# Math 581: Partial Differential Equations II
# **Assignment** 5

Due: April 02, 2010, Friday

1. Prove there is at most one smooth solution of this initial-boundary value solution for the heat equation with Neumann boundary condition:

$$\begin{cases} u_t - \Delta u = f & \text{in } U_T \\ \frac{\partial u}{\partial \nu}|_{\partial U} = 0 \\ u|_{t=0} = g. \end{cases}$$

2. Assume $u$ is a smooth solution of

$$\begin{cases} u_t - \Delta u = 0 & \text{in } U_T \\ u|_{\partial U} = 0 \\ u|_{t=0} = g. \end{cases}$$

   Prove the exponential decay estimate:

$$\|u(\cdot,t)\|_{L^2(U)} \le e^{-\lambda_1 t}\|g\|_{L^2(U)}, \quad t \ge 0,$$

   where $\lambda_1 > 0$ is the principal eigenvalue of $-\Delta$ (with zero boundary conditions) on $U$.

3. Use Galerkin's method to prove that, suppose $f \in L^2(U)$ and assume that $u_m = \sum_{k=1}^m d_m^k w_k$ solves

$$\int_U Du_m \cdot Dw_k dx = \int_U f \cdot w_k dx$$

   for $k = 1, \cdots, m$. Show that a subsequence of $\{u_m\}_{m=1}^\infty$ converges weakly in $H_0^1(U)$ to the weak solution $u$ of

$$\begin{cases} -\Delta u = f & \text{in } U \\ u|_{\partial U} = 0. \end{cases}$$

4. Suppose that $u$ is a smooth solution of

$$\begin{cases} u_t - \Delta u + cu = 0 \\ u|_{\partial U} = 0 \\ u|_{t=0} = g \end{cases}$$

   and the function $c$ satisfies

$$c \ge \gamma > 0.$$

   Prove

$$|u(x,t)| \le Ce^{-\gamma t}, \quad (x,t) \in U_T.$$