Math 581: Partial Differential Equations II
Assignment 1

Due: January 22, 2010, Friday

1. Assume $0 < \beta < \gamma \leq 1$. Prove the interpolation inequality

$$\|u\|_{C^{0,\gamma}(U)} \leq \|u\|_{C^{0,\beta}(U)}^{\frac{1-\gamma}{1-\beta}} \|u\|_{C^{0,1}(U)}^{\frac{\gamma-\beta}{1-\beta}}.$$

2. Denote by U the open square $\{x \in R^2 | |x_1| < 1, |x_2| < 1\}$. Define

$$u(x) = \begin{cases} 1 - x_1, & \text{if } x_1 > 0, |x_2| < x_1 \\ 1 + x_1, & \text{if } x_1 < 0, |x_2| < -x_1 \\ 1 - x_2, & \text{if } x_2 > 0, |x_1| < x_2 \\ 1 + x_2, & \text{if } x_2 < 0, |x_1| < -x_2. \end{cases}$$

For which $1 \leq p \leq \infty$, does u belongs to $W^{1,p}(U)$?

3. Integrate by parts to prove

$$\int_U |Du|^2 dx \leq C\Big(\int_U u^2 dx\Big)^{1/2}\Big(\int_U |D^2u|^2 dx\Big)^{1/2}$$

for all $u \in C_0^\infty(U)$.

Furthermore, if ∂U is smooth, then prove the above inequality for $u \in H^2(U) \cap H_0^1(U)$.

(Hint: Take $\{v_k\}_{k=1}^\infty \in C_0^\infty(U)$ converging to u in $H_0^1(U)$, and $\{w_k\}_{k=1}^\infty \in C^\infty(U)$ converging to u in $H^2(U)$.)

4. Assume $1 \leq p \leq \infty$, and U is bounded, and

$$u^+ = \max\{u, 0\}, \quad u^- = \min\{u, 0\}.$$

(a) Prove $u \in W^{1,p}(U)$ implies u^+, $u^- \in W^{1,p}(U)$, and

$$Du^+ = \begin{cases} Du, & \text{a.e. on } \{u > 0\} \\ 0, & \text{a.e. on } \{u \leq 0\}, \end{cases}$$

$$Du^- = \begin{cases} 0, & \text{a.e. on } \{u \geq 0\} \\ Du, & \text{a.e. on } \{u < 0\}, \end{cases}$$

$$D|u| = \begin{cases} Du, & \text{a.e. on } \{u > 0\} \\ 0, & \text{a.e. on } \{u = 0\} \\ -Du, & \text{a.e. on } \{u < 0\}. \end{cases}$$

(b) Prove that if $u \in W^{1,p}(U)$, then $|u| \in W^{1,p}(U)$.

(c) Prove that if $u \in W^{1,p}(U)$, then

$$Du = 0 \text{ a.e. on the set } \{u = 0\}.$$

(Hint: $u^+ = \lim_{\varepsilon \to 0} F_\varepsilon(u)$, for

$$F_\varepsilon(z) = \begin{cases} (z^2 + \varepsilon^2)^{1/2} - \varepsilon, & \text{if } z \geq 0 \\ 0, & \text{if } z < 0.) \end{cases}$$