Champlain College – St.-Lambert

MATH 201-203: Calculus II

Review Questions for Test # 2

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1. Find integrals:

$$\text{(a)} \int \frac{\ln x}{x^3}\, dx, \qquad \text{(b)} \int (x-3)e^{-x} dx,$$

$$\text{(c)} \int \frac{x+1}{x^2-4x+3}\, dx, \qquad \text{(d)} \int \frac{1}{x^3-4x^2+4x}\, dx.$$

2. Evaluate each integral, and test if it is convergent or divergent:

$$\text{(a)} \int_1^{\infty} e^{-\sqrt{x}}\, dx, \qquad \text{(b)} \int_0^1 \frac{1}{x^2-1} dx.$$

3. Find the solutions to the following differential equations:

   (a) $y' = xye^x$,

   (b) $y' = \dfrac{y^2}{x-2}, \quad y(3) = 1.$

Solutions to Review Questions for Test # 2

1(a). Let $f(x) = \ln x$, $g'(x) = x^{-3}$, then $f'(x) = \frac{1}{x}$ and $g(x) = -\frac{1}{2}x^{-2}$. By using integration by parts, we obtain

$$\begin{aligned}
\int \frac{\ln x}{x^3}\, dx &= \int x^{-3} \ln x\, dx \\
&= f(x)g(x) - \int f'(x)g(x)\, dx \\
&= (\ln x)\Big(-\frac{1}{2}x^{-2}\Big) - \int \frac{1}{x} \cdot \Big(-\frac{1}{2}x^{-2}\Big)\, dx \\
&= -\frac{\ln x}{2x^2} + \frac{1}{2}\int x^{-3} dx \\
&= -\frac{\ln x}{2x^2} - \frac{1}{4x^2} + C.
\end{aligned}$$

1(b). Let $f(x) = x - 3$, $g'(x) = e^{-x}$, then $f'(x) = 1$ and $g(x) = -e^{-x}$, which can be integrated by substituting $u = -x$ (i.e., $du = -dx$) as follows

$$g(x) = \int e^{-x} dx = \int e^u(-du) = -\int e^u du = -e^u = -e^{-x}.$$

By using the integration by parts, we then obtain

$$\begin{aligned}
\int (x-3)e^{-x}\, dx &= (x-3)(-e^{-x}) - \int 1 \cdot (-e^{-x})\, dx \\
&= -(x-3)e^{-x} + \int e^{-x} dx \\
&= -(x-3)e^{-x} - e^{-x} + C.
\end{aligned}$$

1(c). Notice that $x^2 - 4x + 3 = (x-3)(x-1)$, and use the strategy of partial fractions to set

$$\frac{x+1}{x^2-4x+3} = \frac{x+1}{(x-3)(x-1)} = \frac{A}{x-3} + \frac{B}{x-1} = \frac{A(x-1)+B(x-3)}{(x-3)(x-1)},$$

and compare the numerators to have

$$x + 1 = A(x-1) + B(x-3),$$

we then get $A = 2$ by setting $x = 3$ and $B = -1$ by setting $x = 1$. Thus, we can integrate

$$\begin{aligned}
\int \frac{x+1}{x^2-4x+3}\,dx &= \int \Big(\frac{2}{x-3} - \frac{1}{x-1}\Big)\,dx \\
&= 2\int \frac{1}{x-3}\,dx - \int \frac{1}{x-1}\,dx \\
&= 2\ln|x-3| - \ln|x-1| + C.
\end{aligned}$$

1(d). Since $x^3 - 4x^2 + 4x = x(x-2)^2$, we then set

$$\frac{1}{x^3-4x^2+4x} = \frac{1}{x(x-2)^2} = \frac{A}{x} + \frac{B}{x-2} + \frac{C}{(x-2)^2} = \frac{A(x-2)^2 + Bx(x-2) + Cx}{x(x-2)^2},$$

and compare the numerators to have

$$1 = A(x-2)^2 + Bx(x-2) + Cx.$$

By setting $x = 0$, we have $A = \frac{1}{4}$, and set $x = 2$ to have $C = \frac{1}{2}$, and set $x = 1$ to have $B = -\frac{1}{4}$. Thus, we can integrate

$$\begin{aligned}
\int \frac{1}{x^3-4x^2+4x}\,dx &= \int \Big[\frac{1}{4x} - \frac{1}{4(x-2)} + \frac{1}{2(x-2)^2}\Big]\,dx \\
&= \frac{1}{4}\int \frac{1}{x}\,dx - \frac{1}{4}\int \frac{1}{x-2}\,dx + \frac{1}{2}\int (x-2)^{-2}dx \\
&= \frac{1}{4}\ln|x| - \frac{1}{4}\ln|x-2| - \frac{1}{2(x-2)} + C.
\end{aligned}$$

2(a). By substituting $u = -\sqrt{x}$, i.e., $u^2 = (-\sqrt{x})^2 = x$ and $dx = 2udu$, and the new upper-limit of the integral is $u = -\sqrt{x}|_{x=\infty} = -\infty$ and the new lower-limit of the integral is $u = -\sqrt{x}|_{x=1} = -1$, we have

$$\int_1^\infty e^{-\sqrt{x}}dx = \int_{-1}^{-\infty} e^u 2udu = \lim_{t\to-\infty}\int_{-1}^t 2u\, e^u\, du$$

[integration by parts: $f(u) = 2u, g'(u) = e^u, f'(u) = 2, g(u) = e^u$]

$$\begin{aligned}
&= \lim_{t\to-\infty}\Big(2ue^u\Big|_{-1}^t - \int_{-1}^t 2e^u du\Big) \\
&= \lim_{t\to-\infty}\Big(2te^t - 2(-1)e^{-1} - 2e^u\Big|_{-1}^t\Big) \\
&= \lim_{t\to-\infty}\Big(2te^t + 2e^{-1} - [2e^t - 2e^{-1}]\Big) \\
&= \lim_{t\to-\infty}(2te^t) - \lim_{t\to-\infty}(2e^t) + 4e^{-1} \\
&= 0 - 0 + \frac{4}{e} = \frac{4}{e},
\end{aligned}$$

where we got $\lim_{t\to-\infty} e^t = 0$ by the horizontal asymptotic property of the exponential function, and got $\lim_{t\to-\infty}(te^t) = 0$ by the so-called L'Hospital Law as follows

$$\begin{aligned}
\lim_{t\to-\infty}(2te^t) &= \lim_{s\to\infty}(-s)e^{-s} && \text{[set: } s=-t\text{]}\\
&= -\lim_{s\to\infty}\frac{s}{e^s} && \text{[type of "}\tfrac{\infty}{\infty}\text{"]}\\
&= -\lim_{s\to\infty}\frac{(s)'}{(e^s)'} = \lim_{s\to\infty}\frac{1}{e^s}\\
&= -\frac{1}{\infty} = 0.
\end{aligned}$$

So, the improper integral is convergent to $\frac{4}{e}$.

2(b). Since $x^2 - 1 = 0$ at $x = 1$, the integral is improper at the singular point $x = 1$. Thus we have

$$\begin{aligned}
\int_0^1 \frac{1}{x^2-1}\,dx &= \lim_{t\to1^-}\int_0^t \frac{1}{x^2-1}\,dx && \text{[by partial fractions]}\\
&= \lim_{t\to1^-}\int_0^t \left[\frac{1}{2(x-1)} - \frac{1}{2(x+1)}\right]dx\\
&= \lim_{t\to1^-}\left[\frac{1}{2}\ln|x-1| - \frac{1}{2}\ln|x+1|\right]\Big|_0^t\\
&= \lim_{t\to1^-}\left[\left(\frac{1}{2}\ln|t-1| - \frac{1}{2}\ln|t+1|\right) - \left(\frac{1}{2}\ln|0-1| - \frac{1}{2}\ln|0+1|\right)\right]\\
&= \frac{1}{2}\ln|0^+| - \frac{1}{2}\ln 2 = -\infty.
\end{aligned}$$

So, the improper integral is divergent.

3(A). By the separation of variables, from $y' = xye^x$, we have

$$\frac{dy}{y} = xe^x dx.$$

Integrating the above equation

$$\int \frac{1}{y}\,dy = \int xe^x\,dx,$$

yields

$$\ln|y| = xe^x - e^x + C, \tag{0.1}$$

where by setting $f(x) = x$ and $g'(x) = e^x$ which imply $f'(x) = 1$ and $g(x) = e^x$, we used the integration by parts to get

$$\int xe^x dx = xe^x - \int 1\cdot e^x dx = xe^x - \int e^x dx = xe^x - e^x + C.$$

From (0.1), we have

$$e^{\ln|y|} = e^{xe^x - e^x + C},$$

which gives

$$|y| = e^C e^{xe^x - e^x},$$

namely,

$$y = \pm e^C e^{xe^x - e^x}.$$

Denote $C_1 := \pm e^C$, which can be any non-zero number due to the arbitrariness of C, we have

$$y = C_1 e^{xe^x - e^x}, \qquad C_1 \neq 0. \tag{0.2}$$

Notice that, $y = 0$ is a particular solution of the differential equation, which is same to the solution in (0.2) by setting $C_1 = 0$. So, the general solution is

$$y = C_1 e^{xe^x - e^x} \text{ for arbitrary constant } C_1.$$

3(b). By the separation of variables, from $y' = \frac{y^2}{x-2}$, we have

$$\frac{dy}{y^2} = \frac{1}{x-2}dx.$$

Integrating the above equation

$$\int y^{-2}\, dy = \int \frac{1}{x-2}\, dx,$$

yields

$$-\frac{1}{y} = \ln|x-2| + C,$$

where C is an arbitrary constant. So, the general solution is

$$y = -\frac{1}{\ln|x-2| + C}.$$

Using the initial value condition $y(3) = 1$, we can specify

$$1 = -\frac{1}{\ln|3-2| + C} = -\frac{1}{C}, \quad i.e., \ C = -1.$$

Thus, the particular solution is

$$y = -\frac{1}{\ln|x-2| - 1}.$$