McGill University

Midterm Test

Math 581: Partial Differential Equations II

Instructor: *Dr. Ming Mei*

Name: ______________________________

ID #: ______________________________

REMARKS:

- Lecture notes are allowed.
- Read every question carefully and show your work for all questions.

Question	Mark	
1		20
2		20
3		30
4		30
TOTAL		100

1. Prove that, if $u \in W^{1,p}(\Omega)$, then u is absolutely continuous *a.e.* in Ω, where $1 < p < \infty$ and Ω is a bounded set in $\mathbb{R}^1$. MARKS (20)

2. Prove that, if $u \in W^{1,\infty}(\Omega)$, then u is Lipschitz continuous, where Ω is an open and bounded set in $\mathbb{R}^n$. MARKS (20)

3. Consider the Dirichlet boundary problem

$$\begin{cases} Lu = f, \\ u|_{\partial\Omega} = 0, \end{cases}$$

where L is the elliptic operator $Lu = -u_{xx} - u_{yy} + e^{-(x^2+y^2)}(u_x + u_y) + \gamma u,\ \ \gamma > \frac{1}{4}$, and $f \in L^2(\Omega)$, $\Omega \subset \mathbb{R}^2$, and denote the bilinear form as $B[u, v] = (Lu, v)$, $\quad (\cdot, \cdot)$ is the inner product of $L^2(\Omega)$.

(a) Prove that, there exist constants $\alpha = \alpha(\gamma) > 0$ and $\beta = \beta(\gamma) > 0$ such that, for all $u, v \in H_0^1(\Omega)$,

$$|B[u, v]| \leq \alpha \|u\|_{H_0^1(\Omega)} \|v\|_{H_0^1(\Omega)},$$
$$B[u, u] \geq \beta \|u\|^2_{H_0^1(\Omega)}.$$

(b) Use Lax-Milgram theorem to prove that, there exists a unique weak solution $u \in H_0^1(\Omega)$ to the Dirichlet boundary problem.

<u>MARKS</u> (30)

4. Let $L^2_{per}(\mathbb{R}^n)$ be the space with periodic functions in form of

$$L^2_{per}(\mathbb{R}^n) = \{u|\ u(x) = u(x+p),\ u \in L^2(0,p)\}$$

with $x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n$, and $p = (p_1, p_2, \cdots, p_n)$ ($p_i > 0$, $i = 1, 2, \cdots, n$) is the period, and let its norm be

$$\|u\|_{L^2_{per}} = \Big(\int_0^p |u(x)|^2 dx\Big)^{1/2} = \Big(\int_0^{p_1} \cdots \int_0^{p_n} |u(x_1, \cdots, x_n)|^2 dx_1 \cdots dx_n\Big)^{1/2}.$$

Periodic Sobolev space $H^k_{per}(\mathbb{R}^n)$ can be similarly defined. Consider the eigenvalue problem to Laplace's equation with periodic boundary

$$\begin{cases} -\Delta u = \lambda u, \\ u(x) = u(x+p), \quad\quad x \in \mathbb{R}^n. \\ \int_0^p u(x)dx = 0, \end{cases}$$

Prove that

(a) The eigenvalues are real and satisfy

$$0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_k \le \cdots .$$

(b) Let $w_k \in H^1_{per}(\mathbb{R}^n)$ be the corresponding normalized eigenfunctions with the period p. Then $\{w_k\}_{k=1}^{\infty}$ is the orthonormal basis of $L^2_{per}(\mathbb{R}^n)$.

<u>MARKS</u> (30)