Math 141C: **Quiz 2**

1. (4 pts) Evaluate integrals:

a). $\int x^3 e^{x^2} \mathrm{d}x;$

b). $\int \cot^5 \theta \sin^4 \theta \mathrm{d}\theta;$

c). $\int e^t \sqrt{9 - e^{2t}} \mathrm{d}t;$

d). $\int_1^{27} \frac{1}{1 + x^{1/3}} \mathrm{d}x;$

2. (2 pts) Determine whether each integral is convergent or divergent. Evaluate those that are convergent.

a). $\int_{-\infty}^{\infty} x^2 e^{-x^3} dx;$ **b)**. $\int_0^2 z^2 \ln z dz.$

3. (4 pts) **a)**. Find the length of the arc of $y = e^x$ from $(0, 1)$ to $(1, e)$;

b). This arc is rotated about the x-axis. Find the area of the resulting surface.

Solutions to Qiuz 2

1. Solution. **a)**. Using integration by parts and Substitution Rule, we get

$$\int x^3 e^{x^2}\mathrm{d}x = \int x^2\cdot(xe^{x^2})\mathrm{d}x = x^2\cdot\frac{1}{2}e^{x^2} - \int xe^{x^2}dx = \frac{1}{2}x^2e^{x^2} - \frac{1}{2}e^{x^2}.$$

b).

$$\begin{aligned}
&\int \cot^5\theta\sin^4\theta\mathrm{d}\theta = \int \frac{\cos^5\theta}{\sin^5\theta}\sin^4\theta d\theta\\
&= \int \frac{\cos^4\theta}{\sin\theta}\cos\theta d\theta = \int \frac{(1-\sin^2\theta)^2}{\sin\theta}\cos\theta d\theta\\
&= \int \frac{(1-u^2)^2}{u}du \qquad (\text{substituting } u=\sin\theta)\\
&= \int \Big(\frac{1}{u} - 2u + u^3\Big)du = \ln|u| - u^2 + \frac{u^4}{4} + C\\
&= \ln|\sin\theta| - \sin^2\theta + \frac{\sin^4\theta}{4} + C
\end{aligned}$$

c).

$$\begin{aligned}
&\int e^t\sqrt{9-e^{2t}}dt\\
&= \int \sqrt{9-u^2}du \qquad (u=e^t,\ du=e^tdt)\\
&= \int \sqrt{9-9\sin^2\theta}\,3\cos\theta d\theta \qquad (u=3\sin\theta,\ 0<\theta\le\tfrac{\pi}{2})\\
&= \int 9\cos^2\theta d\theta = 9\int\frac{1+\cos2\theta}{2}d\theta\\
&= \frac{9}{2}\theta + \frac{9}{4}\sin2\theta + C\\
&= \frac{9}{2}\sin^{-1}\frac{u}{3} + \frac{9}{2}\frac{u}{3}\sqrt{1-\frac{u^2}{9}} + C\\
&= \frac{9}{2}\sin^{-1}\frac{e^t}{3} + \frac{9}{2}e^t\sqrt{9-e^{2t}} + C
\end{aligned}$$

d).

$$\int_0^{27} \frac{1}{1+x^{1/3}} dx$$
$$= \int_0^3 \frac{1}{1+u} 3u^2 du \quad (u = x^{1/3},\ dx = 3u^2 du)$$
$$= 3\int_0^3 \Big(u - 1 + \frac{1}{u+1}\Big) du = 3\Big(\frac{u^2}{2} - u + \ln(1+u)\Big)\Big]_0^3 = \frac{9}{2} + 6\ln 2.$$

2. Solution. **a)**. Since

$$\int_{-\infty}^{\infty} x^2 e^{-x^3} dx = \lim_{t\to\infty} \int_{-t}^{t} x^2 e^{-x^3} dx$$
$$= \lim_{t\to\infty} \int_{-t^3}^{t^3} \frac{1}{3} e^{-u} du \quad (u = x^3,\ du = 3x^2 dx)$$
$$= \lim_{t\to\infty} \frac{1}{3} e^{-u}]_{-t^3}^{t^3} = \lim_{t\to\infty} \frac{1}{3}(e^{-t^3} - e^{t^3})$$
$$= \frac{1}{3}(0 - \infty) = -\infty,$$

the improper integral $\int_{-\infty}^{\infty} x^2 e^{-x^3} dx$ is divergent.

b). Since

$$\int_0^2 z^2 \ln z dz = \lim_{\epsilon\to 0} \int_\epsilon^2 z^2 \ln z dz$$
$$= \lim_{\epsilon\to 0} \Big\{ \frac{z^3}{3} \ln z\Big]_\epsilon^2 - \int_\epsilon^2 \frac{z^2}{3} dz \Big\} \quad \text{(integration by parts)}$$
$$= \lim_{\epsilon\to 0} \Big\{ \frac{8}{3}\ln 2 - \frac{\epsilon^3}{3}\ln\epsilon - (\frac{8}{9} - \frac{\epsilon^3}{9}) \Big\}$$
$$= \frac{8}{3}\ln 2 - \lim_{\epsilon\to 0} \frac{\epsilon^3}{3}\ln\epsilon - \frac{8}{9} + \lim_{\epsilon\to 0}\frac{\epsilon^3}{9}$$
$$= \frac{8}{3}\ln 2 - \frac{1}{3}\lim_{\epsilon\to 0} \frac{(\ln\epsilon)'}{(\epsilon^{-3})'} - \frac{8}{9} \quad \text{(l'Hospital Law)}$$
$$= \frac{8}{3}\ln 2 + \frac{1}{3}\lim_{\epsilon\to 0}\epsilon^5 - \frac{8}{9} = \frac{8}{3}\ln 2 - \frac{8}{9},$$

the integral is convergent.

3. Solution. **a)**. Since $y = e^x$ and $y' = e^x$, we obtain

$$L = \int_0^1 \sqrt{1 + (y')^2} dx = \int_0^1 \sqrt{1 + e^{2x}} dx.$$

Substituting $u = \sqrt{1+e^{2x}}$, i.e., $1 + e^{2t} = u^2$, we get $2udu = (1+e^{2t})'dt = 2e^{2t}dt = 2(u^2-1)dt$, i.e., $dt = \frac{u}{u^2-1}du$, and

$$\begin{aligned} L &= \int_0^1 \sqrt{1+e^{2x}}dx \\ &= \int_{\sqrt{2}}^{\sqrt{1+e^2}} \frac{u^2}{u^2-1}du = \int_{\sqrt{2}}^{\sqrt{1+e^2}} \left(1 + \frac{1}{u^2-1}\right)du \\ &= \int_{\sqrt{2}}^{\sqrt{1+e^2}} \left(1 + \frac{1}{2(u-1)} - \frac{1}{2(u+1)}\right)du \\ &= \left(u + \frac{1}{2}\ln|u-1| - \frac{1}{2}\ln|u+1|\right)\Big]_{\sqrt{2}}^{\sqrt{1+e^2}} \\ &= \sqrt{1+e^2} - \sqrt{2} + \frac{1}{2}\ln\frac{\sqrt{1+e^2}-1}{\sqrt{1+e^2}+1} - \frac{1}{2}\ln\frac{\sqrt{2}-1}{\sqrt{2}+1}. \end{aligned}$$

b). Using substitution $t = e^x$, we have

$$S = \int_0^1 2\pi e^x\sqrt{1+e^{2x}}dx = \int_1^e 2\pi\sqrt{1+t^2}dt.$$

Using another substitution $t = \tan\theta$ for $\frac{\pi}{4} \le \theta \le \tan^{-1} e$, and noting $dt = \sec^2\theta d\theta$, $\sqrt{1+t^2} = \sqrt{1+\tan^2\theta}$, we then have

$$L = \int_1^e 2\pi\sqrt{1+t^2}dt = 2\pi\int_{\frac{\pi}{4}}^{\tan^{-1} e} \sec^3\theta d\theta. \tag{0.1}$$

Here we integrate by parts with $u = \sec\theta$, $dv = \sec^2\theta d\theta$, which implies $du = \sec\theta\tan\theta d\theta$ and $v = \tan\theta$. Then

$$\begin{aligned} \textstyle\int \sec^3\theta d\theta &= \sec\theta\tan\theta - \int \sec\theta\tan^2\theta d\theta \\ &= \sec\theta\tan\theta - \int \sec\theta(\sec^2\theta - 1)d\theta \\ &= \sec\theta\tan\theta - \int \sec^3\theta d\theta + \int \sec\theta d\theta \\ &= \sec\theta\tan\theta - \int \sec^3\theta d\theta + \ln|\sec\theta + \tan\theta| + C, \end{aligned}$$

which gives

$$\int \sec^3\theta d\theta = \frac{1}{2}(\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|) + C.$$

Applying the above formula to (0.1), and noting that $tan\theta_1 = e$ gives $\sec\theta_1 = \frac{2e^2}{\sqrt{1+4e^2}-1}$, we obtain

$$\begin{aligned} L &= 2\pi\frac{1}{2}(\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|)_{\frac{\pi}{4}}^{\tan^{-1}e} \\ &= \frac{2\pi e^3}{\sqrt{1+4e^2}-1} + \pi\ln\left(\frac{2e^2}{\sqrt{1+4e^2}-1} + e\right) - \sqrt{2}\pi - \pi\ln(\sqrt{2}+1). \end{aligned}$$