McGill University
Math 262: Intermediate Calculus
Solutions to Sample Final

1. (a) If $a_n = \frac{(-1)^n x^{2n}}{n4^n}$ then $|a_{n+1}/a_n| = \frac{|x|^{2n+2}}{(n+1)4^{n+1}} \frac{n4^n}{|x|^{2n}} = \frac{|x|^2}{4} \frac{n}{n+1}$ which converges to $x^2/4$ as $n \to \infty$. Hence, by the ratio test, the series converges absolutely for $|x|^2/4 < 1$ or $|x| < 2$ and diverges for $|x|^2/4 > 1$ or $|x| > 2$. Hence the radius of convergence is 2. At $x = \pm 2$ the series is $\sum_1^\infty \frac{(-1)^n}{n}$ which converges by the alternating series test.

   (b) If $a_n = \frac{(-1)^n x^{2n}}{n4^n}$ then $|a_{n+1}/a_n| = \frac{|x|^{3n+3}}{64^{n+1}\sqrt{n+2}} \frac{64^n \sqrt{n+1}}{|x|^{3n}} = \frac{|x|^3}{64} \sqrt{\frac{n+1}{n+2}}$ which converges to $|x|^3/64$ as $n \to \infty$. The series converges absolutely for $|x| < 4$ and diverges for $|x| > 4$. The radius of convergence is therefore 4. At $x = 4$ the series is $\sum_0^\infty \frac{1}{\sqrt{n+1}} = \sum_1^\infty \frac{1}{\sqrt{n}}$ which is a divergent p-series with $p = 1/2$. At $x = -4$ the series is $\sum_1^\infty \frac{(-1)^n}{\sqrt{n}}$ which is convergent by the alternating series test.

2. (a) $F(x) = \int_0^x \sum_{n=0}^\infty (-1)^n \frac{t^{2n}}{2^n n!}\, dt = \sum_{n=0}^\infty (-1)^n \frac{x^{2n+1}}{(2n+1)2^n n!}$ which is an alternating series if $x > 0$. Hence $F(.1) = .1 - \frac{(.1)^3}{6} + R$ with $|R| < \frac{(.1)^5}{40} = 2.5 \times 10^{-7}$. Since $.1 - \frac{(.1)^3}{6} = .0998333$ to 5 decimal places we see that $F(.1) = .099833$ to six decimal places and hence $.09983$ to 5 places.

   (b) $\lim_{x\to\infty} \frac{e^{2x}-1)^2}{\ln(1+x)-x} = \lim_{x\to\infty} \frac{(2x+O(x^2))^2}{-x^2/2+O(x^3)} = \lim_{x\to\infty} \frac{4x^2+O(x^3)}{-x^2/2+O(x^3)} = \lim_{x\to\infty} \frac{4+O(x)}{-1/2+O(x)} = -4.$

3. Since g, h are continuous if $(x, y) \neq (0, 0)$ and $f(0, 0) = g(0, 0) = 0$, we only have to check whether these functions have limit 0 when $(x, y) \to 0$. Since $|g(x, y)| \leq |y|$ we have $\lim_{(x,y)\to(0,0)} g(x, y) = 0$ and hence g is continuous. Since $h(x, x) = 1/2$ and $h(x, 0) = 0$, we see that $h(x, y)$ does not converge as $(x, y) \to (0, 0)$ and hence h is not continuous.

4. (a) Since $\mathbf{r} = \mathbf{r}'(t) = (2, -\sin t, \cos t)$ we have $\frac{ds}{dt} = \sqrt{5}$ and hence $s = \sqrt{5}t$ so that $\mathbf{r} = \mathbf{r}(s) = (2, -\sin(s/\sqrt{5}), \cos(s/\sqrt{5}))$.

   (b) We have $\mathbf{T} = \frac{d\mathbf{r}}{ds} = \frac{1}{\sqrt{5}}(0, -\cos(s/\sqrt{5}), -\sin(s/\sqrt{5}))$ and $\frac{d\mathbf{T}}{ds} = \frac{1}{5}(0, \sin(s/\sqrt{5}), -\cos(s/\sqrt{5})) = \kappa\mathbf{N}$ which implies $\kappa = 1/5$ and $\mathbf{N} = (0, \sin(s/\sqrt{5}), -\cos(s/\sqrt{5}))$. Then $\mathbf{B} = \mathbf{T} \times \mathbf{N} = (1, 0, 0)$ and hence $\frac{d\mathbf{B}}{ds} = 0$ which implies $\tau = 0$.

5. (a) We have $\frac{\partial z}{\partial x} = 3e^y - 3x^2$, $\frac{\partial z}{\partial y} = 3xe^y - 3e^{3y}$ so that, at $(0, 0)$, we have $\frac{\partial z}{\partial x} = 3$, $\frac{\partial z}{\partial y} = -3$. Hence the equations of the tangent plane and normal line at $(0, 0, -1)$ are respectively $z = -1 + 3x - 3y$ and $x = -3t, y = 3t, z = -1 + t$.

   (b) Let $f(x, y, z) = 2x^2 + 3yz + z^2 - 6$, $g(x, y) = x^2 + xy + xz - 3$. Then $\nabla f = (4x, 3z, 3y + 2z)$, $\nabla g = (2x + y + z, x, x)$ so that $\nabla f(1, 1, 1) = (4, 3, 5)$, $\nabla g(1, 1, 1) = (4, 1, 1)$. The tangent line at $(1, 1, 1)$ to the curve of intersection of $f(x, y, z) = 0$ and $g(x, y, z) = 0$ has direction $\nabla f(1, 1, 1) \times \nabla g(1, 1, 1) = (-2, 1, 1)$. Hence the parametric equations of the tangent line are $x = 1 - 2t, y = 1 + t, z = 1 + t$.

6. Differentiating $x = r^3 - s, y = s^3 - r$ implicitly with respect to x, we get

$$3r^2\frac{\partial r}{\partial x} - \frac{\partial s}{\partial x} = 1$$
$$-\frac{\partial r}{\partial x} + 3s^2\frac{\partial s}{\partial x} = 0.$$

Solving, we get $\frac{\partial r}{\partial x} = \frac{3s^2}{9r^2s^2-1}$, $\frac{\partial s}{\partial x} = \frac{1}{9r^2s^2-1}$. Similarly, differentiating $x = r^3 - s, y = s^3 - r$ with respect to y, we get $\frac{\partial r}{\partial y} = \frac{1}{9r^2s^2-1}$, $\frac{\partial s}{\partial y} = \frac{3r^2}{9r^2s^2-1}$. Hence $\frac{\partial r}{\partial x} = \frac{\partial s}{\partial y} = 3/8$, $\frac{\partial r}{\partial y} = \frac{\partial s}{\partial x} = 1/8$ when $x = y = 0$. Then

$$\frac{\partial^2 r}{\partial x^2} = \frac{(9r^2s^2-1)(-6s)\frac{\partial s}{\partial x} - (3s^2)(16rs^2\frac{\partial r}{\partial x} + 18r^2s\frac{\partial s}{\partial x})}{(9r^2s^2-1)^2} = -\frac{33}{64} \quad \text{when} \quad x = y = 0.$$

7. (a) We have $\nabla T = (3x^2y + z^3, 3y^2z + x^3, 3z^2x + y^3)$ and $\mathbf{u} = \overrightarrow{PQ} = (-1, 2, 2)$ so that, at $(2, -1, 0)$, we have $D_{\mathbf{u}}T = \nabla T \cdot \mathbf{u}/|\mathbf{u}| = (-12, 8, -1) \cdot (-1, 2, 2)/3 = 26/3$.

   (b) Let $\mathbf{r}(t) = (x(t), y(t), z(t))$ be the position of the mosquito at time t. Then $\mathbf{v} = \frac{d\mathbf{r}}{dt}$ is the velocity of the mosquito at time t. We have $|\mathbf{v}|$ = speed of mosquito =5 and the direction of $\mathbf{v}$ is, up to sign, the gradient of f at $(2, -1, 0)$, namely $(8, -6, 0)/10 = (4, -3, 0)/5$ so that, at $(2, -1, 0)$, we have $\mathbf{v} = \pm((4, -3, 0)$. At time t, the temperature of the mosquito is $T(\mathbf{r}(t))$. The rate of change of the temperature of the mosquito per unit time is therefore

$$\frac{d}{dt}T(\mathbf{r}(t) = \nabla T(\mathbf{r}(t)) \cdot \mathbf{v}$$

   which, at the time the mosquito is at $(2, -1, 0)$, is $(-12, 8, -1) \cdot \pm(4, -3, 0) = \mp 72$. Since the mosquito is flying in the direction of increasing temperature, the rate must be positive so that $\mathbf{v} = (-4, 3, 0)$ and the rate is 72. (Things are getting hot for the mosquito!)

8. The point (x, y) is a critical point of the function $f(x, y)$ if and only if $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0$. Now $\frac{\partial f}{\partial x} = 3e^y - 3x^2$ and $\frac{\partial f}{\partial y} = 3xe^y - 3e^{3y}$ so that (x, y) is a critical point if and only if $e^y = x^2$ and $x = e^{2y}$. These two equations have the unique solution $x = 1, y = 0$. Now $A = \frac{\partial^2 f}{\partial x^2} = -6x$, $B = \frac{\partial^2 f}{\partial x \partial y} = 3e^y$, $C = 3e^y - 9e^{3y}$ so that at the critical point $(1, 0)$ we have $A < 0$, $AC - B^2 = (-6)(-6) - 9 = 27 > 0$ which shows that $f(1, 0) = 1$ is a local maximum. Since $f(-3, 0) = 17$ the function f does not have a maximum at $(1, 0)$.

9. The shortest distance occurs as a critical point of $L = x^2 + y^2 - \lambda(xy^2 - 1)$. Hence, we have

$$\frac{\partial L}{\partial x} = 2x - \lambda y^2 = 0, \quad \frac{\partial L}{\partial y} = 2y - 2\lambda xy = 0, \quad \frac{\partial L}{\partial \lambda} = 1 - xy^2 = 0.$$

Since $x, y \neq 0$ by the last equation, we have $\frac{2x}{y^2} = \frac{1}{x} = \lambda$ so that $y^2 = 2x^2$. Since $x = \frac{1}{y^2}$, we get $y^6 = 2$ and hence $y = \pm\sqrt[6]{2}$, $x = 1/\sqrt[3]{2}$ so that the shortest distance is $\sqrt{x^2 + y^2} = \sqrt{2^{-2/3} + 2^{1/3}}$ which occurs at two points.