Math 581: Partial Differential Equations II
Assignment 2

Due: February 5, 2010, Friday

1. Let $u \in H^1(\mathbb{R})$. Then
$$\|u\|_{L^\infty(\mathbb{R})} \leq \sqrt{2}\|u\|_{L^2(\mathbb{R})}^{1/2}\|u_x\|_{L^2(\mathbb{R})}^{1/2}.$$

2. Let
$$w(x) = \begin{cases} 1, & x \geq 0 \\ \sqrt{1+x^2}, & x < 0 \end{cases}$$
be a weighted function, $H^1_w(\mathbb{R})$ be a weighted Sobolev space equipped with a norm
$$\|u\|_{H^1_w(\mathbb{R})} = \Big(\int_{\mathbb{R}} w(x)[|u(x)|^2 + |u_x(x)|^2]dx\Big)^{\frac{1}{2}},$$
and $C^0_w(\mathbb{R})$ be a weighted space of continuous functions with a norm
$$\|u\|_{C^0_w(\mathbb{R})} = \sup_{x\in R} w(x)|u(x)|.$$
Prove
$$H^1_w(\mathbb{R}) \hookrightarrow C^0_w(\mathbb{R}) \hookrightarrow C^0(\mathbb{R})$$
with
$$\|u\|_{C^0(\mathbb{R})} \leq \|u\|_{C^0_w(\mathbb{R})} \leq C\|u\|_{H^1_w(\mathbb{R})}.$$

3. Use Fourier transform to prove that
$$H^s(\mathbb{R}^n) \hookrightarrow C^1(\mathbb{R}^n), \quad s > \frac{n}{2} + 1$$
with
$$\|u\|_{C^1(\mathbb{R}^n)} \leq C\|u\|_{H^s(\mathbb{R}^n)}.$$