Solutions to Midterm Examination
MATH 251:Linear Algebra I

Instructor: Dr. Ming Mei

Solution to Q1. a). W_1 is a subspace of V, because: 1. zero 0 is also an odd function; 2. for any odd functions $f(t)$ and $g(t)$ in W_1, i.e., $f(-t) = -f(t)$ and $g(-t) = -g(t)$, we have

$$af(-t) + bg(-t) = -[af(t) + bg(t)], \quad \text{for any } a, b \in \mathbf{R},$$

which implies $af(t) + bg(t) \in W_1$.

b). W_2 is not a subspace of V. For example, $f(t) = 2 \in W_2$, but $3f(t) = 6 > 2$, which means $3f(t) \notin W_2$.

Solution to Q2. Set a matrix M whose rows are u_1, u_2 and u_3. Taking row operations to M, we then have

$$M = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & -6 & -12 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

The nonzero row vectors are $(1, 2, 3)$ and $(0, 1, 2)$, so a basis of $Span(S)$ is $\{(1, 2, 3), (0, 1, 2)\}$ and the dimension is 2.

Solution to Q3. Since $\dim V = 5$, $\dim U = 1$ and $\dim W = 4$, and $\dim(U \cap W)$ may be 0 or 1, because it is only possible to have $U \cap W = 0$ or $U \cap W = U$, then by the formula

$$\dim(U + W) = \dim U + \dim W - \dim(U \cap W),$$

we get $\dim(U + W) = 5$ or 4. When $\dim(U + W) = 5$, since $U + W \subset V$ and $\dim V = 5$, we must have $U + W = V$. When $\dim(U + W) = 4$, since $W \subset U + W$ and $\dim W = 4$, we must have $U + W = W$.

Solution 1 to Q4. If

$$xw_1 + yw_2 + zw_3 + tw_4 = 0, \quad \text{for some scalars } x, y, z, t,$$

we then have

$$(x + z + 2t)v_1 + (x + y - z + t)v_2 + (x + 3t)v_3 + (x + y + z + t)v_4 = 0.$$

Since v_1, v_2, v_3 and v_4 are linearly independent, we must have

$$\begin{cases} x + z + 2t = 0, \\ x + y - z + t = 0, \\ x + 3t = 0, \\ x + y + z + t = 0, \end{cases}$$

which can be checked to have only zero solution $(x, y, z, t) = (0, 0, 0, 0)$, because, by the row operations, its determinant is

$$\begin{vmatrix} 1 & 0 & 1 & 2 \\ 1 & 1 & -1 & 1 \\ 1 & 0 & 0 & 3 \\ 1 & 1 & 1 & 1 \end{vmatrix} = 2 \neq 0.$$

So, $\{w_1, w_2, w_3, w_4\}$ is linearly independent.

Solution 2 to Q4. Let $S = Span(v_1, v_2, v_3, v_4)$. Since v_1, v_2, v_3, v_4 are linearly independent, So S is a 4-dimensional vector space with a basis $\{v_1, v_2, v_3, v_4\}$, and is isomorphic to R^4, i.e., $S \cong R^4$. Since the coordinates of w_1, w_2, w_3 and w_4 in the vector space S corresponding to the basis $\{v_1, v_2, v_3, v_4\}$ are

$$[w_1]_S = [1, 0, 1, 2], \quad [w_2]_S = [1, 1, -1, 1], \quad [w_3]_S = [1, 0, 0, 3], \quad [w_4]_S = [1, 1, 1, 1]$$

and it can be checked that

$$\begin{pmatrix} [w_1]_S \\ [w_2]_S \\ [w_3]_S \\ [w_4]_S \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 & 2 \\ 1 & 1 & -1 & 1 \\ 1 & 0 & 0 & 3 \\ 1 & 1 & 1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 2 \end{pmatrix},$$

where the row vectors in the echelon matrix all are nonzero, so $[w_1]_S = [1, 0, 1, 2]$, $[w_2]_S = [1, 1, -1, 1]$, $[w_3]_S = [1, 0, 0, 3]$, and $[w_4]_S = [1, 1, 1, 1]$ are linearly independent in R^4. Therefore, $\{w_1, w_2, w_3, w_4\}$ is linearly independent in S, of course, in V.