Math 189-141C: **Deferred Examination** (2001/2002)

Notice:

1. No calculators allowed.
2. No textbooks, classnotes or integral formulas allowed.
3. Show all your work.

1. (16 pts, 4 pts for each) Evaluate integrals:

a). $\int_0^{\pi/2} \sin^3 x \, dx;$ b). $\int x^2 e^x \, dx;$

c). $\int \frac{1}{1+\sqrt{x}} \, dx;$ d). $\int \frac{x}{\sqrt{1-x^2}} \, dx.$

2. (8 pts, 4 pts for each) For each of the following integrals, determine whether it is convergent or divergent. If it is convergent, find its value.

a). $\int_2^{\infty} \frac{1}{x \ln x} \, dx;$ b). $\int_0^2 x \ln x \, dx.$

3. (8 pts, 4 pts for each) For each of the following sequences, determine whether it is convergent or divergent. If it is convergent, find its value.

a). $\left\{(-1)^n \frac{n+1}{n}\right\};$ b). $\{\sqrt{n^2+n} - n\}.$

4. (8 pts, 4 pts for each) For each of the following series, determine whether it is convergent or divergent, conditionally convergent and/or absolutely convergent.

a). $\sum_{n=1}^{\infty} \frac{(-3)^n}{2^{3n}};$ b). $\sum_{n=1}^{\infty} \frac{\cos n\pi}{n\pi}.$

5. (10 pts) Find the area that is inside the circle $r = 2\cos\theta$ and outside the curve $r = \frac{3}{2} - \cos\theta$.

6. (10 pts) For the curve given parametrically by $x = 1 + t$ and $y = t + t^2$, determine
   a). the equation of the tangent line at the point $(x, y) = (1, 0)$;
   b). the concavity of the curve at the point $(x, y) = (1, 0)$.

7. (40 pts, 10 pts for each sub-question) For the arc of the parabola $y = \sqrt{x}$ from $A(0,0)$ to $B(1,1)$, use the methods of the calculus to find
   a). the length of the arc; (**Hint**: $\int \sqrt{a^2+x^2}dx = \frac{1}{2}x\sqrt{a^2+x^2} + \frac{a^2}{2}\ln(x + \sqrt{a^2+x^2}) + C$)
   b). the area of the region bounded by the arc, the x-axis and the line $x = 1$;
   c). the volume of the solid obtained by rotating the region specified in (b) about the x-axis;
   d). the surface area of the solid obtained in (c).

Solutions to Deferred Examination of Math 141C (2001-2002)

1. Solution.

a). Let $u = \cos x$, then $du = -\sin x dx$ and $u = 1$ for $x = 0$, $u = 0$ for $x = \frac{\pi}{2}$. By the substitution rule, we have

$$\int_0^{\pi/2} \sin^3 x dx = \int_0^{\pi/2} \sin^2 x \sin x dx = \int_0^{\pi/2} (1 - \cos^2 x) \sin x dx$$
$$= \int_0^1 (1 - u^2) du = \left(u - \frac{u^3}{3}\right)\Big]_0^1 = \frac{2}{3}.$$

b). We integrat it by parts to have

$$\int x^2 e^x dx = x^2 e^x - \int 2x e^x dx = x^2 e^x - 2xe^x + \int 2e^x dx = x^2 e^x - 2xe^x + 2e^x + C.$$

c). Let $u = \sqrt{x}$, i.e., $x = u^2$ and $dx = 2udu$. Using the substitution rule, we rationalize the irration integral to

$$\int \frac{1}{1+\sqrt{x}} dx = \int \frac{2u}{1+u} du = 2\int \left(1 - \frac{1}{1+u}\right) du = 2u - 2\ln|1+u| + C = 2\sqrt{x} - 2\ln(1+\sqrt{x}) + C.$$

d). Let $x = \sin\theta$ for $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, which gives $\theta = \arcsin x$ and $dx = \cos\theta d\theta$. Then we have

$$\int \frac{x}{\sqrt{1-x^2}} dx = \int \frac{\sin\theta}{\sqrt{1-\sin^2\theta}} \cos\theta d\theta = \int \sin\theta d\theta = -\cos\theta + C = -\sqrt{1-x^2} + C.$$

2. Solution.

a). Integrating it by parts and applying the l'Hospital Rule, we obtain

$$\int_2^\infty \frac{1}{x\ln x} dx = \lim_{t\to\infty} \int_2^t \frac{1}{x\ln x} dx = \lim_{t\to\infty} \int_{\ln 2}^{\ln t} \frac{1}{u} du \quad \text{(substitute } u = \ln x)$$
$$= \lim_{t\to\infty} \ln u\Big]_{\ln 2}^{\ln t} = \lim_{t\to\infty} (\ln\ln t - \ln\ln 2) = +\infty.$$

So, this improper integral diverges.

b).

$$\int_0^2 x\ln x dx = \lim_{t\to 0} \int_t^2 x \ln x dx = \lim_{t\to 0}\left(\frac{x^2 \ln x}{2}\Big]_t^2 - \int_t^2 \frac{x}{2} dx\right)$$
$$= \lim_{t\to 0}\left(\frac{x^2\ln x}{2}\Big]_t^2 - \frac{x^2}{4}\Big]_t^2\right) = \lim_{t\to 0}\left(2\ln 2 - \frac{1}{2}t^2 \ln t - 1 + \frac{1}{4}t^2\right) = 2\ln 2 - 1.$$

So, this improper integral is convergent to $2\ln 2 - 1$.

3. Solution.

a). In the even case: $n = 2k$, $k = 1, 2, 3, \cdots$, it holds $(-1)^n \frac{n+1}{n} = \frac{2k+1}{2k}$ and

$$\lim_{n\to\infty} (-1)^n \frac{n+1}{n} = \lim_{k\to\infty} \frac{2k+1}{2k} = \lim_{k\to\infty} \Big(1 + \frac{1}{2k}\Big) = 1.$$

But, in the odd case: $n = 2k+1$, $k = 1, 2, 3, \cdots$, it holds $(-1)^n \frac{n+1}{n} = -\frac{2k+2}{2k+1}$ and

$$\lim_{n\to\infty} (-1)^n \frac{n+1}{n} = \lim_{k\to\infty} -\frac{2k+2}{2k+1} = -\lim_{k\to\infty} \Big(1 + \frac{1}{2k+1}\Big) = -1.$$

This implies that $\{(-1)^n \frac{n+1}{n}\}$ is divergent.

b).

$$\begin{aligned} \lim_{n\to\infty} \sqrt{n^2+n} - n &= \lim_{n\to\infty} \frac{(\sqrt{n^2+n} - \sqrt{n^2})(\sqrt{n^2+n} + \sqrt{n^2})}{\sqrt{n^2+n} + \sqrt{n^2}} \\ &= \lim_{n\to\infty} \frac{n}{\sqrt{n^2+n} + \sqrt{n^2}} = \lim_{n\to\infty} \frac{1}{\sqrt{1+\frac{1}{n}} + \sqrt{1}} = \frac{1}{2}. \end{aligned}$$

So, the sequence $\{\sqrt{n^2+n} - n\}$ is convergent to $\frac{1}{2}$.

4. Solution. [The solutions to a) and b) are not unique. We present here only one solution to each sub-question.]

a). Notice that $\sum_1^\infty \left|\frac{(-3)^n}{2^{3n}}\right| = \sum_1^\infty \left(\frac{3}{8}\right)^n$ is the geometric series with $r = \frac{3}{8} < 1$, so it is convergent, and hence $\sum_1^\infty \frac{(-3)^n}{2^{3n}}$ is absolutely convergent.

b). Notice that $\sum_1^\infty \frac{\cos n\pi}{n\pi} = \sum_1^\infty (-1)^n \frac{1}{n\pi}$ is an alternating series. Since $a_n := \frac{1}{n\pi}$ is decreasing and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n\pi} = 0$, applying the Alternating Series Test, we obtain the convergence of $\sum_1^\infty \frac{\cos n\pi}{n\pi}$. But it is not absolutely convergent, because $\sum_1^\infty \left|\frac{\cos n\pi}{n\pi}\right| = \sum_1^\infty \frac{1}{n\pi}$ is divergent by the Comparison Test with the divergent p-series $\sum_1^\infty \frac{1}{n}$. Therefore, $\sum_1^\infty \frac{\cos n\pi}{n\pi}$ is conditionally convergent.

5. Solution. Notice that

$$2\cos\theta = \frac{3}{2} - \cos\theta$$

gives $\cos\theta = \frac{1}{2}$, i.e., $\theta = \pm\frac{\pi}{3}$. So, the inserction points are $(1, \frac{\pi}{3})$ and $(1, -\frac{\pi}{3})$. The area of the region inside of $r = 2\cos\theta$ and outside $r = \frac{3}{2} - \cos\theta$ is

$$\begin{aligned} A &= \int_{-\pi/3}^{\pi/3} \frac{1}{2}[(2\cos\theta)^2 - (\frac{3}{2} - \cos\theta)^2]d\theta \\ &= \int_{-\pi/3}^{\pi/3} \frac{1}{2}[3\cos^2\theta - \frac{9}{4} + 3\cos\theta]d\theta \\ &= \frac{1}{2}\int_{-\pi/3}^{\pi/3} [\frac{3}{2}\cos 2\theta - \frac{3}{4} + 3\cos\theta]d\theta \\ &= \frac{1}{2}\Big(\frac{3\sin 2\theta}{4} - \frac{3}{4}\theta + 3\sin\theta\Big)_{-\pi/3}^{\pi/3} = \frac{15}{8}\sqrt{3} - \frac{\pi}{4}. \end{aligned}$$

6. Solution.
a). Since

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{(t+t^2)'}{(1+t)'} = 1+2t,$$

so, the tangent at the point (1,0), i.e., $t=0$, is

$$\left.\frac{dy}{dx}\right|_{(1,0)} = (1+2t)|_{t=0} = 1.$$

The tangent line is $\frac{y}{x-1} = 1$, i.e., $y = x - 1$.

b). Since

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}(\frac{dy}{dt})}{\frac{dx}{dt}} = \frac{(1+2t)'}{(1+t)'} = 2,$$

we have $\left.\frac{d^2y}{dx^2}\right|_{(1,0)} = 2$. So, the curve at the point $(1,0)$ is concave upward.

7. Solution.
a). Since $\frac{dy}{dx} = \frac{1}{2\sqrt{x}}$, we adopt the given integral formula to have

$$\begin{aligned} L &= \int_0^1 \sqrt{1+\left(\frac{dy}{dx}\right)^2}dx = \int_0^1 \sqrt{1+\frac{1}{4x}}dx \\ &= \int_0^1 \frac{\sqrt{1+4x}}{\sqrt{4x}}dx = \frac{1}{2}\int_0^2 \sqrt{1+u^2}du \qquad \text{(substitute } u=\sqrt{4x},\ dx = \frac{u}{2}du) \\ &= \frac{1}{4}(u\sqrt{1+u^2} + \ln|u+\sqrt{1+u^2}|)]_0^2 = \frac{\sqrt{5}}{2} + \frac{1}{4}\ln(2+\sqrt{5}) \end{aligned}$$

b).

$$A = \int_0^1 \sqrt{x}dx = \frac{2}{3}x^{3/2}\Big]_0^1 = \frac{2}{3}.$$

c).

$$V = \int_0^1 \pi(\sqrt{x})^2 dx = \pi\int_0^1 xdx = \frac{\pi}{2}.$$

d).

$$\begin{aligned} S &= \int_0^1 2\pi y\sqrt{1+\left(\frac{dy}{dx}\right)^2}dx = 2\pi\int_0^1 \sqrt{x}\sqrt{1+\frac{1}{4x}}dx = 2\pi\int_0^1 \sqrt{x}\frac{\sqrt{1+4x}}{\sqrt{4x}}dx \\ &= \pi\int_0^1 \sqrt{1+4x}dx = \frac{\pi}{4}\int_1^5 \sqrt{u}du \qquad \text{(substitute } u = 1+4x) \\ &= \frac{\pi}{6}u^{3/2}]_1^5 = \frac{5\sqrt{5}-1}{6}\pi. \end{aligned}$$