Math 581: Partial Differential Equations II
Assignment 3

Due: February 19, 2010, Friday

1. Let
$$Lu = -\sum_{i,j=1}^{n} (a^{ij}u_{x_i})_{x_j} + cu.$$
Prove that there exists a constant $\mu > 0$ such that the corresponding bilinear form $B[\ ,\ ]$ satisfies the hypotheses of the Lax-Milgram Theorem, provided
$$c(x) \geq -\mu, \quad \text{for } x \in U.$$

2. Let $u \in H_0^2(U)$ be the weak solution of the boundary-value problem for the *biharmonic equation*
$$\begin{cases} \Delta^2 u = f & \text{in } U \\ u = \frac{\partial u}{\partial v} = 0 & \text{on } \partial U \end{cases} \tag{1}$$
provided
$$\int_U \Delta u \Delta v dx = \int_U f v dx$$
for all $v \in H_0^2(U)$. Given $f \in L^2(U)$, prove that there exists a unique weak solution of (1).

3. Assume U is connected. A function $u \in H^1(U)$ is a weak solution of Neumann's problem
$$\begin{cases} -\Delta u = f & \text{in } U \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial U \end{cases} \tag{2}$$
if
$$\int_U Du \cdot Dv dx = \int_U f v dx$$
for all $v \in H^1(U)$. Suppose $f \in L^2(U)$. Prove (2) has a weak solution if and only if
$$\int_U f dx = 0.$$

4. Let u be a smooth solution of
$$Lu = -\sum_{i,j=1}^{n} a^{ij} u_{x_i x_j} = 0 \text{ in } U.$$
Set $v := |Du|^2 + \lambda u^2$. Show that
$$Lv \leq 0 \text{ in } U, \text{ if } \lambda \text{ is large enough.}$$
Deduce
$$\|Du\|_{L^\infty(U)} \leq C(\|Du\|_{L^\infty(\partial U)} + \|u\|_{L^\infty(\partial U)}).$$