NA 334 and 354, Midterm 1, FALL 2003

Solutions and Discussions

1. Problem 1: Let $f(x) = \frac{1}{1+x}$ and $x_0 = 0$.

a) Approximate $f(1/4)$ by $T_2(1/4)$, where T_2 is the second Taylor polynomial for f about x_0. Estimate the error.

b) Use four-digit rounding arithmetic to calculate $T_2(1/4)$ and the actual value $f(1/4)$. Compute the absolute error and the relative error of the approximation.

Solution. a)

$$T_n(x) \;=\; \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k$$

and

$$f(x) \;=\; T_n(x) \;+\; \frac{f^{(n+1)}(\xi_x)}{(n+1)!}(x-x_0)^{n+1}\,.$$

So, $n = 2$ in our case and we have to calculate $f^{'}$, $f^{''}$, and $f^{'''}$:

$f(x) = (1{+}x)^{-1},\; f^{'}(x) = -(1{+}x)^{-2},\; f^{''}(x) = 2(1{+}x)^{-3},\; f^{'''}(x) = -6(1{+}x)^{-4}\,.$

From here: ¿

$$f(0) = 1,\; f^{'}(0) = -1,\; f^{''}(0) = 2,\; f^{'''}(x) = -6(1+x)^{-4}$$

and

$$T_2(x) \;=\; x^2 \;-\; x \;+ 1$$

and

$$f(x) \;=\; T_2(x) \;+\; \frac{-6\,(1+\xi_x)^{-4}}{3!}\,(x-0)^3$$

and in view of this:

$$T_2(0.25) \;=\; 0.8125, \quad f(0.25) \;=\; 0.8,$$

$$|f(0.25) \;-\; T_2(0.25)| \;\leq\; (0.25)^3 = 0.015625 \approx\; 10^{-1} \;\times\; 0.15625$$

b) Absolute Error:

$$|f(1/4) \;-\; T_2(1/4)| = |0.8 \;-\; 0.8125| = 0.0125 \;=\; 10^{-1} \;\times\; 0.125$$

Relative Error:

$$\frac{|f(1/4) \;-\; T_2(1/4)|}{|f(1/4)|} \;=\; \frac{|0.8 \;-\; 0.8125|}{|0.8|} \;=\; 0.0156252 \approx 10^{-1} \;\times\; 0.1563$$

by using four-digit rounding arithmetic.

Problem 2.

a) Let $p = \sqrt[3]{9} \in [2, 2.5]$ (please check). Compare the following three fixed-point iterative methods with $p_0 = 2$ to approximate p:

$$\text{a)} \qquad p_n = \frac{1}{2}\left(p_{n-1} + \frac{9}{p_{n-1}^2}\right);$$

$$\text{b)} \qquad p_n = \frac{2p_{n-1}^3 + 9}{3p_{n-1}^2};$$

$$\text{c)} \qquad p_n = 2 + \frac{1}{p_{n-1}^2 + 2\,p_{n-1} + 4}.$$

Order the three methods according to their fastness. What is the order of convergence of each of the methods?

b) By using the fastest method compute (find an approximation of) p with accuracy 10^{-4}.

Solution. a) Obviously, $\sqrt[3]{9} = 2.080083 \in [2, 2.5]$. First we denote by g_1, g_2, and g_3 the iterative functions of the methods a), b), and c), respectively. Then,

$$g_1(x) = \frac{x}{2} + 4.5\,x^{-2}, \; g_2(x) = \frac{2\,x^3 + 9}{3\,x^2} =, \; g_3(x) = 2 + \frac{1}{x^2 + 2\,x + 4}$$

and

$$f(x) = x^3 - 9, \; f(x) = 0.$$

Second, we calculate the first derivatives of g_1, g_2, and g_3 at p by using that $p^3 = 9$ and $p \in [2, 2.5]$. We obtain

$$g_1^{'}(p) = \frac{1}{2} - 9\,p^{-3} = 0.5 - 1 = -0.5 \neq 0$$

so, the first method possesses a linear order of convergence. For the second method, given by b) we can compute directly

$$g_2^{'}(p) = \frac{2\,p^4 - 18\,p}{3\,p^4} = \frac{(2p) \times (9) - 18\,p}{3p^4} = 0$$

and from here it follows that **the iterative method** b) has **at least quadratic order of convergence**. For this method we can proceed also in the following way: First we observe that the method b) is in fact Newton's method:

$$g_2(x) = x - \frac{f(x)}{f'(x)} = x - \frac{x^3 - 9}{3\,x^2} = \frac{2\,x^3 + 9}{3\,x^2}$$

so, **to conclude in this way that b) has at least quadratic order of convergence we must check**

$$f^{'}(p) = 3\,p^2 \neq 0.$$

(Note that if $f^{'}(p) = 0$, then Newton's method has no quadratic order of convergence and that is because we have considered in class its modifications.)

Now, to establish the exact order of convergence of b) we have to check the second derivatives of g_2. We have

$$g_2^{''}(p) = \left(\frac{f(p)f^{''}(p)}{[f^{'}(p)]^2} \right)^{'}$$

$$= \frac{(f^{'}(p)f^{''}(p) + f(p)f^{'''}(p))[f^{'}(p)]^2 - f(p)f^{''}(p)(2f^{'}(p)f^{''}(p))}{[f^{'}(p)]^4}$$

$$= \frac{f^{''}(p)}{f^{'}(p)} = \frac{6\,p}{3\,p^2} = \frac{2}{p} \neq 0$$

so, the order of convergence of b) is exactly 2 (a quadratic order of convergence).

Now, we consider the method c). We have

$$g_3^{'}(x) = -\frac{2(x+1)}{(x^2+2x+4)^2}$$

which is obviously $\neq 0$ for $x = p$ so, **the iterative method c) has linear order of convergence.**

Now, we can conclude immediately that b) is the fastest one. To compare a) and c) we can proceed in two ways:

One of them is to calculate approximately (by using the approximate value of $\sqrt[3]{9}$ which practically is not always possible:

$$|g_3^{'}(p)| = \left| -\frac{2(p+1)}{(p^2+2p+4)^2} \right| = |-2(p+1)(p-2)^2| < 0.03951 < 0.5$$

and from here c) is faster than a).

On the other hand practically we can estimate

$$|g_3^{'}(x)| = \left| \frac{2(x+1)}{(x^2+2x+4)^2} \right| < \frac{2(2.5+1)}{(12)^2} < 0.04862 < 0.5$$

and from here also follows that c) is faster than a). On the other hand, we saw that a) and c) have linear order of convergence and b) has a quadratic order of convergence.

However, the solution is not complete. We have to proof that the three methods a), b), and c) are convergent in $[2, 2.5]$ with an initial approximation $p_0 = 2$

We consider again the first method a) with

$$g_1(x) = \frac{x}{2} + 4.5\,x^{-2},\; g_1^{'}(x) = \frac{1}{2} - 9\,x^{-3},\; g_1^{''}(x) = 27\,x^{-4} > 0\; x \in [2, 2.5]\,.$$

and $g_1^{'}(2) = -0.625,\ g_1^{'}(2.5) = -0.076$ and $g_1^{'}$ is increasing so,

$$\max_{x \in [2,2.5]} |g_1^{'}(x)| = 0.625 \, < \, 1$$

and from here the function g_1 is contractive in $[2, 2.5]$ with coefficient of constructiveness $k_1 \,=\, 0.625$. What is left is to prove that all other approximations p_n where $p_n = g_1(p_{n-1})$ will belong to $[2, 2.5]$. It can be done in many ways by using the materials that we have considered in class and you have copies of these materials. Take the midpoint of $[2, 2.5]$ which is $(a = 2,\ b = 2.5)$ and $(a+b)/2 = 2.25$. Now it is obvious that $a = p_0 = 2 \,<\, p = \sqrt[3]{9} \,<\, (a+b)/2 \,=\, 2.25$. Then, g_1 being contractive we have that all next approximations p_n, $n \,\geq\, 1$ will belong to the interval $[2, 2.5]$. Another method is by using the fact that $g_1^{'}(x) \,<\, 0$ in $[2, 2.5]$ starting from $p_0 < p = \sqrt[3]{9}$ we are sure that the next approximation p_1 will be grater than p. So, it is sufficient to check that p_1 belongs to $[2, 2.5]$. From the theory we expect $p_1 > p$ and this is the case:

$$p_1 \,=\, g(p_0) \,=\, 2.125 \,>\, p \,=\, \sqrt[3]{9}.$$

Let us calculate few more approximations by using a):

$$p_0 \,=\, 2 \,<\, p_2 \,=\, g_1(p_1) \,=\, 2.05903979 \,<\, p \,=\, \sqrt[3]{9} \,=\, 2.080083823\,.$$

$$p_1 \,>\, p_3 = g_1(p_2) \,=\, 2.090929554 > p \,=\, \sqrt[3]{9} \,=\, 2.080083823$$

$$p_4 = 2.074745198,\ p_5 \,=\, 2.082773759,\ p_6 \,=\, 2.078744064,\ p_7 \,=\, 2.080754998,$$

$$p_8 \,=\, 2.07974856\,.$$

and this confirms our theoretical conclusions (please, see the materials that you have):

$$p_0 \,<\, p_2 \,<\, p_4 \,<\, p_6 \,<\, \cdots \,<\, p \,=\, \sqrt[3]{9}$$

and

$$p_1 \,>\, p_3 \,>\, p_5 \,>\, p_7 \,>\, \cdots \,>\, p \,=\, \sqrt[3]{9}$$

and in **this particular case the following stopping rule can be very useful:**

$$|p_n \,-\, p| \,<\, |p_n \,-\, p_{n-1}| \qquad (1)$$

that **can replace very efficiently the standard general estimates** given by the coefficient of constructiveness $k_1 \,=\, 0.625$ from which I would like to mention **the following general stopping rule** for fixed-point iterative methods that can be found in your materials:

$$|p_n \,-\, p| \,<\, \frac{k_1}{1 - k_1}\, |p_n \,-\, p_{n-1}| \quad (k_1 = 0.625) \quad (2).$$

Applying (1) we obtain

$$|p_8 \,-\, p| \,<\, |p_8 \,-\, p_7| \,=\, |2.07974856 \,-\, 2.080754998| \,=\, \mathbf{0}.001006437$$

but applying (2)

$$|p_8 - p| < \frac{k_1}{1-k_1}|p_8 - p_7| = \frac{0.625}{1-0.625}|2.07974856 - 2.080754998|$$

$$= (1.6666) \times 0.001006437 = \mathbf{0}.001677395.$$

Now we consider **the Newton's method b) which is the fastest one between a), b) and c). Here we perform the solution of b) from Problem 2.** First we prove that the method is convergent with $p_0 = 2$. This can be done at least by two ways. The first one is to consider Newton's method as a particular (special) case of fixed-point method with

$$g_2(x) = \frac{2}{3}x + \frac{9}{3}x^{-2},\; g_2^{'}(x) = \frac{2}{3}\left(1 - \frac{9}{x^3}\right),\; g_2^{''}(x) = 18\,x^{-4} > 0 \quad x \in [2, 2.5]\,.$$

Hence, $g_2^{'}(x)$ is increasing in $[2, 2.5], g_2^{'}(p) = 0,\; g_2^{'}(2) = -0.083, g_2^{'}(2.5) = 0.282666$, and from here

$$k_2 = \max_{x \in [2,2.5]} |g_2^{'}(x)| < 0.2827 < 1$$

to conclude that g_2 is contractive in $[2, 2.5]$.

Note that here $g_2^{'}(x)$ is negative in $[2, p)$ and $g_2^{'}(x)$ is positive in $(p, 2.5]$ so, we can not make the same conclusions for monotone behavior of the consecutive approximations $p_n = g_2(p_{n-1}$. However, the geometric interpretation of Newton's method by using the fact that $f(x) = x^3 - 9$ is convex on $[2, 2.5]$ gives very clear picture about the monotone behavior of the approximations p_n:

$$p_0 < p, \quad p < p_1 = g_2(p_0) = 2.0833333 < b = 2.5$$

and all other approximations must satisfy:

$$p_1 > p_2 > p_3 > p_4 > \cdots > p = \sqrt[3]{9}$$

so, they all are in $[2, 2.5]$. **So, the method is convergent.**

Another way to see that all p_n are inside $[2, 2.5]$ is to observe (as it was done with the method a)) that p belongs to the interval $[p_0, (a + b)/2]$ where $a = 2$ and $b = 2.5$ are the end point of the interval $[2, 2.5]$ under consideration. From here and by the fact that g_2 is contractive follows that all other approximations will belong to the interval $[2, 2.5]$. **So, the method b) is convergent.**

Now we shall use the method b) to calculate $p = \sqrt[3]{9}$ with accuracy (error) 10^{-4}. This can be done in many ways. We shall consider three of them:

1. First following the following well known estimates for a fixed-point iterative methods:

$$|p_n - p| < \frac{k_2^n}{1-k_2}|p_1 - p_0| \quad (3)$$

or

$$|p_n - p| < k_2^n \max(|b - p_0|, |p_0 - a|) \quad (4)$$

where $k_2 = 0.2827$, $p = \sqrt[3]{9}$, $a = 2$ and $b = 2.5$. Let us apply (3). We must have

$$\frac{k_2^n}{1-k_2}|p_1 - p_0| = \frac{(0.2827)^n}{1-0.2827}|p_1 - p_0| = \frac{(0.2827)^n}{1-0.2827}|2.0833333 - 2|$$

$$= 0.116176 \times (0.2827)^n < 10^{-4}.$$

For $n = 6$ we have

$$0.116176 \times (0.2827)^6 < 0.0000594 < 10^{-4}$$

and for $n = 5$

$$0.116176 \times (0.2827)^5 = 0.000209 > 10^{-4}$$

to conclude that $p_6$will give the desired accuracy.

2. By using the stopping rule for fixed-point iterative methods (as it was done in method a) estimating the error of p_8)

$$|p_n - p| < \frac{k_2}{1-k_2}|p_n - p_{n-1}| \quad (k_2 = 0.2827) \quad (2)$$

or

$$|p_n - p| < 0.3941\,|p_n - p_{n-1}|.$$

We calculate:

$$p_0 = 2,\ p_1 = 2.0833333,\ p_2 = 2.080088911,\ p_3 = 2.080083823.$$

So, we have

$$|p_2 - p| < 0.3941\,|p_2 - p_1| = 0.3941\,|2.080088911 - 2.0833333| = 0.00127 > 10^{-4}.$$

$$|p_3 - p| < 0.3941\,|p_3 - p_2| = 0.3941\,|2.080083823 - 2.080088911|$$
$$= 0.000002005 < 10^{-4}$$

and from here p_3 approximates p with error 10^{-4}. **Note that here we obtain that we need only 3 approximations not 6 as it was in 1.**

3. Here we shall use **the best stopping rule (error estimate) concerning Newton's method**:

$$|p_n - p| < \frac{M_2}{2\,m_1}|p_n - p_{n-1}|^2 \quad (4)$$

where M_2* is an upper bound for the maximum of $|f''(x)|$ on the interval $[a, b]$ ($a = 2,\ b = 2.5$, and $f(x) = x^3 - 9$ in our case) and m_1 is a lower bound for the minimum of $|f'(x)|$ on the interval $[2, 2.5]$. In other words for M_2 can serve any number such that such that $|f^{''}(x)| \leq M_2$ on $[2, 2.5]$ and for m_1 any number such that $|f^{'}(x)| \geq m_1$ on the interval $[2, 2.5]$. We have

$$f(x) = x^3 - 9,\ f^{'}(x) = 3\,x^2 \geq 12,\ f^{''}(x) = 6x \leq 15$$

so, (4) takes the form

$$|p_n - p| < \frac{15}{2 \times 12}\,|p_n - p_{n-1}|^2 = 0.625\,|p_n - p_{n-1}|^2\,. \quad (5)$$

Now, by using (5) we calculate ($p_0 = 2,\ p_1 = 2.0833333,\ p_2 = 2.080088911,\ p_3 = 2.080083823$):

$$|p_2 - p| < 0.625\,|p_2 - p_1|^2$$

$$= 0.625\,|2.080088911 - 2.0833333|^2 = 0.000006578 < 10^{-4}$$

so, we obtain that p_2 give also the desired approximation of $\sqrt[3]{9} = 2.080083823$ with error 10^{-4}. **Note that we got by the method 2. that p_3 will give the desired approximation but not p_2. This shows also that the way 3. is optimal concerning the Newton's method because it is using the fact that Newton has a quadratic order of convergence. The ways 1. and 2. have used only that Newton's method is a special case, fixed-point method that in particular has at least linear order of convergence.**

c) The method c) can be studied the same way as we studied method a). Here,

$$g_3(x) = 2 + \frac{1}{x^2 + 2x + 4},\ g_3^{'}(x) = -\frac{2(x+1)}{(x^2 + 2x + 4)^2},\ k_3 = 0{,}048611 < 1$$

and because the first derivative of $g_3^{'}(x) < 0$ is negative on $[2, 2.5]$ we have the same monotone behavior as it was in a):

$$p_0 < p_2 < p_4 < p_6 < \cdots < p = \sqrt[3]{9}$$

and

$$p_1 > p_3 > p_5 > p_7 > \cdots > p = \sqrt[3]{9}$$

with $p_0 = 2$ and $p_1 = g_3(p_0) = 2.08333333 \in [2, 2.5]$ and for the same reasons all other approximations will belong to $[2, 2.5]$. **We obtained in the first part of our discussions that this method, being convergent, is faster than a), but slower than b).**

Problem 4. At year 1225, Leonardo de Pisa found that the number $\tilde{p} = 1.3688081$ is a root of the equation $f(x) = 0$, where $f(x) = x^3 + 2x^2 + 10x - 20$. The method to obtain the above approximation of the root is not known.

a) Prove that $f(x) = 0$ has only one real root.

b) Obtain the above approximation of the unique real root of $f(x) = 0$ by using a fixed point iterative scheme (NOT Newton's method and not secant method).

Solution. **a)** We calculate $f(0) = -20 < 0$ and $f(1.5) = 2.875 > 0$ to conclude by the I.V.T. that $f(x)$ at least one real root. On the other hand, $f^{'}(x) = 3x^2 + 4x + 10 > 0$ and assuming that f has at least two real roots we arrive to a contradiction by using Rolle's Theorem. We may use also the fact that f is increasing function on the x-axis, the first derivative of $f(x)$ being positive on the x-axis.

Solution 1 of b) We can obtain the above approximation by using the following fixed-point method:

$$x = g_1(x), \qquad g_1(x) = \frac{20}{x^2 + 2x + 10}$$

on the interval $[1, 1.5]$. We calculate

$$|g_1^{'}(x)| = \left| -40 \frac{x+1}{(x^2 + 2x + 10)^2} \right| < \frac{40 \times 2.5}{13^2} = \mathbf{0.59172} < 1$$

so, $\mathbf{k_1} = \mathbf{0.59172}$. On the other hand the first derivative of g_1 is **negative** and if we started with $p_0 = 1.5$ we shall have

$$p_0 > p_2 > p_4 > p_6 > \cdots > p$$

and

$$p_1 < p_3 < p_5 < p_7 < \cdots < p\,,$$

where p is the the only real root of $f(x) = 0$. Also, because the midpoint of the interval $[1, 1.5]$ namely, 1.25 satisfies $p > 1.25$ the we are sure that all other approximations will belong to the interval $[1, 1.5]$. Let us calculate few approximations:

$$p_0 = 1.5,\ p_1 = 1.31147541,\ p_2 = 1.394416339,\ p_3 = 1.357475621\,.$$

Solution 2 of b). Now we shall obtain faster than the above method by introducing a real parameter α (as it was done in class when studding iterative methods for $x^2 = 2$).
The nonlinear equation of Leonardo is equivalent to the equation:

$$x(x^2 + 2x + 10 + \alpha) = 20 + \alpha x$$

so,

$$g_\alpha(x) = \frac{20 + \alpha x}{x^2 + 2x + 10 + \alpha}$$

and

$$g_\alpha^{'}(x) = \frac{-\alpha x^2 + 10\alpha + \alpha^2 - 40(x+1)}{(x^2 + 2x + 10 + \alpha)^2}$$

Denoting by

$$r_\alpha(x) = -\alpha x^2 + 10\alpha + \alpha^2 - 40(x+1)$$

the nominator of $g_\alpha^{'}(x)$ we obtain

$$r_\alpha^{'}(x) = -2\alpha x - 40 < 0$$

for $x \in [1, 1.5]$ and $\alpha > 0$ so, $r(x)$ is decreasing. Now

$$r_\alpha(1) = 9\alpha + \alpha^2 - 80, \;\; r(1.5) = 7.75\alpha + \alpha^2 - 90 .$$

Take now $\alpha = 5$ to obtain

$$r_5(1) = 45 + 25 - 80 = -10, \;\; r(1.5) = 38.75 + 25 - 90 = -26.25$$

and by using the fact that r_5 is decreasing in $[1, 1.5]$ we obtain

$$\max_{x \in [1,1.5]} |g_5^{'}(x)| \leq \frac{26.25}{18^2} = \mathbf{0.08101}$$

and from here

$$\mathbf{k_{g_5}} = \mathbf{0.08101} .$$

We can make even better. Solving the quadratic equation

$$9\alpha + \alpha^2 - 80 = 0$$

we obtain $\alpha_0 = 5.5124$ and $r_{\alpha_0}(5.5124) = -16.89163$ so,

$$\max_{x \in [1,1.5]} |g_{\alpha_0}^{'}(x)| \leq \frac{16.89163}{(13 + 5.5124)^2} = \mathbf{0.04929}$$

$$\mathbf{k_{g_{\alpha_0}}} = \mathbf{0.04929} .$$

so we shall calculate by using g_5 and g_{α_0}, where $\alpha_0 = 5.5124$.

Calculations by using

$$g_5(x) = \frac{20 + 5x}{x^2 + 2x + 15}$$

$$p_0 = 1.5,\ p_1 = 1.06217,\ p_2 = 1.386721009,\ p_3 = 1.367435383,$$
$$p_4 = 1.368911948,\ p_5 = 1.368800245,\ p_6 = 1.368808703,\ p_7 = 1.368808063,$$
$$\mathbf{p_8 = 1.368808111},\ p_9 = 1.368808108,\ p_{10} = 1.368808108\,.$$

Calculations by using

$$g_5(x) = \frac{20 + 5.5124\,x}{x^2 + 2\,x + 15.5124}$$

$$p_0 = 1.5,\ p_1 = 1.361528532,\ p_2 = 1.36915689,\ p_3 = 1.368791246,$$
$$p_4 = 1.368808923,\ p_5 = 1.368808068,\ \mathbf{p_6 = 1.36880811},\ p_7 = 1.368808108.$$

Problem 3. By the Newton's iterative method b):

$$p_0 = 2,\ p_1 = 2.0833333,\ p_2 = 2.080088911,\ p_3 = 2.080083823$$

so,

$$\Delta\, p_0 = p_1 - p_0 = 0.0833333,\ [\Delta\, p_0]^2 = 0.006944438$$
$$\Delta^2\, p_0 = p_2 - 2\,p_1 + p_0 = -0.08657769$$

and

$$\hat{p}_0 = p_0 - \frac{[\Delta\, p_0]^2}{\Delta^2\, p_0} = \mathbf{2.080210489}\,.$$

Then

$$p_1^{(1)} = g_2(p_0^{(1)}) = g_2(\hat{p}_0) = \mathbf{2.080083831}\ \ (\sqrt[3]{9} = \mathbf{2.080083823})\,.$$