Champlain College – St.-Lambert

MATH 201-103-RE: Calculus I

Sample Questions for Test # 1

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Questions

1. Let $f(x) = \sqrt{x}$ and $g(x) = x - 2$.

   (a) Find $\dfrac{f}{g}$ and its domain.

   (b) Find $(f \circ g)(x)$ and its inverse $(f \circ g)^{-1}(x)$.

2. Evaluate the limits

   (a) $\displaystyle\lim_{x\to\infty} \frac{3x^2+1}{2x-5x^2}$, (b) $\displaystyle\lim_{x\to 1} \frac{x^2+3x-4}{x^2-1}$, (c) $\displaystyle\lim_{x\to 0} \frac{\sqrt{2-x^2}-\sqrt{2}}{x^2}$.

3. Find parameters a and b such that the function

$$f(x) = \begin{cases} x^2 + a, & \text{if } x > 1 \\ b, & \text{if } x = 1 \\ 5x + 3, & \text{if } x < 1 \end{cases}$$

   will be continuous at $x = 1$.

4. Given $f(x) = \sqrt{1-x}$.

   (a) Use the definition of derivative to find $f'(x)$.

   (b) Find the tangent line of $f(x)$ at $x = 0$.

Solutions to Sample Questions

1(a). $\frac{f}{g} = \frac{\sqrt{x}}{x-2}$. For its domain, since the numerator is a square root, we need the inside of the square root to be non-negative, for the denominator, we need it not to be zero. Thus, we restrict

$$\begin{cases} x \geq 0 \\ x - 2 \neq 0, \end{cases}$$

which is solved as $0 \leq x < 2$ and $x > 2$. Namely, the domain is $D = [0, 2) \cup (2, \infty)$.

1(b). $f \circ g = \sqrt{g} = \sqrt{x-2}$. For the inverse of $f \circ g$, let $y = f \circ g = \sqrt{x-2}$, replace x with y and y with x to the previous equation, respectively, we then have $x = \sqrt{y-2}$. Squaring it yields $x^2 = y - 2$, i.e., $y = x^2 + 2$. So, $(f \circ g)^{-1}(x) = x^2 + 2$ with $x \geq 0$.

2(a).

$$\lim_{x\to\infty} \frac{3x^2+1}{2x-5x^2} = \lim_{x\to\infty} \frac{(3x^2+1)/x^2}{(2x-5x^2)/x^2} = \lim_{x\to\infty} \frac{\frac{3x^2}{x^2}+\frac{1}{x^2}}{\frac{2x}{x^2}-\frac{5x^2}{x^2}} = \lim_{x\to\infty} \frac{3+\frac{1}{x^2}}{\frac{2}{x}-5} = \frac{3+0}{0-5} = -\frac{3}{5}.$$

2(b).

$$\lim_{x\to 1} \frac{x^2+3x-4}{x^2-1} = \lim_{x\to 1} \frac{(x+4)(x-1)}{(x+1)(x-1)} = \lim_{x\to 1} \frac{x+4}{x+1} = \frac{1+4}{1+1} = \frac{5}{2}.$$

2(c).

$$\begin{aligned} &\lim_{x\to 0} \frac{\sqrt{2-x^2}-\sqrt{2}}{x^2} = \lim_{x\to 0} \frac{\sqrt{2-x^2}-\sqrt{2}}{x^2} \cdot \frac{\sqrt{2-x^2}+\sqrt{2}}{\sqrt{2-x^2}+\sqrt{2}} \\ &= \lim_{x\to 0} \frac{(\sqrt{2-x^2})^2-(\sqrt{2})^2}{x^2(\sqrt{2-x^2}+\sqrt{2})} = \lim_{x\to 0} \frac{2-x^2-2}{x^2(\sqrt{2-x^2}+\sqrt{2})} \\ &= \lim_{x\to 0} \frac{-x^2}{x^2(\sqrt{2-x^2}+\sqrt{2})} = \lim_{x\to 0} \frac{-1}{\sqrt{2-x^2}+\sqrt{2}} \\ &= \frac{-1}{\sqrt{2-0^2}+\sqrt{2}} = -\frac{1}{2\sqrt{2}}. \end{aligned}$$

3. Since the right-sided limit at $x = 1$ is $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+}(x^2 + a) = 1 + a$, the left-sided limit is $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-}(5x + 3) = 8$, and $f(1) = b$, for the continuity of $f(x)$ at $x = 1$, we need

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^-} f(x) = f(1),$$

which gives, $1 + a = 8 = b$. So, $a = 7$ and $b = 8$.

4(a).

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\begin{aligned}
f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{\sqrt{1-(x+h)}-\sqrt{1-x}}{h} \\
&= \lim_{h\to 0}\frac{\sqrt{1-(x+h)}-\sqrt{1-x}}{h}\cdot\frac{\sqrt{1-(x+h)}+\sqrt{1-x}}{\sqrt{1-(x+h)}+\sqrt{1-x}} \\
&= \lim_{h\to 0}\frac{(\sqrt{1-(x+h)})^2-(\sqrt{1-x})^2}{h[\sqrt{1-(x+h)}+\sqrt{1-x}]} = \lim_{h\to 0}\frac{1-(x+h)-(1-x)}{h[\sqrt{1-(x+h)}+\sqrt{1-x}]} \\
&= \lim_{h\to 0}\frac{-h}{h(\sqrt{1-(x+h)}+\sqrt{1-x})} = \lim_{h\to 0}\frac{-1}{\sqrt{1-(x+h)}+\sqrt{1-x}} \\
&= \frac{-1}{\sqrt{1-(x+0)}+\sqrt{1-x}} = -\frac{1}{2\sqrt{1-x}}.
\end{aligned}
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4(b). For the tangent line of $f(x)$ at $x = 0$, its slope is $m = f'(0) = -\frac{1}{2\sqrt{1-0}} = -\frac{1}{2}$, and the touched point is $(0, f(0)) = (0, 1)$. By the slope-point form, the equation of the tangent line is

$$\frac{y-1}{x-0} = -\frac{1}{2},$$

i.e., $y = -\frac{1}{2}x + 1$.