Math 141C: **Quiz 1**

1. (4 pts) Evaluate integrals:

   a). $\displaystyle\int \frac{x}{\sqrt{1-x}}\mathrm{d}x;$

   b). $\displaystyle\int_0^2 |1-x+x^2-x^3|\mathrm{d}x;$

   c). $\displaystyle\int \frac{\sin 5x}{\cos 5x+2}\mathrm{d}x;$

   d). $\displaystyle\int_2^3 e^{e^x}e^x\mathrm{d}x;$

2. (2 pts) Find the area of the region bounded by the curve $y = x^2$ and the line $y = x$.

3. (4 pts) Use two different methods to find the volume of the solid obtained by rotating the region specified in Question 2 about the x-axis. (Hint: the method of cylinders and the method of cylindrical shells)

Solutions to Qiuz 1

1. Solution. **a)**. Using the substitution $u = 1 - x$, $du = -dx$, we get

$$\int \frac{x}{\sqrt{1-x}}\mathrm{d}x = \int \frac{1-u}{\sqrt{u}}(-\mathrm{d}u) = -\int \Big(\frac{1}{\sqrt{u}} - \sqrt{u}\Big)\mathrm{d}u$$
$$= -\int u^{-\frac{1}{2}}du + \int u^{\frac{1}{2}}du = -2u^{\frac{1}{2}} + \frac{2}{3}u^{\frac{3}{2}} + C$$
$$= -2(1-x)^{\frac{1}{2}} + \frac{2}{3}(1-x)^{\frac{3}{2}} + C.$$

b). Since $1 - x + x^2 - x^3 = (1-x)(1+x^2)$ is positive for $x \in (0,1)$ and negative for $x \in (1,2)$, we slipt the integral into two parts then have

$$\int_0^2 |1 - x + x^2 - x^3|\mathrm{d}x = \int_0^1 |1 - x + x^2 - x^3|\mathrm{d}x + \int_1^2 |1 - x + x^2 - x^3|\mathrm{d}x$$
$$= \int_0^1 (1 - x + x^2 - x^3)\mathrm{d}x + \int_1^2 -(1 - x + x^2 - x^3)\mathrm{d}x$$
$$= \Big(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}\Big)\Big]_0^1 - \Big(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}\Big)\Big]_1^2$$
$$= \frac{5}{2}$$

c). Let $u = \cos 5x$, then $du = -5\sin 5x dx$. Substituting this into the integral gives

$$\int \frac{\sin 5x}{\cos 5x + 2}dx = -\frac{1}{5}\int \frac{1}{u+2}du = -\frac{1}{5}\ln|u+2| + C = -\frac{1}{5}\ln|\cos 5x + 2| + C.$$

d). Let $u = e^x$, then $du = e^x dx$, and $u = e^3$ and $u =^2$ for $x = 3$ and $x = 2$, respectively. Substituting this into the integral leads to

$$\int_2^3 e^{e^x} e^x \mathrm{d}x = \int_{e^2}^{e^3} e^u \mathrm{d}u = e^u]_{e^2}^{e^3} = e^{e^3} - e^{e^2}.$$

2. Solution.

$$\int_0^1 (1 - x^2)dx = \Big(x - \frac{x^3}{3}\Big)\Big]_0^1 = \frac{2}{3}.$$

3. Solution. Method 1: the method of cylinders. Let $f_1(x) = x$ and $f_2(x) = x^2$. The cross-section is a ring, and its area is

$$A(x) = \pi(f_1(x)^2 - f_2(x)^2) = \pi(x^2 - (x^2)^2) = \pi(x^2 - x^4).$$

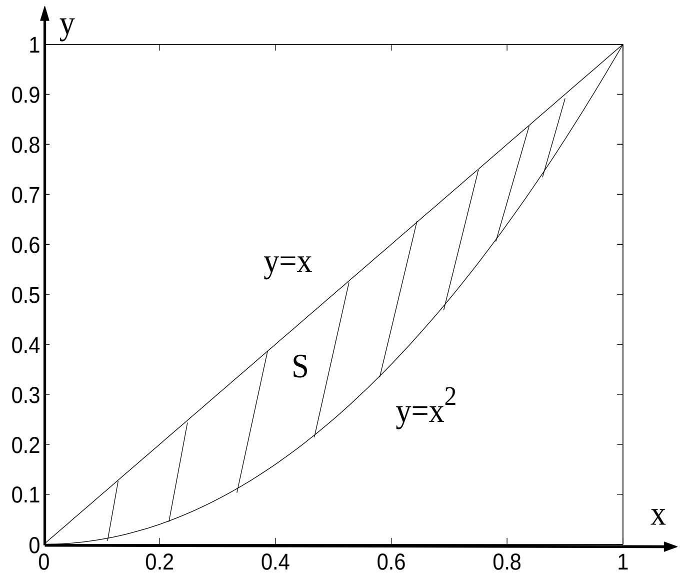

Figure 0.1:

By the method of cylinders, the volume is obtained by

$$V = \int_0^1 A(x)dx = \int_0^1 \pi(x^2 - x^4)dx = \pi\Big(\frac{x^3}{3} - \frac{x^5}{5}\Big)\Big]_0^1 = \frac{2}{15}\pi.$$

Method 2: the method of cylindrical shells. The line $y = x$ and the curve $y = x^2$ can be written in $x = y$ and $x = \sqrt{y}$, respectively. Let $g_1(y) = y$ and $g_2(y) = \sqrt{y}$. $g_2(y)$ is on the right-hand side of $g_1(y)$. By the method of cylindrical shell, the volume of the solid can be evaluated as

$$\begin{aligned} V &= \int_0^1 2\pi y\ [g_2(y) - g_1(y)]dy = \int_0^1 2\pi y\ [\sqrt{y} - y]dy \\ &= 2\pi \int_0^1 [y^{3/2} - y^2]dy = 2\pi\Big(\frac{2y^{5/2}}{5} - \frac{y^3}{3}\Big)\Big]_0^1 = \frac{2}{15}\pi. \end{aligned}$$