Solutions to The First Midterm Examination
EMAT 213: Differential Equations

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Solution to Q1: 1). the first order nonlinear ODE; 2). the first order linear ODE; 3). the second order nonlinear ODE;

Solution to Q2: Let $u = y/x$. Substituting $y = ux$ and $\frac{dy}{dx} = x\frac{du}{dx} + u$ into the original equation, we have

$$x\frac{du}{dx} + u = \frac{u-1}{u+1}, \quad i.e., \quad x\frac{du}{dx} = -\frac{1+u^2}{1+u},$$

which can be written to the equation of separable variable

$$\frac{1+u}{1+u^2}du = -\frac{1}{x}dx.$$

Integrating it gives

$$\int \frac{1+u}{1+u^2}du = -\int \frac{1}{x}dx.$$

Since $\int \frac{1+u}{1+u^2}du = \int \frac{1}{1+u^2}du + \int \frac{u}{1+u^2}du = \arctan u + \frac{1}{2}\ln(1+u^2)$ and $\int \frac{1}{x}dx = \ln x$, we solve the above ODE as

$$\arctan u + \frac{1}{2}\ln(1+u^2) = \ln|x| + C.$$

Substituting $u = y/x$ back into the above solution for u, we finally obtain the solution to the original ODE as

$$\arctan\frac{y}{x} + \frac{1}{2}\ln\left(1 + \frac{y^2}{x^2}\right) = \ln|x| + C.$$

Solution to Q3: Set $u = y^{1-2} = 1/y$. Then substituting $y = \frac{1}{u}$ and $\frac{dy}{dx} = -\frac{1}{u^2}\frac{du}{dx}$ into the original equation, we have

$$-\frac{1}{u^2}\frac{du}{dx} - \frac{1}{u} = \frac{e^x}{u^2}, \quad i.e., \quad \frac{du}{dx} + u = -e^x,$$

which is a first-order linear nonhomogeneous equation. The general solution the homogeneous equation $u' + u = 0$ is $u_c = Ce^{-x}$ and one particular solution to the nonhomogeneous equation $u' + u = -e^x$ is $u_p = \frac{1}{2}e^x$. So, the general solution for u is

$$u = u_c + u_p = Ce^{-x} - \frac{1}{2}e^x.$$

Thus, after substituting $u = 1/y$ back to the above, we get the solution to the original equation as

$$y = \frac{2}{2Ce^{-x} - e^x}.$$

Solution to Q4: Let $M(x, y) = e^x + y$ and $N(x, y) = 2 + x + y$. Since

$$\frac{\partial M}{\partial y} = 1 = \frac{\partial N}{\partial x},$$

the equation is exact. Let $f(x, y)$ be the function such that

$$\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = (e^x + y)dx + (2 + x + y)dy,$$

we then have

$$\begin{cases} \frac{\partial f}{\partial x} = e^x + y \\ \frac{\partial f}{\partial y} = 2 + x + y \end{cases} \tag{0.1}$$

Firstly, we integrate the first equation of (0.1) with respect to x

$$f(x, y) = \int (e^x + y)dx = e^x + xy + g(y) \tag{0.2}$$

where $g(y)$ is the integral "constant" and will be determined below, then differentiate (0.2) with respect to y

$$\frac{\partial f}{\partial y} = x + g'(y), \tag{0.3}$$

finally, compare (0.3) and the second equation of (0.1) to get

$$x + g'(y) = 2 + x + y, \quad i.e., \quad g'(y) = 2 + y.$$

So, $g(y) = 2y + \frac{1}{2}y^2$. Substituting $g(y)$ into (0.2), we obtain the general solution

$$e^x + xy + 2y + \frac{1}{2}y^2 = C.$$

Noting the initial value $y(0) = 1$, we have

$$C = e^0 + 0y(0) + 2y(0) + \frac{1}{2}y(0)^2 = \frac{7}{2}.$$

Thus, the IV solution is

$$e^x + xy + 2y + \frac{1}{2}y^2 = \frac{7}{2}.$$