Champlain College – St.-Lambert

MATH 201-203: Calculus II

Review Questions for Test # 3

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1. Test the convergence or divergence of the following sequence, if it is convergent, find its limit.

   (a) $a_n = \dfrac{n}{2n+1}$, (b) $a_n = \dfrac{10^n}{3^{2n}}$,

   (c) $a_n = \dfrac{(-1)^n n}{2n+1}$, (d) $a_n = \dfrac{(-2)^n}{4^n+1}$.

2. Test convergence or divergence of the following series.

   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{2n}{4n^2-1}$, (b) $\displaystyle\sum_{n=0}^{\infty} \frac{n+1}{\sqrt{n^2+1}}$

   (c) $\displaystyle\sum_{n=1}^{\infty} \frac{3^n}{4^n+1}$, (d) $\displaystyle\sum_{n=0}^{\infty} \frac{n}{e^n}$.

3. Find an exact fraction number to $1.121121\cdots = 1.\overline{121}$.

4. Find the interval of convergence of the power series $\sum_{n=1}^{\infty} \dfrac{(x+1)^n}{n^2}$.

5. Find Maclaurin series of the following function:

   (a) $\ln(1+x)$, (b) $\dfrac{1}{(1+x)^2}$.

Solutions to Review Questions for Test # 3

1(a).

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n}{2n+1} = \lim_{n\to\infty} \frac{n/n}{(2n+1)/n} = \lim_{n\to\infty} \frac{1}{2+\frac{1}{n}} = \frac{1}{2+0} = \frac{1}{2}.$$

It converges to $\frac{1}{2}$.

1(b).

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{10^n}{3^{2n}} = \lim_{n\to\infty} \frac{10^n}{(3^2)^n} = \lim_{n\to\infty} \left(\frac{10}{9}\right)^n = \infty.$$

It diverges to $+\infty$.

1(c). When n is even, then $(-1)^n = 1$, and

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{(-1)^n n}{2n+1} = \lim_{n\to\infty} \frac{n}{2n+1} = \frac{1}{2}.$$

When n is odd, then $(-1)^n = -1$, and

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{(-1)^n n}{2n+1} = \lim_{n\to\infty} \frac{-n}{2n+1} = -\frac{1}{2}.$$

Since the limits of a_n for even n and odd n are different, the limit $\lim_{n\to\infty} a_n$ doesn't exit. So, the sequence is divergent.

1(d). Since $-1 \le (-1)^n \le 1$, we have $-2^n \le (-2)^n = (-1)^n 2^n \le 2^n$, and $-\frac{2^n}{4^n+1} \le \frac{(-2)^n}{4^n+1} \le \frac{2^n}{4^n+1}$. Notice that,

$$\lim_{n\to\infty} \frac{2^n}{4^n+1} = \lim_{n\to\infty} \frac{2^n/4^n}{(4^n+1)/4^n} = \lim_{n\to\infty} \frac{(\frac{1}{2})^n}{1+(\frac{1}{4})^n} = \frac{0}{1+0} = 0,$$

by using the squeeze theorem, $\lim_{n\to\infty} \frac{(-2)^n}{4^n+1} = 0$. So, it is convergent.

2(a)[Method 1: Limit Comparison Test]. Let $a_n = \frac{2n}{4n^2-1}$ and $b_n = \frac{n}{n^2} = \frac{1}{n}$. Since

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{2n}{4n^2-1}\Big/\frac{1}{n} = \lim_{n\to\infty} \frac{2n}{4n^2-1} \times \frac{n}{1} = \lim_{n\to\infty} \frac{2n^2}{4n^2-1} = \frac{1}{2},$$

by the limit comparison test, the series $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{2n}{4n^2-1}$ and $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n}$ both have the same convergence or divergence. Since $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n}$ is divergent, because it is a p-series with $p = 1$, we know that $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{2n}{4n^2-1}$ is also divergent.

2(a)[**Method 2: Integral Test**].

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{2n}{4n^2-1} &\sim \int_1^{\infty} \frac{2x}{4x^2-1}dx \\
&\qquad [\text{substitute: } u = 4x^2-1,\ du = 8dx, \\
&\qquad \ \text{new limits: } u = 3 \text{ for } x = 1,\ \text{and } u = \infty \text{ for } x = \infty] \\
&= \int_3^{\infty} \frac{1}{4u}du \\
&= \lim_{t\to\infty} \int_3^{t} \frac{1}{4u}du \\
&= \lim_{t\to\infty} \frac{1}{4} \ln |u| \Big|_{u=3}^{t} \\
&= \lim_{t\to\infty} [\frac{1}{4} \ln t - \frac{1}{4} \ln 3] \\
&= \infty.
\end{aligned}$$

So it diverges.

2(b). Since

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n+1}{\sqrt{n^2+1}} = \lim_{n\to\infty} \frac{(n+1)/n}{\sqrt{n^2+1}/n} = \lim_{n\to\infty} \frac{1+\frac{1}{n}}{\sqrt{1+\frac{1}{n^2}}} = \frac{1+0}{\sqrt{1+0}} = 1 \neq 0,$$

by the test for divergence, the series is divergent.

2(c). Let $a_n = \frac{3^n}{4^n+1}$ and $b_n = \frac{3^n}{4^n} = (\frac{3}{4})^n$. Notice that,

$$\begin{aligned}
\lim_{n\to\infty} \frac{a_n}{b_n} &= \lim_{n\to\infty} \frac{3^n}{4^n+1} / \frac{3^n}{4^n} = \lim_{n\to\infty} \frac{3^n}{4^n+1} \times \frac{4^n}{3^n} \\
&= \lim_{n\to\infty} \frac{12^n}{12^n+3^n} = \lim_{n\to\infty} \frac{12^n/12^n}{(12^n+3^n)/12^n} \\
&= \lim_{n\to\infty} \frac{1}{1+(\frac{1}{4})^n} = \frac{1}{1+0} = 1,
\end{aligned}$$

by the limit comparison test, the series $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{3^n}{4^n+1}$ and $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} (\frac{3}{4})^n$ both have the same convergence or divergence. Since $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} (\frac{3}{4})^n$ is convergent, because it is a geometric series with $r = \frac{3}{4} < 1$, we know that $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{3^n}{4^n+1}$ is also convergent.

2(d).

$$\begin{aligned}
\sum_{n=0}^{\infty}\frac{n}{e^n} &\sim \int_0^{\infty}\frac{x}{e^x}dx \\
&= \lim_{t\to\infty}\int_0^t xe^{-x}dx \\
&\qquad\qquad [\text{integration by parts: } u=x,\ dv=e^{-x}dx,\ du=dx,\ v=-e^{-x}] \\
&= \lim_{t\to\infty}\Big[-xe^{-x}\Big|_{x=0}^{t}+\int_0^t e^{-x}dx\Big] \\
&= \lim_{t\to\infty}\Big[-xe^{-x}\Big|_{x=0}^{t}-e^{-x}\Big|_{x=0}^{t}\Big] \\
&= \lim_{t\to\infty}[-te^{-t}-e^{-t}+e^{0}] = 1-\lim_{t\to\infty}\frac{t}{e^t} \\
&= 1-\lim_{t\to\infty}\frac{(t)'}{(e^t)'} \qquad [\text{by l'Hospital Law}] \\
&= 1-\lim_{t\to\infty}\frac{1}{e^t} \\
&= 1-\frac{1}{\infty} = 1-0 = 1.
\end{aligned}$$

So, it converges.

3.

$$\begin{aligned}
1.\overline{121} &= 1+0.121+0.000121+0.000000121+\cdots \\
&= 1+\frac{121}{1000}+\frac{121}{1000^2}+\frac{121}{1000^3}+\cdots \\
&= 1+\frac{121}{1000}\Big(1+\frac{1}{1000}+\frac{1}{1000^2}+\frac{1}{1000^3}+\cdots\Big) \\
&= 1+\frac{121}{1000}\cdot\frac{1}{1-\frac{1}{1000}} \\
&= \frac{1120}{999}.
\end{aligned}$$

4. The radius of convergence is

$$R=\lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty}\frac{1}{n^2}\Big/\frac{1}{(n+1)^2} = \lim_{n\to\infty}\frac{(n+1)^2}{n^2} = 1.$$

So, the series $\sum_{n=1}^{\infty}\frac{(x+1)^n}{n^2}$ is convergent for x in $(a-R,a+R)=(-1-1,-1+1)=(-2,0)$. Furthermore, at the endpoint $x=0$, the series becomes $\sum_{n=1}^{\infty}\frac{1}{n^2}$, which is convergent, because it is a p-series with $p = 2\ (> 1)$. While, at the other endpoint $x = -2$, the series becomes $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$, which is absolutely convergent, because $\left|\frac{(-1)^n}{n^2}\right| = \frac{1}{n^2}$, and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is the p-series with $p = 2\ (> 1)$. Therefore, the interval of convergence for $\sum_{n=1}^{\infty} \frac{x^n}{n^2}$ is $[-2, 0]$.

5 (a).

$$\begin{aligned}
\ln(1+x) &= \int_0^x \frac{1}{1+y}dy \\
&= \int_0^x \frac{1}{1-(-y)}dy \\
&= \int_0^x \sum_{n=0}^{\infty}(-y)^n dy \\
&= \sum_{n=0}^{\infty} \int_0^x (-y)^n dy \\
&= \sum_{n=0}^{\infty} \int_0^x (-1)^n y^n dy \\
&= \sum_{n=0}^{\infty}(-1)^n \frac{x^{n+1}}{n+1}, \qquad x \in (-1, 1).
\end{aligned}$$

5 (b).

$$\begin{aligned}
\frac{1}{(1+x)^2} &= -\frac{d}{dx}\Big(\frac{1}{1+x}\Big) \\
&= -\frac{d}{dx}\Big(\frac{1}{1-(-x)}\Big) \\
&= -\frac{d}{dx}\sum_{n=0}^{\infty}(-x)^n \\
&= -\sum_{n=0}^{\infty} \frac{d}{dx}(-x)^n dy \\
&= -\sum_{n=1}^{\infty} n(-x)^{n-1}(-1) \\
&= \sum_{n=1}^{\infty}(-1)^n n x^{n-1}, \qquad x \in (-1, 1).
\end{aligned}$$