is homotopic in $\widehat{\mathsf{X}}_{22}$ to $(e_2^{-1}e_3^{-1}e_4^{-2}e_3^{-1}e_2^{-1})^{k_1}e_2^{2m}(e_2e_3e_4^2e_3e_2)^{k_1}$. Indeed, since $k_1 + k_2 = 0$, by Lemma 3.10(1), the terms from Lemma 3.10(2) involving Δ will cancel. Since P_ω is homotopically trivial in $\widehat{\mathsf{X}}_{22}$, and the inclusion $\widehat{e}_2 \cup \widehat{e}_3 \cup \widehat{e}_4 \to \widehat{\mathsf{X}}_{22}$ is π_1-injective (Lemma 3.8), we obtain that $P_1 = (e_2^{-1}e_3^{-1}e_4^{-2}e_3^{-1}e_2^{-1})^{k_1}e_2^{2m}(e_2e_3e_4^2e_3e_2)^{k_1}$ and $P_2 = e_2^*e_3^*e_4^*e_3^*e_4^*e_3^*e_2^*$ are homotopic in the graph $\Gamma_{234} = \widehat{e}_2 \cup \widehat{e}_3 \cup \widehat{e}_4$. Given an edge-path P in Γ_{234}, its *reduced representative* is the unique locally embedded edge-path in Γ_{234} homotopic to P in Γ_{234}. However, the reduced representative of P_1 is distinct from P_2, which is already reduced, which is a contradiction.

In the second case, by Lemma 4.9 we obtain a locally embedded edge-loop in $\widehat{\mathsf{X}}_{22}$ of form $P_\omega = e_1^{2k_1}e_2^{2m}e_1^{2k_2}e_2^*e_3^*e_4^*e_5^{2k_3}e_4^*e_3^*e_2^*$, where k_1, m, k_2 and $*$ are non-zero integers. By considering $\Pi_{\widehat{e}_1}\colon \widehat{\mathsf{X}}_{22} \to \widehat{e}_1$, we obtain $k_1 + k_2 + k_3 = 0$. By Lemma 3.10, P_ω is homotopic in $\widehat{X}_{22}$ to

$$(e_2^{-1}e_3^{-1}e_4^{-2}e_3^{-1}e_2^{-1})^{k_1}e_2^{2m}(e_2^{-1}e_3^{-1}e_4^{-2}e_3^{-1}e_2^{-1})^{k_2}e_2^*e_3^*e_4^*(e_4^{-1}e_3^{-1}e_2^{-2}e_3^{-1}e_4^{-1})^{k_3}e_4^*e_3^*e_2^*.$$

This edge-loop is homotopically trivial in $\widehat{X}_{22}$, hence it is homotopically trivial in Γ_{234}. Thus the reduced representative of

$$e_4^*e_3^*e_2^*(e_2^{-1}e_3^{-1}e_4^{-2}e_3^{-1}e_2^{-1})^{k_1}e_2^{2m}(e_2^{-1}e_3^{-1}e_4^{-2}e_3^{-1}e_2^{-1})^{k_2}e_2^*e_3^*e_4^*$$

is $(e_4e_3e_2^2e_3e_4)^{k_3}$, which contradicts $m \neq 0$. $\square$

Lemma 5.6. *Let $D \to \mathbb{V}_{234}$ be a minimal disc diagram with an edge xy of type $\hat{s}_1\widehat{\mathsf{X}}_{32}$ lying in triangles xyz, xyz' of D. Then x, y, z, z' cannot be simultaneously interior vertices of D with degrees $8, 4, 6, 6$.*

Proof. For triangles δ_1, δ_2 of $\mathbb{V}_{234}$ sharing an edge τ, and corresponding vertices x_1, x_2 of $\widetilde{K}_{234}$, let $P(\delta_1, \delta_2)$ denote the image in $\widehat{\Sigma}_{234}$ of the embedded edge-path from x_1 to x_2 in the line of $\widetilde{K}_{234}$ corresponding to τ. We assume without loss of generality that x has face type $\widehat{\mathsf{X}}_{22}$. We argue by contradiction and refer to Figure 8(I). Without loss of generality, we can assume that z' has face type $\widehat{\mathsf{X}}_{33}$. Then Lemma 5.2 implies that z has face type $\widehat{\mathsf{X}}_{31}$. Moreover, either $P(\delta_0, \delta_1) = e_4$, $P(\delta_1, \delta_2) = e_3$, and $P(\delta_2, \delta_3) = e_2$; or $P(\delta_0, \delta_1) = e_4^{-1}$, $P(\delta_1, \delta_2) = e_3^{-1}$, and $P(\delta_2, \delta_3) = e_2^{-1}$. We only discuss the former case, since the latter is similar. Applying Lemma 5.4 to the 6-cycles in the link of z and z' implies that $P(\delta_4, \delta_2) = e_7$ and $P(\delta_1, \delta_5) = e_9$. On the other hand, since $\widehat{\mathsf{X}}_{32}$ is a product of two oriented circles, and the degree of y in D is 4, $P(\delta_4, \delta_2) = e_7$ implies $P(\delta_5, \delta_1) = e_9$, which is a contradiction. $\square$

5.2. **Filling special cycles in $\mathbb{V}_{234}$.** We induce the partial order on the vertex set $\mathbb{V}_{234}^0$ from Δ_{234}^0 via the inclusion $\mathbb{V}_{234} \subset \Delta_{234}$. The map $\pi\colon \Delta_{234} \to \mathfrak{C}$ sends $\mathbb{V}_{234}$ to V_{234}.

Lemma 5.7. *$\mathbb{V}_{234}^0$ is bowtie free.*

Proof. Given distinct $x_1, x_2, y_1, y_2 \in \mathbb{V}_{234}^0$ with $x_i \leq y_j$ for $1 \leq i, j \leq 2$, there is $z \in \Delta_{234}^0$ such that $x_1, x_2 \leq z \leq y_1, y_2$, by Theorem 2.9. Since $\pi(z)$ is a neighbour or equal to each of $\pi(x_1), \pi(x_2), \pi(y_1), \pi(y_2) \in V_{234}^0$, we have $\pi(z) \in V_{234}^0$. Hence $z \in \mathbb{V}_{234}^0$, as desired. $\square$

Lemma 5.8. *Let $\omega = (x_i)_{i=1}^6$ be a locally embedded cycle in $\mathbb{V}_{234}$ of type $\hat{s}_1\hat{s}_3\hat{s}_1\hat{s}_3\hat{s}_2\hat{s}_3$ or $\hat{s}_1\hat{s}_3\hat{s}_2\hat{s}_3\hat{s}_2\hat{s}_3$, angle π at x_5, and angle $\geq \frac{3\pi}{4}$ at x_6. Then ω is embedded and bounds a diagram in $\mathbb{V}_{234}$ as in Figure 8 (II). Furthermore, there is no locally embedded cycle in $\mathbb{V}_{234}$ of type $\hat{s}_2\hat{s}_3\hat{s}_2\hat{s}_3\hat{s}_2\hat{s}_3$.*

Proof. For the first assertion, by the upward flag property in Theorem 2.8, there is $z \in \Delta_{234}^0$ of type $\hat{s}_3$ that is a common upper bound for x_1, x_3, and x_5. If $z \neq x_6$, then by the bowtie free property in Theorem 2.8 applied to $x_1zx_5x_6$, we obtain that x_1 is a neighbour of x_5, contradicting the angle assumption at x_6. Thus $z = x_6$. By Lemma 5.7 applied to $x_3x_4x_5x_6$ and to $x_1x_2x_3x_6$, we obtain that x_3 is a neighbour of x_5, and there is $w \in \mathbb{V}_{234}^0$ of type $\hat{s}_2$ that is neighbour of each of x_1, x_2, x_3, x_6. Then the first assertion follows. The furthermore assertion is proved similarly. $\square$

Lemma 5.9. *Let $\omega = (x_i)_{i=1}^6$ be a cycle in $\mathbb{V}_{234}$ of type $\hat{s}_1\hat{s}_3\hat{s}_1\hat{s}_3\hat{s}_1\hat{s}_3$ such that the face type of x_3 is distinct from that of x_1 and x_5. Then*

(1) $x_2 = x_4$, or
(2) x_2 and x_4 are connected in $\mathrm{lk}(x_3, \mathbb{V}_{234})$ by a locally embedded path of length 2 with middle vertex of face type $\widehat{\mathsf{X}}_{32}$, or
(3) There is a vertex z of $\mathbb{V}_{234}$ such that the cycle obtained from ω by replacing x_6 with z bounds a reduced disc diagram in Figure 9 on the right, with the interior vertices of face type $\widehat{\mathsf{X}}_{32}$.

Proof. Suppose $x_2 \neq x_4$. By the upward flag property in Theorem 2.8, there is a vertex z of Δ_{234} of type $\hat{s}_3$ that is a neighbour of all x_1, x_3, x_5. Since $\pi(z)$ is a neighbour of $\pi(x_1)$ and $\pi(x_3)$, it has face type $\widehat{\mathsf{X}}_{31}$ or $\widehat{\mathsf{X}}_{33}$, and so it belongs fo V^0_{234}. Consequently, we have $z \in \mathbb{V}^0_{234}$. If $z = x_2$ or x_4, then we have (2) by Lemma 5.7. Otherwise, still by Lemma 5.7, we have the disc diagram in Figure 9 on the left. If the subdiagram on the right is not reduced, then the two interior vertices are equal and so we have (2). Otherwise, we have (3). □

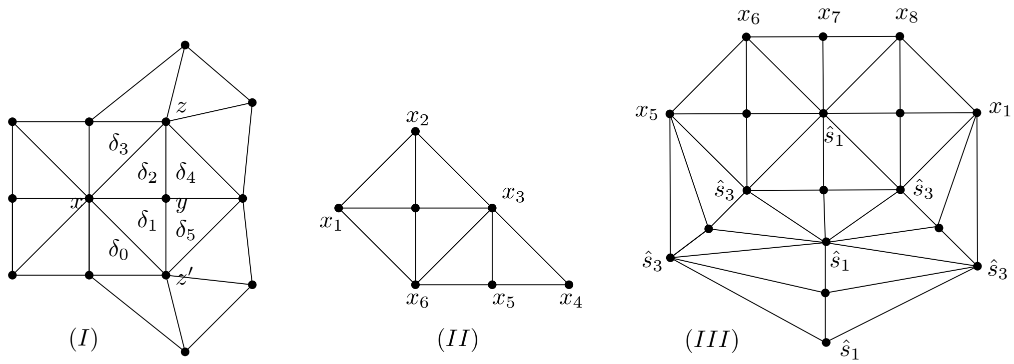


FIGURE 8.

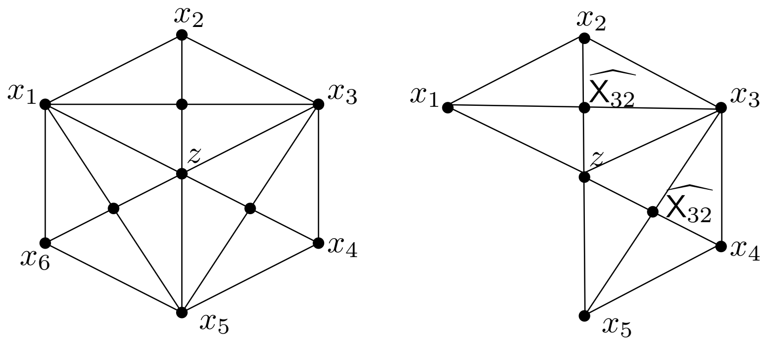


FIGURE 9.

The following propositions will be proved simultaneously.

Proposition 5.10. *Let $\omega = (x_i)_{i=1}^8$ be a cycle in $\mathbb{V}_{234}$ of type $\hat{s}_1\hat{s}_3\hat{s}_1\hat{s}_3\hat{s}_1\hat{s}_3\hat{s}_2\hat{s}_3$. Suppose that*

(1) ω has angle $\geq \frac{3\pi}{4}$ at x_6 and x_8,
(2) ω has angle π at x_7, and
(3) ω has angle $\geq \frac{\pi}{2}$ at x_1 and x_5.

Then ω is embedded, and it bounds a minimal disc diagram $D \to \mathbb{V}_{234}$ such that D embeds as a subdiagram of Figure 8(III) with $x_5x_6x_7x_8x_1$ mapping to the indicated path.

Proposition 5.11. *Let $\omega = (x_i)_{i=1}^{10}$ be a cycle in $\mathbb{V}_{234}$ of type $\hat{s}_1\hat{s}_3\hat{s}_2\hat{s}_3\hat{s}_2\hat{s}_3\hat{s}_1\hat{s}_3\hat{s}_2\hat{s}_3$. Then the following properties cannot hold simultaneously:*

(1) ω has angle $\geq \frac{\pi}{2}$ at x_1 and x_7,
(2) ω has angle π at x_3, x_4, x_5, and x_9, and
(3) ω has angle $\geq \frac{3\pi}{4}$ at x_2, x_6, x_8 and x_{10}.

Proposition 5.12. *Let $\omega = (x_i)_{i=1}^{10}$ be a locally embedded cycle in $\mathbb{V}_{234}$ of type $\hat{s}_3\hat{s}_2\hat{s}_3\hat{s}_2\hat{s}_3\hat{s}_2\hat{s}_3\hat{s}_1\hat{s}_3\hat{s}_1$. Assume that x_2 has face type $\widehat{\mathsf{X}}_{32}$, but x_4 and x_6 do not have* *face type* $\widehat{\mathsf{X}}_{32}$*. Suppose that* ω *has angle* π *at* x_2, x_3*,* x_4 *and* x_6*. Then* ω *has angle* $\frac{\pi}{4}$ *at* x_1*.*

To prove Propositions 5.10, 5.11, and 5.12, we will consider minimal disc diagrams $D \to \mathbb{V}_{234}$ with boundary ω. First note that ω is embedded by Lemma 5.8. Thus D is homeomorphic to a disc.

Definition 5.13. A *splitting system* of a minimal disc diagram $D \to \mathbb{V}_{234}$ is the preimage under $D \to \mathbb{V}_{234}$ of all straight line segments in the triangles xyz of $\mathbb{V}_{234}$ of type $\hat{s}_1\hat{s}_2\hat{s}_3$ joining the midpoint of xz with the midpoint of xy, for y of face type $\widehat{\mathsf{X}}_{32}$, or with the midpoint of yz, for y not of face type $\widehat{\mathsf{X}}_{32}$. Equivalently, we can define the splitting system in the following way. Consider the complex V_{234} illustrated in Figure 10(I), where the vertices of type $\hat{s}_i$ are labelled i and the vertices of face type $\widehat{\mathsf{X}}_{32}$ are circled. Then the splitting system of $D \to \mathbb{V}_{234}$ is the preimage of the dashed lines under the composition $D \to \mathbb{V}_{234} \to V_{234}$. Note that the splitting system is a union of arcs, starting and ending on ∂D, and (possibly) circles.

The union of all the edges of D disjoint from the splitting is the *core graph* of $D \to \mathbb{V}_{234}$. In other words, the core graph of $D \to \mathbb{V}_{234}$ is the preimage of the thickened lines in Figure 10(I) under the composition $D \to \mathbb{V}_{234} \to V_{234}$.

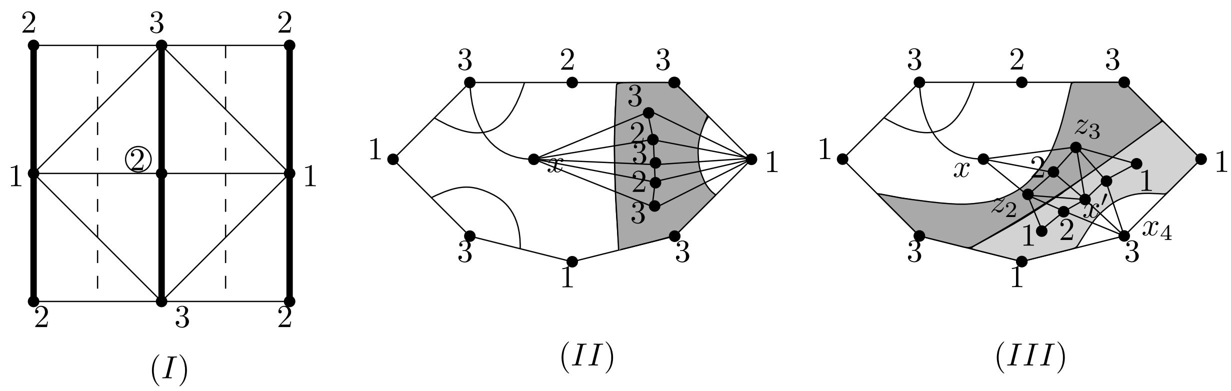


Figure 10.

Remark 5.14. By Lemma 5.1, each vertex of the core graph lying in $\mathrm{int}D$ has degree ≥ 2 in the core graph. In other words, all leaves of the core graph lie in ∂D.

Lemma 5.15. *(i) The splitting system contains no circles.*

(ii) The core graph is a forest.

(iii) Let x_i *be a vertex of* ∂D *of type* $\hat{s}_2$ *with distinct neighbours* x_{i-1}, x_{i+1} *both of type* $\hat{s}_1$ *or* $\hat{s}_3$*. Then there is no arc in the splitting system joining the midpoints of* $x_{i-1}x_i$ *and* x_ix_{i+1}*.*

(iv) Let x_i *be a vertex of* ∂D *of type* $\hat{s}_1$ *or* $\hat{s}_3$*. If there is an arc* β *in the splitting system joining the midpoints of* $x_{i-1}x_i$ *and* x_ix_{i+1}*, then the intersection of the core graph with the connected component* R *of* $D \setminus \beta$ *containing* x_i *consists only of* x_i*.*

(v) Let $x_{i-1}x_ix_{i+1}$ *be a path of* ∂D *of type* $\hat{s}_3\widehat{\mathsf{X}}_{32}\hat{s}_3$*. If there is an arc* β *in the splitting system joining the midpoints of* $x_{i-2}x_{i-1}$ *and* $x_{i+1}x_{i+2}$*, then the intersection of the core graph with the connected component of* $D \setminus \beta$ *containing* x_i *consists only of* $x_{i-1}x_ix_{i+1}$*. Similarly, if* $x_{i-2} \cdots x_{i+2}$ *is a path of* ∂D *of type* $\hat{s}_3\widehat{\mathsf{X}}_{32}\hat{s}_3\widehat{\mathsf{X}}_{32}\hat{s}_3$*, and there is an arc* β *in the splitting system joining the midpoints of* $x_{i-3}x_{i-2}$ *and* $x_{i+2}x_{i+3}$*, then the intersection of the core graph with the connected component of* $D \setminus \beta$ *containing* x_i *consists only of* $x_{i-2} \cdots x_{i+2}$*.*

(vi) *If a connected component Q of the complement in D of the splitting system contains exactly two vertices of ∂D and both of them are of type $\hat{s}_3$, then the intersection of the core graph with Q is an arc ending at these vertices.*

Proof. To prove (i) and (ii), consider an innermost cycle β in either the splitting system or the core graph. Note that the open region $R \subset D$ bounded β contains a point of the core graph or the splitting system. Since all connected components of the splitting system in R are circles, by the innermost assumption we have that β lies in the splitting system, and each connected component of the core graph in R is a tree. This contradicts Remark 5.14.

For (iii), assume without loss of generality that x_i has type $\hat{s}_2$ but not $\widehat{\mathsf{X}}_{32}$ and x_{i-1}, x_{i+1} are of type $\hat{s}_3$. If β were such an arc, consider the connected component R of $D \setminus \beta$ containing x_i. By (ii), each connected component of the core graph in R is a tree. By Remark 5.14, this connected component equals x_i. Hence x_i does not have a neighbour of type $\hat{s}_1$, which is impossible for $x_{i-1} \neq x_{i+1}$. The proofs of (iv),(v), and (vi) are analogous. □

Lemma 5.15(i) gives a bound on the number of the connected components of the splitting system, since each of them is an arc starting and ending in ∂D. In Propositions 5.10, 5.11, and 5.12, the number of points in the intersection of the splitting system with ω is ≤ 10. Up to a homeomorphism of D, each splitting system corresponds to a perfect non-crossing matching of these points. We illustrate the ones satisfying Lemma 5.15(ii,iii) in Figure 11 and Figure 12 below. In Proposition 5.10 we consider cases A and F, depending on whether the vertex x_7 has type $\widehat{\mathsf{X}}_{32}$ (then it is circled) or not. Similarly, in Proposition 5.11 we distinguish cases B, C, D, G,H, and I, depending on which vertices of ω are of face type $\widehat{\mathsf{X}}_{32}$ (they are circled). We will now gradually analyse all these 42 diagrams, excluding most of them.

Proof of Propositions 5.10, 5.11, and 5.12. In Proposition 5.11, assume by contradiction that all (1)-(3) hold. In Proposition 5.12, assume that ω has angle $\geq \frac{3\pi}{4}$ at x_1. Consider a minimal disc diagram $D \to \mathbb{V}_{234}$ with boundary ω. We will reach a contradiction for all the diagrams illustrated in Figures 11 and 12, except for diagrams A6, F3, and F4.

In diagram C3, the core graph in the shaded region cannot have a leaf at x_8 (or at x_{10}). Otherwise, considering the triangle yx_7x_8 of D, by assumption (3) of Proposition 5.11, the vertex y would not be of face type $\widehat{\mathsf{X}}_{32}$. Consequently, the edge x_8y would intersect a splitting curve that also intersects the edge x_7y, and so the edge x_7y would intersect two splitting curves, which is a contradiction. Thus, by Lemma 5.15(ii) the core graph in the shaded region is of the form indicated by the thickened line in Figure 11.

In most of the diagrams, we indicated an edge (or edges) x_ix with $x \neq x_{i-1}, x_{i+1}$ of type $\hat{s}_1$, which exists by the assumption on the angles. In diagrams D1, D3, E1, E2, E4, G2, and G3, there are at least two such edges and we denote by x_4x the first one in the order around x_4 indicated in Figure 11 and Figure 12. Let x_4y be the second such edge. Note that x and y lie on the same side of the arc of the splitting system intersecting the edge x_4x_5 (which is clear for diagrams E1, E4, and G3). Otherwise, for diagrams D1, D3, E2, and G2, considering the arc of the splitting system intersecting x_4y, we would obtain $y = x_1$, contradicting Lemma 5.7.

If $x \in \partial D$, then we can appeal to Lemma 5.7 and Lemma 5.8 to reach the conclusion of Proposition 5.10, or a contradiction with one of the assumptions on the angles. Thus from now on we assume $x \in \mathrm{int}D$. This excludes diagrams F1 and I1, where x_8 cannot have an interior neighbour x of type $\hat{s}_1$. By Lemma 5.5, the degree of x is at least 12. In other words, x has at least 6 neighbours of type $\hat{s}_3$.

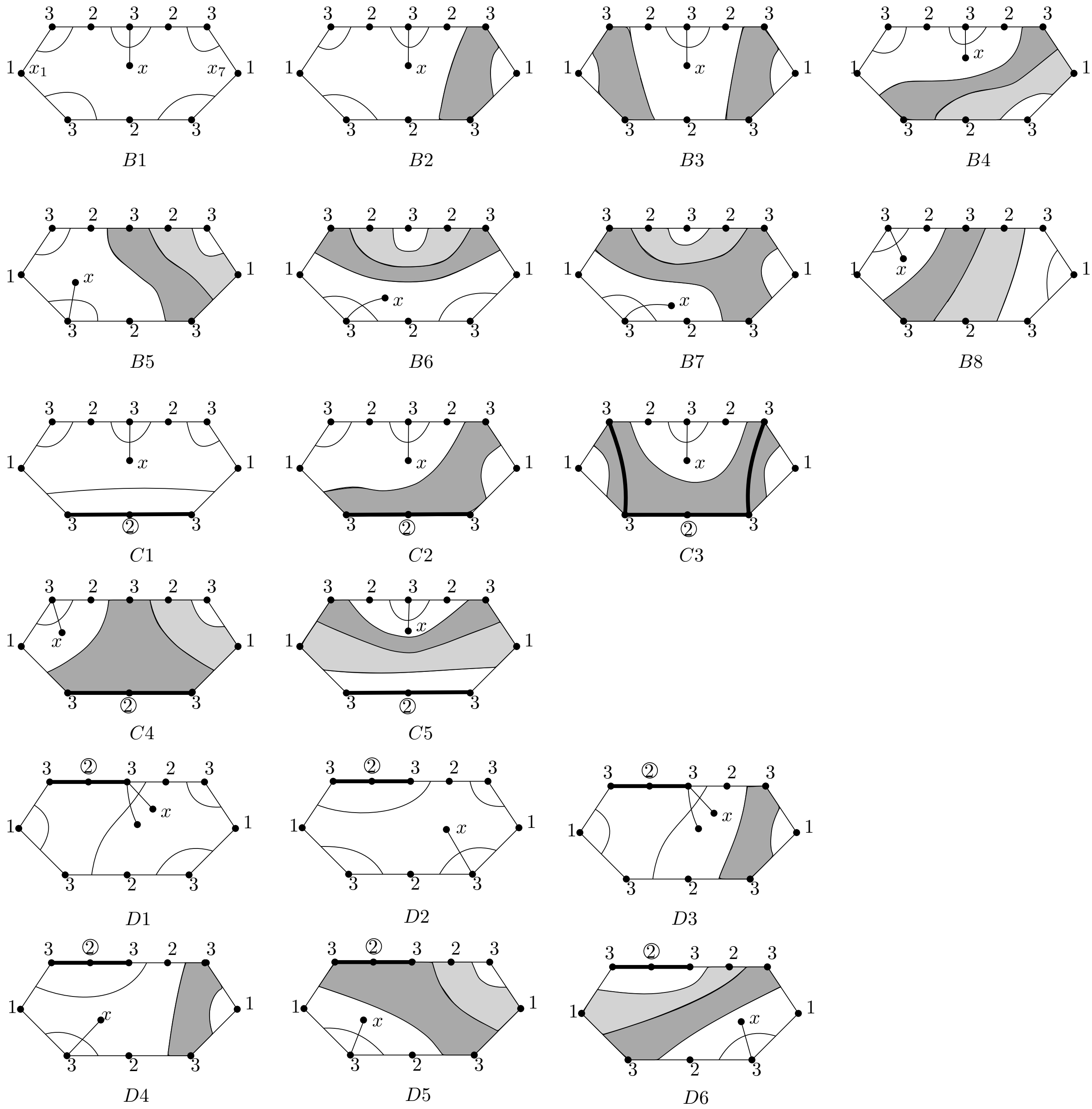


FIGURE 11.

By Lemma 5.15(iv,v), since any edge between x and a vertex of type $\hat{s}_3$ intersects an arc of the splitting system, in diagrams A1, B1, C1, D1, D2, E1, E2, G1, G2, and H1, the vertex x can have at most 5 neighbours of type $\hat{s}_3$, which is a contradiction.

In diagram E3, we consider the first two edges x_3x', x_3x'' of type $\hat{s}_3\hat{s}_1$ in the order around x_4 indicated in Figure 12. We claim that the vertex x'' equals to x_9. Otherwise, x'' lies in the light-shaded region. Since all the vertices in the light-shaded region are neighbours of x_6, the vertex x has 4 neighbours, which contradicts the $n \geq 8$ part of Lemma 5.2, and justifies the claim. Since x' has at least 8 neighbours,

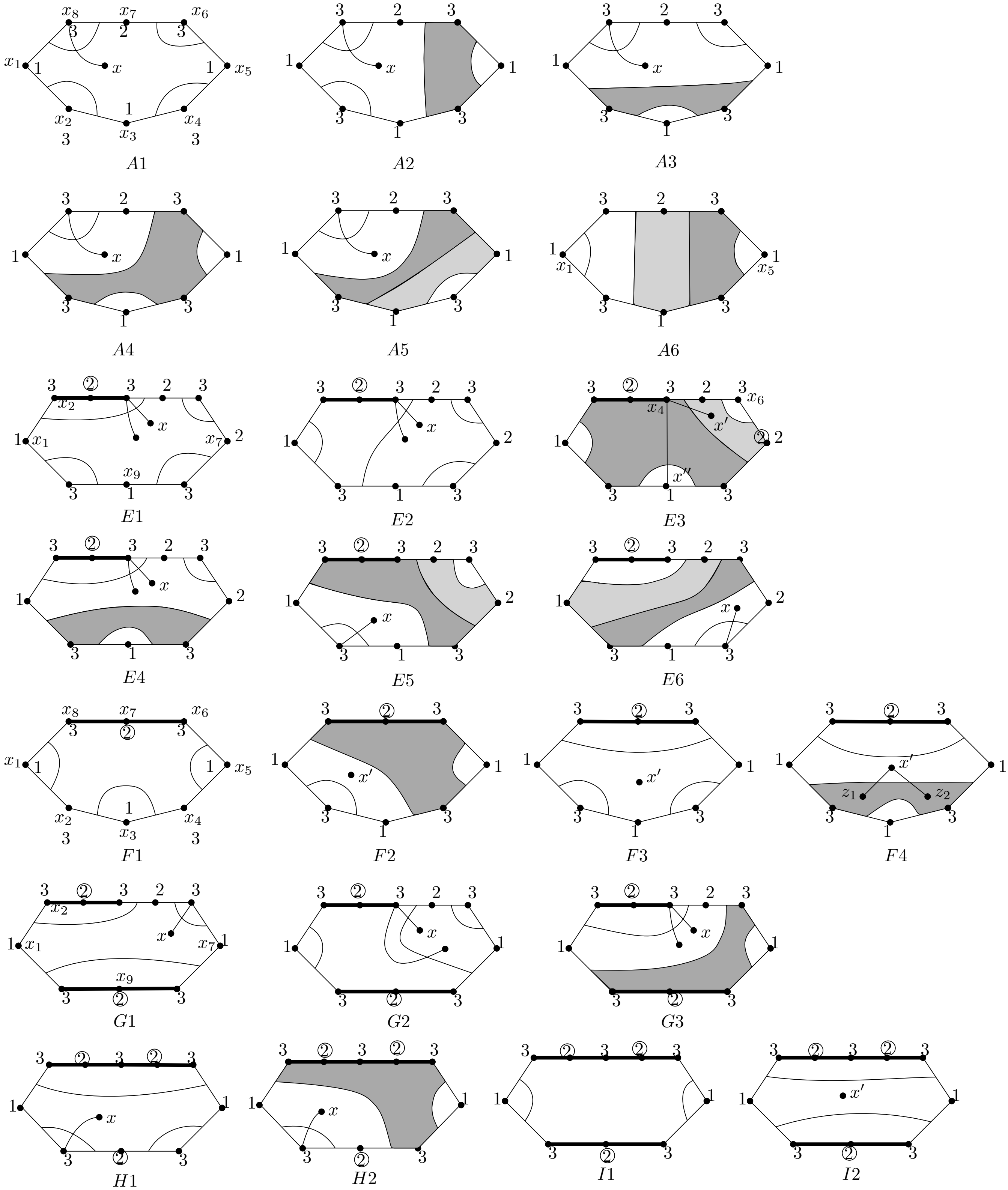


FIGURE 12.

it has at least 4 neighbours of type $\hat{s}_3$, all of which, except for x_6, lie in the shaded region. Consequently, x' has a neighbour z of type $\hat{s}_3$ in the interior of the shaded region. Since the two neighbours of z in the core graph are neighbours of both x'' and x', the vertex z has 4 neighbours, contradicting the $n \geq 6$ part of Lemma 5.4.

In diagrams A2, A3, D3, E4 (resp. B2, B3, D4), the vertex x has at least 4 (resp. at least 3) consecutive type $\hat{s}_3$ neighbours z_j in one of the shaded regions, labelled according to their order around x. By Lemma 5.15(vi), only the first and the last of z_j might lie in ∂D. Thus, except for the first and the last one, any z_j has at most two type $\hat{s}_1$ neighbours (one of which is x), and so z_j has 4 neighbours, contradicting Lemma 5.4. See Figure 10(II) for the A2 case. Similarly, in diagrams C2 (resp. A4, C3, G3, H2), the vertex x has at least 4 (resp. at least 5) consecutive type $\hat{s}_3$ neighbours in the shaded region, one of which contradicts Lemma 5.4. Note that in diagram C3 such a type $\hat{s}_3$ neighbour cannot be simultaneously a neighbour of both x_1 and x_7, by the shape of the connected component of the core graph we established earlier.

In diagrams A5, B4, B7, C4, D5, E5, (resp. B5, B6) the vertex x has at least 5 (resp. at least 4) consecutive type $\hat{s}_3$ neighbours z_j in the shaded region. Except for the first and the last one, and the second or next-to-last one in diagrams C4, D5, E5, and B7, all z_j lie in the interior of the shaded region, and so there are at least two such consecutive z_j. By Lemma 5.4, each such z_j has at least 6 neighbours, so it has at least two type $\hat{s}_1$ neighbours in the light-shaded region. Thus we can find x' of type $\hat{s}_1$ in the light-shaded region that is a common neighbour of two such z_j. See Figure 10(III) for the A5 case. Then x' has at most 3 neighbours of type $\hat{s}_3$ (two of which are among z_j), implying that x' has at most 6 neighbours, which contradicts Lemma 5.2.

In diagrams C5, D6, and E6, the vertex x has at least 5 consecutive type $\hat{s}_3$ neighbours z_j in the shaded region. Except for the first and the last one, each z_j is interior and by the $n \geq 6$ part of Lemma 5.4 has at least two type $\hat{s}_1$ neighbours in the light-shaded region. We can assume that they all have exactly two such neighbours. Indeed, if z_j had 3 consecutive type $\hat{s}_1$ neighbours in the light-shaded region, then the middle one would have at most 6 neighbours (one in the shaded region, two in the light-shaded region, and 3 in the thickened part of ∂D), contradicting Lemma 5.2. Let $x' \neq x$ be a common neighbour of type $\hat{s}_1$ of two such z_j. Then x' has degree 8, which contradicts Lemma 5.6.

Consider now diagram A6. We claim that x_1 has at most one interior neighbour of type $\hat{s}_3$. Otherwise, if we had such consecutive z, z', by Lemma 5.4 each of them would have at least two type $\hat{s}_1$ neighbours in the light-shaded region. If one of them, say z, had degree > 6, then it would have at least 3 type $\hat{s}_1$ neighbours in the light-shaded region. Except for the first and last one, any such neighbour x' would have degree ≥ 12 by Lemma 5.5. Then one if its type $\hat{s}_3$ neighbours in the shaded region would have degree 4, contradicting Lemma 5.4. We can thus assume that the degrees of z and z' are equal to 6. Let x' be the common neighbour of type $\hat{s}_1$ of z, z' in the light-shaded region. Then x' has no common neighbours of type $\hat{s}_3$ with x_1 except for z, z'. By Lemma 5.6, we have that x' has at least 3 common neighbours of type $\hat{s}_3$ with x_5. This contradicts Lemma 5.4 for the middle one of these neighbours and justifies the claim. Analogously, x_5 has at most one interior neighbour of type $\hat{s}_3$. This implies that the length of the core graph component in the light shaded region is ≤ 5 and so it implies the conclusion of Proposition 5.10.

In diagram B8, the vertex x has at least 5 consecutive type $\hat{s}_3$ neighbours z_j in the shaded region. Except for the first and the last one, each z_j is interior and by Lemma 5.4 has at least two type $\hat{s}_1$ neighbours in the light-shaded region. We can assume that they all have exactly two such neighbours since otherwise by Lemma 5.5 one of these neighbours would have degree ≥ 12 and it would have a neighbour of type $\hat{s}_3$ outside the shaded region violating Lemma 5.4. Let $x' \neq x$ be a common neighbour of type $\hat{s}_1$ of two such z_j. By Lemma 5.6, the vertex x' has degree > 8 and so it contains at least 3 type $\hat{s}_3$ neighbours outside the shaded region. This contradicts Lemma 5.4 for the middle one of these neighbours.

In diagrams I2, F2, F3, and F4, since the angle at x_8 is $\geq \frac{3\pi}{4}$, there is neighbour x' of x_8, of type $\hat{s}_1$, in the indicated region. In diagram F2 we have that x' is also a neighbour of x_2 and so we obtain a contradiction as in diagram H2. In diagrams I2, F3, and F4, by Lemma 5.2, the vertex x' has ≥ 4 neighbours of type $\hat{s}_3$. In diagram F3, these can be only x_2, x_4, x_6, and x_8, which, by Lemma 5.7, implies the conclusion of Proposition 5.10. In diagram I2, by Lemma 5.5, both x_8 and x_{10} are neighbours of x'. Since the same holds for the interior neighbours of type $\hat{s}_1$ of x_2 and x_6, we have that both of these neighbours equal x'. But then x_4 has at most one interior vertex of type $\hat{s}_1$, contradiction. In diagram F4, x' has at most two neighbours z_j in the shaded region, since otherwise one of them would have only 4 neighbours, contradicting Lemma 5.4. Thus, by Lemma 5.2, x' must be also a neighbour of x_6 and x_8. In particular, x' is the only interior vertex of type $\hat{s}_1$ in its region. Consequently, if we have $z_1 \neq x_2$, then, by Lemma 5.4, z_1 is a neighbour of x_1 and x_3. Analogously, if we have $z_2 \neq x_4$, then z_1 is a neighour of x_3 and x_5. By Lemma 5.7, this implies the conclusion of Proposition 5.10. □

Corollary 5.16. *Proposition 5.11 remains valid without assumption (1).*

Proof. Suppose $x_6 = x_8$. If $x_5 = x_9$, then, by Lemma 5.8, we have $x_2 = x_{10}$ or $x_4 = x_{10}$. By Lemma 5.7, this contradicts assumptions (2) or (3). Thus we can assume $x_5 \neq x_9$. If $x_2 = x_{10}$, then, by Lemmas 5.8 and 5.7, this contradicts assumption (2). Thus we can assume $x_2 \neq x_{10}$. Let x_3', x_9' be type $\hat{a}$ neighbours of x_3, x_9. The cycle $\omega_8 = x_9'x_{10}x_1x_2x_3'x_4x_5x_6$ satisfies assumptions (2) and (3) of Proposition 5.10. For assumption (1), if ω_8 has angle $\frac{\pi}{4}$ at x_4, then this contradicts assumption (2) for ω. If ω_8 has angle $\frac{\pi}{4}$ at x_6, then by Lemma 5.7 applied to $x_9'x_{10}x_1x_2x_3x_4$, we obtain that x_9' is a neighbour of x_3, contradicting (2) for ω as well. Thus by Proposition 5.10 we have that ω_8 bounds a minimal disc diagram that is a subdiagram of Figure 8(III). By Lemma 5.2, the vertex x_3 lies in the image of that disc diagram. Thus there is a neighbour of type $\hat{a}$ of x_3 and x_5, which again contradicts (2) for ω. □

6. Critical 8-cycles

Let Λ be the linear graph abc with $m_{ab} = 3$, $m_{bc} = 5$, as in Section 4. A *critical 8-cycle* in Δ has type $\hat{a}\hat{c}\hat{a}\hat{c}\hat{a}\hat{c}\hat{b}\hat{c}$ (or, shortly, $(\hat{a}\hat{c})^3\hat{b}\hat{c}$).

Definition 6.1. An embedded critical 8-cycle (x_i) is *admissible* if x_7 is a neighbour of

(1) x_1, x_5, or
(2) x_3.

Note that in Case (2), the vertex x_3 is a neighbour of both x_6 and x_8 by Remark 2.1.

Lemma 6.2. *Let ω be an embedded critical 8-cycle. Under any of the following conditions, ω is admissible.*

(1) The vertex x_3 is a neighbour of x_8 (or x_6).
(2) There is a vertex x of type $\hat{a}$ that is a neighbour of x_2, x_4, and x_7.
(3) Replacing in ω the vertex x_2 by z_2 results in a critical cycle ω_0 that is not embedded or is admissible.
(4) There is a vertex x of type $\hat{a}$ that is a neighbour of x_4 and x_8 (or of x_2 and x_6).
(5) There is a vertex z of type $\hat{c}$ and a vertex x of type $\hat{a}$ such that z a neighbour of x_3 and x, and x is a neighbour of x_7.

(6) *Replacing in ω the vertex x_1 by x results in a critical cycle ω_0 that is not embedded or is admissible.*
(7) *Replacing in ω the vertex x_3 by x results in a critical cycle ω_0 that is not embedded or is admissible.*

Proof. For (1), note that x_3, x_5, x_7 are pairwise upper bounded. By Theorem 2.9, there is $z \in \Delta^0$ of type $\hat{c}$ that is their common upper bound. Applying the bowtie freeness from Theorem 2.9 to $x_3zx_7x_8$, we obtain that x_3 is a neighbour of x_7 or $z = x_8$. In the latter case, x_8 is a neighbour of x_5. Applying the bowtie freeness to $x_5x_6x_7x_8$, we obtain that x_5 is a neighbour of x_7, as desired.

For (2), we can assume $x \neq x_1, x_3, x_5$. By Remark 2.1, x is a neighbour of both x_8 and x_6. Applying the bowtie freeness to $x_1x_2xx_8$, we obtain their common neighbour y_1. Analogously, we obtain a common neighbour y_2 of x, x_4, x_5, x_6, and a common neighbour y of x_2, x_3, x_4, x. Then we have an 8-cycle $x_8y_1x_2yx_4y_2x_6x_7$ in $\mathrm{lk}(x, \Delta)$. Since $\mathrm{lk}(x, \Delta)$ has girth ≥ 10, this 8-cycle is not locally embedded at one of y_1, x_8, x_7, x_6, y_2. Since ω is embedded, this 8-cycle is not locally embedded at x_8 or x_6, which implies that ω is admissible.

For (3), if ω_0 is not embedded, then since ω is embedded, the only possibility is that z_2 equals x_6, x_8, or x_4. In the first two cases, ω is admissible by (1). If $z_2 = x_4$, then, since x_1, x_5, x_7 are pairwise upper bounded, they have a common upper bound z of type $\hat{c}$. If $z \neq x_6$, then by the bowtie freeness applied to $x_5x_6x_7z$ we obtain that x_5 and x_7 are neighbours. If $z = x_6$, then $z \neq x_8$, and analogously x_1 and x_7 are neighbours. If ω_0 is admissible, then so is ω since they share the vertices x_1, x_3, x_5, x_7.

For (4), since x, x_5, x_7 are pairwise upper bounded, they have a common upper bound z of type $\hat{c}$. We can assume that x_5 and x_7 are not neighbours, and so applying the bowtie freeness to $zx_5x_6x_7$, we obtain $z = x_6$, i.e. x is a neighbour of x_6. Applying the bowtie freeness to $x_6x_7x_8x$, we obtain that x_7 is a neighbour of x. Since x_1, x_3, x are pairwise upper bounded, they have a common upper bound z_2 of type $\hat{c}$. Since x is a neighbour of z_2, x_4, x_7, the critical cycle obtained from ω by replacing x_2 with z_2 is either not embedded or is admissible by (2). Thus we are done by (3).

For (5), since x_3, x_5, x are pairwise upper bounded, they have a common upper bound z_4 of type $\hat{c}$. The critical cycle obtained from ω by replacing x_4 with z_4 is either not embedded or is admissible by (4). Thus we are done by (3).

For (6), if ω_0 is not embedded, then either $x = x_3$, in which case ω is admissible by (1), or $x = x_5$, in which case x_5 is a neighbour of x_7 by applying the bowtie freeness to $x_5x_6x_7x_8$. Now assume that ω_0 is embedded. Then x_7 is a neighbour of one of x_3, x_5, x. In the last case, x is a neighbour of x_6, and so ω is admissible by (4).

For (7), if ω_0 is not embedded, then, say, $x = x_1$, and so ω is admissible by (4). If ω_0 is admissible and satisfies Definition 6.1(1), then so does ω. If ω_0 satisfies Definition 6.1(2), then ω is admissible by (2). □

In the remaining part of this section, let $\omega = x_1 \cdots x_8$ be an embedded critical 8-cycle. Let w_i, C_i, P_i be as in Construction 4.1.

The goal of this section is to prove:

Proposition 6.3. *Each embedded critical 8-cycle is admissible.*

Proposition 6.3 follows from Propositions 6.4, 6.5, and 6.6, which are proved in Subsections 6.1, 6.2, and 6.3.

6.1. Case of one decagon.

Proposition 6.4. *Let ω be an embedded critical 8-cycle with only one decagon among the C_i. Then ω is admissible.*

Since $C_1 = C_3 = C_5$, we have that C_i intersects C_1 for all even i. We will also assume that C_7 intersects C_1. Indeed, otherwise we have $C_6 = C_8$, and there is a hyperplane dual to an edge of C_7 and disjoint from all the remaining C_i. Thus, by Lemma 3.5, we have $\Pi_{\widehat{C}_7}(P_i) \subset \widehat{C}_7 \cap \widehat{C}_6$ for $i \neq 7$. Since $\Pi_{\widehat{C}_7}(P)$ is homotopically trivial in $\widehat{C}_7$, this implies that P_7 is homotopic in $\widehat{C}_7$ to a path inside $\widehat{C}_7 \cap \widehat{C}_6$. Thus $x_6 = x_8$, contradicting the assumption that ω is embedded.

Case 1: $C_2 = C_4 = C_6 = C_8$. Let $B \neq C_7$ be the other square intersecting C_1 and C_2. Let $\mathcal{H}$ be the type I sub-arrangement of $\mathcal{A}$ with $\kappa_{\mathcal{H}}(C_1) = X_{22}, \kappa_{\mathcal{H}}(C_2) = X_{11}$, and $\kappa_{\mathcal{H}}(C_7) = X_{21}$. Let $\omega^{\mathcal{H}} = \kappa^*_{\mathcal{H}}(\omega)$. Since ω is locally embedded, and C_i are non-collapsed, $\omega^{\mathcal{H}}$ is locally embedded by Lemma 4.11. In particular, the angle of $\omega^{\mathcal{H}}$ at $x_7^{\mathcal{H}}$ equals π. If the angle of $\omega^{\mathcal{H}}$ at $x_6^{\mathcal{H}}$ or $x_8^{\mathcal{H}}$ equals $\frac{\pi}{4}$, then $x_7^{\mathcal{H}}$ is a neighbour of $x_5^{\mathcal{H}}$ or $x_1^{\mathcal{H}}$. By Lemma 4.11, x_7 is a neighbour of x_5 or x_1, as desired. Thus we can assume that the angles of $\omega^{\mathcal{H}}$ at $x_6^{\mathcal{H}}$ or $x_8^{\mathcal{H}}$ are $\geq \frac{3\pi}{4}$.

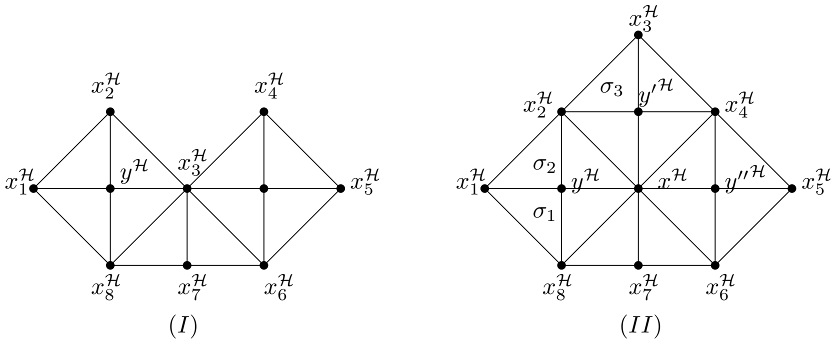


Figure 13.

Since $\mathbb{U}_{12}$ is CAT(0) (Lemma 4.6), $\omega^{\mathcal{H}}$ bounds one of the diagrams in Figure 13. Suppose first that $y^{\mathcal{H}}$ has face type $\widehat{X}_{21}$. Then, by Lemma 4.11, we can lift $y^{\mathcal{H}}$ to $y \in \mathrm{lk}(x_1, \Delta)^0$ that is a neighbour of x_2, x_8. Furthermore, we can lift $x_3^{\mathcal{H}}$ (in case (I)) or $x^{\mathcal{H}}$ (in case (II)) to $x \in \mathrm{lk}(y, \Delta)^0$ that is a neighbour of x_2 and x_8. Thus we can replace in ω the vertex x_1 by x to form another critical 8-cycle ω_0. Note that ω_0 is not embedded or is admissible, since x and x_7 are neighbours by Lemma 4.11. Thus by Lemma 6.2(6), ω is admissible. Hence we can assume that $y^{\mathcal{H}}$ has face type $\widehat{X}_{12}$. Thus, by Lemma 4.11, the vertices x_1 and x_7 are connected in $\mathrm{lk}(x_8, \Delta^*_{\mathcal{H}})$ by a locally embedded path of length three with an interior vertex of face type B.

Let $\mathcal{J}$ be the type I sub-arrangement of $\mathcal{A}$ with $\kappa_{\mathcal{J}}(C_1) = X_{22}, \kappa_{\mathcal{J}}(C_2) = X_{11}$, and $\kappa_{\mathcal{J}}(B) = X_{21}$. Since $\widehat{\kappa}_{\mathcal{J}}$ maps $\widehat{C}_7$ π_1-injectively into $\widehat{X}_{12}$, we have $x_6^{\mathcal{J}} \neq x_8^{\mathcal{J}}$. Thus we can again assume that $\omega^{\mathcal{J}}$ bounds a minimal disc diagram in Figure 13, with $\mathcal{H}$ replaced by $\mathcal{J}$. As before, we can assume that $y^{\mathcal{J}}$ has face type $\widehat{X}_{12}$. Thus by Lemma 4.11, the vertices x_1 and x_7 are connected in $\mathrm{lk}(x_8, \Delta^*_{\mathcal{J}})$ by a locally embedded path of length three with an interior vertex of face type C_7. Since $B \neq C_7$, this contradicts the fact that the girth of $\mathrm{lk}(x_8, \Delta^*_{\mathcal{J}}) = \mathrm{lk}(x_8, \Delta^*_{\mathcal{H}})$ is 8.

Case 2: There are exactly two distinct hexagons among the C_i. Denote these hexagons by D_1 and D_2. To start with, we consider the case where D_1 and D_2 do not intersect a common square. Then $C_6 = C_8$. Assume without loss of generality $C_6 = D_1$. Let B be the square such that $B \cap C_1$ and $D_2 \cap C_1$ are opposite edges of C_1. Note that B intersects D_1. Let $\mathcal{H}$ be the type I sub-arrangement of $\mathcal{A}$ with $\kappa_{\mathcal{H}}(C_1) = X_{22}, \kappa_{\mathcal{H}}(D_1) = X_{11}$, and $\kappa_{\mathcal{H}}(B) = X_{21}$. Note that the vertices of ω of face type D_2 do not belong to $\Delta^*_{\mathcal{H}}$. However, $\kappa_{\mathcal{H}}(D_2)$ is an edge of X_{22}, and thus $\widehat{\kappa}_{\mathcal{H}}(P) \subset \widehat{X}_1 \cup \widehat{X}_2$. Thus declaring that $\widehat{\kappa}_{\mathcal{H}}(P_i)$ is hosted by $\widehat{X}_{22}$ whenever $i \in \{2,4\}$ satisfies $C_i = D_2$, we obtain $\omega^{\mathcal{H}}$ in $\mathbb{U}_{12}$ as in Definition 4.7. If $C_2 = D_2$, then $x_1^{\mathcal{H}} = x_3^{\mathcal{H}}$, and if $C_4 = D_2$, then $x_3^{\mathcal{H}} = x_5^{\mathcal{H}}$. Since at least one of C_2, C_4 equals D_2, we have $|x_1^{\mathcal{H}}, x_5^{\mathcal{H}}| \leq 2\sqrt{2}$. On the other hand, if $\omega^{\mathcal{H}}$ has angle $\geq \frac{3\pi}{4}$ at both $x_6^{\mathcal{H}}$ and $x_8^{\mathcal{H}}$, since by Lemma 4.11 it also has angle π at $x_7^{\mathcal{H}}$, and $\mathbb{U}_{12}$ is CAT(0), the endpoints of the path $x_5^{\mathcal{H}}x_6^{\mathcal{H}}x_7^{\mathcal{H}}x_8^{\mathcal{H}}x_1^{\mathcal{H}}$ are at distance ≥ 4, which is a contradiction. Thus $\omega^{\mathcal{H}}$ has angle $\frac{\pi}{4}$ at one of $x_6^{\mathcal{H}}, x_8^{\mathcal{H}}$. As before we can deduce that x_7 is a neighbour of x_5 or x_1, as desired.

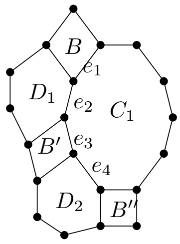


FIGURE 14.

It remains to assume that D_1 and D_2 intersect a common square B', see Figure 14. Assume without loss of generality $C_7 \in \{B', B\}$. Let $\mathcal{K}$ be the type II sub-arrangement of $\mathcal{A}$ with $\kappa_{\mathcal{K}}(D_1) = \mathsf{X}_{31}, \kappa_{\mathcal{K}}(B') = \mathsf{X}_{32}$, and $\kappa_{\mathcal{K}}(C_1) = \mathsf{X}_{42}$. Let $\omega^{\mathcal{K}} = \kappa^*_{\mathcal{K}}(\omega) \subset \mathbb{V}_{34}$. As before, we can assume that the angles at $x_6^{\mathcal{K}}, x_8^{\mathcal{K}}$ are $\geq \frac{3\pi}{4}$. Since $\mathbb{V}_{34}$ is CAT(0) (Corollary 5.3), we obtain that $\omega^{\mathcal{K}}$ bounds, up to a symmetry, one of the minimal disc diagrams in Figure 13, with $\mathcal{H}$ replaced by $\mathcal{K}$, or Figure 15. If ω is not admissible, consider such ω with the smallest area of the disc diagram.

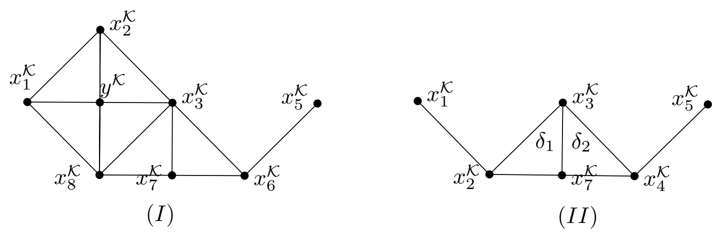


FIGURE 15.

We claim that $x_2^{\mathcal{K}}$ does not have a single interior neighbour $y^{\mathcal{K}}$. Indeed, otherwise by Lemma 4.15 we can lift $y^{\mathcal{K}}$ to $y \in \mathrm{lk}(x_2, \Delta)^0$, and we can lift the neighbour of $y^{\mathcal{K}}$ opposite to $x_2^{\mathcal{K}}$ to $z_2 \in \mathrm{lk}(y, \Delta)^0$. The critical cycle obtained from ω by replacing x_2 with z_2 has appropriate diagram with smaller area and so it is not embedded or it is admissible. Thus ω is admissible by Lemma 6.2(3). This justifies the claim and excludes Figures 13(I) and 15(I). In Figure 13(II), by Lemma 5.2, we have, up to a symmetry,

(i) $C_2 = C_8 = D_1, C_7 = B', C_4 = C_6 = D_2$, or
(ii) $C_6 = C_8 = D_1, C_7 = B, C_2 = C_4 = D_2$.

In Figure 15(II) we must have case (i).

Let $\mathcal{H}$ be the type I sub-arrangement of $\mathcal{A}$ with $\kappa_{\mathcal{H}}(C_1) = X_{22}, \kappa_{\mathcal{H}}(D_2) = X_{11}$, and $\kappa_{\mathcal{H}}(B'') = X_{21}$. Note that, to achieve that, we need to reflect Figure 4 with respect to, say, the line h_2, and then apply an orientation-preserving isometry carrying it appropriately to Figure 14. In case (i), let $\omega^{\mathcal{H}} = \kappa^*_{\mathcal{H}}(\omega) \subset \mathbb{U}_{12}$. Note that $x_7^{\mathcal{H}} = x_8^{\mathcal{H}}$ and so $x_1^{\mathcal{H}}$ is a neighbour of $x_6^{\mathcal{H}}$, and $x_2^{\mathcal{H}}$ has type $\hat{b}$ (since it has face type $\widehat{X}_{12}$).

Furthermore, the 6-cycle $x_1^{\mathcal{H}}x_2^{\mathcal{H}}x_3^{\mathcal{H}}x_4^{\mathcal{H}}x_5^{\mathcal{H}}x_6^{\mathcal{H}}$ is locally embedded at $x_4^{\mathcal{H}}, x_5^{\mathcal{H}}$, and $x_6^{\mathcal{H}}$ by Lemma 4.11. Since $\mathbb{U}_{12}$ is CAT(0), there is a common neighbour $y^{\mathcal{H}}$ of $x_4^{\mathcal{H}}$ and $x_6^{\mathcal{H}}$ in $\mathrm{lk}(x_5^{\mathcal{H}}, \mathbb{U}_{12})$ (otherwise $x_4^{\mathcal{H}}x_5^{\mathcal{H}}x_6^{\mathcal{H}}$ is a geodesic and this 6-cycle cannot 'close up' in $\mathbb{U}_{12}$). See Figure 16 for all the possible minimal disc diagrams bounded by this 6-cycle.

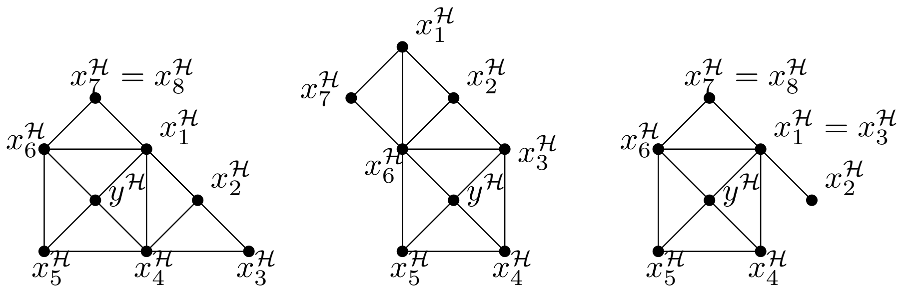


FIGURE 16.

If $y^{\mathcal{H}}$ is of face type $\widehat{X}_{21}$, then by Lemma 4.11 we can lift it to a common neighbour $y \in \mathrm{lk}(x_5, \Delta)$ of x_4 and x_6. If $x_3^{\mathcal{H}}$ is a neighbour of $x_6^{\mathcal{H}}$, then by Lemma 4.11 x_3 is a neighbour of x_6, and so ω is admissible by Lemma 6.2(1). Otherwise, $x_4^{\mathcal{H}}$ is a neighbour of $x_1^{\mathcal{H}}$, and so we can lift $x_1^{\mathcal{H}}$ to a common neighbour $x \in \mathrm{lk}(y, \Delta)$ of x_4 and x_6. Since $x_1^{\mathcal{H}}$ is a neighbour of $x_7^{\mathcal{H}}$, by Lemma 4.11 we have that x is a neighbour of x_7. Hence the cycle obtained from ω by replacing x_5 with x is not embedded or is admissible, and so ω is admissible by Lemma 6.2(6).

If $y^{\mathcal{H}}$ is of face type $\widehat{X}_{12}$, then, by Lemma 4.11:

(1) x_3 and x_5 are connected in $\mathrm{lk}(x_4, \Delta)$ by a locally embedded path of face type $C_1B'C_1$ or $C_1B'C_1B'C_1$, and
(2) x_4 and x_6 are connected in $\mathrm{lk}(x_5, \Delta)$ by a locally embedded nontrivial path all of whose interior vertices have face types B, D_1, or B'.

In Figure 13(II), x_3 and x_5 are connected in $\mathrm{lk}(x_4, \Delta)$ by a locally embedded path of face type $C_1B'C_1B''C_1$, which contradicts (1) or Lemma 3.9. To conclude discussing case (i), we consider Figure 15(II). Let $f_i = \kappa_{\mathcal{K}}(e_i)$. Then $\widehat{\kappa}_{\mathcal{K}}(P_5)$ is homotopic in $\widehat{\mathsf{X}}_{42}$ into $\widehat{f}_4$. On the other hand, by (2), after possibly replacing the w_i by equivalent words, we can choose P_5 to be an edge-loop in $\widehat{e}_1 \cup \widehat{e}_2 \cup \widehat{e}_3$ that is homotopically nontrivial in $\widehat{C}_1$. This contradicts Lemma 3.12.

In case (ii), note that $\kappa_{\mathcal{H}}(B)$ is an edge of X_{12}. Thus declaring that $\widehat{\kappa}_{\mathcal{H}}(P_7)$ is hosted by $\widehat{X}_{12}$, we obtain $\omega^{\mathcal{H}}$ in $\mathbb{U}_{12}$ as in Definition 4.7. Note that $x_6^{\mathcal{H}} = x_7^{\mathcal{H}} = x_8^{\mathcal{H}}$. The 6-cycle $x_1^{\mathcal{H}}x_2^{\mathcal{H}}x_3^{\mathcal{H}}x_4^{\mathcal{H}}x_5^{\mathcal{H}}x_6^{\mathcal{H}}$ is locally embedded at $x_2^{\mathcal{H}}, x_3^{\mathcal{H}}$, and $x_4^{\mathcal{H}}$. As before, since $\mathbb{U}_{12}$ is CAT(0), there is a common neighbour $y^{\mathcal{H}}$ of $x_2^{\mathcal{H}}$ and $x_4^{\mathcal{H}}$ in $\mathrm{lk}(x_3^{\mathcal{H}}, \mathbb{U}_{12})$. Moreover, this 6-cycle has angle $\frac{\pi}{2}$ at $x_4^{\mathcal{H}}$, in which case $x_2^{\mathcal{H}}, x_5^{\mathcal{H}}$ are neighbours, or angle π at $x_4^{\mathcal{H}}$, in which case $x_1^{\mathcal{H}}, x_4^{\mathcal{H}}$ are neighbours.

If $y^{\mathcal{H}}$ is of face type $\widehat{X}_{21}$, then by Lemma 4.11 we can lift it to a common neighbour $y \in \mathrm{lk}(x_5, \Delta)$ of x_4 and x_6. The link of $y^{\mathcal{H}}$ contains neighbours $x_2^{\mathcal{H}}, x_5^{\mathcal{H}}$ or $x_1^{\mathcal{H}}, x_4^{\mathcal{H}}$. By Lemma 4.11, x_2, x_5 are neighbours or x_1, x_4 are neighbours, and so ω is admissible by Lemma 6.2(4).

If $y^{\mathcal{H}}$ is of face type $\widehat{X}_{12}$, then x_3 and x_5 are connected in $\mathrm{lk}(x_4, \Delta)$ by a locally embedded path of face type $C_1B'C_1$ or $C_1B'C_1B'C_1$. On the other hand, in Figure 13(II), x_3 and x_5 are connected in $\mathrm{lk}(x_4, \Delta)$ by a locally embedded path of face type $C_1B''C_1B'C_1$, which contradicts Lemma 3.9 as before.

Case 3: There are at least three distinct hexagons among the C_i. Hexagons intersecting C_1 are *consecutive* if they intersect a common square. We claim that either

(i) among C_2, C_4, C_6, C_8 there is C that equals C_i for a unique i, and such that for C', C'' consecutive with C, there is at most one j with $C_j \in \{C', C''\}$ or there are two such j, and they equal 6 and 8, or
(ii) up to a symmetry, C_6, C_8 are as in Figure 17(I) and $\{C_2, C_4\} = \{C_6, C\}$.

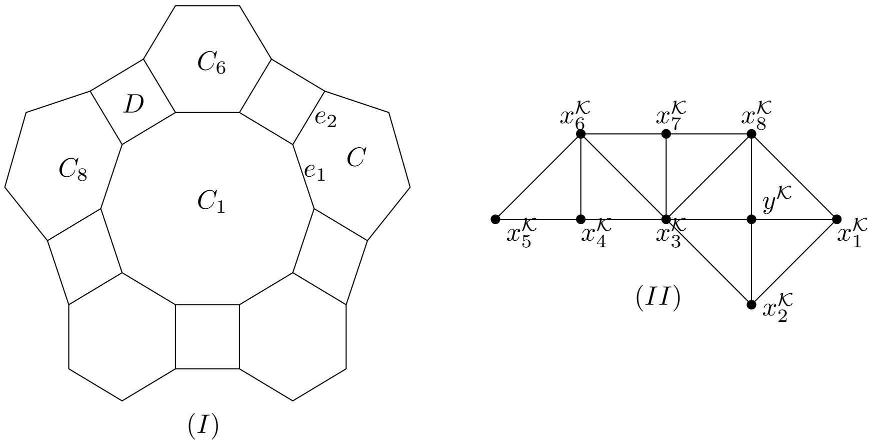


FIGURE 17.

To justify the claim, start with any C that equals C_i for a unique i. If both C', C'' equal to some C_j, then (i) is satisfied with C replaced by C' or C''. If, say, C'' is distinct from all C_j, but C is not as required in (i), then without loss of generality $C' = C_6$ and $C' = C_2$ or C_4. If the remaining C_k is consecutive with C', this brings us to (ii). Otherwise, we have (i) with C replaced by C_k. This justifies the claim.

If (i) holds, first note that if $\{C_6, C_8\} \subset \{C', C''\}$, then $C_6 = C_8$ since these faces are equal or consecutive. Furthermore, P_i is not homotopic in $\widehat{C} = \widehat{C}_i$ to a path in $\widehat{C}_1$, since ω is embedded. Consequently, by considering $\Pi_{\widehat{C}}(P) = \Pi_{\widehat{C}}(P_1) \cdots \Pi_{\widehat{C}}(P_8)$, which is homotopically trivial in $\widehat{C}$, we deduce, via Lemma 3.5, that there exists C_j consecutive with C. Moreover, we obtain $P_i = e^*$, where $e \subset C$ is the edge contained in the square B intersecting C and C_j. If $i = 6$ or 8, then x_7 is a neighbour of x_5 or x_1, as desired. Otherwise, if, say, $i = 2$, then the critical cycle obtained from ω by denting x_2 to C_j (see Definition 4.2) is not embedded or is admissible by Case 2. Consequently, ω is admissible by Lemma 6.2(3).

If (ii) holds, then by considering $\Pi_{\widehat{C}}(P)$, after possibly replacing the w_i by equivalent words, we can choose $P_i = e_2^* e_1^* e_2^*$ for $C_i = C$. Let $\mathcal{K}$ be the type II subarrangement of $\mathcal{A}$ with $\kappa_{\mathcal{K}}(C_6) = \mathsf{X}_{31}, \kappa_{\mathcal{K}}(D) = \mathsf{X}_{32}, \kappa_{\mathcal{K}}(C_1) = \mathsf{X}_{42}$. Since $\kappa_{\mathcal{K}}(e_1)$ is a vertex, we have $\widehat{\kappa}_{\mathcal{K}}(P_i) \subset \widehat{\mathsf{X}}_{41}$. Declaring that $\widehat{\kappa}_{\mathcal{K}}(P_i)$ is hosted by $\widehat{\mathsf{X}}_{41}$, we obtain $\omega^{\mathcal{K}}$ in $\mathbb{V}_{34}$ as in Definition 4.7. As before, we can assume that $\omega^{\mathcal{K}}$ has angle $\geq \frac{3\pi}{4}$ at both $x_6^{\mathcal{K}}$ and $x_8^{\mathcal{K}}$. If $C_2 = C$, then $x_2^{\mathcal{K}}$ has face type $\widehat{\mathsf{X}}_{41}$. Since $x_8^{\mathcal{K}}$ has face type $\widehat{\mathsf{X}}_{33}$, we have that $\omega^{\mathcal{K}}$ has angle $\geq \frac{3\pi}{4}$ at $x_1^{\mathcal{K}}$. Since $\mathbb{V}_{34}$ is CAT(0), the endpoints of $x_5^{\mathcal{K}} x_6^{\mathcal{K}} x_7^{\mathcal{K}} x_8^{\mathcal{K}} x_1^{\mathcal{K}} x_2^{\mathcal{K}}$ are at distance ≥ 4. This contradicts the fact that the path $x_2^{\mathcal{K}} x_3^{\mathcal{K}} x_4^{\mathcal{K}} x_5^{\mathcal{K}}$ has length $2\sqrt{2}+1$. Thus we have $C_2 = C_6$. Since $x_2^{\mathcal{K}}$ and $x_8^{\mathcal{K}}$ have distinct face types, $\omega^{\mathcal{K}}$ is locally embedded at $x_1^{\mathcal{K}}$. Since $\mathbb{V}_{34}$ is CAT(0), we obtain that $\omega^{\mathcal{K}}$ bounds the minimal disc diagram in Figure 17(II). Since $x_2^{\mathcal{K}}$ and $x_8^{\mathcal{K}}$ have face types $\widehat{\mathsf{X}}_{31}$ and $\widehat{\mathsf{X}}_{33}$, the vertex $y^{\mathcal{K}}$ has face type $\widehat{\mathsf{X}}_{32}$. By Lemma 4.15, we can lift $y^{\mathcal{K}}$ to $y \in \mathrm{lk}(x_2, \Delta)^0$ and $x_8^{\mathcal{K}}$ to $z \in \mathrm{lk}(y, \Delta)^0$ of face type C_8. Thus the critical cycle obtained from ω by replacing x_2 by z satisfies (i), and so it is not embedded or it is admissible. Consequently, ω is admissible by Lemma 6.2(3).

6.2. **Case of two decagons.**

Proposition 6.5. *Let ω be an embedded critical 8-cycle with exactly two decagons among the C_i. Then ω is admissible.*

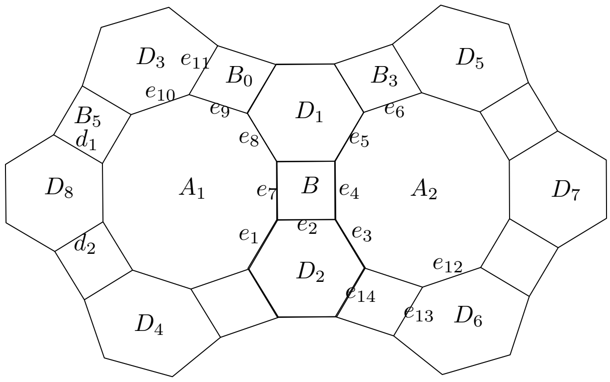


FIGURE 18.

Denote these decagons A_1, A_2. There is a square B intersecting A_1 and A_2, see Figure 18. Note that any C_i intersects A_1 or A_2. This is immediate for all C_i except for C_7, where otherwise we can obtain $x_6 = x_8$ by a similar argument as at the beginning of Section 6.1.

Case 1: All C_i belong, up to a symmetry, to $\{A_1, A_2, D_1, D_2, B_0, B, B_3\}$. Let Ω be the collection of all embedded critical 8-cycles with all their C_i belonging to the above set, and satisfying an extra condition $(*)$:

- if $C_8 = D_2$, then $C_1 \neq C_3$, and
- if $C_6 = D_2$, then $C_3 \neq C_5$.

Note that condition $(*)$, and hence the class Ω, is invariant under the involution $\mathcal{I}$ on the set of critical 8-cycles sending $x_1 \cdots x_8$ to $x_5 x_4 \cdots x_1 x_8 x_7 x_6$, which still has type $\hat{a}\hat{c}\hat{a}\hat{c}\hat{a}\hat{c}\hat{b}\hat{c}$.

It suffices to show that the critical cycles in Ω are admissible. Indeed, if ω is a critical cycle with, say, $C_1 = C_3$ (we cannot have simultaneously $C_3 = C_5$) and $C_8 = D_2$, then $C_7 = B$. Thus we can apply a symmetry of Σ interchanging D_1 with D_2, which fixes B, to send ω to an element of Ω.

Let $\mathcal{K}$ be the type II sub-arrangement of $\mathcal{A}$ with $\kappa_{\mathcal{K}}(D_1) = \mathsf{X}_{31}, \kappa_{\mathcal{K}}(B) = \mathsf{X}_{32}$, $\kappa_{\mathcal{K}}(A_2) = \mathsf{X}_{42}$. Then $\omega^{\mathcal{K}} = \kappa^*_{\mathcal{K}}(\omega) \subset \mathbb{V}_{234}$. Denote $f_i = \kappa_{\mathcal{K}}(e_i)$. As before, we can assume that the angles at $x_6^{\mathcal{K}}, x_8^{\mathcal{K}}$ are $\geq \frac{3\pi}{4}$. If there is $\omega \in \Omega$ that is not admissible, consider such ω with $\omega^{\mathcal{K}}$ bounding a minimal disc diagram in $\mathbb{V}_{234}$ of the smallest possible area. Proposition 5.10 and Lemmas 5.8 and 5.7 imply that, up to the involution $\mathcal{I}$, $\omega^{\mathcal{K}}$ bounds one of the minimal disc diagrams in Figures 13 (with $\mathcal{H}$ replaced by $\mathcal{K}$), 15, or 19.

We claim that $x_2^{\mathcal{K}}$ does not have a single interior neighbour $y^{\mathcal{K}}$. To justify the claim, we first verify that such $y^{\mathcal{K}}$ would have face type distinct from $\widehat{\mathsf{X}}_{23}$ and $\widehat{\mathsf{X}}_{43}$.

For contradiction, suppose that such $y^{\mathcal{K}}$ has face type $\widehat{\mathsf{X}}_{23}$ or $\widehat{\mathsf{X}}_{43}$. In Figure 19(I,II), by Lemma 5.4 applied with $x = z_2^{\mathcal{K}}$, the two vertices labelled $y'^{\mathcal{K}}$ would have face type $\widehat{\mathsf{X}}_{32}$, contradicting Lemma 5.2 applied with $x = x'^{\mathcal{K}}$. In Figures 13(I) and 15(I), the vertex $y^{\mathcal{K}}$ is a neighbour of $x_8^{\mathcal{K}}$, $x_1^{\mathcal{K}}$ and $x_3^{\mathcal{K}}$. Thus $C_1 = C_3 = A_1$, and x_8 has face type D_2. This contradicts condition $(*)$.

This confirms that the face type of $y^{\mathcal{K}}$ is distinct from $\widehat{\mathsf{X}}_{23}$ and $\widehat{\mathsf{X}}_{43}$. By Lemma 4.15, we can lift $y^{\mathcal{K}}$ to $y \in \mathrm{lk}(x_2, \Delta)^0$, and we can lift the neighbour of $y^{\mathcal{K}}$ opposite to $x_2^{\mathcal{K}}$ to $z_2 \in \mathrm{lk}(y, \Delta)^0$. The critical cycle obtained from ω by replacing x_2 with z_2 has appropriate diagram with smaller area, and still satisfies condition $(*)$, and so it

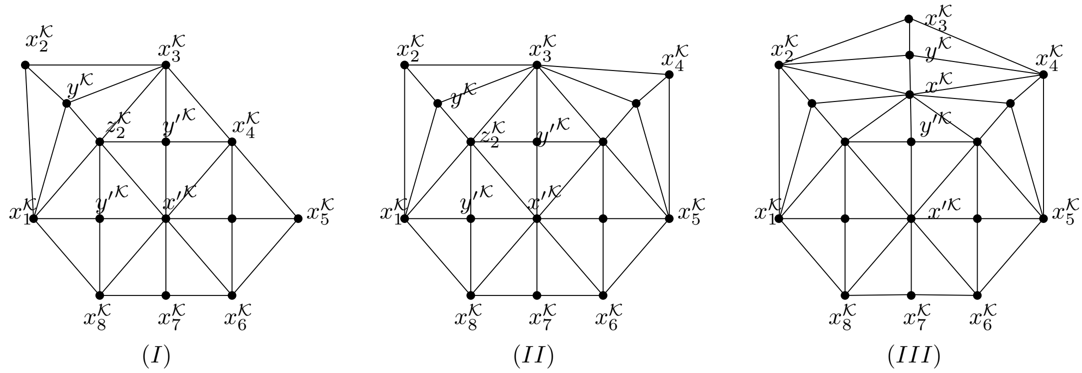


FIGURE 19.

is not embedded or it is admissible. Thus ω is admissible by Lemma 6.2(3). This justifies the claim and excludes Figures 13(I), 15(I), and 19(I,II).

Up to a symmetry, we can assume $A_2 = C_i$ for exactly one i. By considering $\Pi_{\widehat{A}_2}(P)$, for any choice of the P_i we have $P_i \subset \widehat{e}_3 \cup \widehat{e}_4 \cup \widehat{e}_5$ or $P_i \subset \widehat{e}_4 \cup \widehat{e}_5 \cup \widehat{e}_6$, or we are in a special case with $C_7 = B_3, C_6 = C_8 = D_1$, and $D_2 = C_2$ or C_4.

In this special case, we have that $x_2^{\mathcal{K}}$ or $x_4^{\mathcal{K}}$, say $x_2^{\mathcal{K}}$, has face type distinct from that of $x_6^{\mathcal{K}}, x_8^{\mathcal{K}}$, which excludes Figure 15(II). In Figure 19(III), by Lemma 5.2, $y'^{\mathcal{K}}$ (resp. $y^{\mathcal{K}}$) has face type $\widehat{\mathsf{X}}_{43}$ (resp. $\widehat{\mathsf{X}}_{41}$). Thus $x_2^{\mathcal{K}}$ and $x_4^{\mathcal{K}}$ have the same face type as $x_6^{\mathcal{K}}, x_8^{\mathcal{K}}$, which is a contradiction. Consequently, we have Figure 13(II). Again, by Lemma 5.2, $y'^{\mathcal{K}}$ has face type $\widehat{\mathsf{X}}_{43}$ and so $x_3^{\mathcal{K}}$ has face type $\widehat{\mathsf{X}}_{42}$, which implies $C_3 = A_2$, and consequently $C_1 = C_5 = A_1$. After possibly replacing the w_i by equivalent words, we can assume that $P_1^{\mathcal{K}}$ starts at σ_1 and ends at σ_2, and $P_2^{\mathcal{K}}$ starts at σ_2 and ends at σ_3 (see Definition 4.8). Since $y^{\mathcal{K}}$ has face type $\widehat{\mathsf{X}}_{32}$, by Lemma 4.9 we have $P_1^{\mathcal{K}} = f_7^*$ and $P_2^{\mathcal{K}} = f_2^* f_3^* f_{14}^*$ with $*$ non-zero, where $f_j = \kappa_{\mathcal{K}}(e_j)$. Since $\Pi_{\widehat{f}_1}(P_1^{\mathcal{K}})$ is homotopically trivial, we obtain that $\Pi_{\widehat{e}_1}(P_1)$ is homotopically trivial. Since $x_2^{\mathcal{K}}$ is non-collapsed, we have $P_2 = e_2^* e_3^* e_{14}^*$ with $*$ non-zero. Thus $\Pi_{\widehat{e}_1}(P_1 P_2)$ is homotopically nontrivial. Then $\Pi_{\widehat{D}_6}(P) = e_{13}^* e_{12}^* e_{13}^* e_{12}^*$, where the first e_{13}^* and last e_{12}^* come from $\Pi_{\widehat{D}_6}(P_1 P_2)$ and $\Pi_{\widehat{D}_6}(P_7)$, which are homotopically nontrivial. This contradicts Lemma 3.8 applied to $\widehat{e}_{13} \cup \widehat{e}_{12} \subset \widehat{D}_6$, and finishes the discussion of the special case.

In Figure 19(III), by Lemma 5.6 applied with the edge $x^{\mathcal{K}} y'^{\mathcal{K}}$ playing the role of xy, we have that $y'^{\mathcal{K}}$ is not of face type $\widehat{\mathsf{X}}_{32}$. Hence, by Lemma 5.2, the vertices $y^{\mathcal{K}}, x_7^{\mathcal{K}}$ have the same face types (which are thus $\widehat{\mathsf{X}}_{41}$) and the vertices $x^{\mathcal{K}}, x'^{\mathcal{K}}$ have the same face types. Thus, by Lemma 5.4, the vertices $x_1^{\mathcal{K}}, x_5^{\mathcal{K}}$ have the same face types (distinct from that of $x^{\mathcal{K}}, x'^{\mathcal{K}}$), implying $C_1 = C_5$, and consequently $C_3 = A_2$. Thus $x_i^{\mathcal{K}} = x_3^{\mathcal{K}}$ has a single interior neighbour $y^{\mathcal{K}}$, which has face type $\widehat{\mathsf{X}}_{41}$. Since we have covered already the special case, we have $P_3 \subset \widehat{e}_3 \cup \widehat{e}_4 \cup \widehat{e}_5$ or $P_3 \subset \widehat{e}_4 \cup \widehat{e}_5 \cup \widehat{e}_6$. Furthermore, $P_3^{\mathcal{K}}$ has the form $f_j^* f_6^* f_l^*$, where the first and the third term can be removed after possibly replacing the w_i by equivalent words. By Lemma 3.12, we can lift $y^{\mathcal{K}}$ to $y \in \mathrm{lk}(x_3, \Delta)^0$. We also lift $x^{\mathcal{K}}$ to $x \in \mathrm{lk}(y, \Delta)^0$, which still has face type A_2. The critical cycle obtained from ω by replacing x_3 with x has appropriate diagram with smaller area, and still satisfies condition $(*)$, and so it is not embedded or it is admissible. Thus ω is admissible by Lemma 6.2(7).

In Figure 13(II), suppose first $i = 3$. If x_7 has face type B_3, then, by Lemma 5.2, $y'^{\mathcal{K}}$ has face type $\widehat{\mathsf{X}}_{43}$. Thus C_2 has face type D_2, which is the special case that we have already covered. If x_7 has face type B_0, then $y'^{\mathcal{K}}$ has face type $\widehat{\mathsf{X}}_{23}$, and so $x_3^{\mathcal{K}}$ has face type $\widehat{\mathsf{X}}_{22}$, which is a contradiction. If x_7 has face type B, then $y^{\mathcal{K}}$ has face type $\widehat{\mathsf{X}}_{21}$ or $\widehat{\mathsf{X}}_{23}$, and $y'^{\mathcal{K}}, x^{\mathcal{K}}$ have face types $\widehat{\mathsf{X}}_{32}, \widehat{\mathsf{X}}_{22}$. By the same reasoning as in the previous paragraph, we can lift $y'^{\mathcal{K}}$ to $y' \in \mathrm{lk}(x_3, \Delta)^0$. We also lift $x^{\mathcal{K}}$ to $x \in \mathrm{lk}(y, \Delta)^0$, which has face type A_1. The critical cycle obtained from ω by replacing x_3 with x is not embedded or is admissible by Proposition 6.3. Thus ω is admissible by Lemma 6.2(7).

Second, suppose in Figure 13(II) that we have $i \neq 3$, say $i = 1$. If $y^{\mathcal{K}}$ has face type $\widehat{\mathsf{X}}_{32}$, then we lift $y^{\mathcal{K}}, x^{\mathcal{K}}$, and we proceed as in the last case with y and y' interchanged. If $y^{\mathcal{K}}$ has face type distinct from $\widehat{\mathsf{X}}_{32}$, then, by Lemma 5.2, $y''^{\mathcal{K}}$ has face type $\widehat{\mathsf{X}}_{41}$ or $\widehat{\mathsf{X}}_{43}$, thus $x_1^{\mathcal{K}}, x_5^{\mathcal{K}}$ have the same face type, which is a contradiction.

It remains to consider Figure 15(II). Since ω is embedded, by Lemma 3.12 we have $i = 3$. Furthermore, if $C_7 = B$, then the critical cycle obtained from ω by denting x_3 to A_1 is not embedded or is admissible by Proposition 6.4. Thus ω is admissible by Lemma 6.2(7).

Thus we can assume $C_7 = B_3$, where all even C_i equal D_1. Moreover, we can choose $P_3 = e_6^*$ with $P_3^{\mathcal{K}}$ starting at δ_1 and ending at δ_2. Let H be the hyperplane in $\mathcal{A}$ dual to e_8, and let $K \subset \Sigma$ be the union of the faces on the side of H containing e_6. We will justify that we can choose all P_j inside $\widehat{K}$. Indeed, except $j = 3$, all C_j intersect H. Starting with $j = 4$, and applying Lemma 3.11, P_j is homotopic in $\widehat{C}_j$, relative to the endpoints, to $P_{j1}P_{j2}$ with $P_{j1} \subset \widehat{K}$ and $P_{j2} \subset \widehat{C}_j \cap \widehat{C}_{j+1}$. We replace P_j by P_{j1} and P_{j+1} by $P_{j2}P_{j+1}$. We repeat the same procedure for $j = 5, 6, 7, 8, 1$. Since $\Pi_{\widehat{e}_8}(P)$ is homotopically trivial, ending this procedure with $j = 1$ yields $P_2 \subset \widehat{K}$.

Then $P_4^{\mathcal{K}}$ starts at δ_2 and ends at a triangle δ_3 of $\mathbb{V}_{234}$ containing the edge $x_4^{\mathcal{K}} x_5^{\mathcal{K}}$. Since $x_4^{\mathcal{K}} = x_6^{\mathcal{K}}$, we have that $P_5^{\mathcal{K}}$ is homotopic in $\widehat{\mathsf{X}}_{22}$ to f_8^*. Since $P_5 \subset \widehat{K}$, we conclude that $P_5^{\mathcal{K}}$ is homotopically trivial in $\widehat{\mathsf{X}}_{22}$ and so it both starts and ends at δ_3. Then $P_6^{\mathcal{K}}$ starts at δ_3 and ends at a triangle δ_4 containing the edge $x_6^{\mathcal{K}} x_7^{\mathcal{K}}$. Note that $\delta_4 = \delta_2$, since otherwise $P_4^{\mathcal{K}} P_6^{\mathcal{K}}$ is a path in $\widehat{\mathsf{X}}_{31}$ with nontrivial image under $\Pi_{\widehat{f}_8}$ contradicting $P_4, P_6 \subset \widehat{K}$. Thus we can assume $P_6^{\mathcal{K}} = P_4^{\mathcal{K}-1}$, and so $P_6 = P_4^{-1}$ by Lemma 4.15. Analogously, $P_8 = P_2^{-1}$, and similarly $P_7 = P_3^{-1}$. Considering $\Pi_{\widehat{A}_1}(P)$, and noticing that $\Pi_{\widehat{A}_1}(P_i)$ are homotopically trivial loops in $\widehat{A}_1$ for $i \neq 1, 5$, we obtain $P_1 = P_5^{-1}$. It follows that w_1 commutes with $w_2 w_3 w_4$.

Let P be the parabolic closure of w_1 (i.e. the smallest parabolic subgroup of A_Λ containing w_1, which exists by [CGGMW19]). Note that $\mathsf{P} = A_{bc}$, since otherwise we would have $w_1 = gb^*g^{-1}$ or gc^*g^{-1} for some $g \in A_{bc}$. Hence there would exist $j \in \{1, 7, 8, 9, 10\}$ with $\Pi_{\widehat{e}_j}(P_1)$ homotopically nontrivial. However, $j \neq 8$ since $P_1 \subset \widehat{K}$. Furthermore, since $\Pi_{\widehat{D}}(P) = k_1^* k_2^* k_1^* k_2^*$ is homotopically trivial in $\widehat{D}$ (see Figure 20), hence in $\widehat{k}_1 \cup \widehat{k}_2$ by Lemma 3.8, and the $*$ over k_2 are non-zero, we have that the remaining $*$ are zero, and so $j \neq 10$. The remaining j are excluded since $\omega^{\mathcal{K}}$ is not locally embedded at $x_1^{\mathcal{K}}$.

Since $w_2 w_3 w_4$ commutes with w_1, it normalises $\mathsf{P} = A_{bc}$. By [Par97, Thm 5.2 (5)], the edge-loop $P_2 P_3 P_4$ is homotopic in $\widehat{\Sigma}$ to an edge-loop of the form $Q^{n_1}(Q_1 Q_2)^{n_2} Q_3$ defined as follows. Let p_2 be the antipodal vertex to the basepoint p_1 of P_1 in Σ. Let $A_1' \subset \Sigma$ be the opposite face to A_1, and let p_3 be the projection of p_1 onto A_1'. We define Q_1 to be the minimal positive path from p_1 to p_3, Q_2 to be the minimal positive path from p_3 to p_1, and Q to be the concatenation of a minimal positive path from p_1 to p_2 and a minimal positive path from p_2 to p_1. We allow any $Q_3 \subset \widehat{A}_1$. Since $\Pi_{\widehat{e}_6}(P_2 P_3 P_4) = \Pi_{\widehat{e}_6}(P_7^{-1})$ is homotopically nontrivial, we have $n_1 + n_2 \neq 0$. On the other hand, for an edge e whose dual hyperplane does not intersect A_1, D_1 or B_3, we have that $\Pi_{\widehat{e}}(P_2 P_3 P_4)$ is trivial, implying $n_1 + n_2 = 0$, contradiction.

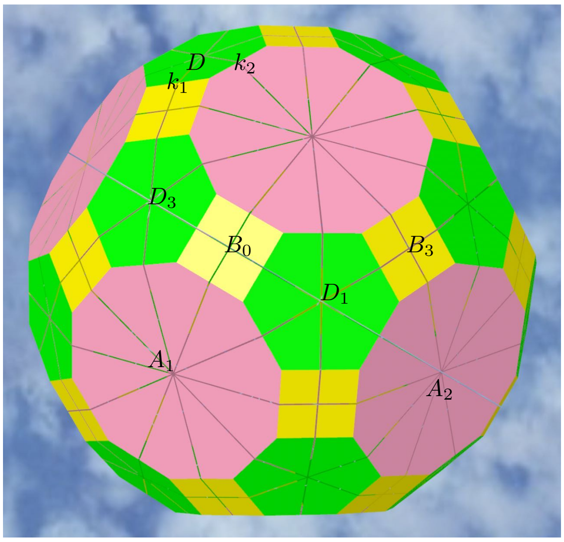


FIGURE 20.

Suppose now that the condition of Case 1 is not satisfied. We assume without loss of generality $C_5 = A_1$.

<u>Case 2: $C_1 = C_5$.</u> Then all of the C_i intersect A_1. Suppose that one of the C_i, say C_6, equals D_8. If we also have $C_8 = D_8$, then by considering $\Pi_{\widehat{A}_2}(P)$, we can choose $P_3 = e_4^*$. By Proposition 6.4, the critical cycle obtained from ω by denting x_3 to A_1 is not embedded or is admissible. Thus ω is admissible by Lemma 6.2(7). If $C_8 \neq D_8$, then by considering $\Pi_{\widehat{D}_8}(P)$, we obtain that x_7 is a neighbour of x_5. Thus we can assume that none of the C_i equals D_8.

Up to a symmetry, it remains to assume $C_6 = D_3$. Suppose first $C_8 = D_1$. By considering $\Pi_{\widehat{D}_3}(P)$, we can choose $P_6 \subset \widehat{e}_{10} \cup \widehat{e}_{11}$. Let $\mathcal{K}$ be the sub-arrangement of $\mathcal{A}$ as in Case 1. All the vertices of ω lie in $\Delta^*_{\mathcal{K}}$, except for x_6. However, $\kappa_{\mathcal{K}}(e_{10})$ is a vertex, so we have $\widehat{\kappa}_{\mathcal{K}}(P_6) \subset \widehat{\mathsf{X}}_{21}$. Thus declaring that $\widehat{\kappa}_{\mathcal{K}}(P_6)$ is hosted by $\widehat{\mathsf{X}}_{21}$, we obtain $\omega^{\mathcal{K}}$ in $\mathbb{V}_{234}$ as in Definition 4.7. Note that $x_6^{\mathcal{K}} = x_7^{\mathcal{K}}$, and so $x_5^{\mathcal{K}}$ and $x_8^{\mathcal{K}}$ are neighbours.

By considering $\Pi_{\widehat{A}_2}(P)$, we obtain that P_3 is contained in $\widehat{e}_3 \cup \widehat{e}_4 \cup \widehat{e}_5$, which will allow us in a moment to apply Lemma 3.12 to P_3. We apply Lemma 5.9 to the 6-cycle $x_1^{\mathcal{K}} \cdots x_5^{\mathcal{K}} x_8^{\mathcal{K}}$. If we have Lemma 5.9(1), i.e. $x_2^{\mathcal{K}} = x_4^{\mathcal{K}}$, then, by Lemma 3.12, we obtain $x_2 = x_4$, which is a contradiction. If we have Lemma 5.9(2), i.e. $x_2^{\mathcal{K}} \neq x_4^{\mathcal{K}}$ have a common neighbour $y^{\mathcal{K}}$ in $\mathrm{lk}(x_3^{\mathcal{K}}, \mathbb{V}_{234})$, then, as in Case 1, by Lemma 3.12, the vertex $y^{\mathcal{K}}$ can be lifted to a neighbour y of x_2, x_4 in $\mathrm{lk}(x_3, \Delta)$ of face type B. Let x be a neighbour of y of face type A_1. The critical cycle obtained from ω by replacing x_3 with x is not embedded or is admissible by Proposition 6.4. Thus ω is admissible by Lemma 6.2(7).

If we have Lemma 5.9(3), then let $y'^{\mathcal{K}}$ be the interior vertex of the disc diagram in (3) that is a neighbour of $x_2^{\mathcal{K}}$. By Lemma 4.15, we can lift $y'^{\mathcal{K}}$ to a neighbour y' of x_1, x_3 in $\mathrm{lk}(x_2, \Delta)$ of face type B. Let z be a lift of $z^{\mathcal{K}}$ to a neighbour of x_1, x_3 in $\mathrm{lk}(y', \Delta)$. Then the critical cycle obtained from ω by replacing x_2 with z is not embedded or is admissible, since it satisfies Lemma 5.9(2). Hence ω is admissible by Lemma 6.2(3).

Second, suppose $C_8 = D_3$. Let $\mathcal{H}$ be the type I sub-arrangement of $\mathcal{A}$ with $\kappa_{\mathcal{H}}(A_1) = X_{22}, \kappa_{\mathcal{H}}(D_3) = X_{11}$, and $\kappa_{\mathcal{H}}(B_5) = X_{21}$. Note that again we need to reflect Figure 4 before comparing it with Figure 18. Then all x_i belong to $\Delta^*_{\mathcal{H}}$ (note that $\kappa(A_2)$ is a square of X_{12}), except for the ones of face type D_2. However, declaring that such $\widehat{\kappa}_{\mathcal{H}}(P_i)$ are hosted by X_{22}, we obtain $\omega^{\mathcal{H}}$ in $\mathbb{U}_{12}$ as in Definition 4.7. Note that $x_3^{\mathcal{H}}$ is a neighbour of $x_1^{\mathcal{H}}$ and $x_5^{\mathcal{H}}$. Since $\mathbb{U}_{12}$ is CAT(0), we have that $\omega^{\mathcal{H}}$ has angle $\frac{\pi}{4}$ at $x_8^{\mathcal{H}}$ or $x_6^{\mathcal{H}}$, and so ω is admissible as before.

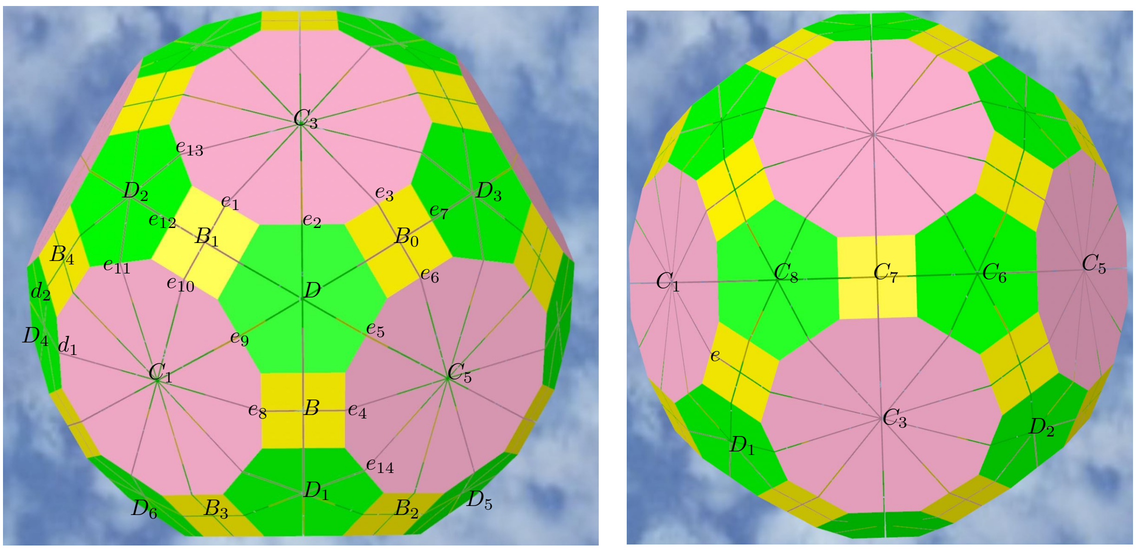
Case 3: $C_3 = C_5$. Then none of the C_i equals D_7. If one of the C_i equals D_5, then $i = 8$ and, considering $\Pi_{\widehat{D}_5}(P)$, we deduce that x_7 is a neighbour of x_1. Thus we can assume that none of the C_i equals D_5 or D_6. We can also assume $C_i \neq D_8$, since otherwise $i = 4$, and, by considering $\Pi_{\widehat{D}_8}(P)$, we can choose $P_4 = d_1^*$ or d_2^*, say d_1^*. Then, by Lemma 6.2(3), we can replace ω by the critical cycle obtained by denting x_4 to D_3.

Up to a symmetry, it remains to assume $D_3 \in \{C_4, C_6\}$. If $C_4 = D_3 \neq C_6$, then $\Pi_{\widehat{D}_3}(P_6 \cup P_7 \cup P_8) = e_{10}^*$ or e_{11}^*. Thus, by considering $\Pi_{\widehat{D}_3}(P)$, we can choose $P_4 = e_{11}^*$. By denting x_4 to D_1 we obtain a critical cycle that is admissible either by Case 1 or the case $C_6 = D_3 \neq C_4$, which will be discussed in a moment. Thus ω is admissible by Lemma 6.2(3). If $C_6 = D_3 \neq C_4$, then, by considering $\Pi_{\widehat{D}_3}(P)$, we can choose $P_6 \subset \widehat{e}_{10} \cup \widehat{e}_{11}$. Let $\mathcal{K}$ be the sub-arrangement of $\mathcal{A}$ as in Case 1. Declaring that $\widehat{\kappa}_{\mathcal{K}}(P_6)$ is hosted by $\widehat{\mathsf{X}}_{21}$, we obtain $\omega^{\mathcal{K}}$ in $\mathbb{V}_{234}$ as in Definition 4.7. As before, $x_5^{\mathcal{K}}$ and $x_8^{\mathcal{K}}$ are neighbours. We apply Lemma 5.9 to the 6-cycle $x_5^{\mathcal{K}}x_8^{\mathcal{K}}x_1^{\mathcal{K}}x_2^{\mathcal{K}}x_3^{\mathcal{K}}x_4^{\mathcal{K}}$, with $x_1^{\mathcal{K}}$ playing the role of x_3, and we finish as in Case 2.

If $C_6 = C_4 = D_3$, then let $\mathcal{H}$ be the type I sub-arrangement of $\mathcal{A}$ as in Case 2. Then all x_i belong to $\Delta_{\mathcal{H}}^*$, except possibly for x_2 if it has face type D_2. However, declaring then that $\widehat{\kappa}_{\mathcal{H}}(P_2)$ is hosted by X_{22}, we obtain $\omega^{\mathcal{H}}$ in $\mathbb{U}_{12}$ as in Definition 4.7. Note that $x_3^{\mathcal{H}}$ is a neighbour of $x_1^{\mathcal{H}} = x_8^{\mathcal{H}} = x_7^{\mathcal{H}}$, and hence of $x_6^{\mathcal{H}}$. Since $\mathbb{U}_{12}$ is CAT(0), we have that the 4-cycle $x_3^{\mathcal{H}}x_4^{\mathcal{H}}x_5^{\mathcal{H}}x_6^{\mathcal{H}}$ has a common neighbour $y^{\mathcal{H}}$. If $y^{\mathcal{H}}$ has face type X_{21}, then by Lemma 4.11 we obtain that x_3 and x_6 are neighbours, and so ω is admissible by Lemma 6.2(1). If $y^{\mathcal{H}}$ has face type X_{12}, then by Lemma 4.11 x_4 can be dented to D_1, reducing to the case $C_4 \neq D_3$ by Lemma 6.2(3).

6.3. **Case of three decagons.**

Proposition 6.6. *Let ω be an embedded critical 8-cycle with three decagons among the C_i. Then ω is admissible.*



(*I*) (*II*)

FIGURE 21.

Up to a symmetry, we have Figure 21(I) or (II). In (II), we have $C_2 \in \{C_8, D_1\}$ and $C_4 \in \{C_6, D_2\}$. By considering $\Pi_{\widehat{C}_1}(P)$, we can choose $P_1 = e^*$. The critical cycle obtained from ω by denting x_1 to C_3 is not embedded or is admissible by Proposition 6.5. Thus ω is admissible by Lemma 6.2(6). Hence in the remainder of the subsection, we assume (I).

Case 1: One of C_6, C_8 belongs to $\{D_2, D_3\}$. Then we have either $C_8 = D_2$ and $C_6 = D$, or $C_6 = D_3$ and $C_8 = D$. However, in the latter case, reflecting along the hyperplane of Σ intersecting C_3 and D, and applying the involution $\mathcal{I}$ from Case 1, brings us to the former one. Thus we can assume $C_8 = D_2$ and $C_6 = D$. First, we assume $C_4 = D$. By considering $\Pi_{\widehat{C}_5}(P)$, we can choose $P_5 = e_6^* e_5^* e_4^*$. We can assume that the last $*$ is non-zero, since otherwise we can choose $P_5 = e_6^*$ and by Lemma 6.2(6) reduce to Proposition 6.5 by denting x_5 to C_3. Then $\Pi_{\widehat{D}_4}(P_5) = d_1^*$ is nontrivial. Since $\Pi_{\widehat{D}_4}(P) = d_1^* d_2^* d_1^* d_2^*$ is homotopically trivial in $\widehat{D}_4$, by Lemma 3.8 we obtain that the $*$ over d_2 are zero. In other words, $\Pi_{\widehat{e}_{13}}(P_2P_3)$ and $\Pi_{\widehat{e}_{13}}(P_8)$ are homotopically trivial. By considering $\Pi_{\widehat{C}_3}(P)$, we can assume $P_3 \subset \widehat{e}_1 \cup \widehat{e}_2 \cup \widehat{e}_3$.

Let $\mathcal{H}$ be the type I sub-arrangement of $\mathcal{A}$ with $\kappa_{\mathcal{H}}(C_1) = X_{22}, \kappa_{\mathcal{H}}(D_2) = X_{11}$, and $\kappa_{\mathcal{H}}(B_4) = X_{21}$. Here again we reflect Figure 4. Let $P^{\mathcal{H}} = \widehat{\kappa}_{\mathcal{H}}(P)$. Note that $\kappa_{\mathcal{H}}(e_3)$ is a vertex. Since $C_4 = D$, declaring that $P_i^{\mathcal{H}}$ is hosted by $\widehat{X}_{12}$ for $i = 3, \ldots, 7$, and for $i = 2$ when $C_2 = D$, we obtain a 4- or 3-cycle $\omega^{\mathcal{H}}$ in $\mathbb{U}_{12}$ as in Definition 4.7. By Lemma 4.11, $\omega^{\mathcal{H}}$ is locally embedded at $x_1^{\mathcal{H}}$. Since $\mathbb{U}_{12}$ is CAT(0), we obtain that $\omega^{\mathcal{H}}$ has angle $\frac{\pi}{4}$ at $x_8^{\mathcal{H}}$. Thus $x_1^{\mathcal{H}}$ and $x_7^{\mathcal{H}}$ are neighbours and so x_1 and x_7 are neighbours. If $C_4 = D_3$, then, by considering $\Pi_{\widehat{D}_3}(P)$, we can choose $P_4 = e_7^*$. Thus, by Lemma 6.2(3), we can reduce to the case $C_4 = D$ by denting x_4 to D.

In the remaining cases, we assume that none of C_6, C_8 belongs to $\{D_2, D_3\}$.

Case 2: $C_2 = C_4 = D$, and at least one of C_6, C_8 equals D.

Case 2.1: $C_7 = B$. By considering $\Pi_{\widehat{C}_3}(P)$, we can choose $P_3 = e_1^* e_2^* e_3^*$. Let $\mathcal{K}$ be the type II sub-arrangement of $\mathcal{A}$ with $\kappa_{\mathcal{K}}(D_1) = \mathsf{X}_{31}, \kappa_{\mathcal{K}}(B) = \mathsf{X}_{32}, \kappa_{\mathcal{K}}(C_1) = \mathsf{X}_{42}$. Then $\kappa_{\mathcal{K}}(e_1)$ and $\kappa_{\mathcal{K}}(e_3)$ are vertices. Declaring that $P_3^{\mathcal{H}}$ is hosted by $\widehat{\mathsf{X}}_{33}$ we obtain a cycle $\omega^{\mathcal{H}}$ in $\mathbb{V}_{234}$ as in Definition 4.7 with $x_2^{\mathcal{H}} = x_3^{\mathcal{H}} = x_4^{\mathcal{H}}$. Note that $\omega^{\mathcal{H}}$ is locally embedded at $x_6^{\mathcal{H}}, x_7^{\mathcal{H}}$, and $x_8^{\mathcal{H}}$. Thus $\omega^{\mathcal{H}}$ is locally embedded at one of $x_1^{\mathcal{H}}, x_5^{\mathcal{H}}$, say $x_1^{\mathcal{H}}$. If $\omega^{\mathcal{H}}$ is not locally embedded at $x_5^{\mathcal{H}}$ or $x_4^{\mathcal{H}}$, then by Lemma 5.7 applied to $x_1^{\mathcal{H}} x_6^{\mathcal{H}} x_7^{\mathcal{H}} x_8^{\mathcal{H}}$ we have that $x_1^{\mathcal{H}}$ is a neighbour of $x_7^{\mathcal{H}}$ and so x_1 is a neighbour of x_7. If $\omega^{\mathcal{H}}$ is locally embedded at $x_5^{\mathcal{H}}$ and $x_4^{\mathcal{H}}$, then by Lemma 5.8 applied to $x_1^{\mathcal{H}} x_2^{\mathcal{H}} x_5^{\mathcal{H}} x_6^{\mathcal{H}} x_7^{\mathcal{H}} x_8^{\mathcal{H}}$ we obtain that $x_7^{\mathcal{H}}$ is a neighbour of $x_1^{\mathcal{H}}$ or $x_5^{\mathcal{H}}$ and we finish as before.

Case 2.2: $C_7 = B_0$ or B_1. By the boldface assumption at the end of Case 1, we have $C_6 = C_8 = D$. By the argument similar to the one at the beginning of Case 1, we can assume $C_7 = B_0$. By considering $\Pi_{\widehat{C}_1}(P)$, we can choose $P_1 = e_8^* e_9^* e_{10}^*$. We can assume that the $*$ are non-zero, since otherwise we can reduce to Proposition 6.5 by denting x_1 to C_3 or C_5. Consider the path $\eta = x_8 x_{11} x_{12} x_{13} x_2$ in $\mathrm{lk}(x_1, \Delta)$ of face type $DBDB_1D$ corresponding to $e_8^* e_9^* e_{10}^*$. Let $\mathcal{K}$ be the type II sub-arrangement of $\mathcal{A}$ with $\kappa_{\mathcal{K}}(D) = \mathsf{X}_{31}, \kappa_{\mathcal{K}}(B_0) = \mathsf{X}_{32}, \kappa_{\mathcal{K}}(C_3) = \mathsf{X}_{42}$. Then the cycle ω_0 obtained from ω by replacing $x_8 x_1 x_2$ with η lies in $\Delta^*_{\mathcal{K}}$. Let $\omega_0^{\mathcal{K}} = \kappa^*_{\mathcal{K}}(\omega_0)$.

We first consider the case where $\omega_0^{\mathcal{K}}$ has angle π at $x_8^{\mathcal{K}}$. By considering $\Pi_{\widehat{C}_3}(P)$, we obtain $P_3 \subset \widehat{e}_1 \cup \widehat{e}_2 \cup \widehat{e}_3$. Thus by Lemma 3.12 $\omega_0^{\mathcal{K}}$ is locally embedded at $x_3^{\mathcal{K}}$. Analogously $\omega_0^{\mathcal{K}}$ is locally embedded at $x_5^{\mathcal{K}}$ and so it is locally embedded. By Proposition 5.12, with $x_6^{\mathcal{K}} x_7^{\mathcal{K}} x_8^{\mathcal{K}} x_{11}^{\mathcal{K}} x_{12}^{\mathcal{K}} \cdots$ playing the role of $x_1 x_2 x_3 x_4 x_5 \cdots$ (note that $x_7^{\mathcal{K}}$ has face type $\widehat{\mathsf{X}}_{32}$, but $x_{11}^{\mathcal{K}}$ and $x_{13}^{\mathcal{K}}$ do not have face type $\widehat{\mathsf{X}}_{32}$), $\omega_0^{\mathcal{K}}$ has angle $\frac{\pi}{4}$ at $x_6^{\mathcal{K}}$, and so x_5 and x_7 are neighbours.

Second, assume that $\omega_0^{\mathcal{K}}$ has angle $\frac{\pi}{2}$ at $x_8^{\mathcal{K}}$, and so there is a common neighbour $x^{\mathcal{K}}$ of $x_7^{\mathcal{K}}$ and $x_{11}^{\mathcal{K}}$ in $\mathrm{lk}(x_8^{\mathcal{K}}, \mathbb{V}_{234})$. By Lemma 4.15, we can lift $x^{\mathcal{K}}$ to a a common neighbour x of x_7 and x_{11} in $\mathrm{lk}(x_8, \Delta)$. Since $x_7x_8x_{11}$ has face type B_0DB, we have that x has face type C_5. Let $\omega' = x_3x_4x_5x_6xx_{12}x_{13}x_2$. If ω' is not embedded, then either $x = x_5$, in which case x_5 and x_7 are neighbours, or $x_{12} = x_6$, in which case x_1 is a neighbour of both x_6 and x_2 and ω is admissible by Lemma 6.2(4), or $x_{12} = x_4$, in which case x_{12} is a neighbour of x_3 and ω is admissible by Lemma 6.2(5). If ω' is embedded, then by Proposition 6.5 it is admissible, and so x_{13} is a neighbour of x_3 (other possibilities are excluded since the faces B_1 and C_5 are disjoint). The critical cycle obtained from ω by replacing x_2 with x_{12} is not embedded or is admissible since it fits the case of $P_1 = e_8^*e_9^*e_{10}^*$ with one of the $*$ zero. Thus ω is admissible by Lemma 6.2(3).

Case 3: $C_2 = C_4 = D$, and none of C_6, C_8 equals D. If $C_6 = D_5$ and $C_8 = D_1$, then by considering $\Pi_{\widehat{D}_5}(P)$ we obtain that x_7 is is a neighbour of x_5. The case $C_6 = D_1$ and $C_8 = D_6$ is analogous. It remains to assume $C_6 = C_8 = D_1$. If $C_7 = B_3$, then by considering $\Pi_{\widehat{C}_5}(P)$ we can choose $P_5 = e_6^*e_5^*e_4^*$. Let $\mathcal{H}$ be the type I subarrangement of $\mathcal{A}$ with $\kappa_{\mathcal{H}}(C_1) = X_{22}, \kappa_{\mathcal{H}}(D_1) = X_{11}, \kappa_{\mathcal{H}}(B_3) = X_{21}$. Declaring that $P_i^{\mathcal{H}}$ are hosted by $\widehat{X}_{12}$ for $i = 2, \ldots, 5$, we obtain a cycle $\omega^{\mathcal{H}}$ in $\mathbb{U}_{12}$ as in Definition 4.7 with $x_2^{\mathcal{H}} = x_3^{\mathcal{H}} = x_4^{\mathcal{H}} = x_5^{\mathcal{H}}$. By Lemma 4.11, $x_6^{\mathcal{H}}x_7^{\mathcal{H}}x_8^{\mathcal{H}}$ is a geodesic. Since $\mathbb{U}_{12}$ is CAT(0), $\omega^{\mathcal{H}}$ has angle $\frac{\pi}{4}$ at $x_8^{\mathcal{H}}$ and so x_7 and x_1 are neighbours as usual. The case $C_7 = B_2$ is analogous.

We now assume $C_7 = B$. By considering $\Pi_{\widehat{C}_5}(P)$, we can assume $P_5 = P_{51}P_{52}P_{53}$ with $P_{51} = e_6^*e_5^*e_4^*$, $P_{52} = e_{14}^*$ and $P_{53} = e_4^*$. We assume that P_{52} and P_{53} are nontrivial, since otherwise we can proceed as in the previous paragraph. Declaring that $P_i^{\mathcal{H}}$ are hosted by $\widehat{X}_{12}$ for $i = 2, 3, 4, 51, 53$, and $P_{52}^{\mathcal{H}}$ is hosted by $\widehat{X}_{11}$, we obtain a cycle $\omega^{\mathcal{H}}$ in $\mathbb{U}_{12}$ as in Definition 4.7, with $x_2^{\mathcal{H}} = x_3^{\mathcal{H}} = x_4^{\mathcal{H}} = x_{51}^{\mathcal{H}}$. Since $P_{53}^{\mathcal{H}}$ is nontrivial, $\omega^{\mathcal{H}}$ has angle π at $x_{53}^{\mathcal{H}}$. Thus $x_6^{\mathcal{H}}x_7^{\mathcal{H}}x_8^{\mathcal{H}}$ and $x_6^{\mathcal{H}}x_{53}^{\mathcal{H}}x_{52}^{\mathcal{H}}$ are geodesics meeting at an angle $\geq \frac{\pi}{2}$. Since $\mathbb{U}_{12}$ is CAT(0), it follows that the angle of $\omega^{\mathcal{H}}$ at $x_8^{\mathcal{H}}$ is $\frac{\pi}{4}$, and so x_7 and x_1 are neighbours as usual.

Case 4: $C_2 = D_2$ or $C_4 = D_3$. If $C_2 = D_2$, then $\Pi_{\widehat{D}_2}(P_6 \cup P_7 \cup P_8) = e_{12}^*$ or e_{11}^*. Thus, by considering $\Pi_{\widehat{D}_2}(P)$, we can choose $P_2 = e_{12}^*$. By Lemma 6.2(3), by denting x_2 to D we can reduce to the case where $C_2 = D$. Analogously, if $C_4 = D_3$, then by denting x_4 to D we can reduce to the case where $C_4 = D$.

7. Critical 10-cycles

Let Λ be the linear graph abc with $m_{ab} = 3$, $m_{bc} = 5$, as in Section 4. A *critical* 10-*cycle* in Δ has type $\hat{a}\hat{c}\hat{b}\hat{c}\hat{b}\hat{c}\hat{a}\hat{c}\hat{b}\hat{c}$ (or, shortly, $\hat{a}\hat{c}(\hat{b}\hat{c})^2\hat{a}\hat{c}\hat{b}\hat{c}$).

Definition 7.1. An embedded critical 10 cycle (x_i) is *admissible* if

(1) x_1 is a neighbour of x_3 or x_9, or x_7 is a neighbour of x_5 or x_9, or
(2) there is a vertex of type $\hat{a}$ that is a neighbour of x_3, x_5, and x_9.

The goal of this section is to prove:

Proposition 7.2. *Each embedded critical* 10*-cycle is admissible.*

Proposition 7.2 follows from Propositions 7.4 and 7.5 below, which are proved in Subsections 7.1 and 7.2. In the remaining part of this section, let $\omega = x_1 \cdots x_{10}$ be an embedded critical 10-cycle. Let w_i, C_i, P_i be as in Construction 4.1.

Lemma 7.3. *Let ω be an embedded critical* 10*-cycle. Under any of the following conditions ω is admissible.*

(1) There is a vertex x of type $\hat{a}$ that is a neighbour of x_3 and x_5.
(2) There is a vertex x of type $\hat{a}$ that is a neighbour of x_3 and x_9.

(3) Replacing in ω the vertex x_1 by x'_1 results in a critical cycle ω_0 that is not embedded or is admissible.

(4) Replacing in ω the vertex x_2 or x_{10} results in a critical cycle ω_0 that is not embedded or is admissible.

Proof. In (1), by Remark 2.1, x is a neighbour of x_2 and x_6. Let ω_8 be the critical 8-cycle $x_1x_2xx_6x_7x_8x_9x_{10}$. By Proposition 6.3, ω_8 is not embedded, or is admissible. If ω_8 is not embedded, then, since ω is embedded, we have $x = x_1$ or $x = x_7$, which implies that ω is admissible. If ω_8 is embedded and satisfies Definition 6.1(1), then ω satisfies Definition 7.1(1). If ω_8 satisfies Definition 6.1(2), then ω satisfies Definition 7.1(2).

In (2), by Remark 2.1, x is a neighbour of x_4 and x_8. By Theorem 2.9, there is a common upper bound $z \in \Delta^0$ of type $\hat{c}$ of x, x_5, x_7. If $z = x_6$, then applying the bowtie freeness from Theorem 2.9 to $xx_4x_5x_6$, we obtain that x is a neighbour of x_5, and so ω satisfies Definition 7.1(2). If $z \neq x_6$, then applying the bowtie freeness to $zx_5x_6x_7$, we obtain that x_7 is a neighbour of x_5, and so ω satisfies Definition 7.1(1).

In (3), if ω_0 is not embedded, then $x'_1 = x_7$. Applying the bowtie freeness from Theorem 2.9 to $x_7x_8x_9x_{10}$, we obtain that x_7 is a neighbour of x_9. Thus we can assume that ω_0 is embedded. The admissibility of ω follows immediately from the admissibility of ω_0 unless x'_1 is a neighbour of (i) x_3 or (ii) x_9.

In (i), let x'_9 be a neighbour of type $\hat{a}$ of x_9 (and hence of x_8 and x_{10} by Remark 2.1). Let ω_8 be the critical 8-cycle $x_7x_8x'_9x_{10}x'_1x_4x_5x_6$. Note that ω_8 is embedded, since otherwise $x_7 = x'_9$, which is a neighbour of x_9, or $x'_9 = x'_1$, and so ω is admissible by (2) applied with $x = x'_1$. By Proposition 6.3, ω_8 is admissible, and so x_5 is a neighbour of x_7, x'_1, or x'_9. In the second case, ω is admissible by (1) applied with $x = x'_1$. In the third case, ω is admissible by (2) applied with $x = x'_9$.

In (ii), let x'_3 be a neighbour of type $\hat{a}$ of x_3. We consider the critical 8-cycle $\omega_8 = x_7x_8x'_1x_2x'_3x_4x_5x_6$, and we proceed analogously as in (i).

In (4), if ω_0 is not embedded, then, by Lemma 5.7, ω satisfies Definition 7.1(1). If ω_0 is admissible and satisfies Definition 7.1(1) (resp. (2)), then ω satisfies Definition 7.1(1) (resp. (2)). □

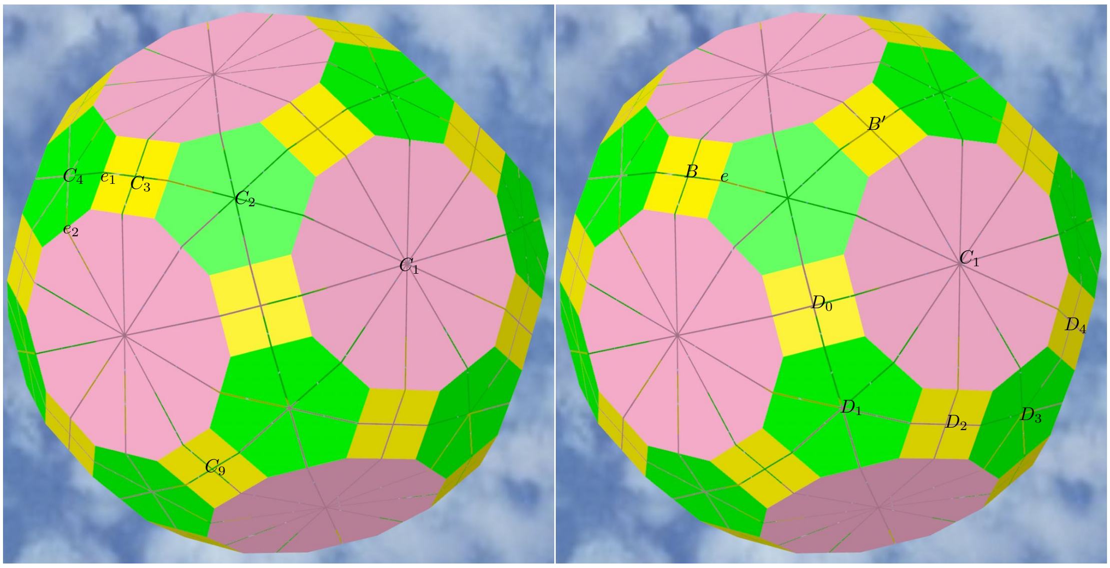
7.1. **Case of one decagon.**

Proposition 7.4. *Let ω be an embedded critical* 10*-cycle with $C_1 = C_7$. Then ω is admissible.*

In the discussion below, we consider two kinds of symmetries. One kind are the symmetries of Σ. The second is the involution $\mathcal{I}'$ on the set of critical 10-cycles sending $x_1 \cdots x_{10}$ to $x_7x_6 \cdots x_1x_{10}x_9x_8$, which still has type $\hat{a}\hat{c}\hat{b}\hat{c}\hat{b}\hat{c}\hat{a}\hat{c}\hat{b}\hat{c}$.

Note that C_4 intersects C_1. Otherwise, up to a symmetry, C_1 and C_4 are as in Figure 22(I), and we have $\Pi_{C_4}(C_i) \subset e_1$ for $i \neq 4, 9$. If $\Pi_{C_4}(C_9)$ is not contained in e_1, then, up to a symmetry, C_9 is as in Figure 22(I). Since $\Pi_{\widehat{C}_4}(P)$ is homotopically trivial in $\widehat{C}_4$, by Lemma 3.5 we obtain that P_4 is homotopic in $\widehat{C}_4$ to e_1^* or $e_1^*e_2^*e_1^*$. The former is impossible, since it implies $x_3 = x_5$. The latter implies that x_3 and x_5 have a common neighbour of type $\hat{a}$, and so ω is admissible by Lemma 7.3(1).

We now show that C_3, C_5, C_9 intersect C_1. Otherwise, up to a symmetry, we can assume that one of them equals B in Figure 22(II). If exactly one of them equals B, then, by Lemma 3.5, we have $\Pi_{\widehat{B}}(P) = e^*$, implying that ω is not locally embedded at one of x_3, x_5, x_9. Thus we can assume that at least two of them equal B. Then, up to a symmetry, we have $C_3 = B$ and at least one of C_5, C_9 equals B. Up to a symmetry of Σ interchanging B' with D_0, we can also assume $\{C_5, C_9\} \subset \{B, D_0, D_2, D_4\}$.



(I) (II)

FIGURE 22.

First assume that C_9 equals B or D_0. Let $\mathcal{K}$ be the type II sub-arrangement of $\mathcal{A}$ with $\kappa_{\mathcal{K}}(C_2) = \mathsf{X}_{31}, \kappa_{\mathcal{K}}(D_0) = \mathsf{X}_{32}, \kappa_{\mathcal{K}}(C_1) = \mathsf{X}_{42}$. Since $C_5 \neq B'$, we have $\kappa_{\mathcal{K}}(C_6) = \mathsf{X}_{31}$ or X_{33}. By Lemma 7.3(1), we can suppose that x_3 and x_5 do not have a common neighbour of type $\hat{a}$. If ω is not admissible, then, by Lemma 4.15, $\omega^{\mathcal{K}} = \kappa^*_{\mathcal{K}}(\omega)$ satisfies the assumptions (2) and (3) of Proposition 5.11. This contradicts Corollary 5.16.

Second, assume $C_9 = D_2$, and $C_8 = C_{10} = D_1$. Let $\mathcal{L}$ be the type II sub-arrangement of $\mathcal{A}$ with $\kappa_{\mathcal{L}}(D_1) = \mathsf{X}_{31}, \kappa_{\mathcal{L}}(D_0) = \mathsf{X}_{32}, \kappa_{\mathcal{L}}(C_1) = \mathsf{X}_{42}$. Note that, to achieve that, we need to reflect Figure 7 with respect to, say, the line h_3, and then apply an orientation-preserving isometry carrying it appropriately to Figure 22(II). Then $\kappa_{\mathcal{L}}(C_3)$ is an edge. Declaring that $P_i^{\mathcal{L}}$ are hosted by $\widehat{\mathsf{X}}_{33}$ for $i = 2, \ldots, 6$, we obtain a 6-cycle $x_1^{\mathcal{L}} x_2^{\mathcal{L}} x_7^{\mathcal{L}} x_8^{\mathcal{L}} x_9^{\mathcal{L}} x_{10}^{\mathcal{L}}$ in $\mathbb{V}_{34}$ as in Definition 4.7. Since $\mathbb{V}_{34}$ is CAT(0), the angle at $x_8^{\mathcal{L}}$ or $x_{10}^{\mathcal{L}}$ is $\frac{\pi}{4}$, and so, by Lemma 4.15, the angle at x_8 or x_{10} is $\frac{\pi}{4}$. Thus ω is admissible.

Third, assume $C_8 = D_3$. If $C_{10} \neq D_3$, then, by considering $\Pi_{\widehat{C}_8}(P)$, we obtain that the angle at x_8 is $\frac{\pi}{4}$, and so ω is admissible. If $C_{10} = D_3$, then let $\mathcal{J}$ be the type I sub-arrangement of $\mathcal{A}$ with $\kappa_{\mathcal{J}}(C_1) = X_{22}, \kappa_{\mathcal{J}}(D_3) = X_{11}, \kappa_{\mathcal{J}}(D_4) = X_{21}$. Note that again we reflect Figure 4. Since $\kappa_{\mathcal{J}}(C_3) = \kappa_{\mathcal{J}}(C_2)$ is an edge of X_{11}, declaring that $P_i^{\mathcal{J}}$ are hosted by $\widehat{X}_{11}$ for $i = 1, \ldots, 7$, we obtain a 4-cycle $x_7^{\mathcal{J}} x_8^{\mathcal{J}} x_9^{\mathcal{J}} x_{10}^{\mathcal{J}}$ in $\mathbb{U}_{12}$ as in Definition 4.7. Since $\mathbb{U}_{12}$ is CAT(0), the angle at $x_8^{\mathcal{H}}$ is $\frac{\pi}{4}$, and so, by Lemma 4.11, the angle at x_8 is $\frac{\pi}{4}$ and ω is again admissible.

Up to a symmetry, this exhausts all the possibilities, since in particular the case $C_9 = D_2$, $C_8 = D_1$, and $C_{10} = D_3$ is sent to the case $C_9 = D_2, C_8 = D_3$, and $C_{10} = D_1$ under the involution $\mathcal{I}'$. Thus we can assume that all the C_i intersect C_1.

Let $\mathcal{C}$ be the family of hexagons appearing among the C_i. Recall that two hexagons intersecting C_1 are *consecutive* if they intersect a common square. We can assume that there is no hexagon C that equals C_i for a unique i, and such that for C', C'' consecutive with C, there is at most one j with $C_j \in \{C', C''\}$ or there are two such j_1, j_2, and $\{i, j_1, j_2\} = \{2, 4, 6\}$. Otherwise, we consider $\Pi_{\widehat{C}}(P)$, and the argument is similar to Case 3(i) of Proposition 6.4, except no denting is necessary here and we possibly need Lemma 7.3(1) when $i = 4$.

We claim that then $\mathcal{C}$ is contained in a sequence F_1, F_3, F_5 of consecutive hexagons, see Figure 23, left. Indeed, if $|\mathcal{C}| = 5$, then C_2, C_4, C_6 are distinct and consecutive, and we can take $i = 4$ above. If $|\mathcal{C}| = 4$, then we can take C_i to be one of the two hexagons that have only one consecutive hexagon among the C_j. If $\mathcal{C}$ consists of three hexagons that are not consecutive, then two of them are consecutive. If this consecutive pair equals $\{C_8, C_{10}\}$, then we can take $i = 8$ or 10. If this pair equals $\{C_2, C_4, C_6\}$, then we can take $i = 2, 4$, or 6. This justifies the claim.

Case 1: C_2, C_4, C_6 are distinct. Since $\{C_8, C_{10}\} \neq \{F_1, F_5\}$, using a symmetry of Σ we can assume $F_1 \neq C_8, C_{10}$ and $F_2 \neq C_9$. Using the involution $\mathcal{I}'$, we can assume $F_1 = C_2$. Furthermore, one of C_8, C_{10} equals F_3, since otherwise we could take $C_i = C_2$ above. In particular, we have $C_9 \neq F_6$.

Let $\mathcal{K}$ be the type II sub-arrangement of $\mathcal{A}$ with

$$\kappa_{\mathcal{K}}(F_1) = \mathsf{X}_{31}, \kappa_{\mathcal{K}}(F_2) = \mathsf{X}_{32}, \kappa_{\mathcal{K}}(C_1) = \mathsf{X}_{42}.$$

Here we reflect Figure 7 before comparing it with Figure 23, left. We construct the following cycle $\omega^{\mathcal{K}}$ in $\mathbb{V}_{34}$, which is CAT(0). We declare that $P_i^{\mathcal{K}}$ is hosted by $\widehat{X}_{43}$ for x_i of face type F_5, and that $P_5^{\mathcal{K}}$ (resp. $P_9^{\mathcal{K}}$) has the same host as $P_6^{\mathcal{K}}$ (resp. $P_{10}^{\mathcal{K}}$). Then $x_5^{\mathcal{K}} = x_6^{\mathcal{K}}$ and $x_9^{\mathcal{K}} = x_{10}^{\mathcal{K}}$, see Figure 23, right. By Lemma 7.3(1) and Lemma 4.15, we can assume that the path $x_2^{\mathcal{K}} x_3^{\mathcal{K}} x_4^{\mathcal{K}} x_6^{\mathcal{K}}$ is a geodesic with angle $\geq \frac{3\pi}{4}$ at $x_2^{\mathcal{K}}$, and so $|x_1^{\mathcal{K}}, x_7^{\mathcal{K}}| \geq 4$. However, the length of the path $x_7^{\mathcal{K}} x_8^{\mathcal{K}} x_{10}^{\mathcal{K}} x_1^{\mathcal{K}}$ is $\leq 2 + \sqrt{2}$, which is a contradiction.

Case 2: C_2, C_4, C_6 are not distinct. Then we can assume that none of them equals F_5. Let $\mathcal{K}$ be the type II sub-arrangement of $\mathcal{A}$ as in Case 1. Declaring that $P_i^{\mathcal{K}}$ are hosted by $\widehat{X}_{43}$ for $C_i = F_5$, by $\widehat{X}_{42}$ for $C_i = F_6$, and by $\widehat{X}_{43}$ or $\widehat{X}_{33}$ for $C_i = F_4$, we obtain a cycle $\omega^{\mathcal{K}}$ in $\mathbb{V}_{34}$. Suppose first $C_5 \neq F_4$. Then by Lemma 7.3(1) and Lemma 4.15, we can assume that the path $x_2^{\mathcal{K}} x_3^{\mathcal{K}} x_4^{\mathcal{K}} x_5^{\mathcal{K}} x_6^{\mathcal{K}}$ is a geodesic with angles $\geq \frac{3}{4}$ at $x_2^{\mathcal{K}}, x_6^{\mathcal{K}}$, which implies $|x_1^{\mathcal{K}}, x_7^{\mathcal{K}}| \geq 6$. However, the length of the path $x_7^{\mathcal{K}} x_8^{\mathcal{K}} \cdots x_1^{\mathcal{K}}$ is $\leq 2 + 2\sqrt{2}$, which is a contradiction. If $C_5 = F_4$, then we obtain a contradiction exactly as in Case 1.

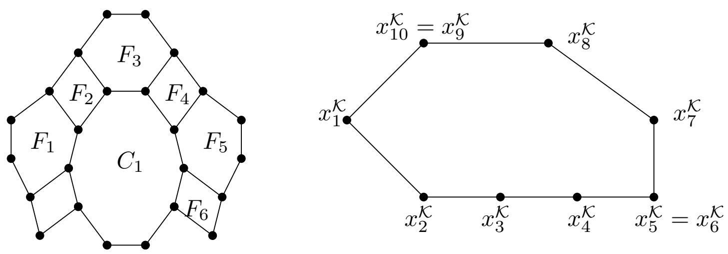
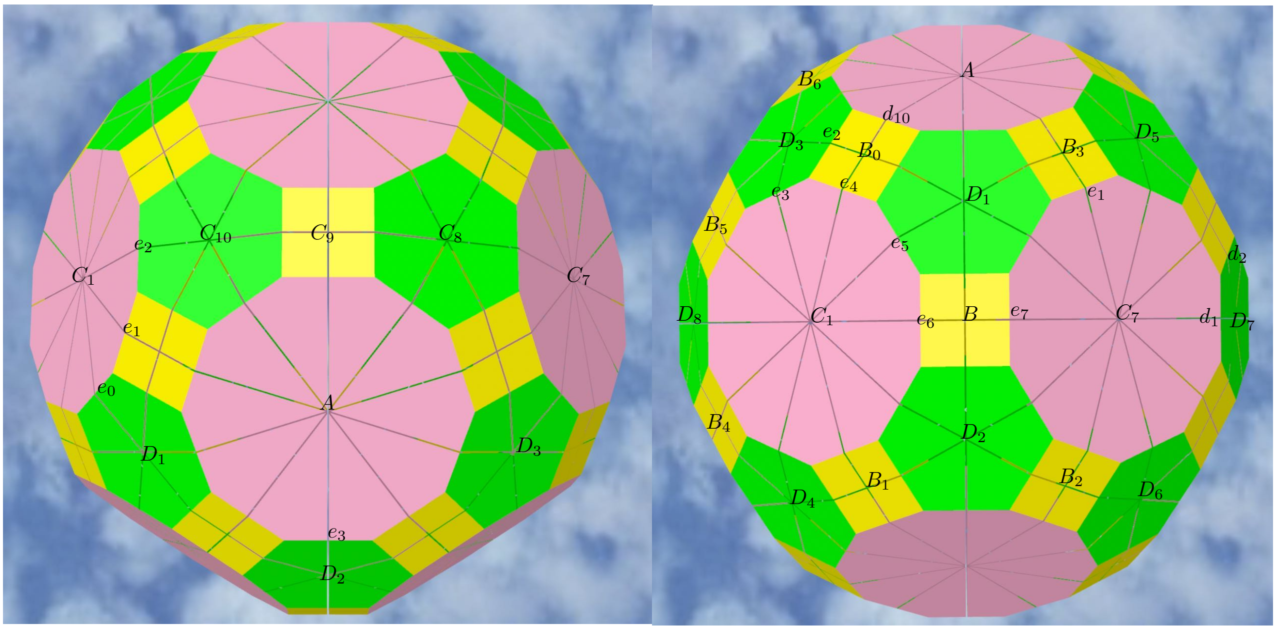


FIGURE 23.

7.2. **Case of two decagons.**

Proposition 7.5. *Let ω be an embedded critical 10-cycle with $C_1 \neq C_7$. Then ω is admissible.*

There are two possible configurations for the pair C_1, C_7, illustrated in Figure 24(I,II). Consider first Figure 24(I). If $C_4 \neq D_2$, and all C_2, C_4, C_6 belong to $\{C_8, C_{10}\}$, then, by considering $\Pi_{\widehat{C}_1}(P)$, we obtain that $P_1 = e_2^*$, and so ω is not locally embedded at x_1, which is a contradiction. Thus, if $C_4 \neq D_2$, then one of C_2, C_4, C_6 is distinct from C_8, C_{10}. Using a symmetry, we can assume without loss of generality $C_2 = D_1, C_4 = C_{10}$, and $C_6 = C_8$. By considering $\Pi_{\widehat{C}_1}(P)$, we obtain that P_1 is homotopic in $\widehat{C}_1$, relative to the endpoints, to $e_2^* e_1^* e_0^*$. After possibly



(I) (II)

FIGURE 24.

replacing the w_i by equivalent words, we can assume $P_1 = e_1^*$, and so we can dent x_1 to A. By considering $\Pi_{\widehat{C}_7}(P)$, and arguing similarly, we can also dent x_7 to A. By Lemma 7.3(3) and Proposition 7.4 it follows that ω is admissible. If $C_4 = D_2$, then $C_2 = D_1$ and $C_6 = D_3$. From considering $\Pi_{\widehat{C}_4}(P)$ it follows that we can choose $P_4 = e_3^*$. Thus there is a neighbour of type $\hat{a}$ of x_3 and x_5 and so ω is admissible by Lemma 7.3(1).

In the remaining part of the subsection, we will consider Figure 24(II). Then C_9 and all even C_i intersect C_1 or C_7. If C_3 is disjoint from both C_1, C_7, then, by considering $\Pi_{\widehat{C}_3}(P)$, we obtain that ω is not locally embedded at x_3, except in one case, where, up to a symmetry, we have $C_1 \cdots C_{10} = C_1D_3B_6D_3B_0D_1C_7D_5B_3D_1$. Note that $\Pi_{\widehat{D}_7}(P) = d_1^*d_2^*d_1^*d_2^*$ is homotopically trivial in $\widehat{D}_7$. Since ω is locally embedded at x_3, the $*$ over d_1 are nonzero. By Lemma 3.8, we obtain that the $*$ over d_2 are zero. It follows that $\Pi_{\widehat{C}_7}(P_1)$ is a homotopically trivial loop in $\widehat{C}_7$. Thus, by considering $\Pi_{\widehat{C}_7}(P)$, we can choose $P_7 = e_1^*$. This allows us to dent x_7 to A, which, by Lemma 7.3(3), brings us to the case where all the C_i intersect C_1 or C_7. An analogous discussion applies to C_5. Thus we can assume that all the C_i intersect C_1 or C_7.

Case 1: One of the C_i equals D_3, D_4, D_5, or D_6. This includes the case where C_2 or C_6 equals D_7 or D_8. We say that $D_j \in \{D_3, D_4, D_5, D_6\}$ is *good*, if there is a unique i with $C_i = D_j$.

Case 1.1: One of the D_j, say $D_3 = C_i$, is good. If $i = 4$, then $C_6 = D_1$ and $C_2 = D_1$ or D_8. Furthermore, $\Pi_{\widehat{D}_3}(P_8 \cup P_9 \cup P_{10})$ equals e_3^* or e_2^*. Considering $\Pi_{\widehat{D}_3}(P)$, it follows that we can choose $P_4 = e_3^*$. Thus there is a neighbour of type $\hat{a}$ of x_3 and x_5 and so ω is admissible by Lemma 7.3(1). If $i = 2$, then $C_4 = D_1$ and $C_6 = D_1, D_2$ or D_5, and again $\Pi_{\widehat{D}_3}(P_8 \cup P_9 \cup P_{10})$ equals e_3^* or e_2^*. Considering $\Pi_{\widehat{D}_3}(P)$, we can choose P_2 to be trivial. Thus ω has angle $\frac{\pi}{4}$ at x_2, and so ω is admissible.

If $i = 10$, then $C_9 = B_0$, and $C_8 = D_1$. Considering $\Pi_{\widehat{D}_3}(P)$, we can choose P_{10} to be trivial, which leads to angle $\frac{\pi}{4}$ at x_{10}, except the special cases where $C_2C_3C_4C_5C_6 = D_1BD_2B_1D_2$ or $D_8B_4D_4B_1D_2$. In the second special case, by considering $\Pi_{\widehat{C}_7}(P)$, we can choose $P_7 = e_7^*$, and denting x_7 to C_1, by Lemma 7.3(3), reduce to Proposition 7.4.

In the first special case, let $\mathcal{K}$ be the type II sub-arrangement of $\mathcal{A}$ with $\kappa_{\mathcal{K}}(D_2) = \mathsf{X}_{31}, \kappa_{\mathcal{K}}(B) = \mathsf{X}_{32}, \kappa_{\mathcal{K}}(C_1) = \mathsf{X}_{42}$. Declaring that $P_9^{\mathcal{K}}$ is hosted by $\widehat{X}_{43}$, we obtain a cycle $\omega^{\mathcal{K}}$ in $\mathbb{V}_{234}$, see Figure 25(I). Since $x_9^{\mathcal{K}} = x_{10}^{\mathcal{K}}$, we have that $x_8^{\mathcal{K}}$ is a neighbour of $x_1^{\mathcal{K}}$. By Lemmas 7.3(1) and 4.15, we can assume that $x_3^{\mathcal{K}}, x_4^{\mathcal{K}}, x_5^{\mathcal{K}}$ do not have a common neighbour and the angles at $x_2^{\mathcal{K}}, x_6^{\mathcal{K}}$ are $\geq \frac{3\pi}{4}$. By Lemma 5.8, we have $x_6^{\mathcal{K}} \neq x_8^{\mathcal{K}}$. Let ${x'_3}^{\mathcal{K}}$ be a type $\hat{a}$ neighbour of $x_3^{\mathcal{K}}$, and let $\omega_8^{\mathcal{K}} = x_7^{\mathcal{K}} x_8^{\mathcal{K}} x_1^{\mathcal{K}} x_2^{\mathcal{K}} {x'}_3^{\mathcal{K}} x_4^{\mathcal{K}} x_5^{\mathcal{K}} x_6^{\mathcal{K}}$. Then $\omega_8^{\mathcal{K}}$ satisfies the hypotheses of Proposition 5.10. Thus ω_8 bounds a minimal disc diagram $\mathcal{D}$ that is a subdiagram of Figure 8(III). By Lemma 5.2, the link $\mathrm{lk}({x'_3}^{\mathcal{K}}, \mathbb{V}_{234})$ has girth ≥ 8. Thus, since ${x'_3}^{\mathcal{K}}$ in Figure 25(I) corresponds to x_5 in Figure 8(III), the vertex $x_3^{\mathcal{K}}$ lies in the image of $\mathcal{D}$. Considering the simplicial structure of $\mathcal{D}$, we obtain a neighbour of type $\hat{a}$ of $x_3^{\mathcal{K}}, x_4^{\mathcal{K}}$, and $x_5^{\mathcal{K}}$, which is a contradiction.

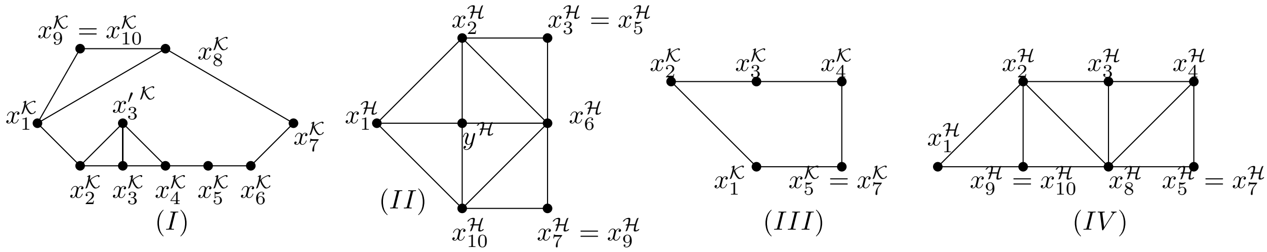


FIGURE 25.

Now we assume that none of D_3, D_4, D_5, D_6 are good, and in particular $C_i = D_3$ is not good. Consider sequences $\Theta_1 = (C_2, C_3, C_4, C_5, C_6)$, and $\Theta_2 = (C_8, C_9, C_{10})$. Note that D_3 cannot occur twice in Θ_2. Hence D_3 occurs at least once in both Θ_1 and Θ_2, or D_3 occurs twice in Θ_1 but does not occur in Θ_2.

Case 1.2: D_3 occurs at least once in both Θ_1 and Θ_2. Then $C_8 = D_1$, and $C_{10} = D_3$. If $C_5 = B_0$, then, by considering $\Pi_{\widehat{C}_7}(P)$, we can choose $P_7 = e_7^*$. Denting x_7 to C_1, we reduce to Proposition 7.4 by Lemma 7.3(3).

Otherwise, since D_5 is not good, we have $C_2 = D_3, C_4 = D_1, C_5 = B$ or B_3, and $C_6 = D_1$ or D_2. Let $\mathcal{H}$ be the type I sub-arrangement of $\mathcal{A}$ with $\kappa_{\mathcal{H}}(C_1) = X_{22}, \kappa_{\mathcal{H}}(D_3) = X_{11}, \kappa_{\mathcal{H}}(B_5) = X_{21}$ (we reflect Figure 4). We declare that P_6 is hosted by X_{22} if $C_6 = D_2$, and P_i is hosted by X_{12} if $C_i \in \{B, B_3, C_7\}$.

If $C_6 \neq D_2$, then we have $x_3^{\mathcal{H}} = x_4^{\mathcal{H}} = x_5^{\mathcal{H}} = x_6^{\mathcal{H}} = x_7^{\mathcal{H}} = x_8^{\mathcal{H}} = x_9^{\mathcal{H}}$. By Lemma 5.7, the cycle $x_{10}^{\mathcal{H}} x_1^{\mathcal{H}} x_2^{\mathcal{H}} x_3^{\mathcal{H}}$ has angle $\frac{\pi}{4}$ at $x_2^{\mathcal{H}}$, and so ω is admissible by Lemma 4.11.

If $C_6 = D_2$, then we have $x_3^{\mathcal{H}} = x_4^{\mathcal{H}} = x_5^{\mathcal{H}}$, and $x_7^{\mathcal{H}} = x_8^{\mathcal{H}} = x_9^{\mathcal{H}}$. We can assume that $\omega^{\mathcal{H}}$ has angle $\frac{3\pi}{4}$ at $x_2^{\mathcal{H}}$ and $x_{10}^{\mathcal{H}}$, as before. Since $\mathbb{U}_{12}$ is CAT(0), we obtain that $\omega^{\mathcal{H}}$ bounds a minimal disc diagram in Figure 25(II). If $y^{\mathcal{H}}$ has type $\widehat{X}_{12}$, then we can choose $P_1^{\mathcal{H}}$ homotopic in $\widehat{X}_{22}$, relative to the endpoints, to a path in $\widehat{X}_{22} \cap \widehat{X}_{12}$. Thus we can assume $P_1 \subset \widehat{e}_6 \cup \widehat{e}_5 \cup \widehat{e}_4$. Consequently, $\Pi_{\widehat{D}_2}(P_1)$ is trivial. By considering $\Pi_{\widehat{D}_2}(P)$, it follows that we can choose P_6 to be trivial, which implies that the angle at x_6 is $\frac{\pi}{4}$ and so ω is admissible as before. If $y^{\mathcal{H}}$ has face type $\widehat{X}_{21}$, then we can dent x_1 to the decagon distinct from C_1 intersecting B_5 and reduce to the configuration from Figure 24(I).

Case 1.3: D_3 occurs at least twice in Θ_1 but does not occur in Θ_2. Then $C_2 = C_4 = D_3$ and $C_6 = D_1$. We have $\{C_8, C_{10}\} \subset \{D_1, D_2\}$, since D_4, D_5, D_6 are not good. Let $\mathcal{H}$ be the sub-arrangement as in Case 1.2. Then P_5, P_6, P_7 are hosted by X_{12}. We declare that P_i with $C_i = D_1$ (resp. D_2) are hosted by X_{12} (resp. X_{22}), and P_9 has the same host as P_{10}. If $C_8 \neq D_2$ or $C_{10} \neq D_1$, then $\omega^{\mathcal{H}}$ is a 5-cycle $x_1^{\mathcal{H}} x_2^{\mathcal{H}} x_3^{\mathcal{H}} x_4^{\mathcal{H}} x_5^{\mathcal{H}}$ as in Figure 25(III) with angle π at $x_3^{\mathcal{H}}$ and angle $\frac{\pi}{2}$ at $x_5^{\mathcal{H}}$. Since $\mathbb{U}_{12}$ is CAT(0), $\omega^{\mathcal{H}}$ has angle $\frac{\pi}{4}$ at $x_2^{\mathcal{H}}$, and we finish as before. If $C_8 = D_2$ and $C_{10} = D_1$, then $x_5^{\mathcal{H}} = x_6^{\mathcal{H}} = x_7^{\mathcal{H}}$ and so $\omega^{\mathcal{H}}$ is a 7-cycle $x_1^{\mathcal{H}} x_2^{\mathcal{H}} x_3^{\mathcal{H}} x_4^{\mathcal{H}} x_5^{\mathcal{H}} x_8^{\mathcal{H}} x_{10}^{\mathcal{H}}$, where $x_4^{\mathcal{H}}$ and $x_8^{\mathcal{H}}$ are neighbours. If $x_1^{\mathcal{H}}, x_3^{\mathcal{H}}$ are not neighbours, then, since $\mathbb{U}_{12}$ is CAT(0), the path $\omega^{\mathcal{H}}$ bounds the reduced disc diagram in Figure 25(IV). Thus $x_3^{\mathcal{H}}, x_4^{\mathcal{H}}, x_5^{\mathcal{H}}$ have a common neighbour, and by Lemma 4.11 so do x_3, x_5. Hence, by Lemma 7.3(1), ω is admissible.

Case 2: All even C_i belong to $\{D_1, D_2\}$. Without loss of generality, we can assume that neither C_3 nor C_5 equals B_1 or B_2. Let $\mathcal{L}$ be the type II sub-arrangement of $\mathcal{A}$ with $\kappa_{\mathcal{K}}(D_1) = \mathsf{X}_{31}, \kappa_{\mathcal{K}}(B) = \mathsf{X}_{32}, \kappa_{\mathcal{K}}(C_7) = \mathsf{X}_{42}$. If $C_9 \notin \{B_1, B_2\}$, and ω is not admissible, then $\widehat{\kappa}_{\mathcal{L}}(\omega)$ contradicts Corollary 5.16.

If $C_9 \in \{B_1, B_2\}$, then $C_8 = C_{10}$, and we consider the 8-cycle $\omega_8^{\mathcal{L}}$ obtained from $x_1^{\mathcal{L}} \cdots x_8^{\mathcal{L}}$ by replacing $x_3^{\mathcal{L}}$ with its neighbour of type $\hat{a}$. Using Proposition 5.10, we obtain that ω is admissible by the same argument as in the special case of Case 1.1.

8. 353 square complexes

We refer to Definition 1.5 for the notion of a 353-square complex.

Definition 8.1. A 353-square complex is *stable* if for any set $S \subset \mathcal{A}$ or $\mathcal{D}$ of pairwise close vertices, there is a finite subset $S' \subset S$, such that if a vertex v is close to or a neighbour of the entire S', then v is close (or equal) to or a neighbour of the entire S.

Definition 8.2. A 353-square complex is *wide* if any simplex of $X^{\boxtimes}$ is contained in a simplex σ with $|\sigma^0 \cap \mathcal{A}| \geq 2$, and $|\sigma^0 \cap \mathcal{D}| \geq 2$.

The goal of this section is to prove Theorem 1.3.

8.1. Disc diagrams.

Remark 8.3. From Definition 1.5(1) it follows that if $ad_1a_1d_2$, $ad_1a_2d_2$, and $da_1d_1a_2$ are squares, then $d_1a_1d_2a_2$ and $da_1d_2a_2$ are squares (see Figure 26(1)). Consequently, if a minimal disc diagram D in X contains a cube corner, then it cannot contain the additional two squares in Definition 1.5(3), since otherwise we could replace the five squares by three squares.

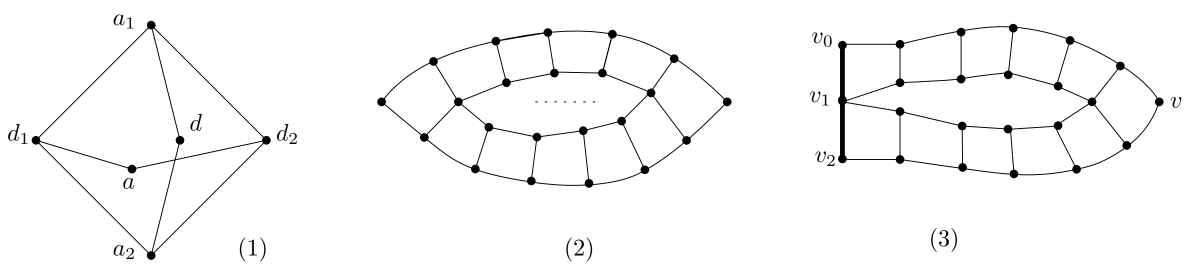


FIGURE 26.

Definition 8.4. Let D be a square disc diagram homeomorphic to a disc. We say that D is n-*bordered*, for $n \geq 0$, if

- there are exactly n vertices of ∂D not contained in any interior edge, and
- each of the remaining vertices of ∂D is contained in exactly one interior edge.

See Figure 26(2) for an example of a 2-bordered disc diagram.

We say that D is an *inter-osculation* (see Figure 26(3)) if there are consecutive vertices $v_0, v_1, v_2 \in \partial D$, and another vertex $v \in \partial D$, such that

- v_0, v_2, v are not contained in any interior edge, and
- v_1 is contained in at least one interior edge, and
- each of the vertices of $\partial D \setminus \{v_0, v_1, v_2, v\}$ is contained in exactly one interior edge.

A *hyperplane* in a square disc diagram is a maximal immersed 1-manifold obtained by connecting the midpoints of opposite edges (called *dual edges*) in consecutive squares, see for example [Sag95, §2.4]. The *carrier* of an embedded hyperplane h is the union of squares intersecting h.

Lemma 8.5. *Let $f\colon D \to X$ be a minimal disc diagram in a 353-square complex. We equip D^1 with the path metric such that each edge has length 1.*

(i) D is not 1- or 2-bordered. If D is an inter-osculation, then it is a cube corner.
(ii) Hyperplanes in D are embedded, not homeomorphic to circles, and pairwise intersect at most once.
(iii) A geodesic in D^1 intersects each hyperplane at most once.
(iv) If $\partial D = \alpha\beta^{-1}$, where α, β are geodesics in D^1 with first edges da_1, da_2, then either d, a_1, a_2 lie in a square of D, or not in a square but in a cube corner of D. In the latter case $f(a_1), f(a_2)$ are not close.
(v) In (iv), the vertex u opposite to d in the top square (resp. cube corner) lies on a geodesic in D^1 with the same endpoints as α.

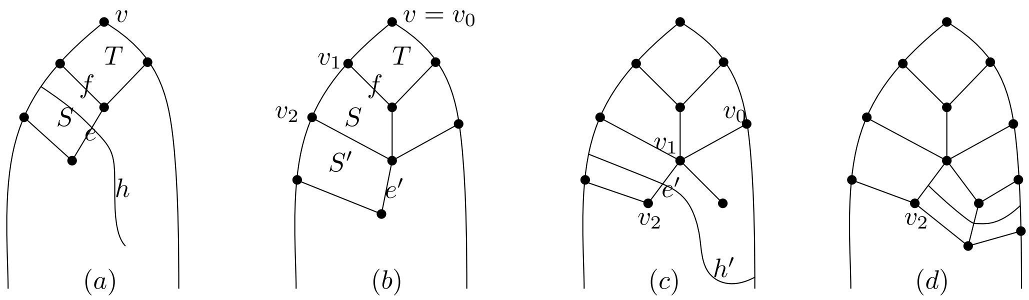


FIGURE 27.

Proof. For part (i), to reach a contradiction, let $D \to X$ be a minimal disc diagram with the smallest area that is

- 1-bordered, or
- 2-bordered, or
- is an inter-osculation but not a cube corner.

Let v be the vertex of ∂D from Definition 8.4 in the third case, or a vertex not contained in an interior edge, in the first two cases. Let $T \subset D$ be the square containing v, and let $f \subset T$ be an edge not containing v. Let $S \subset D$ be the second square containing f, and let e be the second edge in S containing the vertex of T opposite to v (see Figure 27(a)). By Definition 1.5(1), we have that e is not contained in T. Consider the hyperplane h of D dual to e. Note that h cannot self-intersect, since we would obtain a smaller area 1-bordered diagram. Let p be the intersection point of h and ∂D outside S. Note that when D is an inter-osculation, p is not the midpoint of a thickened edge in Figure 26(3). If there would be a path in ∂D from p to $f \cap \partial D$ whose all vertices are contained in the interior edges of D, then we would obtain a smaller area 2-bordered diagram. Thus there is a path in ∂D from p to

$f \cap \partial D$ whose all vertices except for v are contained in the interior edges of D. This gives us an inter-osculation with $v_0 = v, v_1 = f \cap \partial D$, and v_2 the other vertex of $S \cap \partial D$. By the minimality assumption on D, this inter-osculation is a cube corner and so e lies in a square sharing an edge with T. See Figure 27(b). Note that this cube corner is minimal, since it is a subdiagram of a minimal diagram.

Let S' be the remaining square of D sharing an edge with S, and let e' be its edge intersecting e but not contained in S. By Definition 1.5(2,3), and Remark 8.3, the degree of the vertex $v_1 = e \cap e'$ is distinct from 3 and 4. Let h' be the hyperplane of D dual to e'. As before, h' does not self-intersect, and we obtain an inter-osculation with v_0, v_1, v_2 as in Figure 27(c). By the minimality assumption on D this inter-osculation is a cube corner and so D contains the subdiagram in Figure 27(d). Analogically, interchanging the left and the right side of the diagram, we obtain that the degree at v_2 equals 3. This contradicts Definition 1.5(4).

Hyperplanes not satisfying part (ii) give rise to disc diagrams excluded by part (i).

For part (iii), note that the 1-skeleta of hyperplane carriers are isometrically embedded in D^1. Indeed, if vertices of the carrier were closer in D^1 than in the 1-skeleton of the carrier, then they would be separated in the carrier by two hyperplanes contained in the same hyperpane of D, which would contradict part (ii). Thus a path intersecting twice a hyperplane can be shortened by replacing its subpath by a path in the carrier.

In part (iv), if d, a_1, a_2 do not lie in a square, then, by part (iii), the hyperplanes dual to da_1, da_2 need to intersect elsewhere. Thus they form an inter-osculation, which by part (i) is a cube corner. Consequently, a_1 and a_2 are not close by Definition 1.5(2) and Corollary 8.6 below.

Part (v) follows from part (iii), since if the geodesic from u to the last vertex of α and β was intersected by any hyperplane h of the top square or cube corner, then h would intersect twice α or β. □

Corollary 8.6. *Let X be a 353-square complex. Then all 4-cycles in X are squares.*

Proof. Let α be a 4-cycle in X and $D \to X$ a minimal disc diagram bounded by α. By Lemma 8.5(ii), D has only 2 hyperplanes, which moreover intersect at most once. Thus D consists of a single square. □

Corollary 8.7. *Let X be a 353-square complex and let a_1, a_2, a_3 be pairwise close. Then there exists d that is a neighbour of all a_i.*

Proof. Assume by contradiction that a_1, a_2, a_3 do not have a common neighbour. Then we have an embedded 6-cycle in X passing through a_1, a_2, a_3. By Lemma 8.5(ii), its minimal disc diagram $D \to X$ has 3 hyperplanes and 3 squares, and so it is a cube corner. Moreover, a_i are not contained in the interior edges of D. By Lemma 8.5(iv), a_1, a_2 are not close, which is a contradiction. □

8.2. **Structure of downward links.** We fix from now on a 'base' vertex w of a 353-square complex X. Let $S^k = S^k(w)$ denote the set of vertices of X at distance k from w in X^1, and let B^k be the subgraph of X^1 induced on the union of S^l over $0 \leq l \leq k$. We also suppose that X is stable. Given $d \in S^k$, we define $p(d)$ as the set of all neighbours of d in S^{k-1}.

Lemma 8.8. *Let $A \subset p(d)$ be a set of pairwise close vertices. Then there exists $d' \in S^{k-2}$ that is a neighbour of all the elements of A.*

Proof. Assume first that A is finite, that is, $A = \{a_i\}_{i=1}^n$. For $n = 1$, there is nothing prove. For $n = 2$, choose geodesics α, β from d to the base vertex w passing through a_1, a_2, and let $D \to X$ be a minimal disc diagram with boundary $\alpha\beta^{-1}$. By

Lemma 8.5(iv), we have that d, a_1, a_2 lie in a square of D, whose remaining vertex belongs to S^{k-2} by Lemma 8.5(v).

Suppose now $n = 3$. By $n = 2$ case, there are $d_1, d_2, d_3 \in S^{k-2}$ that are neighbours of both a_i distinct from a_1 (resp. a_2, a_3). Suppose without loss of generality that all d_i are distinct. Consider a minimal disc diagram D bounded by geodesics from a_2 to w via d_1 and d_3. By Lemma 8.5(iv,v), we have one of the following options. Suppose first that there is a square containing d_1, a_2, d_3 and a vertex $a \in S^{k-3}$. Then, by Definition 1.5(2), the cube corner with vertices d, a_i, d_i, is not minimal, and so for some i there is an edge $a_i d_i$, as desired. Second, suppose that there are two squares containing a_2 (and some d_0) in D. By Lemma 8.5(v), we have $d_0 \in S^{k-2}$. By Definition 1.5(3), we have either again an edge $a_i d_i$, or d_0 is a neighbour of both a_1 and a_3 and so d_0 is the required d'.

If $n \geq 4$, arguing by induction, there are again distinct $d_1, d_2, d_3 \in S^{k-2}$ that are neighbours of all a_i except for a_1 (resp. a_2, a_3). In particular, we have a square $d_1 a_2 d_3 a_4$ and so d_1, d_3 are close. By $n = 2$ case, there is a vertex $a \in S^{k-3}$ that is a neighbour of both d_1, d_3. As in the first case of the previous paragraph, applying Definition 1.5(2), we deduce that for some i there is an edge $a_i d_i$.

The case of infinite A follows from the stability of X. □

Note that the following result would be trivial if we had assumed that X is wide.

Corollary 8.9. *Let $A \subset \mathcal{A}$ be a set of pairwise close vertices. Then there exists d that is a neighbour of all the elements of A.*

Proof. By the stability of X, we can assume $A = \{a_i\}_{i=1}^n$, and we proceed by the induction on n. For the induction step, assume that we have already a common neighbour d of all a_i distinct from a_1. If d is not a neighbour of a_1, then it belongs to $S^3(a_1)$. We then apply Lemma 8.8 with $w = a_1$ and $k = 3$. □

In view of Corollary 8.9, in Lemma 8.8 instead of assuming that there exists d with $a_i \in p(d)$, we can just assume $a_i \in S^{k-1}$. Indeed, if d is a common neighbour of a_i from Corollary 8.9, then d either belongs to S_{k-2}, as required in Lemma 8.8 or d belongs to S^k, implying $a_i \in p(d)$.

Lemma 8.10. *Suppose that $a_1, a_2 \in p(d)$ are not close, but are both close to a neighbour a of d.*

(i) Then $a \in p(d)$ (i.e. $a \in S^{k-1}$).
(ii) If a_1, a_2 are both close to another neighbour a' of d, then a and a' are close.
(iii) If $d_1, d_2 \in S^{k-2}$ are neighbours of a' and a_1, a_2, respectively, then d_1 and d_2 are close.

Proof. By Lemma 8.5(iv,v), there is a minimal cube corner C with boundary vertices $d_1 a_1 d a_2 d_2 a_3$, where $d_1 \in S^{k-2}$. Since a_1 and a are close and a and a_2 are close, they belong to the remaining squares needed to apply Definition 1.5(3). This shows that a is a neighbour of d_1 (and d_2) and so $a \in S^{k-1}$, proving (i). For (ii), we analogously obtain that a' is a neighbour of d_1 and d_2. Thus $a d_1 a' d_2$ is a square and so a and a' are close.

For (iii), If d_1, d_2 are not close, then, by Lemma 8.5(iv,v), there is a minimal cube corner C with boundary vertices $a^- d_1 a' d_2 a^+ d'$ with $a^- \in S^{k-3}$. By Definition 1.5(3), we have that $d \in S^k$ is a neighbour of a^-, which is a contradiction. □

Lemma 8.11. *Let G be a simplicial graph*

(1) of diameter 2,
(2) without induced embedded 4-cycles, and
(3) whose all induced embedded 5-cycles have a common neighbour.

Let A_1, A_2 be the vertex sets of finite complete subgraphs of G. Then there is a vertex a of G that is a neighbour of or equal to all of the elements of $A_1 \cup A_2$.

Proof. First note that we can assume that each element $a_1 \in A_1$ is not a neighbour of some element $a_2' \in A_2$, since otherwise we can take $a = a_1$. Second, note that we can assume that a_1 is not a neighbour of **each** element $a_2 \in A_2$. Indeed, otherwise we replace A_2 by $A_2 \setminus \{a_2\}$, and we find by induction a neighbour a of all the elements of $A_1 \cup A_2 \setminus \{a_2\}$. Applying assumption (2) to $a_1 a_2 a_2' a$ with a_2' as above, we obtain that a is a neighbour of a_2, as desired.

If $|A_1| = |A_2| = 1$, then the lemma follows from assumption (1). Assume now $A_1 = \{a_1\}$ and $A_2 = \{a_3, a_4\}$. By assumption (1), a_1 and a_3 have a neighbour a_2, and a_1 and a_4 have a neighbour a_5. We can assume that a_2 and a_4 are not neighbours, since otherwise we can take $a = a_2$. Analogously, we can assume that a_3 and a_5 are not neighbours, since otherwise we can take $a = a_5$. Then a_2 and a_5 are not neighbours by assumption (2). Thus $a_1a_2a_3a_4a_5$ is an induced embedded 5-cycle and it remains to apply assumption (3).

We now suppose $A_1 = \{a_1\}$ and $3 \leq m = |A_2|$, and we argue by induction on m. By induction, there is a vertex a (resp. a') that is a neighbour of or equal to all the elements of $A_1 \cup A_2$ except possibly for a_2 (resp. a_2') in A_2. Applying assumption (2) to $aa_1a'a_2''$ for some $a_2'' \in A_2 \setminus \{a_2, a_2'\}$, we obtain that a and a' are neighbours. Applying assumption (2) to $aa'a_2a_2'$, we obtain that a is a neighbour of a_2 or a' is a neighbour of a_2'.

Finally, assume $2 \leq |A_1|, |A_2|$. Choose $a_1 \neq a_1' \in A_1, a_2 \neq a_2' \in A_2$. By induction, there is a vertex a^1 (resp. a'^1, a^2, a'^2) that is a neighbour of or equal to all the elements of $A_1 \cup A_2$ except possibly for a_1 (resp. a_1', a_2, a_2'). By assumption (2), both a^1 and a'^1 are neighbours of both a^2, a'^2. Applying assumption (2) to $a^1a^2a'^1a'^2$ we obtain that, say, a^1 is a neighbour of a'^1. Applying again assumption (2) to $a^1a'^1a_1a_1'$, we obtain that $a = a^1$ or $a = a'^1$ satisfies the lemma. □

Lemma 8.12. *Let $A_1, A_2 \subset p(d)$ be sets of pairwise close vertices. Then there is $a \in p(d)$ that is a close (or equal) to all the elements of $A_1 \cup A_2$. In particular, two maximal sets of pairwise close vertices in $p(d)$ have non-empty intersection.*

Proof. Assume to start with that both A_1 and A_2 are finite. Let G be the simplicial graph with vertex set $p(d)$, and edges between close elements. It suffices to verify that G satisfies the assumptions of Lemma 8.11. Assumption (1) follows from Lemma 8.5(iv,v). Assumption (2) follows from Lemma 8.10(ii). To verify assumption (3), let $a_1 \cdots a_5$ be an induced embedded 5-cycle. By Lemma 8.8, for $i = 1, \ldots, 5$ there are $d_i \in S^{k-2}$ that are neighbours of a_i and a_{i-1} (mod 5). By Lemma 8.10(iii), each d_i is close to d_{i+1}. Again by Lemma 8.8, this leads to the existence of squares $d_ia_id_{i+1}a_i'$ with $a_i' \in S^{k-3}$. Note that the cube corners with centres a_i and boundaries $a_{i-1}da_{i+1}d_{i+1}a_i'd_i$ are minimal. By Definition 1.5(4), there is $a \in p(d)$ that is close to all a_i, as desired.

The cases of infinite $|A_1|$ or $|A_2|$ follow from the stability of X. □

8.3. **Contractibility.**

Lemma 8.13. *Let L be a simplicial complex containing a simplex M satisfying the following properties.*

(i) *Every maximal simplex σ in L intersects M.*
(ii) *For every set V of vertices in L pairwise connected by edges and such that $V \setminus M$ spans a simplex, we have that V spans a simplex.*

Then L is contractible.

Proof. Assume to start with that L is finite. Let L' be the barycentric subdivision of L. We equip the vertex set of L' with the poset structure coming from the inclusion between the simplices of L. Let K be the subcomplex of L' spanned on the barycentres of all the simplices of L intersecting M. Let $M' \subset K$ be the barycentric subdivision of M.

First, we justify that K is contractible. Indeed, assign to each vertex x of K corresponding to a simplex σ of L, the barycentre $F(x)$ of $\sigma \cap M$. For $x \leq x'$, we have $F(x) \leq F(x')$. In particular, we have that F extends to a simplicial map from K to M', which is homotopically trivial. Finally, since $F(x) \leq x$, by [Seg83, Prop 2.1], we have that F is homotopic to the identity map on K. Consequently, the identity map on K is homotopically trivial.

Second, we justify that L is contractible. For that, let L'' be the barycentric subdivision of L'. Let v be a vertex of L'', which is a chain $x_0 \leq x_1 \leq \cdots \leq x_n$ of vertices of L'. We consider two maps F_1, F_2 assigning to each such v a vertex of L'. Let $F_1(v) = x_0$. For F_2, let τ_j be the simplex of L corresponding to x_j. Let π_n be the set of all the vertices of M that are neighbours of, or equal to, all the vertices of τ_n. By (i), we have that π_n is non-empty and by (ii) we have that $\tau_n \cup \pi_n$ spans a simplex. We consider its subsimplex spanned on $\tau_0 \cup \pi_n$, the barycentre of which we denote $F_2(v)$. Note that for $v \subseteq v'$, we have $F_1(v) \geq F_1(v')$ and $F_2(v) \geq F_2(v')$, and so F_i extend to simplicial maps from L'' to L'. In fact, we have that F_1 is homotopic to the identity map on L'' (see [Prz09, Prop 4.2]). Furthermore, since $F_1(v) \subseteq F_2(v)$, by [Seg83, Prop 2.1], we have that F_1 and F_2 are homotopic. Finally, the image of F_2 is contained in the subcomplex corresponding to K. Since K is contractible, we obtain that the identity map on L'' is homotopically trivial.

If L is infinite, then each finite subcomplex of L is contained in a finite subcomplex $L' \subset L$ satisfying the assumptions of the lemma. This implies that all the homotopy groups of L vanish, and so L is contractible by Whitehead theorem. □

Remark 8.14. Suppose that L and M satisfy conditions (i) and (ii) from Lemma 8.13. Then any induced subcomplex of L containing M also satisfies (i) and (ii) with the same M.

Proof of Theorem 1.3. Inside the thickening $X^{\boxtimes}$, we consider the full subcomplexes $\operatorname{Span} B^k$ spanned on the graphs B^k (see the beginning of Section 8.2). Suppose without loss of generality $S^k \subset \mathcal{D}$. To start with, suppose that S^k is finite. We call a simplex τ of $\operatorname{Span} B^k$ *peelable* if $|\mathcal{A} \cap \tau| = 1$ and τ is not contained in a simplex of $\operatorname{Span} B^k$ with another element of $\mathcal{A}$. Note that if τ is peelable, and τ is contained in a simplex ρ of $\operatorname{Span} B^k$, then ρ is peelable.

Since X is wide, there are no peelable simplices in $X^{\boxtimes}$. This implies that peelable τ in $\operatorname{Span} B^k$ satisfies $\tau \cap \mathcal{D} \subset S^k$ (since a common neighbour of $\tau \cap \mathcal{D}$ must be missing from B^k, hence belongs to S^{k+1}).

For τ peelable, let $\tau' = \tau \cap \mathcal{D}$. We claim that if $\tau_1' = \tau_2'$, then $\tau_1 = \tau_2$. Indeed, suppose $\tau_1 = \tau_1' \cup \{a_1\}, \tau_2 = \tau_1' \cup \{a_2\}$. If $|\tau_1'| \geq 2$, then a_1, a_2 are close, and so τ_1 is contained in $\operatorname{Span} B^k$ in the simplex $\tau_1 \cup \{a_2\}$, contradicting peelability. If $\tau_1' = \{d\}$, then we have $d \in S^k, a_1, a_2 \in S^{k-1}$. Thus $|p(d)| \geq 2$, and so by Lemma 8.5(iv,v) there is a square in $\operatorname{Span} B^k$ containing $a_1 d$, which is a contradiction.

We denote by P^k the subcomplex of $\operatorname{Span} B^k$ obtained from removing the peelable open simplices τ and their 'free' open faces τ'. Note that by the claim above, we have that P^k is obtained from performing successive collapses on $\operatorname{Span} B^k$ (starting with τ of the maximal dimension), so they are homotopy equivalent.

Let now $d \in P^k \cap S^k$. We will describe the link L of d in P^k. Let A_λ be the maximal subsets of $p(d)$ of pairwise close elements, over $\lambda \in \Lambda$. Since $d \in P^k$, we have $|A_\lambda| \geq 2$ for each $\lambda \in \Lambda$. For each $\lambda \in \Lambda$, let $D_\lambda \subset S^{k-2}$ be the set of common neighbours of A_λ, which is non-empty by Lemma 8.8. Since $|A_\lambda| \geq 2$, any $d_\lambda, d'_\lambda \in D_\lambda$ are close. Furthermore, by Lemma 8.12 and Lemma 8.10(iii), any $d_\lambda \in D_\lambda, d_\mu \in D_\mu$ are close. Since $|A_\lambda| \geq 2$, we have that all $d_\lambda \in D_\lambda$ are also close to d. Thus the union of D_λ over $\lambda \in \Lambda$ spans a simplex M in the link L of d in P^k. We will now show that L and M satisfy the conditions of Lemma 8.13.

By Lemma 8.13, this will imply that P^k deformation retracts to its subcomplex obtained from removing the open star of d. Repeating this procedure with d replaced by other elements of $P^k \cap S^k$, which can be done by Remark 8.14, shows that P^k deformation retracts to $\operatorname{Span} B^{k-1}$.

Condition (ii) of Lemma 8.13 follows from the fact that $\operatorname{Span} B^k$ is flag, and no peelable simplices have a vertex in M, since M is contained in S^{k-2}. For condition (i), since $d \cup \sigma$ is a maximal simplex of P^k, it contains at least one $a \in A$ by Lemma 8.8 and Corollary 8.9. Since P^k does not contain peelable simplices, in fact we have $\sigma^0 = A \cup D$, where $|A| \geq 2$. We have $A \subset p(d)$, and so we can pick λ satisfying $A \subseteq A_\lambda$. Then any $d_\lambda \in D_\lambda$ is close to all the elements of D, and so by the maximality of σ we have $d_\lambda \in \sigma$. Thus $M \cap \sigma$ contains d_λ, as desired.

If S^k is infinite, then each finite subcomplex of $\operatorname{Span} B^k$ is contained in the span of $\operatorname{Span} B^{k-1}$ and a finite subset of S^k. Hence, by the above discussion, we can homotope this subcomplex into $\operatorname{Span} B^{k-1}$. This implies that all the homotopy groups of $\operatorname{Span} B^k$ vanish, and so $\operatorname{Span} B^k$ is contractible by the Whitehead theorem. □

9. 353 Simplicial complexes

Let Δ be a simplicial complex of type $S = \{\hat{a}, \hat{b}, \hat{c}, \hat{d}\}$, see Section 2.2. We equip S with the total order $\hat{a} < \hat{b} < \hat{c} < \hat{d}$, which induces a relation $<$ on the vertex set of Δ as in Definition 2.3. We denote by $\mathcal{A}$ the set of vertices of type $\hat{a}$ etc.

Definition 9.1. Δ is a 353-*simplicial complex* if it is simply connected and satisfies the following properties.

(1) The relation $<$ on Δ^0 is a partial order.
(2) Each $\operatorname{lk}(d, \Delta)^0$ (resp. $\operatorname{lk}(a, \Delta)^0$) is bowtie free and upward (resp. downward) flag.
(3) Each cycle $a_1c_1a_2c_2a_3c_3bc_4$ (resp. $d_1b_1d_2b_2d_3b_3cb_4$) in some $\operatorname{lk}(d, \Delta)$ (resp. $\operatorname{lk}(a, \Delta)$) is not embedded or not induced.
(4) If $\gamma = c_1b_1c_2a_2c_3b_3c_4b_4c_5a_5$ is an induced embedded cycle in some $\operatorname{lk}(d, \Delta)$, then there is a neighbour $a \in \operatorname{lk}(d, \Delta)$ of all the vertices of γ in $\mathcal{B}$. An analogous condition holds for $\mathcal{A}, \mathcal{B}$ interchanged with $\mathcal{D}, \mathcal{C}$.

Δ is *wide* if each vertex in $\mathcal{B} \cup \mathcal{C}$ has at least two neighbours in $\mathcal{A}$ and two neighbours in $\mathcal{D}$.

We will provide the main example of a 353-simplicial complex in Theorem 9.12. Our goal for the moment is to prove the following.

Theorem 9.2. *Let Δ be a 353-simplicial complex. Let $X^1 \subset \Delta^1$ be the subgraph induced on $\mathcal{A} \cup \mathcal{D}$. Let X be the square complex with 1-skeleton X^1 and squares that are the 4-cycles γ having a common neighbour in Δ, called an* apex *of γ. Then X is a 353-square complex.*

9.1. **Links.** We need a series of preparatory observations on the link $\Gamma = \operatorname{lk}(d, \Delta)$. By reversing the order $<$ we have the obvious analogues of all the results in this subsection for $\operatorname{lk}(a, \Delta)$. From Definition 9.1(1,2), we obtain:

Remark 9.3. Let γ be an induced embedded n-cycle in Γ.

(i) If $n = 4$, then $\gamma = a_1c_1a_2c_2$ and there is a common neighbour b of all a_1, c_1, a_2, c_2.
(ii) If $n = 6$, then γ has three vertices in $\mathcal{A}$ and they have a common neighbour.

Corollary 9.4. *There is no cycle $\gamma = u_1c_1bc_2u_2c$ in Γ with $u_1c_1bc_2u_2$ embedded and induced.*

Proof. Since $u_1c_1bc_2u_2$ is embedded and induced, we have that γ is embedded. By Remark 9.3(ii), we have that γ is not induced. Thus c is a neighbour of b, which contradicts Remark 9.3(i). □

Corollary 9.5. *Let $\gamma = a_1c_1bc_2a_2c_3ac_4$ be a cycle in Γ with $a_1c_1bc_2a_2$ embedded and induced. Then b is a neighbour of a.*

Proof. By Definition 9.1(3), we have that γ is not embedded or not induced.

Suppose first that γ is not embedded. Since $a_1c_1bc_2a_2$ was embedded and induced, we have $c_3 \neq c_1$, and by Corollary 9.4, we have $a \neq a_1, a_2$ and $c_3 \neq c_4$. Thus without loss of generality we can assume $c_3 = c_2$. If $c_1 = c_4$, then the corollary follows from Remark 9.3(i) applied to the cycle c_1bc_2a. If $c_1 \neq c_4$, then we argue in the same way as in the proof of Lemma 5.8, using Definition 9.1(1,2).

Suppose now that γ is embedded but not induced. If a is a neighbour of c_2 (or c_1), then we argue as above. Note that c_3 is not a neighbour of b, since this would contradict Remark 9.3(i) applied to the cycle $bc_2a_2c_3$. Finally, if c_3 is a neighbour of a_1, then this contradicts Corollary 9.4. Up to replacing c_3 by c_4, this exhausts all the possibilities. □

Corollary 9.6. *Let $\gamma = c_1b_1c_2a_2c_3b_3c_4b_4c_5a_5$ be a cycle in Γ with the paths of length 4 centred at each b_i embedded and induced. Then there is a that is a neighbour of all b_i.*

Proof. Suppose by contradiction that such a does not exist. Then by Definition 9.1(4), we have that γ is not embedded or not induced.

Suppose first that γ is not embedded. If $c_1 = c_3, c_2 = c_5$, or $b_1 = b_3$ (or b_4), then this contradicts Corollary 9.4. If $c_4 = c_1$ or $c_3 = c_5$, then this contradicts the assumption that $b_3c_4b_4c_5a_5$ is embedded and induced. We obtain an analogous contradiction for $c_2 = c_4$. If $c_2 = c_3$, then let a_i be a neighbour of b_i for $i = 1, 3$. If $a_3c_4b_4c_5a_5$ is embedded and induced, then, applying Corollary 9.5 to the cycle $a_3c_4b_4c_5a_5c_1a_1c_2$, we obtain that a_1 is a neighbour of b_4. Hence a_1 is a neighbour of c_4 by Definition 9.1(1). By Definition 9.1(2) applied to $a_1c_4b_3c_2$, we obtain that a_1 is a neighbour of b_3, and so we can take $a = a_1$. If $a_3c_4b_4c_5a_5$ is not embedded, then $a_3 = a_5$, contradicting the hypothesis that $b_3c_4b_4c_5a_5$ is induced. If $a_3c_4b_4c_5a_5$ is embedded but not induced, then a_3 is a neighbour of b_4 or c_5, which implies that a_3 is a neighbour of both b_4 and c_5 by Definition 9.1(1,2). By Corollary 9.4 applied to the 6-cycle $a_3c_2b_1c_1a_5c_5$, we obtain that $a_3c_2b_1c_1a_5$ is not induced. Since $c_2b_1c_1a_5$ is embedded and induced, the only possibility is that a_3 is a neighbour of b_1 or c_1, which implies that a_3 is a neighbour of both b_1 and c_1 as before. Hence we can take $a = a_3$. The case of $c_1 = c_5$ is analogous.

Second, suppose that γ is embedded but not induced. By Corollary 9.4, we have that a_2 is not a neighbour of c_5 and b_3 is not a neighbour of c_1. If b_3 is a neighbour of c_2 or b_1 is a neighbour of c_3, then Remark 9.3(i) implies $c_2 = c_3$, which is a contradiction. If b_1 is a neighbour of c_4, but not of c_3, then the path $a_2c_3b_3c_4b_1$ is embedded and induced, and so this contradicts Corollary 9.4 applied to $a_2c_3b_3c_4b_1c_2$. Up to symmetries, this exhausts all the possibilities. □

9.2. **Proof of Theorem 9.2.**

Remark 9.7. Let $ada'd'$ be an embedded cycle in Δ with common neighbour v. Suppose that a, d and a' have a common neighbour b. Then b is a neighbour of d'. Indeed, we apply Remark 9.3(i) to the link of d, and the 4-cycle $aba'v$. If $v \in \mathcal{B}$, then $b = v$, which is a neighbour of d'. If $v \in \mathcal{C}$, then b is a neighbour of v. Since v is a neighbour of d', we have $b < v < d'$, and so $b < d'$ by Definition 9.1(1).

Proof. First note that since Δ is a simplicial complex of type $\{a, b, c, d\}$, it follows from Definition 9.1(1) that X is connected. We now justify that X is simply connected. Let α be a cycle in X^1, and view it as a cycle in Δ^1. Let $D \to \Delta$ be a minimal disc diagram bounded by α. We first justify that we can assume that there is no edge bc in D. Suppose that there were such an edge, in triangles v_0bc, v_nbc. By Definition 9.1(1), the link of bc is complete bipartite, which is connected, and so it contains a path $v_0v_1 \cdots v_n$. We can then replace in D the above two triangles by the union of the triangles $v_iv_{i+1}b, v_iv_{i+1}c$ over $0 \leq i < n$. Repeating this procedure removes each edge bc from D. Then the set of the 2-cells of D can be partitioned into subsets consisting of the 2-cells belonging to each of the stars around the vertices in $\mathcal{B}$ and $\mathcal{C}$. Each link of such a vertex x in D has vertices in $\mathcal{A} \cup \mathcal{D}$. Since in X^1 there is an edge between any such a and d, we can replace the open star of x in D by a square or a union of squares with apex x. Consequently, $D \to \Delta$ can be replaced by a disc diagram in X.

It remains to verify parts (1)-(4) of Definition 1.5. For part (1), consider squares $ada'd_1, ada'd_2$. First assume that they have apices b_1, b_2. Then applying Remark 9.3(i) to the link of d and the 4-cycle $ab_1a'b_2$, we see that $b_1 = b_2$, as desired. Second, suppose that they have apices c_1, c_2. Then, by Remark 9.3(i), there is a neighbour b of all a, c_1, a', c_2, which then is a neighbour of all a, d_1, a', d_2. Third, suppose that they have apices c_1, b_2. Then, by Remark 9.3(i), we have that c_1 and b_2 are neighbours and so b_2 is a neighbour of all a, d_1, a', d_2.

For the remaining parts, consider a minimal cube corner C with centre a, boundary $d_1a_3d_2a_1d_3a_2$ and square apices v_1, v_2, v_3. Then $d_1v_3d_2v_1d_3v_2$ is a cycle in the link of a. Thus, by Definition 9.1(2), there is b that is a common neighbour of all d_i and a.

We claim that for each $j \neq i$ the apex v_i is neither a neighbour of d_i nor of a_j. Indeed, if, say, v_2 is a neighbour of d_2, then by Definition 9.1(1) a_2 is a neighbour of d_2. Hence $a_2d_2ad_1$ and $a_2d_2ad_3$ are squares of X (with apex v_2). By part (1), $a_2d_2a_3d_1$ and $a_2d_2a_1d_3$ are squares of X, contradicting the minimality of C. If, say, v_2 is a neighbour of a_3, then $a_2d_3a_3d_1, ad_3a_3d_1$ are squares of X (with apex v_2). By considering the sequence $ad_3a_3d_1, ad_2a_3d_1, ad_2a_3d_3, ad_2a_1d_3, a_3d_2a_1d_3$ and repeatedly applying part (1), we obtain that all these 4-cycles are squares. This contradicts again the minimality of C, and justifies the claim. In particular, all v_i are distinct.

Then we can assume that v_i belong to $\mathcal{C}$, since if, say, $v_1 = b'$, then it would have to be distinct from b, and so, by Remark 9.3(i) applied to the link of a, there would be c that is a neighbour of b, d_2, b', d_3, and a. Then c would be also an apex of $ad_2a_1d_3$ by Remark 9.7 with the roles of $\mathcal{A}, \mathcal{D}$ interchanged. We will thus write c_i instead of v_i. By Remark 9.3(i), we have that c_i is a neighbour of b. Note that since c_i is not a neighbour of a_j, for $j \neq i$, we have by Definition 9.1(1) that b is not a neighbour of a_j. Consequently, the path $a_1c_1bc_2a_2$ is an induced embedded path in the link of d_3.

For part (2), let v be an apex of a_1da_2d'. Let c_1, c_2 be apices of ada_1d_1, ada_2d_2 guaranteed by the above paragraph (note that the labelling of the $\mathcal{D}$ vertices of C changed). Since $a_1c_1bc_2a_2$ is an induced embedded path in the link of d, this contradicts Corollary 9.4 applied to $a_1c_1bc_2a_2v$.

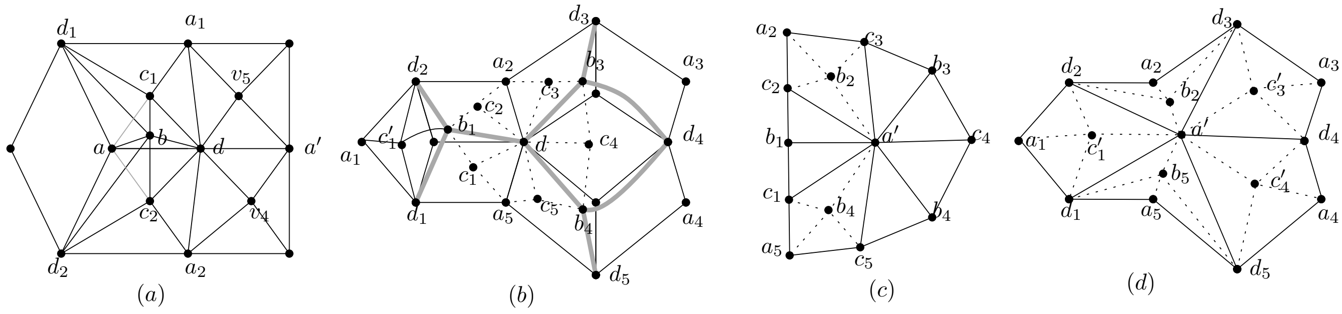


FIGURE 28.

For part (3), and v_4, v_5 the apices of the two last squares, we similarly obtain in the link of d a cycle $\gamma = a_1c_1bc_2a_2v_4a'v_5$, see Figure 28(a). After possibly replacing the v_i by the elements of $\mathcal{C}$, by Corollary 9.5 we obtain that b is a neighbour of a'. Thus d_1, d_2 are neighbours of a' by Definition 9.1(1), as desired. The required 4-cycles are squares with apices b.

In part (4), we label the boundary 10-cycle of E by $d_1a_1\cdots d_5a_5$. We similarly obtain the indicated 10-cycle $c_1b_1c_2a_2c_3b_3c_4b_4c_5a_5$, see Figure 28(b). By Corollary 9.6, b_1, b_3, b_4 have a common neighbour a' in the link of d. By Definition 9.1(1) a' is a neighbour of all the c_i, see Figure 28(c). Again by Definition 9.1(1), a' is a neighbour of all the d_i, see Figure 28(d). For $i = 1, 3, 4$, the 4-cycles $a'd_ia_id_{i+1}$ are squares, since they have common neighbours c_i', where c_i' is a common neighbour of a_i, b_i, d_i, d_{i+1} in Figure 28(b). As for the 4-cycle $a'd_2a_2d_3$, it has either a common neighbour c_2, in the case where $c_2 = c_3$, or, in the case where $c_2 \neq c_3$, a common neighbour b_2, where b_2 is a common neighbour of a', c_2, a_2, c_3 in the link of d, guaranteed by Remark 9.3(i). Analogously, the 4-cycle $a'd_5a_5d_1$ is a square. Part (5) is proved analogously. $\square$

9.3. **Stability and contractibility.**

Lemma 9.8. *Let Δ be a wide 353-simplicial complex. Then the relation $<$ on Δ^0 is a partial order that is bowtie free. Furthermore, $\mathcal{P} = \mathcal{A} \cup \mathcal{B} \cup \mathcal{C}$ is bowtie free and upward flag. Moreover, if in $K \subset \mathcal{P}$ each pair has an upper bound in $\mathcal{P}$, then K has the join in $\mathcal{P}$.*

Proof. By Corollary 8.6 and Theorem 9.2, each embedded 4-cycle $ada'd'$ in Δ has a common neighbour in $\mathcal{B} \cup \mathcal{C}$. Thus the first assertion of the lemma follows from [Hua24a, Lem 8.1]. In particular, $\mathcal{P}$ is also bowtie free.

Now we show that $\mathcal{P}$ is upward flag. Let $u_1, u_2, u_3 \in \mathcal{P}$ be pairwise upper bounded. We can assume that none of them are neighbours. For each i, let a_i be a neighbour of or equal to u_i. Since Δ is wide, we have that a_i are pairwise close in X, and so, by Corollary 8.7, there is a common neighbour d of a_1, a_2, a_3.

We claim that each u_i is also a neighbour of d. To justify the claim for, say, $u_1 = b_1$, let $b_1, u_2 \leq c_2$ and $b_1, u_3 \leq c_3$. Applying the bowtie freeness of Δ^0 to $da_1c_ia_i$, for $i = 2, 3$, we obtain that either c_i is a neighbour of d or $da_1c_ia_i$ have a common neighbour b_i. In each case, d, c_2, c_3 are pairwise lower bounded in $\mathrm{lk}(a_1, \Delta)^0$, hence they have a common lower bound b' by Definition 9.1(2). Considering the cycle $c_2b_1c_3b'$, we obtain $b_1 = b'$, justifying the claim.

Since Δ^0 is bowtie free, u_1, u_2, u_3 are pairwise upper bounded in $\mathrm{lk}(d, \Delta)^0$. By Definition 9.1(2), they have a common upper bound in $\mathrm{lk}(d, \Delta)^0$, as desired.

The last assertion of the lemma follows from Lemma 2.5 and [Hae24, Lem 6.2]. $\square$

Lemma 9.9. *Let Δ be a wide 353-simplicial complex and let X be its 353-square complex constructed in Theorem 9.2.*

(1) X is stable.
(2) The vertices of each simplex of $X^{\boxtimes}$ have a common neighbour b or c in Δ. In particular, X is wide.

Proof. For (1), assume without loss of generality $S = A \subset \mathcal{A}$ and $|A| = \infty$. Then A is pairwise upper bounded in $\mathcal{P}$. By Lemma 9.8, A has a join $u \in \mathcal{B} \cup \mathcal{C}$ in $\mathcal{P}$. Hence u is also the join of A in Δ^0.

Suppose first $u = b \in \mathcal{B}$. Choose $a_1 \neq a_2$ from A and let $S' = \{a_1, a_2\}$. If a is close to each element of S', then a, a_1, a_2 are pairwise upper bounded in $\mathcal{P}$, hence they have a join $u' \in \mathcal{P}$. If $u' \in \mathcal{B}$, then applying the bowtie freeness of Lemma 9.8 to a_1ba_2u' we obtain $u' = b$. Then a is a neighbour of b, and so a is close to each element of A since b has at least two neighbours in $\mathcal{D}$. If $u' = c' \in \mathcal{C}$, then applying the bowtie freeness to a_1ba_2c' we obtain that b is a neighbour of c'. Then, by Definition 9.1(1), each element of A is a neighbour of c'. Thus a is close to each element of A. If d is a neighbour of each element of S', then applying the bowtie freeness to a_1ba_2d, we obtain that b is a neighbour of d, and by Definition 9.1(1) d is a neighbour of each element of A.

Second, suppose $u = c \in \mathcal{C}$. Assume first that the join of each three element subset of A belongs to $\mathcal{B}$. Then for each $A_1, A_2 \subset A$ with $|A_1| = |A_2| = 3$ and $|A_1 \cap A_2| = 2$, the joins of A_1, A_2 are equal. Consequently, $u \in \mathcal{B}$, which is a contradiction. Finally, suppose that there is $S' \subset A$ with $|S'| = 3$ and join $c' \in \mathcal{C}$. Since $c' \leq c$, we have $c' = c$. If d is a neighbour of each element of S', then $c < d$, and so d is a neighbour of each element of A. If a is close to each element of S', then Lemma 9.8 implies that $S' \cup \{a\}$ has a join $u' \in \mathcal{P}$. Since $c \leq u'$, we have $u' = c$ and so a is a neighbour of c. Hence a is close to each element of A.

Part (2) is proved similarly. □

A 353-simplicial complex Δ is *non-degenerate* if for each edge bc there is $d \in \mathrm{lk}(b, \Delta)^0$ that is not a neighbour of c, and there is $a \in \mathrm{lk}(c, \Delta)^0$ that is not a neighbour of b.

Proposition 9.10. *Let Δ be a wide non-degenerate 353-simplicial complex. Then $X^{\boxtimes}$ is homotopy equivalent to Δ.*

Proof. For each $u \in \mathcal{B} \cup \mathcal{C}$, let $\phi(u)$ be the simplex of $X^{\boxtimes}$ spanned by all the neighbours of u in $\mathcal{A} \cup \mathcal{D}$. Note that $\phi(u)$ is a maximal simplex in $X^{\boxtimes}$. Indeed, otherwise by Lemma 9.9(2) we would have $\phi(u) \subsetneq \phi(u')$. Then u and u' would be neighbours contradicting the non-degeneracy for the edge uu'. Similarly, the function $u \to \phi(u)$ is a bijection from $\mathcal{B} \cup \mathcal{C}$ to the family of the maximal simplices of Δ.

We claim that for any subset $U \subset \mathcal{B} \cup \mathcal{C}$, the intersection $\bigcap_{u \in U} \mathrm{St}(u, \Delta)$ is empty or contractible. Indeed, if $v \in \bigcap_{u \in U} \mathrm{St}(u, \Delta)^0$, then v is a lower bound or an upper bound for U, say the latter. By Lemma 2.5 and Lemma 9.8, U has a join M, which belongs to $\bigcap_{u \in U} \mathrm{St}(u, \Delta)$. Furthermore, any vertex of $\bigcap_{u \in U} \mathrm{St}(u, \Delta)$ is $\leq M$, justifying the claim.

Thus for any $U \subset \mathcal{B} \cup \mathcal{C}$, $\bigcap_{u \in U} \mathrm{St}(u, \Delta) \neq \emptyset$ if and only if $\bigcap_{u \in U} \phi(u) \neq \emptyset$. Since Δ is covered by the closed stars of the vertices in $\mathcal{B} \cup \mathcal{C}$, and $X^{\boxtimes}$ is covered by its maximal simplices, it remains to invoke the Nerve Theorem [Bor48], see also the version in [Bjö03, Thm 6]. □

By Lemma 9.9, Theorems 9.2 and 1.3, and Proposition 9.10, we have the following.

Corollary 9.11. *Let Δ be a non-degenerate wide 353-simplicial complex. Then Δ is contractible.*

9.4. **353 Artin complex.**

Theorem 9.12. *Let $\Lambda = abcd$ be the Coxeter diagram that is the linear graph with $m_{ab} = m_{cd} = 3$ and $m_{bc} = 5$. Then the Artin complex $\Delta = \Delta_\Lambda$ is a wide non-degenerate* 353*-simplicial complex.*

As usual, we denote by $\mathcal{A}$ the set of vertices of type $\hat{a}$, etc.

Proof. The simple connectedness of Δ follows from [CMV20, Lem 4]. By Remark 2.1, we have Definition 9.1(1). By [CMV20, Lem 6], we can identify each $\mathrm{lk}(a,\Delta), \mathrm{lk}(d,\Delta)$, with the Artin complex for the Coxeter subdiagram bcd or abc. Hence Definition 9.1(2) follows from Theorem 2.9, Definition 9.1(3) follows from Proposition 6.3, and Definition 9.1(4) follows from Proposition 7.2. Thus Δ is a 353-simplicial complex.

Each vertex in $\mathcal{B} \cup \mathcal{C}$ has infinitely many neighbours in $\mathcal{A}$ and in $\mathcal{D}$, and so Δ is wide.

Let $x \in \Delta^0$ be a vertex of type $\hat{b}$. Then, by Remark 2.1, $\mathrm{lk}(x,\Delta)$ is a join $K_1 * K_2$ where K_1 is the full subcomplex spanned by the vertices of type $\hat{a}$, and K_2 is the full subcomplex spanned by the vertices of type $\hat{c}$ and $\hat{d}$. By [CMV20, Lem 6], we have $K_2 \cong \Delta_{\Lambda'}$, where $\Lambda' \subset \Lambda$ is the edge cd. By considering the simplicial map π from the Artin complex $\Delta_{\Lambda'}$ to the Coxeter complex $\mathfrak{C}_{\Lambda'}$, which is a circle formed of 6 edges, we obtain that for each vertex z of type $\hat{c}$ in K_2, there is a vertex of type $\hat{d}$ in K_2 that is not a neighbour of z. This confirms the first part of the definition of the non-degeneracy of Δ. The second part is analogous. □

10. Relative Artin complexes and related background

10.1. **Relative Artin complexes.**

Definition 10.1 ([Hua24b]). Let $\Lambda' \subset \Lambda$ be an induced subdiagram. The (Λ,Λ')-*relative Artin complex* $\Delta_{\Lambda,\Lambda'}$ is the induced subcomplex of the Artin complex Δ_Λ spanned by vertices of type $\hat{s}$ with s a vertex of Λ'.

Lemma 10.2 ([Hua24b, Lem 6.2] and [CMV20, Lem 4]). *If $|\Lambda'| \geq 3$, then $\Delta_{\Lambda,\Lambda'}$ is simply connected (in particular, it is connected).*

Note that $\Delta_{\Lambda,\Lambda'}$ is a simplicial complex of type S (see Section 2.2) with $S = \{\hat{s}\}_{s\in\Lambda'}$.

Definition 10.3. An induced subdiagram Λ' of Λ is *admissible* if for any vertex x of Λ', if the vertices x_1, x_2 of Λ' are in distinct connected components of $\Lambda' \setminus \{x\}$, then they are in distinct connected components of $\Lambda \setminus \{x\}$.

Lemma 10.4 ([Hua24b, Lem 6.6]). *Suppose that $\Lambda' = s_1 \cdots s_n$ is an admissible linear subgraph of a Coxeter diagram Λ. Let Δ' be the (Λ,Λ')-relative Artin complex, with the relation $<$ on its vertex set induced from $s_1 < \cdots < s_n$ or $s_n < \cdots < s_1$. Then $(\Delta'^0, <)$ is a weakly graded poset.*

Definition 10.5. Let Δ' be as in Lemma 10.4. We say that Δ' is *bowtie free* (resp., *flag*, or *weakly flag*) if $(\Delta'^0, <)$ is bowtie free, (resp. flag, or weakly flag). Note that these definitions do not depend on the choice of one or the other total order on Λ'.

Lemma 10.6 ([Hua24b, Lem 6.4(1)]). *Let $v \in \Delta' = \Delta_{\Lambda,\Lambda'}$ be a vertex of type $\hat{s}$. Then there is a type-preserving isomorphism between $\mathrm{lk}(v,\Delta')$ and $\Delta_{\Lambda\setminus\{s\},\Lambda'\setminus\{s\}}$.*

We have the following consequence.

Lemma 10.7. *Let $\Lambda' \subset \Lambda$ be an induced subdiagram.*

(1) *Let s be a vertex of Λ'. If $\Delta_{\Lambda\setminus\{s\},\Lambda'\setminus\{s\}}$ is contractible, then $\Delta_{\Lambda,\Lambda'}$ deformation retracts onto $\Delta_{\Lambda,\Lambda'\setminus\{s\}}$.*
(2) *More generally, let T be a subset of the vertex set of Λ'. If $\Delta_{\Lambda\setminus R,\Lambda'\setminus R}$ is contractible for each non-empty subset R of T, then $\Delta_{\Lambda,\Lambda'}$ deformation retracts onto $\Delta_{\Lambda,\Lambda'\setminus T}$.*

Proof. Part (1) is [Hua24b, Lem 7.1] in view of Lemma 10.6. We prove part (2) by induction on $|T|$. For $s \in T$, by the inductive assumption, $\Delta_{\Lambda,\Lambda'}$ deformation retracts onto $\Delta_{\Lambda,\Lambda'\setminus T\cup\{s\}}$. It remains to prove that $\Delta_{\Lambda,\Lambda'\setminus T\cup\{s\}}$ deformation retracts onto $\Delta_{\Lambda,\Lambda'\setminus T}$. This will follow from part (1) once we verify that $\Delta_{\Lambda\setminus\{s\},\Lambda'\setminus T}$ is contractible. This follows from the assumption that $\Delta_{\Lambda\setminus\{s\},\Lambda'\setminus\{s\}}$ is contractible, since by the inductive assumption it deformation retracts to $\Delta_{\Lambda\setminus\{s\},\Lambda'\setminus T}$. □

10.2. **Properties of some relative Artin complexes.**

Lemma 10.8 ([Hua24b, Lem 6.16])**.** *Suppose that Λ is an arbitrary Coxeter diagram. Let $\omega = x_1\cdots x_4$ be an embedded 4-cycle in Δ_Λ of type $\hat{s}_1\cdots\hat{s}_4$. Suppose $\hat{s}_1 \neq \hat{s}_3$. Then there exists a vertex $x_3' \in \Delta_\Lambda$ of type $\hat{s}_1$ that is a common neighbour of x_2, x_3, and x_4.*

Corollary 10.9. *Suppose that Λ is an arbitrary Coxeter diagram with an edge s_1s_2 such that Δ_{Λ,s_1s_2} has girth ≥ 6. Let ω be an embedded 4-cycle in Δ_Λ with an edge of type $\hat{s}_1\hat{s}_2$. Then ω not induced.*

Proof. Let $\omega = x_1\cdots x_4$ with x_i of type $\hat{s}_i$. Since ω is embedded, the girth hypothesis implies that we cannot have simultaneously $s_1 = s_3$ and $s_2 = s_4$. Assume first $s_1 \neq s_3$ and $s_2 = s_4$. By Lemma 10.8, there is a vertex x_3' of Δ_Λ of type $\hat{s}_1$ that is a common neighbour of x_2, x_3 and x_4. Then $x_1x_2x_3'x_4$ is a 4-cycle in Δ_{Λ,s_1s_2}. Since $x_2 \neq x_4$, we must have $x_1 = x_3'$, implying that x_1 is a neighbour of x_3 and so ω is not induced. The case $s_1 = s_3$ and $s_2 \neq s_4$ is analogous.

Now assume $s_1 \neq s_3$ and $s_2 \neq s_4$. Let x_3' of type $\hat{s}_1$ be chosen as before. If $x_1x_2x_3'x_4$ is not embedded, then we deduce that x_1 is a neighbour of x_3 as before. If $x_1x_2x_3'x_4$ is embedded, then x_2 is a neighbour of x_4 by the case of $s_1 = s_3$ and $s_2 \neq s_4$. □

Theorem 10.10 ([Hua24b, Thm 8.1])**.** *Let Λ be irreducible spherical, and let $\Lambda' \subset \Lambda$ be a linear subdiagram. Then $\Delta_{\Lambda,\Lambda'}$ is bowtie free.*

10.3. **Haettel contractibility criteria.** Let $S = \{s_1,\ldots,s_n\}$ be a totally ordered set. Let X be a simplicial complex of type S with the induced relation on its vertex set, as defined in Section 2.2: for vertices $x, x' \in X^0$ we write $x < x'$ if x, x' are neighbours and $\mathrm{Type}(x) < \mathrm{Type}(x')$. The following is a consequence of [Hae22b, §4.3, Thm B] and [Hae24, Thm 1.15].

Theorem 10.11. *Let X be a simplicial complex of type S. Assume that*

(1) *X is simply connected,*
(2) *the relation $<$ on X^0 is a partial order,*
(3) *for each $x \in X^0$, the collection of vertices $\geq x$ is bowtie free and upward flag, and*
(4) *for each $x \in X^0$, the collection of vertices $\leq x$ is bowtie free and downward flag.*

Then X is contractible.

Proposition 10.12 ([Hua24a, Lem 5.1])**.** *Suppose that X satisfies the assumptions of Theorem 10.11. Then $(X^0, <)$ is bowtie free and flag.*

Now we discuss a variation of Theorem 10.11.

Definition 10.13. Let S be cyclically ordered with $s_1 < s_2 < \cdots < s_n < s_1$. For each vertex x of X of type s_i, we consider the relation $<_x$ on $\mathrm{lk}(x, X)^0$ as follows. The cyclic order induces an order on $S \setminus \{s_i\}$ by declaring $s_{i+1} < \cdots < s_n < s_1 < \cdots < s_{i-1}$. For vertices $y, z \in \mathrm{lk}(x, X)^0$, define $y <_x z$ if y, z are neighbours and $\mathrm{Type}(y) < \mathrm{Type}(z)$ in $S \setminus \{s_i\}$.

We say that X is an *$\widetilde{A}_n$-like complex* if

(1) X is simply connected,
(2) for each $x \in X^0$, the relation $<_x$ on $\mathrm{lk}(x, X)^0$ is a partial order, and
(3) for each $x \in X^0$, the relation $<_x$ on $\mathrm{lk}(x, X)^0$ is bowtie free.

For example, the Coxeter complex of type $\widetilde{A}_n$ is an $\widetilde{A}_n$-like complex.

The following is a consequence of [Hae22b, §4.2, Thm A]. It also follows from [Bes06, Thm 3.3 and §8] and [Bes99], since $\widetilde{A}_n$-like complexes are Bestvina complexes for a certain Garside groupoid.

Theorem 10.14. *Each $\widetilde{A}_n$-like complex is contractible.*

Lemma 10.15. *Let X be an $\widetilde{A}_n$-like complex. Then any induced 4-cycle in X^1 has a common neighbour. In particular, X has no embedded cycles of type $\hat{s}_1\hat{s}_2\hat{s}_1\hat{s}_2$.*

Proof. The first assertion is [Hua24b, Lem 4.8]. For the second assertion, if a cycle had type $\hat{s}_1\hat{s}_2\hat{s}_1\hat{s}_2$ and common neighbour x, then it would be a bowtie in the link of x. □

11. Bestvina convexity in $\widetilde{A}_n$-like complexes

11.1. Garside complexes.

Definition 11.1 ([HH24, Def 4.6])**.** Let $\widehat{X}$ be a simply connected flag simplicial complex. Suppose that we have a binary relation $<$ on $\widehat{X}^0$ (not necessarily a partial order) such that vertices x, y are neighbours exactly when $x < y$ or $y < x$. Furthermore, suppose that the transitive closure of $<$ is a partial order that is weakly graded with rank function r. We write $x \leq y$ when $x < y$ or $x = y$.

Assume that we have an automorphism φ of $(\widehat{X}^0, <)$ such that

- $r \circ \varphi = t \circ r$, for a translation $t\colon \mathbb{Z} \to \mathbb{Z}$, and
- $x \leq y$ if and only if $y \leq \varphi(x)$, for all $x, y \in \widehat{X}^0$, and
- the interval $[x, \varphi(x)] = \{z \in \widehat{X}^0 \mid x \leq z \text{ and } z \leq \varphi(x)\}$ is a lattice for all $x \in \widehat{X}^0$ (in particular, the relation $<$ restricted to $[x, \varphi(x)]$ is transitive).

We then call $\widehat{X}$ a *Garside flag complex.*

Let X be an $\widetilde{A}_n$-like complex of type S, with cyclic order $s_1 < s_2 < \cdots < s_n < s_1$. Consider the type function $\tau : X^0 \to \mathbb{Z}/n\mathbb{Z}$ defined by $\tau(x) = i$ for x of type s_i. We define a following simplicial complex structure on $\widehat{X} = X \times \mathbb{R}$. The vertex set $\widehat{X}^0$ of $\widehat{X}$ is
$$\{(x, i) \in X^0 \times \mathbb{Z} \mid \tau(x) = i\},$$
The vertices (x, i) and (x', j) are neighbours if x and x' are equal or neighbours in X, and $|i - j| \leq n$. Let $\widehat{X}$ be the flag simplicial complex with that 1-skeleton. Note that any maximal simplex of $\widehat{X}$ has vertices
$$(x_i, kn+i), (x_{i+1}, kn+i+1), \ldots, (x_n, kn+n), (x_1, kn+n+1), \ldots, (x_i, kn+n+i),$$
where $k \in \mathbb{Z}$, $1 \leq i \leq n$, and $x_1, x_2, \ldots, x_n$ are vertices of a maximal simplex of X with $\tau(x_i) = i$.

Note that the map $\widehat{X}^0 \to X^0$ sending (x,i) to x extends to a simplicial map, denoted by $\pi : \widehat{X} \to X$. Define $r\colon \widehat{X}^0 \to \mathbb{Z}$ by $r(x,i) = i$, and $\varphi\colon \widehat{X}^0 \to \widehat{X}^0$ by $\varphi(x,i) = (x, i+n)$.

We define a binary relation $<$ on $\widehat{X}^0$ by requiring $(x,i) < (y,j)$ exactly when these two vertices are neighbours in $\widehat{X}$ and $i < j$. Note that the transitive closure $\leq_t$ of $\leq$ on $\widehat{X}^0$ is a partial order. The following shows that $\widehat{X}$ is a Garside flag complex with the automorphism φ.

Lemma 11.2. *For each $(x,i) \in \widehat{X}^0$, the interval $[(x,i),(x,i+n)]$ is a lattice.*

Below, the partial order $<_x$ on $\mathrm{lk}(x,X)^0$ was introduced in Definition 10.13.

Proof. Note that the poset $[(x,i),(x,i+n)]$ is isomorphic with the poset obtained from $(\mathrm{lk}(x,X)^0, <_x)$ by adding the smallest and the greatest element. This poset is a lattice by Lemma 2.5. □

By [HH24, Thm 1.3], given $x \in \widehat{X}^0$, the poset $\{w \in \widehat{X}^0 \mid w \geq_t x\}$ is a lattice, and so we can discuss the meet $\wedge$ in that poset. By [HH24, Thm 4.7], a Garside flag complex $\widehat{X}$ is an instance of a *homogeneous categorical Garside structure*. We decided not to give here the definition, since we will be only using [HH24, Prop 4.2] on the *Deligne normal form* (term introduced in [CMW04]), which is more convenient for us to state directly in the terms of $\widehat{X}$:

Theorem 11.3. *For each $x, y \in \widehat{X}^0$, there is a unique edge-path $x_1 \cdots x_l \cdots x_n$ from $x_1 = x$ to $x_n = y$ such that*

- *$x_i < x_{i+1} \neq \varphi(x_i)$ for $1 \leq i < l$, and*
- *$x_i = x_{i+1} \wedge \varphi(x_{i-1})$ for $1 < i < l$ in $[x_i, \varphi(x_i)]$, and*
- *$x_i = \varphi^{\pm(i-l)}(x_l)$ for $l \leq i \leq n$, with all signs positive or all signs negative.*

Note that, as all the notions in this section, the Deligne normal form depends on the cyclic order on the set of the types of X^0.

11.2. **Bestvina-convexity.**

Definition 11.4. Given an edge-path $P = x_1 \cdots x_n$ in X, an *admissible lift* of P is an edge-path $\widehat{P} = \hat{x}_1 \cdots \hat{x}_n$ in $\widehat{X}$ such that $\pi(\hat{x}_i) = x_i$, for $1 \leq i \leq n$, and $\hat{x}_i < \hat{x}_{i+1}$, for $1 \leq i \leq n-1$. Note that for each edge-path P in X, once a lift $\hat{x}_1$ of x_1 has been chosen, there is a unique admissible lift of P starting at $\hat{x}_1$. Different admissible lifts of P differ by the translation by φ^k for some $k \in \mathbb{Z}$.

Let $a, b \in X^0$. Following [Bes99, CMW04], we say that an edge-path P from a to b is a *geodesic*, (or *B-geodesic*) if some (hence all) admissible lift of P to $\widehat{X}$ has Deligne normal form with $n = l$.

Lemma 11.5. *For any $a, b \in X^0$, there is a unique B-geodesic in X from a to b.*

Proof. Let $\hat{a}$ and $\hat{b}$ be lifts of a and b, respectively, i.e. $\pi(\hat{a}) = a$ and $\pi(\hat{b}) = b$. Let $P = \hat{a} \cdots x_l \cdots x_n$ be the path in $\widehat{X}$ from $\hat{a}$ to $\hat{b}$ that has Deligne normal form. Then $\pi(x_l) = \pi(x_n) = b$, and so $\pi(x_1) \cdots \pi(x_l)$ is a B-geodesic from a to b, which proves the existence.

Suppose that there are two B-geodesics P_1 and P_2 from a to b. Let $\widehat{P}_1$ and $\widehat{P}_2$ be admissible lifts of P_1 and P_2 starting at the same point. Then the endpoints $\hat{b}_i$ of $\widehat{P}_i$ differ by φ^k for some $k \in \mathbb{Z}$. Since $\widehat{P}_1$ has Deligne normal form, we have that the concatention of $\widehat{P}_1$ with $\hat{b}_1 \varphi(\hat{b}_1) \cdots \varphi^k(\hat{b}_1)$ also has Deligne normal form. But since the Deligne normal form is unique (Theorem 11.3), the latter path equals $\widehat{P}_2$, and so $k = 0$ and $\widehat{P}_1 = \widehat{P}_2$, hence $P_1 = P_2$. □

Lemma 11.6. *Let $P = x_1 \cdots x_n$ be an edge-path in X. Then P is a B-geodesic if and only if for each $2 \le i \le n-1$, the vertices x_{i-1} and x_{i+1} do not have a common lower bound in $(\mathrm{lk}(x_i, X)^0, <_{x_i})$.*

Proof. Let $\widehat{P} = \hat{x}_1 \cdots \hat{x}_n$ be an admissible lift of P. By the definition of the Deligne normal form, P is a B-geodesic if and only if $\hat{x}_i = \hat{x}_{i+1} \wedge \varphi(\hat{x}_{i-1})$ for $1 < i < n$. This means exactly that $\hat{x}_{i+1}$ and $\varphi(\hat{x}_{i-1})$ have meet $\hat{x}_i$ in the interval $[\hat{x}_i, \varphi(\hat{x}_i)]$. Under the isomorphism of $[\hat{x}_i, \varphi(\hat{x}_i)]$ with the augmented $(\mathrm{lk}(x_i, X)^0, <_{x_i})$ from the proof of Lemma 11.2, this means that x_{i-1} and x_{i+1} have only a trivial lower bound, as desired. $\square$

Definition 11.7. Let X be an $\widetilde{A}_n$-like complex as before, of type S with given cyclic order. Let $Y \subset X$ be a full subcomplex that is also a simplicial complex of type S with the induced type function from X. We say that Y is *locally B-convex* if for each vertex $y \in Y^0$ and any vertices y_1, y_2 of $\mathrm{lk}(y, Y)$, if the meet $y_1 \wedge y_2$ in the poset $(\mathrm{lk}(y, X)^0, <_y)$ exists, then $y_1 \wedge y_2 \in \mathrm{lk}(y, Y)^0$.

The property of being locally B-convex depends on the choice of the cyclic order on the set of types of $\widehat{X}^0$. Reversing the cyclic order gives a different $\widetilde{A}_n$-like complex structure on X, with simplicially isomorphic $\widehat{X}$, but with different collection of locally B-convex subcomplexes.

Proposition 11.8. *Let X be an $\widetilde{A}_n$-like complex, and let $Y \subset X$ be a connected locally B-convex subcomplex. Then Y is simply connected, and for any pair of vertices $y_1, y_2 \in Y^0$, the B-geodesic in X from y_1 to y_2 is contained in Y.*

Proof. Let $\widetilde{Y}$ be the universal cover of Y. We first show that $\widetilde{Y}$ is an $\widetilde{A}_n$-like complex. We induce the type function and the cyclic order on the types from X. It suffices to show for each $y \in Y^0$, the restriction of the relation $<_y$ to the vertex set of $\mathrm{lk}(y, Y)$ satisfies conditions (2) and (3) of Definition 10.13. Condition (2) holds since Y is a full subcomplex of X. To check Condition (3), let $x_1 y_1 x_2 y_2$ be a bowtie in $\mathrm{lk}(y, Y)$. Since y_1, y_2 have a lower bound in $\mathrm{lk}(y, X)^0$, they have a meet $z \in \mathrm{lk}(y, X)^0$ by Lemma 2.5. Then $x_i \le z \le y_j$ for $i, j \in \{1, 2\}$. By the local B-convexity, we have $z \in \mathrm{lk}(y, Y)^0$, as desired.

Let $\theta \colon \widetilde{Y} \to Y$ be the covering map. We claim that if $\widetilde{P}$ is a B-geodesic in $\widetilde{Y}$ from $\tilde{y}_1$ to $\tilde{y}_2$, then $P = \theta(\widetilde{P})$ is the B-geodesic in X from $y_1 = \theta(\tilde{y}_1)$ to $y_2 = \theta(\tilde{y}_2)$.

Let us assume the claim for the moment and finish the proof of the proposition. We first justify that Y is simply connected and $\widetilde{Y} = Y$. Otherwise, we have distinct lifts $\tilde{y}, \tilde{y}' \in \widetilde{Y}$ of a vertex $y \in Y$. By Lemma 11.5, there is a B-geodesic $\widetilde{P}$ from $\tilde{y}$ to $\tilde{y}'$ in $\widetilde{Y}$. By the claim, $\theta(\widetilde{P})$ is a nontrivial B-geodesic in X from y to y. This contradicts the uniqueness of the B-geodesic in Lemma 11.5. Thus Y is simply connected. The remaining assertion of the proposition follows from the claim and the uniqueness of B-geodesics in X.

It remains to prove the claim. Let $\tilde{y}_{i-1}, \tilde{y}_i, \tilde{y}_{i+1}$ be three consecutive vertices in $\widetilde{P}$ with y_{i-1}, y_i, y_{i+1} their images under θ. Since $\widetilde{P}$ is a B-geodesic in $\widetilde{Y}$, Lemma 11.6 implies that $\tilde{y}_{i-1}$ and $\tilde{y}_{i+1}$ do not have a common lower bound in $\mathrm{lk}(\tilde{y}_i, \widetilde{Y})^0$. Since $\mathrm{lk}(\tilde{y}_i, \widetilde{Y}) \cong \mathrm{lk}(y_i, Y)$, we have that y_{i-1} and y_{i+1} do not have a common lower bound in $\mathrm{lk}(y_i, Y)^0$. By the local B-convexity of Y, y_{i-1} and y_{i+1} do not have a common lower bound in $\mathrm{lk}(y_i, X)^0$. Hence P is a B-geodesic in X by Lemma 11.6, and the claim follows. $\square$

Corollary 11.9. *Let X be an $\widetilde{A}_n$-like complex. Let Y_1 and Y_2 be connected locally B-convex subcomplexes of X. If $Y_1 \cap Y_2 \neq \emptyset$, then $Y_1 \cap Y_2$ is connected.*

Proof. Let y and y' be vertices in $Y_1 \cap Y_2$, and let P be a B-geodesic in X from y to y'. By Proposition 11.8, we have that P is contained in $Y_1 \cap Y_2$. $\square$

12. 3-dimensional Artin groups

Definition 12.1. We say that a Coxeter diagram Λ satisfies the *girth condition* if for each edge st of Λ, the graph $\Delta_{\Lambda,\, st}$ has girth ≥ 6.

The goal of this section is to prove the following.

Theorem 12.2. *Let A_Λ be an Artin group of dimension ≤ 3. Then Λ satisfies the girth condition. If Λ is non-spherical, then its Artin complex Δ is contractible.*

A vertex of a graph is *isolated* if it has no neighbours. An *n-cycle* is the graph with vertices $s_1, \ldots, s_n$ and edges $s_1s_2, \ldots, s_{n-1}s_n, s_ns_1$. Thus what we called an 'embedded n-cycle' in a graph Λ is a subgraph isomorphic to an n-cycle. Given a simplicial graph Λ, let Λ^c denote the *complement graph*, i.e. the graph with the same vertex set as Λ and st an edge exactly when there is no edge st in Λ. Note that if A_Λ is 3-dimensional with Coxeter diagram Λ, then Λ^c has no embedded 4-cycles (though the converse might not be true). So Theorem 12.2 follows from the following.

Theorem 12.3. *Let A_Λ be an Artin group such that Λ^c has no embedded 4-cycles. Then Λ satisfies the girth condition. If Λ is non-spherical, then its Artin complex Δ is contractible.*

Corollary 12.4. *Let A_Λ be an Artin group such that Λ^c has no embedded 4-cycles. Then A_Λ satisfies the $K(\pi,1)$ conjecture. In particular, each Artin group of dimension ≤ 3 satisfies the $K(\pi,1)$ conjecture.*

Proof. This follows from Theorems 12.3 and 1.7, by induction on the number of the vertices of Λ (recall that all spherical Artin groups satisfy the $K(\pi,1)$ conjecture [Del72]). $\square$

It remains to prove Theorem 12.3. As a preparation, we establish the following graph-theoretic result. Below $K_{k,l}$ denotes the complete bipartite graph with the parts of size k and l, and $K^-_{k,l}$ denotes $K_{k,l}$ with one edge removed.

Lemma 12.5. *Let Λ be a simplicial graph with at least 5 vertices, no isolated vertices, no embedded 3-cycles, and such that Λ^c has no embedded 4-cycles. Then Λ equals the 5-cycle, $K_{2,3}$, $K^-_{2,3}$, $K_{3,3}$, or $K^-_{3,3}$.*

Proof. Assume first that Λ is not bipartite. Let γ be the shortest odd embedded cycle in Λ. If γ has length ≥ 7, then γ^c contains an embedded 4-cycle, which is a contradiction. Consequently, γ is an induced 5-cycle. We will prove $\Lambda = \gamma$. Assume for contradiction that Λ has a vertex s outside γ. Since Λ has no embedded 3-cycles, s is a neighbour of at most two (non-adjacent) vertices of γ. Then the remaining vertices of γ together with s form an embedded 4-cycle in Λ^c, which is a contradiction.

Second, assume that Λ is bipartite with parts V, W. Since Λ^c has no embedded 4-cycles, we have $|V|, |W| \leq 3$. It remains to prove that there is at most one edge in Λ^c from V to W. Suppose that there are two such edges v_1w_1, v_2w_2. Then they must intersect, since otherwise $v_1w_1w_2v_2$ would be an embedded 4-cycle in Λ^c. Suppose without loss of generality $w_1 = w_2$. Then $V = \{v_1, v_2, v_3\}$, since otherwise w_1 would be isolated in Λ. But then $v_1w_1v_2v_3$ is an embedded 4-cycle in Λ^c, a contradiction. $\square$

We will verify Theorem 12.3 gradually, starting from the simplest Λ. We set $\Delta = \Delta_\Lambda$.

Remark 12.6. By [AS83, Lem 6], if Λ is an edge labelled by m, then Δ_Λ has girth $\geq 2m$. In particular, Λ satisfies the girth condition.

Thus if Λ is a 3-cycle, then by Lemma 10.6 all vertex links of Δ have girth ≥ 6. Since Δ is simply connected (Lemma 10.2), it satisfies the definition of a systolic complex [JŚ06, page 9]. In particular, Δ is contractible [JŚ06, Thm 4.1(1)] and all $\Delta_{\Lambda,\, st}$ have girth ≥ 6 [JŚ06, Prop 1.4].

If Λ is a length 2 linear graph, then Δ is bowtie free by [Cha04, Lem 4.1] (or Theorem 10.10), when both labels are equal to 3, and [Hua24a, Lem 11.5] otherwise. In particular, Λ satisfies the girth condition. The contractibility of Δ for non-spherical Λ follows from [CD95b, Thm B].

If Λ is a 4-cycle, then by Remark 2.1 the relation $<_x$ on each vertex link of Δ described in Definition 10.13 is a partial order. By the previous paragraph, $<_x$ is bowtie free. Thus, by Lemma 10.2, Δ is an $\widetilde{A}_3$-like complex. By Theorem 10.14, Δ is contractible. By Lemma 10.15, we have that Λ satisfies the girth condition.

Remark 12.7. Let Λ be a length 2 linear graph with labels m, n such that $m, n \geq 4$ or $m \geq 6$. Then equipping each triangle of Δ with the Euclidean metric of angles $\frac{\pi}{4}, \frac{\pi}{2}, \frac{\pi}{4}$, or $\frac{\pi}{6}, \frac{\pi}{2}, \frac{\pi}{3}$, respectively, Δ is a CAT(0) metric space. Indeed, by Remark 12.6 the vertex links of Δ have girth $\geq 2\pi$. Thus by the Cartan–Hadamard theorem [BH99, Thm 4.1(2)], we obtain that Δ is CAT(0).

Corollary 12.8. *Let $\Lambda = stp$ be a length 2 linear diagram with $m_{st} \geq 4$. Then $\Delta_{stp,\, st}$ has girth ≥ 8.*

Proof. If $m_{tp} = 3$ and $m_{st} = 4$ or 5, then the lemma follows from Theorems 2.8 or 2.9. If $m_{tp} \geq 4$ or $m_{st} \geq 6$, then, by Remark 12.7, we have that Δ_{stp} is CAT(0), with triangles of angles $\frac{\pi}{4}, \frac{\pi}{2}, \frac{\pi}{4}$ or $\frac{\pi}{6}, \frac{\pi}{2}, \frac{\pi}{3}$. Then the lemma follows from Lemma 2.12(iii) or (ii). □

Lemma 12.9. *Let Λ be a length 3 linear graph. Then Δ is bowtie free. In particular, Λ satisfies the girth condition. If Λ is not spherical, then Δ is contractible.*

Proof. If Λ is spherical, then the lemma follows from Theorem 10.10, so we can assume that Λ is not spherical. If all proper induced subdiagrams of Λ are spherical, then its consecutive edges have labels $353, 434, 435$, or 535. In the 353 case, the lemma follows from Theorem 9.12, Lemma 9.8, and Corollary 9.11. In the remaining cases, the lemma follows from Proposition 10.12 and Theorem 10.11, whose hypotheses are satisfied by Theorems 2.8 and 2.9.

Otherwise, $\Lambda = stpr$ contains a non-spherical subdiagram, say $\Lambda' = stp$. We have either $m_{st}, m_{tp} \geq 4$, or $m_{st} \geq 6, m_{tp} = 3$, or $m_{st} = 3, m_{tp} \geq 6$. Let $\Delta' = \Delta_{\Lambda,\Lambda'}$. By Lemma 10.7 and Remark 12.6, we have that Δ deformation retracts onto Δ', and so Δ' is simply connected.

Assume first $m_{st}, m_{tp} \geq 4$. By Corollary 12.8, we have that $\Delta_{tpr,\, tp}$ has girth ≥ 8. By Lemma 10.6, the complex $\Delta_{tpr,\, tp}$ is the link of a vertex of type $\hat{s}$ in $\Delta' = \Delta_{stpr,\, stp}$. The vertex of type $\hat{p}$ in Δ' has link $\Delta_{str,\, st} = \Delta_{st}$, which has girth ≥ 8 by Remark 12.6. Consequently, equipping each triangle of Δ' with the Euclidean metric of angles $\frac{\pi}{4}, \frac{\pi}{2}, \frac{\pi}{4}$, the complex Δ' is a locally CAT(0) metric space. By the Cartan–Hadamard theorem, we obtain that Δ' is CAT(0), in particular Δ is contractible.

By Lemma 2.12(i), each induced 4-cycle in Δ' has type $\hat{s}\hat{p}\hat{s}\hat{p}$ and has a common neighbour of type $\hat{t}$. This shows that there are no bowties without vertices of type $\hat{r}$.

Let v be a vertex of type $\hat{r}$ and let $C_v = \mathrm{lk}(v, \Delta) \subset \Delta'$. We claim that C_v is convex in Δ'. By Lemma 10.6, we have that C_v is isomorphic to $\Delta_{\Lambda'}$, which is connected. Thus to justify the claim we only need to prove that C_v is locally convex

[BH99, Prop 4.14]. Suppose first that we have $w \in C_v$ of type $\hat{s}$ with neighbours $u_1, u_3 \in C_v$ connected in $\mathrm{lk}(w, \Delta') \cong \Delta_{tpr,tp}$ by a path γ of length $d < \pi$, that is, $d \leq \frac{3\pi}{4}$. Note that if one of u_1, u_3 has type $\hat{p}$, then its neighbour on γ is also a neighbour of v, so we can assume that both u_1, u_3 are of type $\hat{t}$ and γ has only one interior vertex u_2, which has type $\hat{p}$. By Remark 12.6 and Corollary 10.9 applied to Δ_{tpr} we have that $u_2 \in C_v$, as desired. If w has type $\hat{t}$, then the local convexity condition is empty. If w has type $\hat{p}$, then C_v contains all the neighbours of w by Remark 2.1, and again there is nothing to prove. This justifies the claim.

Consider now a possible bowtie $vu_1u_2u_3$ with only v of type $\hat{r}$ and $u_2 > u_1, u_3$, where $\hat{r} > \hat{p} > \hat{t} > \hat{s}$. If u_2 lies in the simplicial span of the CAT(0) convex hull of u_1, u_3 in Δ', then by the convexity of C_v we obtain that v and u_2 are neighbours, as desired. If u_2 lies outside the span of the convex hull of u_1, u_3, then since $\mathrm{St}(u_2, \Delta')$ is convex in Δ', we obtain that the CAT(0) geodesic $\alpha = u_1u_3$ lies in the boundary of that star. Thus α consists of edges u_1w and wu_3 with w a neighbour of u_2 of type $\hat{t}$. Then $w \in C_v$ and so $v, u_2 > w > u_1, u_3$, as desired.

Finally, consider a possible bowtie $v_1u_1v_2u_k$ with both v_i of type $\hat{r}$. Since C_{v_i} are convex, we have that $C = C_{v_1} \cap C_{v_2}$ is convex as well. In particular, C is connected, and we denote by $u_1u_2\cdots u_k$ an edge-path from u_1 to u_k in C with the least number of edges. If some u_i with $2 \leq i \leq k$ has type $\hat{p}$, then $i = 2$, since otherwise the 4-cycle $v_1u_{i-2}v_2u_i$ in $\mathrm{lk}(u_{i-1}, \Delta)$ would contradict Corollary 10.9 (because $\mathrm{lk}(u_{i-1}, \Delta)$ satisfies the girth condition by Remark 12.6). Analogously we have $i = k-1$ and so $k = 3$. Thus $v_1, v_2 > u_2 > u_1, u_k$ are as desired. If there is no u_i of type $\hat{p}$, there must be u_i of type $\hat{t}$. Then we have $i \leq 2$ since otherwise the 4-cycle $v_1u_{i-2}v_2u_i$ in $\mathrm{lk}(u_{i-1}, \Delta) \cong \Delta_{tpr}$, which is not a bowtie by Remark 12.6, would allow us to replace u_{i-1} by a vertex of type $\hat{p}$ and to proceed as before. Analogously we have $i \geq k-1$ and so $k = 3$ and again $v_1, v_2 > u_2 > u_1, u_k$.

Consider now the second case, where $m_{st} \geq 6$ and $m_{tp} = 3$. By Remark 12.6, the vertex links in Δ' of the vertices of types $\hat{s}$ and $\hat{p}$ have girth 6 and $2m_{st} \geq 12$, respectively. Consequently, equipping each triangle with the Euclidean metric of angles $\frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{6}$, the complex Δ' is a CAT(0) metric space as before. Again, by Lemma 2.12(i), there are no bowties without vertices of type $\hat{r}$. The proof of the convexity of C_v and that there are no bowties with vertices of type $\hat{r}$ is the same as before.

Finally, the case $m_{st} = 3$ and $m_{tp} \geq 6$ follows from [Hua24b, Cor 9.14 and Lem 6.14]. ☐

Corollary 12.10. *If Λ is a 5-cycle, then Δ is contractible, and Λ satisfies the girth condition.*

Proof. By Theorem 10.14 and Lemma 10.15, it is enough to show that Δ is an $\widetilde{A}_4$-like complex. By Lemma 10.2, it suffices to prove that the vertex links of Δ satisfy the partial order condition and are bowtie free. By Lemma 10.6, each such link is isomorphic to $\Delta_{\Lambda'}$, where Λ' is a linear diagram of length 3. Thus the partial order condition follows from Remark 2.1. Bowtie freeness follows from Lemma 12.9. ☐

Definition 12.11. Consider a decomposition of the vertex set of Λ into a disjoint union $\bigsqcup_i S_i$. Let Δ^* be the subdivision of Δ obtained by subdividing each simplex σ of type $\widehat{S}_i$ into a cone over $\partial\sigma$ with apex at the barycentre of σ, and by subdividing each join $*_i\sigma_i$ of simplices of type $\widehat{S}_i$ into the join of the subdivisions of $\widehat{S}_i$. We call Δ^* the *$\mathcal{S}$-subdivision*, where the collection $\mathcal{S}$ is obtained from $\{S_i\}_i$ be removing all the elements of size 1. For example, if $\mathcal{S} = \{\{s,t\}\}$, then Δ^* is obtained from Δ by subdividing each edge of type $\hat{s}\hat{t}$, and each simplex containing it, into two. We denote this, shortly, *$\{s,t\}$-subdivision*.

Lemma 12.12. *Let $\Lambda = K_{1,3}$ and let $L \subset \Lambda$ be a length 2 linear subdiagram. Then $\Delta_{\Lambda,L}$ is bowtie free. In particular, Λ satisfies the girth condition. If Λ is not spherical, then Δ is contractible.*

Proof. The contractibility of Δ follows from [Hua24a, Thm 10.3] and [GP12b, Thm 3.1]. Suppose that α is a bowtie in $\Delta_{\Lambda,L}$. Let $L = stp$ and let s' be the remaining vertex of Λ. We define Δ^* to be the $\{s, s'\}$-subdivision of Δ. Let m be the type of the new vertices, and identify the types $\hat{s}$ and $\hat{s}'$. We order the types so that $\hat{s} = \hat{s}' < m < \hat{t} < \hat{p}$, which gives rise to a relation on the vertex set of Δ^* by Definition 2.3. By [Hua24a, Prop 11.34], we have that Δ^* is bowtie free. Thus there is a vertex x of Δ^* that is a common neighbour of α. But x cannot have type m since then α would have two equal vertices of type $\hat{s}$. Thus x belongs to $\Delta_{\Lambda,L}$, as desired. □

By Lemma 12.12 and [Hua24b, Lem 6.14 and Prop 6.17], we have the following.

Corollary 12.13. *Let $\Lambda = K_{1,3}$ with parts $\{s_1, s_2, s_3\}$ and $\{t\}$. Then each induced cycle of type $\hat{s}_1\hat{s}_2\hat{s}_1\hat{s}_2$ or $\hat{s}_1\hat{s}_2\hat{s}_1\hat{s}_3$ in Δ has a common neighbour of type $\hat{t}$.*

Lemma 12.14. *If $\Lambda = K_{k,l}$ with $k, l \geq 2$, then Δ is contractible and Λ satisfies the girth condition.*

Proof. Let $\{s_1, \ldots, s_k\}, \{t_1, \ldots, t_l\}$ be the parts of Λ. To prove the girth condition for the edge s_1t_1, consider the $\{\{s_2, \ldots s_k\}, \{t_2, \ldots, t_l\}\}$-subdivision Δ^* of Δ. Let m, m' be the types of the barycentres of the simplices of types $s_2 \cdots s_k, t_2 \cdots t_l$. By [Hua24a, Lem 11.10] (which relies on [Hua24c, Theorem 1.4]), the subcomplex $\Delta^*_{s_1t_1mm'}$ of Δ^* spanned on the vertices of types $\hat{s}_1, \hat{t}_1, m$, and m' is an $\widetilde{A}_3$-like complex. Thus, by Lemma 10.15, the complex Δ^* has no cycles of type $\hat{s}_1\hat{t}_1\hat{s}_1\hat{t}_1$. The contractibility of Δ can be deduced from [Hua24a, Lem 11.11]. □

Lemma 12.15. *Let $\Lambda = K_{2,3}$ with parts $\{s_1, s_2, s_3\}$ and $\{t_1, t_2\}$. Then each induced cycle of type $\hat{s}_1\hat{s}_2\hat{s}_1\hat{s}_2$ in Δ has common neighbours of type $\hat{t}_1$ and $\hat{t}_2$.*

Proof. Let Δ^* be the $\{s_2, s_3\}$-subdivision of Δ. Let m be the type of the new vertices. By [Hua24a, Lem 11.10], the subcomplex $\Delta^*_{s_1t_1mt_2}$ of Δ^* spanned on the vertices of types $\hat{s}_1, \hat{t}_1, m, \hat{t}_2$, is an $\widetilde{A}_3$-like complex. We fix any of the two cyclic orders on $s_1t_1mt_2$ to be able to discuss meets and joins in the links.

Let v be a vertex of type $\hat{s}_2$ and let $C_v = \mathrm{lk}(v, \Delta^*) \subset \Delta^*_{s_1t_1mt_2}$. We claim that C_v is B-convex in $\Delta^*_{s_1t_1mt_2}$. Note that C_v is connected, since it is isomorphic to $\mathrm{lk}(v, \Delta)$, which, by Lemma 10.6, is in turn isomorphic to $\Delta_{s_1t_1s_3t_2}$. For the local B-convexity, let $w \in C_v$, and let $u_1, u_2 \in C_v$ be neighbours of w. Assume that the meet u of u_1 and u_2 in $\mathrm{lk}(w, \Delta^*_{s_1t_1mt_2})^0$ exists and is distinct from u_1, u_2. We need to show $u \in C_v$, so we can assume that none of w, u_1, u_2 has type m. If u_1 or u_2 has type $\hat{t}_i$, and u is not of type m, then this follows immediately from applying Corollary 10.9 to the 4-cycle vu_1uu_2 in $\mathrm{lk}(w, \Delta)$, which satisfies the girth condition by Lemmas 12.12 and 12.14. If u_1 or u_2 has type $\hat{t}_i$, and u is the midpoint of an edge u^+u^- of Δ of type $\hat{s}_2\hat{s}_3$, then applying as before Corollary 10.9 to the 4-cycle $vu_1u^+u_2$, we obtain $u^+ = v$, and so u is a neighbour of v. It remains to assume that u_1 and u_2 are of type s_1. Again by Corollary 10.9, we can assume that u is not of type t_i, so it is the midpoint of an edge u^+u^- in Δ of type $\hat{s}_2\hat{s}_3$. Applying Corollary 12.13 to $vu_1u^+u_2$ in $\mathrm{lk}(w, \Delta)$, we obtain that $u^+ = v$ as before or there is a common neighbour of type $\hat{t}_i$ of u_1, u_2 in $\mathrm{lk}(w, \Delta)$, contradicting the assumption that $u = u_1 \wedge u_2$ in $\mathrm{lk}(w, \Delta^*_{s_1t_1mt_2})^0$.

Let $u_1v_1u_kv_2$ be a cycle of type $\hat{s}_1\hat{s}_2\hat{s}_1\hat{s}_2$ in Δ. By Corollary 11.9, we have that $C = C_{v_1} \cap C_{v_2}$ is connected. Let $u_1 \cdots u_k$ be an edge-path in C from u_1 to u_k with the least number of edges. None of the u_i can have type m, since $v_1 \neq v_2$. Thus one of the u_i has type t_1 or t_2, and then by the girth condition in the links of u_{i-1}, u_{i+1}, (see Lemmas 12.12 and 12.14) and Corollary 10.9, we have that u_i is a neighbour of u_1 and u_k. Then the lemma follows from Corollary 12.13 applied to the link of u_i. □

Proposition 12.16. *If $\Lambda = K^-_{2,3}$, then Δ is contractible and Λ satisfies the girth condition.*

Proof. Denote the parts to be $\{s,p\}, \{t_1,t_2,r\}$ with missing edge sr. Let $\Delta' = \Delta_{\Lambda,\Lambda'}$ with $\Lambda' = st_1pt_2$. By Lemma 10.7 and Remark 12.6, we have that Δ deformation retracts onto Δ', and so the latter is simply connected. Note that Δ' is an $\widetilde{A}_3$-like complex. In other words, the links $\Delta_{t_1st_2r,\,t_1st_2} = \Delta_{t_1st_2}, \Delta_{t_1pt_2r,\,t_1pt_2}$, and $\Delta_{st_1pr,\,st_1p}$ satisfy the partial order condition and are bowtie free. The partial order condition follows from Remark 2.1. The bowtie freeness of the first link follows from Remark 12.6. For the middle one, this is Lemma 12.12. For the last one, this follows from Lemma 12.9. By Theorem 10.14, we have that Δ' is contractible, and so is Δ. We fix any of the two cyclic orders on st_1pt_2 to be able to discuss meets and joins in the links.

By Lemma 10.15, each induced 4-cycle in Δ' is contained in the link of a vertex, and so the girth condition for the edges in Λ' follows from the girth condition in Lemmas 12.9 and 12.12.

It remains to consider a 4-cycle with vertices of types $\hat{p}$ and $\hat{r}$. First we justify the following.

Claim. *For v of type $\hat{r}$, the subcomplex $C_v = \mathrm{lk}(v,\Delta) \subset \Delta'$ is B-convex.*

Note that C_v is isomorphic to $\Delta_{\Lambda'}$ and so it is connected. For the local B-convexity, let $w \in C_v$, and let $u_1, u_2 \in C_v$ be neighbours of w. Assume that $u = u_1 \wedge u_2$ exists in $\mathrm{lk}(w,\Delta')^0$ and is distinct from u_1, u_2. We need to show $u \in C_v$. If w has type $\hat{p}$, then this is immediate, so without loss of generality we only need to consider the cases where w has type $\hat{t}_1$ and $\hat{s}$.

In the first case, the link of w is isomorphic to Δ_{st_2pr}. If the 4-cycle u_1uu_2v contains an edge whose type lies on the path $\hat{s}\hat{t}_2\hat{p}\hat{r}$, then uv is an edge by the girth condition in Lemma 12.9 and Corollary 10.9. Otherwise, the type of the cycle is $\hat{s}\hat{p}\hat{s}\hat{r}$. Since the link of w is bowtie free by Lemma 12.9, we have that u is a neighbour of v, or there is a common neighbour of type $\hat{t}_2$ of all w, u_1, u, u_2, v. But then it is this vertex and not u that is the meet of u_1, u_2, contradiction.

Now consider the case, where w has type $\hat{s}$. If the 4-cycle u_1uu_2v contains an edge whose type lies in the Coxeter diagram of t_1t_2pr, then uv is an edge by the girth condition in Lemma 12.12 and Corollary 10.9. Otherwise, without loss of generality, the type of the cycle is $\hat{t}_1\hat{t}_2\hat{t}_1\hat{r}$. By Corollary 12.13, we have that u is a neighbour of v, or there is a common neighbour of type $\hat{p}$ of all w, u_1, v, u_2, u, which contradicts the assumption that u is the meet of u_1, u_2. This ends the proof of the claim.

Let $v_1u_1v_2u_k$ be a cycle with both v_i of type $\hat{r}$ and u_1, u_k of type $\hat{p}$. By the B-convexity of C_{v_i} and Corollary 11.9, we have that $C = C_{v_1} \cap C_{v_2}$ is connected. Let $u_1u_2\cdots u_k$ be an edge-path in C from u_1 to u_k with the least number of edges. By the girth condition in the link of u_2 (see Lemmas 12.9 and 12.12), and Corollary 10.9 applied to the 4-cycle $u_1v_1u_3v_2$, we obtain that u_1 is a neighbour of u_3, contradiction. □

Proposition 12.17. *If $\Lambda = K^-_{3,3}$, then Δ is contractible and Λ satisfies the girth condition.*

Proof. Denote the parts of Λ by $\{s_1,s_2,s_3\},\{t_1,t_2,t_3\}$ with the missing edge s_3t_3. Consider the $\{t_1,t_2\}$-subdivision Δ^* of Δ. Let m be the type of the new vertices. We claim that the subcomplex $\Delta^*_{s_1t_3s_2m}$ of Δ^* spanned on the vertices of types $\hat{s}_1,\hat{t}_3,\hat{s}_2$, and m is an $\widetilde{A}_3$-like complex. The partial order condition follows from Lemma 2.2. The bowtie freeness of the link of a vertex of type m in $\Delta^*_{s_1t_3s_2m}$ follows from Remark 12.6, since such link is isomorphic to $\Delta_{s_1t_3s_2}$. To obtain the bowtie freeness of the link $\mathrm{lk}(z,\Delta^*_{s_1t_3s_2m})$ of a vertex z of type $\hat{s}_1$ (or $\hat{s}_2$), for all the possible bowties except for the type $\hat{t}_3m\hat{t}_3m$, it suffices to use Proposition 12.16 and Corollary 10.9.

Consider now a possible bowtie $vev'e'$ of type $\hat{t}_3m\hat{t}_3m$, where e,e' are midpoints of edges $uw,u'w'$ of type $\hat{t}_1\hat{t}_2$. Our goal is to find a vertex of type $\hat{s}_2$ in $L=\mathrm{lk}(z,\Delta)$ that is a neighbour of all z,u,w,u',w',v,v'. Let

$$\Delta'=\mathrm{lk}(z,\Delta_{\Lambda,\,s_1s_2t_1s_3t_2})\cong\Delta_{t_3s_2t_1s_3t_2,\,s_2t_1s_3t_2},$$

which was studied in Proposition 12.16. Let $C_v=\mathrm{lk}(v,\Delta)\cap\Delta'$. By the Claim in the proof of Proposition 12.16 and Corollary 11.9, we have that $C_v\cap C_{v'}\subset\Delta'$ is connected. Let α be an edge-path in $C_v\cap C_{v'}$ from uw to $u'w'$ with the least number of edges.

If α is a single vertex, say $u=u'$, then by Lemma 12.9 applied to $\mathrm{lk}(u,L)$, the cycle $vwv'w'$ has a common neighbour of type $\hat{s}_2$ that is also a neighbour of z and $u=u'$, as desired. If α is a single edge, say uw', then analogously the cycle $vwv'w'$ has a common neighbour y of type $\hat{s}_2$ that is also a neighbour of z and u. By Corollary 10.9, applied to the cycle $vyv'u'$ in $\mathrm{lk}(w',L)$, we obtain that y is also a neighbour of u', as desired. We can now assume that α has at least two edges.

If α contains a vertex of type $\hat{s}_2$, then by Corollary 10.9 it is a neighbour of u,u',w,w', as desired. Otherwise, if α contains a subpath xyx' with y of type $\hat{s}_3$, then applying Corollary 12.13 to $xvx'v'$ in $\mathrm{lk}(y,L)$, we can replace y by a vertex of type $\hat{s}_2$ and we conclude as before. Otherwise, let $x_0x_1x_2$ be any subpath of α. By Lemma 12.9 applied to $\mathrm{lk}(x_1,L)$, which is isomorphic to $\Delta_{t_3s_2t_is_3}$, for $i=1$ or 2, we have that there is a common neighbour of type $\hat{s}_2$ of x_0vx_2v' and we conclude as before.

To obtain the bowtie freeness of the link of a vertex of type $\hat{t}_3$, we apply Lemmas 12.14, 12.15, and Corollaries 12.13 and 10.9. This finishes the proof of the claim.

Note that $\Delta^*_{s_1t_3s_2m}$ is obtained from $\Delta_{\Lambda,\,s_1t_3s_2t_1t_2}$ by removing the disjoint stars in Δ^* of the vertices of type $\hat{t}_1$ and $\hat{t}_2$. These stars are isomorphic to the stars of the vertices of type $\hat{t}_1$ and $\hat{t}_2$ in $\Delta_{\Lambda,\,s_1t_3s_2t_1t_2}$, and hence they are cones over links isomorphic to Δ' whose contracibility we established in the proof of Proposition 12.16. Consequently, $\Delta^*_{s_1t_3s_2m}$ is a deformation retract of $\Delta_{\Lambda,\,s_1t_3s_2t_1t_2}$, and thus of Δ by Lemma 10.7. In particular, since Δ is simply connected, this completes the proof of the claim that $\Delta^*_{s_1t_3s_2m}$ is an $\widetilde{A}_3$-like complex.

By Theorem 10.14, we have that $\Delta^*_{s_1t_3s_2m}$ is contractible, and so is Δ.

The girth condition for the edges s_1t_3 and s_2t_3 follows from Lemma 10.15 applied to $\Delta^*_{s_1t_3s_2m}$. Using a symmetry of Λ, we also obtain the girth condition for the edges t_1s_3 and t_2s_3.

Using a symmetry again, it remains to verify the girth condition for s_1t_2. Let v be a vertex of type $\hat{t}_2$ and let $D_v=\mathrm{lk}(v,\Delta^*)\cap\Delta^*_{s_1t_3s_2m}$. We claim that D_v is B-convex.

Note that D_v is isomorphic to $\Delta_{t_1s_1t_3s_2s_3,\,t_1s_1t_3s_2}$ and so it is connected. For the local B-convexity, all the cases follow from Corollary 10.9 and the girth conditions for smaller Coxeter diagrams, except for the case where w has type $\hat{s}_1$ or $\hat{s}_2$, say $\hat{s}_1$, with u_1uu_2 of type $\hat{t}_3m\hat{t}_3$, where $u=u_1\wedge u_2$ in $\mathrm{lk}(w,\Delta^*_{s_1t_3s_2m})^0$. Then, as in the verification of the bowtie freeness in the second, third and fourth paragraph of the proof, there is a vertex of type $\hat{s}_2$ in $\mathrm{lk}(w, \Delta)$ that is a neighbour of u_1 and u_2, which contradicts the assumption that u is the meet of u_1, u_2. This justifies the claim.

Consider an induced 4-cycle $v_1u_1v_2u_k$ with v_i of type $\hat{t}_2$ and u_1, u_k of type $\hat{s}_1$. Since D_{v_i} are B-convex, by Corollary 11.9 we have that $D = D_{v_1} \cap D_{v_2}$ is connected. Let $\alpha = u_1u_2 \cdots u_k$ be an edge-path from u_1 to u_k in D with the least number of edges. None of the u_i has type m, since otherwise $v_1 = v_2$. Since α is not a single vertex, we obtain a contradiction by applying Proposition 12.16 or Lemma 12.14 to $\mathrm{lk}(u_2, \Delta)$, and Corollary 10.9.

□

Proposition 12.18. *Let $\mathcal{C}$ be a class of Coxeter diagrams closed under taking induced subdiagrams. Suppose that we have $\mathcal{C}_1 \subset \mathcal{C}$ such that each diagram in $\mathcal{C} - \mathcal{C}_1$ contains a triangle. Then*

(1) if each diagram in $\mathcal{C}_1$ satisfies the girth condition, then each diagram in $\mathcal{C}$ satisfies the girth condition, and

(2) if in addition for each nonspherical $\Lambda_1 \in \mathcal{C}_1$ the Artin complex Δ_{Λ_1} is contractible, then for each nonspherical $\Lambda \in \mathcal{C}$ the Artin complex Δ_Λ is contractible. In particular, each diagram in $\mathcal{C}$ satisfies the $K(\pi, 1)$ conjecture.

Proof. We prove assertion (1) by induction on the number of the vertices of $\Lambda \in \mathcal{C}$. We can assume that Λ contains a triangle stp. By the inductive hypothesis, the vertex links of $\Delta' = \Delta_{\Lambda, stp}$ have girth ≥ 6. We have that Δ' is simply connected by Lemma 10.2, and so it is systolic. In particular, $\Delta_{\Lambda, st}, \Delta_{\Lambda, tp}, \Delta_{\Lambda, sp}$ have girth ≥ 6 [JŚ06, Prop 1.4], which verifies part of assertion (1) for Λ. Furthermore, equipping each triangle with the Euclidean metric of an equilateral triangle, Δ' is CAT(0).

Consider now an edge pq of Λ with $q \neq s, t$. We will justify that $\Delta_{\Lambda, pq}$ has girth ≥ 6. Let v be a vertex of Δ of type $\hat{q}$.

We claim that $C_v = \mathrm{lk}(v, \Delta) \cap \Delta'$ is convex in Δ' with respect to the CAT(0) metric. We have that C_v is isomorphic to $\Delta_{\Lambda \setminus \{q\}, stp}$, which is connected. Thus to justify the claim we only need to prove that C_v is locally convex [BH99, Prop 4.14]. Suppose that we have $w \in C_v$ with neighbours $u_1, u_2 \in C_v$ and $u \in \Delta'$ such that u_1u, uu_2 are edges but u_1 and u_2 are not neighbours. Then vu_1uu_2 is a 4-cycle in the link of w, to which we can apply Corollary 10.9 by the inductive hypothesis. Thus $u \in C_v$, which justifies the claim.

Suppose for contradiction that $v_1u_1v_2u_k$ is a 4-cycle in $\Delta_{\Lambda, pq}$ with both v_i of type $\hat{q}$. Then $u_1, u_k \in C_{v_1} \cap C_{v_2}$, which is convex in Δ'. In particular, $C_{v_1} \cap C_{v_2}$ is connected. Consider an edge-path $u_1u_2 \cdots u_k$ from u_1 to u_k in $C_{v_1} \cap C_{v_2}$ with the least number of edges. Note that we have $k \geq 3$. The 4-cycle $u_1v_1u_3v_2$ in the link of u_2 violates Corollary 10.9 by the inductive hypothesis.

Consider now an edge qr of Λ with $q, r \notin \{s, t, p\}$. We will justify that $\Delta_{\Lambda, qr}$ has girth ≥ 6. Suppose for contradiction that $v_1u_1v_2u_2$ is a 4-cycle in $\Delta_{\Lambda, qr}$ with both v_i of type $\hat{q}$ and both u_i of type $\hat{r}$. Consider disc diagrams $D \to \Delta'$ with boundary cycle $\alpha_0\alpha_1\alpha_2\alpha_3$ such that $\alpha_0 \subset C_{v_1}, \alpha_1 \subset C_{u_1}, \alpha_2 \subset C_{v_2}, \alpha_3 \subset C_{u_2}$. Choose D of minimal area, and among such D, choose D with minimal perimeter. Then the boundary cycle of D is a concatenation of paths I_i embedded in D that are the domains of α_i, and the intersections $x_i = I_i \cap I_{i+1}$ (mod 4) are single vertices by the minimality assumption. We can assume that D is not a single vertex, since then we would obtain a contradiction with the inductive hypothesis. In particular, two consecutive I_i cannot be trivial. Thus, up to a symmetry, we have one of the following:

- all x_i are distinct, and so all I_i are nontrivial,

- $x_0 = x_1, x_2, x_3$ are distinct, and so only I_1 is trivial,
- $x_0 = x_1 \neq x_2 = x_3$, and so only I_1, I_3 are trivial.

The x_i equal to x_j for $i \neq j$ are called *singular*. We apply Theorem 2.11 to D. Since Δ' is CAT(0), the curvature at each interior vertex of D is non-positive. Consider now an interior vertex y of one of the I_i, with $\alpha_i \subset C_v$. If the curvature at y was positive, then y would be contained in exactly one or two triangles of D. By the convexity of C_v, the images in Δ' of these triangles would be contained in C_v. Consequently, we could alter α_i be removing these triangles from D, which would contradict the minimality of the area of D. Thus the curvature is non-positive also at each interior vertex of I_i. Consequently, the curvature can be positive only at the x_i, and it then equals $\frac{\pi}{3}, \frac{2\pi}{3}$, or π. Since their sum equals 2π, there must be

(i) a non-sigular x_i with curvature $\geq \frac{2\pi}{3}$, or
(ii) a singular x_i with curvature π.

In case (i), x_i is contained in only one triangle od D. Thus if, say, $x_i = x_2$, then we have a 4-cycle in the link of x_2 containing v_2u_2. By the inductive hypothesis and Corollary 10.9, this 4-cycle has a diagonal, which contradicts the minimality of D. In case (ii), x_i is not contained in any trangle od D. Thus if, say, $x_i = x_0 = x_1$, then we have a 4-cycle in the link of x_0 containing $v_1u_1v_2$ and contradicting the minimal perimeter assumption on D in view of Corollary 10.9 and the inductive hypothesis. This finishes the proof of assertion (1).

For assertion (2), we first show by induction on the number of the vertices of $\Lambda \in \mathcal{C}$ that Δ_Λ is contractible whenever Λ is not spherical. Indeed, we can assume that Λ contains a triangle *stp*. Let Δ' be as in the proof of assertion (1). We proved that Δ' is CAT(0), and so it is contractible. By Lemma 10.7, and the inductive hypothesis, we have that Δ deformation retracts onto Δ', and so Δ is contractible. The last part of assertion (2) follows from Theorem 1.7. □

Proof of Theorem 12.3. If Λ has multiple connected components, then the associated Artin complex is a join of several smaller Artin complexes, one for each connected component of Λ. Thus it suffices to consider the case where Λ is connected. If $|S| \leq 4$, then the theorem follows from Remark 12.6, and Lemmas 12.9 and 12.12. Otherwise, if Λ is complete bipartite, then the theorem follows from Lemma 12.14. Consequently, the theorem follows from Lemma 12.5, Corollary 12.10, Propositions 12.16, 12.17, and 12.18. □

By Theorem 1.7, we have the following consequences.

Corollary 12.19. *Suppose that all non-spherical Λ without triangles satisfy the girth condition and have contractible Δ_Λ. Then all Artin groups satisfy the $K(\pi, 1)$ conjecture.*

Theorem 12.20. *Let $\mathcal{C}$ be a class of Coxeter diagrams closed under taking induced subdiagrams. Suppose that each $\Lambda \in \mathcal{C}$ not containing a triangle satisfies at least one of the following conditions:*

(1) A_Λ is spherical, or more generally Λ satisfies the assumption of [Hua24b, Thm 1.1],
(2) Λ^c does not contain embedded 4-cycles,
(3) Λ is locally reducible.

Then each A_Λ with $\Lambda \in \mathcal{C}$ satisfies the $K(\pi, 1)$ conjecture.

Proof. By Theorem 1.7, and Proposition 12.18, it suffices to show that each Λ in one of the above classes satisfies the girth condition and, if it is not spherical, then Δ_Λ is contractible. For class (2), this is Theorem 12.3. For class (1), this follows from

[Hua24b, Prop 9.11 and 9.12]. For class (3), this follows from [Hua24b, Cor 9.14] (as stated, this result only treats the case where Λ is a locally reducible tree, but the same argument works for any locally reducible diagram, and it also gives the contractibility of Δ_Λ). □

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