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> # Assignment #2 , Math 354/ Mast 334 Solutions:
> # Problem 12 page 54 #################################
> # By  Theorem 2.1 we have |p_n - p|<=(b-a)/2^n.
> # We want the error to be < 10^(-3) working on the
# interval [1,4]. Thus, we need
#             3/2^n <= 10^(-3)
# or       2^n >= 3000.
# The smallest such n is n=12. Below is the printout
# of the program bisectional for our problem:
> a:=1;b:=4;
f:=x->evalf(x^3+x-4);
maxsteps:=15:er:=100:required_precision:=.001:
for i from 1 to maxsteps  while er >required_precision do
  p:=(a+b)/2:
  if evalf(f(a)*f(p))<0 then b:=p else a:=p end if:
ans:=evalf((a+b)/2);
  er:=abs(a-b)/2;
  print("(",a,b,")","answer= ",ans, " error= ",evalf(er));
end do:
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$$a := 1$$

$$b := 4$$

$$f := x \rightarrow \text{evalf}(x^3 + x - 4)$$

$$\text{"(", } 1, \frac{5}{2}, \text{")", "answer= ", } 1.750000000, \text{" error= ", } 0.750000000($$

$$\text{"(", } 1, \frac{7}{4}, \text{")", "answer= ", } 1.375000000, \text{" error= ", } 0.375000000($$

$$\text{"(", } \frac{11}{8}, \frac{7}{4}, \text{")", "answer= ", } 1.562500000, \text{" error= ", } 0.187500000($$

$$\text{"(", } \frac{11}{8}, \frac{25}{16}, \text{")", "answer= ", } 1.468750000, \text{" error= ", } 0.09375000000($$

$$\text{"(", } \frac{11}{8}, \frac{47}{32}, \text{")", "answer= ", } 1.421875000, \text{" error= ", } 0.04687500000$$

$$\text{"(", } \frac{11}{8}, \frac{91}{64}, \text{")", "answer= ", } 1.398437500, \text{" error= ", } 0.0234375000($$

$$\text{"(", } \frac{11}{8}, \frac{179}{128}, \text{")", "answer= ", } 1.386718750, \text{" error= ", } 0.01171875000$$

$$\text{"(", } \frac{11}{8}, \frac{355}{256}, \text{")", "answer= ", } 1.380859375, \text{" error= ", } 0.00585937500($$

$$"(", \frac{11}{8}, \frac{707}{512}, ")", "answer= ", 1.377929688, " error= ", 0.00292968750($$

$$"(", \frac{1411}{1024}, \frac{707}{512}, ")", "answer= ", 1.379394531, " error= ", 0.00146484375($$

$$"(", \frac{1411}{1024}, \frac{2825}{2048}, ")", "answer= ", 1.378662109, " error= ", 0.0007324218750$$

```
> evalf(5647/4096);
```

1.378662109

```
> # The answer is 1.378 with the precision of 0.001.
# One can see that we have reached the required precision
in 11 steps,
# so Theorem 2.1 gives less than optimal estimate of the
error.

> # Problem 18 page 55
################################################
g:=32.17;
xx:=(t,w)->(-g/(2*w^2))*((exp(w*t)-exp(-w*t))/2-sin(w*t));
```

$$g := 32.17$$

$$xx := (t, w) \to -\frac{1}{2}\frac{g\left(\frac{1}{2}e^{(w\,t)} - \frac{1}{2}e^{(-w\,t)} - \sin(w\,t)\right)}{w^2}$$

```
> # we have xx(1,w)-xx(0,w)=1.7 . Thus let us define f(w)
> f:=w->xx(1,w)-xx(0,w)-1.7 ; # and find the 0 of function h
```

$$f := w \to \mathrm{xx}(1, w) - \mathrm{xx}(0, w) - 1.7$$

```
> expand(f(w));
```

$$-\frac{8.042500000e^w}{w^2} + \frac{8.042500000}{w^2\,e^w} + \frac{16.08500000\sin(w)}{w^2} - 1.7$$

```
> # We need to find an interval in which f(w) changes sign:
evalf(f(-1));evalf(f(-0.3));
```

3.66805040

-0.091484469

```
> # We can to start the bisectional algorithm with a=-1 , b=
-0.3
# The error after n steps will be 0.7/2^(n+1)
a:=-1;b:=-0.1;

maxsteps:=25:er:=100:required_precision:=.00001:# 10^(-5)
```

```
for i from 1 to maxsteps  while er >required_precision do
  p:=(a+b)/2:
  if evalf(f(a)*f(p))<0 then b:=p else a:=p end if:
ans:=evalf((a+b)/2);
  er:=abs(a-b)/2;
  print("(",a,b,")","answer= ",ans, " error= ",evalf(er));
end do:
```

$$a := -1$$

$$b := -0.1$$

"(", -0.5500000000 , -0.1, ")", "answer= ", -0.3250000000 , " error= ", 0.2250000000

"(", -0.3250000000, -0.1, ")", "answer= ", -0.2125000000, " error= ", 0.1125000000

"(", -0.3250000000 , -0.2125000000 , ")", "answer= ", -0.2687500000 , " error= ",
0.05625000000

"(", -0.3250000000 , -0.2687500000 , ")", "answer= ", -0.2968750000 , " error= ",
0.02812500000

"(", -0.3250000000 , -0.2968750000 , ")", "answer= ", -0.3109375000 , " error= ",
0.01406250000

"(", -0.3250000000 , -0.3109375000 , ")", "answer= ", -0.3179687500 , " error= ",
0.007031250000

"(", -0.3179687500 , -0.3109375000 , ")", "answer= ", -0.3144531250 , " error= ",
0.003515625000

"(", -0.3179687500 , -0.3144531250 , ")", "answer= ", -0.3162109375 , " error= ",
0.001757812500

"(", -0.3179687500 , -0.3162109375 , ")", "answer= ", -0.3170898438 , " error= ",
0.0008789062500

"(", -0.3170898438 , -0.3162109375 , ")", "answer= ", -0.3166503907 , " error= ",
0.0004394531500

"(", -0.3170898438 , -0.3166503907 , ")", "answer= ", -0.3168701173 , " error= ",
0.0002197265500

"(", -0.3170898438 , -0.3168701173 , ")", "answer= ", -0.3169799805 , " error= ",
0.0001098632500

"(", -0.3170898438 , -0.3169799805 , ")", "answer= ", -0.3170349121 , " error= ",
0.00005493165000

"(", -0.3170898438 , -0.3170349121 , ")", "answer= ", -0.3170623779 , " error= ",
0.00002746585000

"(", -0.3170623779 , -0.3170349121 , ")", "answer= ", -0.3170486450 , " error= ",
0.00001373290000

"(", -0.3170623779 , -0.3170486450 , ")", "answer= ", -0.3170555115 , " error= ",
$0.6866450000 \; 10^{-5}$

```
> # The answer is w=-0.3170555

# Problem 7 page 64
###############################################
g:=x->Pi+0.5*sin(x/2);
```

$$g := x \to \pi + 0.5 \sin\left(\frac{1}{2} x\right)$$

```
> # g'(x)=cos(x/2)/4 and on interval [0,2Pi] , we have
|g'(x)|<= 1/4.
# Thus by theorem 2.2,  g has a unique fixed point in
[0,2Pi].
# let k=1/4. The error after n steps is
#               |p_n - p| <= k^n/(1-k)|p_1 - p_0|
# If p_0=Pi then p_1=Pi+0.5 and |p_1 - p_0|=0.5
# We want |p_n - p|< 10^(-2) so we need
#  (1/4)^n /(3/4) *0.5 < 1/100 or
#  100*(4/3)*0.5 < 4^n
100*(4/3)*0.5;
```

66.66666666

```
> # The smallest such n is n=4. ###########################
# Below we have the printout of Fixed_Point routine:

g:=x->evalf(Pi+0.5*sin(x/2));
a:=evalf(Pi);# initial point

exact_value:=3.6269; # obtained from the same program
                     #with higher precision
MaxSteps:=20:Er:=100:MaxError:=.01:# 10^(-2)
for i from 1 to MaxSteps  while (Er >MaxError) and
(abs(a)<10000) do
  anew:=g(a):
  Er:=abs(exact_value-anew):
  a:=anew:
  print(a, " error= ",evalf(Er));
end do:
```

$$g := x \to \text{evalf}\left(\pi + 0.5 \sin\left(\frac{1}{2} x\right)\right)$$

$$a := 3.141592654$$

$$exact\_value := 3.6269$$

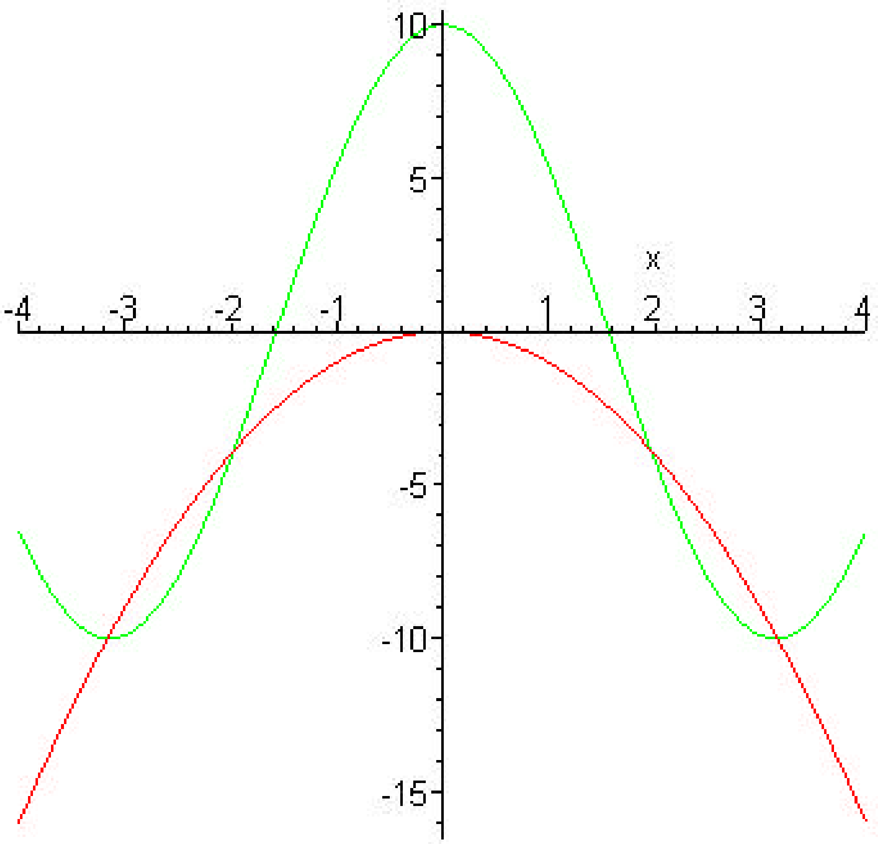
3.641592654 , " error= " , 0.014692654

3.626048865 , " error= " , 0.000851135

```
> #Answer x=3.6260 after 2 steps instead of theoretically
predicted 4.
>
> # problem 13 page 65
##########################################
f1:=x->-x^2;
f2:=x->10*cos(x);
plot([f1(x),f2(x)],x=-4..4);
```

$$f1 := x \to -x^2$$

$$f2 := x \to 10\cos(x)$$




```
> # It is enough to find the positive roots ( by symmetry)
# We try to represent  equation x^2+10*cos(x)=0 in the form
# x=g(x): for example x=sqrt(-10*cos(x))
g:=x->sqrt(-10*cos(x));
a:=3.0;# initial point
MaxSteps:=20:Er:=100:MaxError:=.00001:# 10^(-4)
```

```
for i from 1 to MaxSteps  while (Er >MaxError) and
(abs(a)<10000) do
  anew:=g(a):
  Er:=abs(a-anew):# This is not really an error, just an
indication
  a:=anew:
  print(a, " error= ",evalf(Er));
end do:
```

$$g := x \rightarrow \sqrt{-10\cos(x)}$$

$$a := 3.0$$

3.146414621, " error= ", 0.146414621

3.162259278, " error= ", 0.015844657

3.161939995 , " error= " , 0.000319283

3.161950347 , " error= " , 0.000010352

3.161950014 , " error= " , $0.333\ 10^{-6}$

```
> # We have repeating 5 digits so the error is < 10 ^(-4)
# answer x=3.16195 and x=-3.16195

# Now we try with different initial point:
# say 2.0 but the result is always the same (I am not
# giving the printouts here.
# We have to try another form, say
#          x=x-(x^2+10*cos(x))/(2*x-10*sin(x))
# (Newton-Raphson form)
g:=x->x-(x^2+10*cos(x))/(2*x-10*sin(x));
a:=2.0;# initial point
MaxSteps:=20:Er:=100:MaxError:=.00001:# 10^(-4)
for i from 1 to MaxSteps  while (Er >MaxError) and
(abs(a)<10000) do
  anew:=g(a):
  Er:=abs(a-anew):# This is not really an error, just an
indication
  a:=anew:
  print(a, " error= ",evalf(Er));
end do:
```

$$g := x \rightarrow x - \frac{x^2 + 10\cos(x)}{2x - 10\sin(x)}$$

$$a := 2.0$$

1.968295861 , " error= " , 0.031704139

1.968872753 , " error= " , 0.000576892

$1.968872938, \text{" error= "}, 0.185\ 10^{-6}$

```
> # We have repeating 5 digits so the answer is precise with precision 10^(-4)
# answer x=1.968872938 and x=-1.968872938.
# It is worth noting that the second method with initial point a=3.0
# gives the previous solution x=3.161950014.
```