Champlain College – St.-Lambert

MATH 201-NYB: Calculus II

Review Questions for Test # 1

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Questions

1. Sketch the graph of the function $f(x) = \sqrt{4-x^2}$, and integrate $\int_{-2}^{2} f(x)dx$ by interpreting it in terms of area.

2. Find integrals

   (a) $\int_0^{\pi/2} \sin x \; e^{\cos x} dx$

   (b) $\int_1^e x^3 \ln x dx$

   (c) $\int \frac{1}{x^2-x-2} dx$

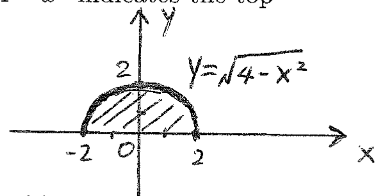
   (d) $\int \sin^3 x \cos^2 x dx$

   (e) $\int x^2\sqrt{4-x^2} dx$

Solutions

Solution to Q.1:

Let $y = \sqrt{4-x^2}$, then $y^2 = 4 - x^2$, i.e., $x^2 + y^2 = 2^2$. This is a circle centred at the origin $(0,0)$ with radius $r = 2$. Note that, $y = \sqrt{4-x^2} \geq 0$, so $y = \sqrt{4-x^2}$ indicates the top half-circle. And then the definite integral is



$$\int_{-2}^{2} f(x)dx = \frac{1}{2}\pi 2^2 = 2\pi.$$

Solution to Q.2 (a):

Substitute $u = \cos x$, then $du = -\sin x dx$ and the new upper-limit and lower-limit for u are 0 and 1, respectively. Thus,

$$\int_0^{\pi/2} \sin x e^{\cos x} dx = \int_1^0 e^u(-du) = -e^u\Big|_1^0 = e - 1.$$

Solution to Q.2 (b):

Take $f(x) = \ln x$ and $g'(x) = x^3$, then $f'(x) = \frac{1}{x}$ and $g(x) = \frac{x^4}{4}$, then apply the integration by parts, we have

$$\begin{aligned}\int_1^e x^3 \ln x dx &= f(x)g(x)\Big|_1^e - \int_1^e f'(x)g(x)dx = \ln x \cdot \frac{x^4}{4}\Big|_1^e - \int_1^e \frac{1}{x}\cdot\frac{x^4}{4}dx \\ &= \frac{x^4}{4}\ln x\Big|_1^e - \frac{1}{4}\int_1^e x^3 dx = \frac{x^4}{4}\ln x\Big|_1^e - \frac{1}{4}\cdot\frac{x^4}{4}\Big|_1^e = \frac{3e^4}{16}.\end{aligned}$$

Solution to Q.2 (c):

Since $Q(x) = x^2 - x - 2 = (x-2)(x+1)$, then we try

$$\frac{1}{x^2 - x - 2} = \frac{A}{x-2} + \frac{B}{x+1} = \frac{A(x+1) + B(x-2)}{(x-2)(x+1)}.$$

Comparing the corresponding numerators, we have

$$1 = A(x+1) + B(x-2).$$

Take $x = 2$, we get $A = \frac{1}{3}$, and take $x = -1$ we get $B = -\frac{1}{3}$. Thus, we can integrate the original integral as

$$\int \frac{1}{x^2-x-2}dx = \int \Big(\frac{\frac{1}{3}}{x-2} - \frac{\frac{1}{3}}{x+1}\Big)dx = \frac{1}{3}\int \frac{1}{x-2}dx - \frac{1}{3}\int\frac{1}{x+1}dx = \frac{1}{3}\ln|x-2| - \frac{1}{3}\ln|x+1| + C.$$

Solution to Q.2 (d):

$$\begin{aligned}
\int \sin^3 x \cos^2 x dx &= \int \sin^2 x \cos^2 x \sin x dx \\
&= \int (1-\cos^2 x)\cos^2 x \sin x dx \qquad [\text{ use: } \sin^2 x = 1-\cos^2 x\ ] \\
&= \int (1-u^2)u^2(-du) \qquad [\text{ substitute: } u=\cos x, \text{then } du = -\sin x dx\ ] \\
&= -\int (u^2-u^4)du \\
&= -\Big(\frac{u^3}{3}-\frac{u^5}{5}\Big)+C \\
&= -\frac{1}{3}\cos^3 x + \frac{1}{5}\cos^5 x + C.
\end{aligned}$$

Solution to Q.2 (e):

$$\begin{aligned}
&\int\ x^2\sqrt{4-x^2}dx \\
&= \int (2\sin\theta)^2\sqrt{4-4\sin^2\theta}\ \ 2\cos\theta\ d\theta \qquad [\ \text{set: } x=2\sin\theta,\ \text{then } dx = 2\cos\theta d\theta] \\
&= 16\int \sin^2\theta\sqrt{1-\sin^2\theta}\ \ \cos\theta\ d\theta \\
&= 16\int \sin^2\theta\cos^2\theta d\theta \qquad [\text{ use: } 1-\sin^2\theta = \cos\theta] \\
&= 4\int \sin^2(2\theta)d\theta \qquad [\text{ use: } 2\sin\theta\cos\theta = \sin 2\theta] \\
&= 4\int \frac{1-\cos 4\theta}{2}d\theta \qquad \Big[\text{ use: } \sin^2 u = \frac{1-\cos 2u}{2}, \text{so } \sin^2 2\theta = \frac{1-\cos 4\theta}{2}\Big] \\
&= 2\int(1-\cos 4\theta)d\theta = 2\int 1d\theta - 2\int \cos 4\theta d\theta \\
&= 2\theta - \frac{1}{2}\int \cos u du \qquad [\text{ substitute: } u = 4\theta,\ du = 4d\theta] \\
&= 2\theta - \frac{1}{2}\sin u + C = 2\theta - \frac{1}{2}\sin 4\theta + C \\
&= 2\theta - \sin 2\theta\cos 2\theta + C \qquad [\text{ use: } \sin 4\theta = 2\sin 2\theta\cos 2\theta\ ] \\
&= 2\theta - 2\sin\theta\cos\theta(1-2\sin^2\theta) + C \quad [\text{ use: } \sin 2\theta = 2\sin\theta\cos\theta,\ \cos 2\theta = 1-2\sin^2\theta\ ] \\
&= 2\sin^{-1}x - x\sqrt{1-\Big(\frac{x}{2}\Big)^2}\Big(1-\frac{1}{2}x^2\Big) + C \qquad \Big[\text{ Here: } \sin\theta = \frac{x}{2}, \cos\theta = \sqrt{1-\frac{x^2}{2^2}}\ \Big] \\
&= 2\sin^{-1}x - \frac{1}{4}x(2-x^2)\sqrt{4-x^2} + C
\end{aligned}$$