McGill University
Department of Mathematics and Statistics

Final Exam

Math 581: Partial Differential Equations II **Winter Semester of 2010**

Instructor: *Dr. Ming Mei*

Name: ______________________________

ID #: ______________________________

REMARKS:

- Read every question carefully and show all your work.
- No cell phones are allowed during the exam.

Question	**Mark**	
1		25
2		25
3		25
4		25
TOTAL		100

1. Prove that, if $u \in H^2(\Omega) \cap H^1_0(\Omega)$, where Ω is an open and bounded set in $\mathbb{R}^n$ and $\partial\Omega$ is smooth, then
$$\|Du\|_{L^2(\Omega)} \leq C\|u\|^{1/2}_{L^2(\Omega)}\|D^2u\|^{1/2}_{L^2(\Omega)}.$$
MARKS (25)

2. Let u be the smooth solution to the initial periodic boundary value problem

$$\begin{cases} u_t - u_{xx} = 0, & x \in \mathbb{R},\ t \in \mathbb{R}_+, \\ u(x,t) = u(x+p,t), & x \in \mathbb{R},\ t \in \mathbb{R}_+, \\ \int_0^p u(x,t)dx = 0, & t \in \mathbb{R}_+, \\ u|_{t=0} = u_0(x), & x \in \mathbb{R}, \end{cases}$$

where $p > 0$ is the period, and $u_0(x)$ satisfies the compatibility condition

$$u_0(x) = u_0(x+p) \text{ for } x \in \mathbb{R}, \text{ and } \int_0^p u_0(x)dx = 0.$$

Prove

$$\|u(t)\|_{L^2_{per}(\mathbb{R})} \le e^{-\lambda_1 t}\|u_0\|_{L^2_{per}(\mathbb{R})},$$

where $\lambda_1 > 0$ is the principal eigenvalue given by

$$\begin{cases} -u_{xx} = \lambda u, \\ u(x) = u(x+p), \\ \int_0^p u(x)dx = 0, \end{cases}$$

and the periodic L^2-norm is defined as $\|f\|_{L^2_{per}(\mathbb{R})} = (\int_0^p |f|^2 dx)^{1/2}$. MARKS (25)

3. Let u be the solution of the IBVP to the nonlinear diffusion equation

$$\begin{cases} u_t - \Delta u = u^{2p}, & (x,t) \in \Omega \times \mathbb{R}_+ \\ \frac{\partial u}{\partial \nu}|_{\partial\Omega} = 0, & (x,t) \in \partial\Omega \times \mathbb{R}_+ \\ u|_{t=0} = u_0(x) > 0, & x \in \Omega \end{cases}$$

where $p \geq 1$ is an integer, and Ω is an open and bounded set in $\mathbb{R}^n$. Prove that u will blow up:

$$\lim_{t \to T_0^-} \|u(t)\|_{L^1(\Omega)} = \infty \quad \text{with} \quad T_0 \leq \frac{|\Omega|^{2p-1}}{(2p-1)\|u_0\|_{L^1(\Omega)}^{2p-1}}.$$

<u>MARKS</u> (25)

4. Assume u is a smooth solution of the IBVP to the damped wave equation

$$\begin{cases} u_{tt} + u_t - \Delta u = 0, & \text{in } \Omega \times \mathbb{R}_+ \\ u|_{\partial\Omega} = 0, & \text{on } \partial\Omega \times \mathbb{R}_+ \\ u|_{t=0} = u_0(x), & \text{in } \Omega \\ u_t|_{t=0} = u_1(x), & \text{in } \Omega, \end{cases}$$

where Ω is an open and bounded set in $\mathbb{R}^n$. Use Galerkin's method to prove the exponential estimate

$$\|u(t)\|_{L^2(\Omega)} \le e^{-\mu t}(\|u_0\|_{L^2(\Omega)} + \|u_1\|_{L^2(\Omega)}) \quad \text{for } 0 < \mu < \frac{1}{2}.$$

MARKS (25)