Champlain College – St.-Lambert

MATH 201-103-RE: Calculus I

Review Questions for Test # 3

Instructor: Dr. Ming Mei

Questions

1. Let $f(x)$ be a function satisfying:

   ♠ $f(-2) = 3, \quad f(-1) = 2, \quad f(0) = 1, \quad f(2) = -2,$

   ♠ $\lim_{x\to 1^-} f(x) = +\infty, \quad \lim_{x\to 1^+} f(x) = -\infty, \quad \lim_{x\to +\infty} f(x) = 0,$

   ♠ $f'(-2) = 0, \quad f'(0) = 0, \quad f''(-1) = 0$

   ♠ $f'(x) > 0$ in $(-\infty, -2) \cup (0, 1) \cup (1, \infty)$, $\quad f'(x) < 0$ in $(-2, 0)$,

   ♠ $f''(x) < 0$ in $(-\infty, -1) \cup (1, \infty)$, $\quad f''(x) > 0$ in $(-1, 1)$.

   Sketch the graph of $f(x)$.

2. Let $f(x) = \dfrac{2x}{x+1}$.

   (a) Find the domain of f;

   (b) Find the horizontal asymptotes and vertical asymptotes, if any;

   (c) Find the critical numbers, if any;

   (d) Find the intervals where $f(x)$ is increasing and where it is decreasing;

   (e) Find the relative extrema;

   (f) Find the intervals where $f(x)$ is concave upward and where it is concave downward;

   (g) Point out the inflection points, if any;

   (h) Sketch the graph of $f(x)$.

3. The estimated monthly profit (in dollars) realizable by Cannon Precision Instruments for manufacturing and selling x units of its model M1 digital camera is

   $$P(x) = -0.04x^3 + 1200x - 10,000.$$

   To maximize its profit, how many cameras should Cannon produce each month?

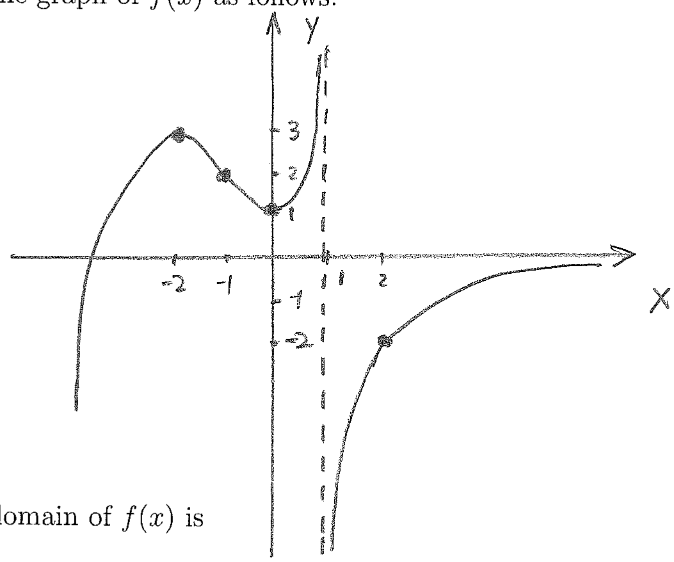
4. A box with an open top is to be constructed from a square piece of cardboard, 10 inches wide, by cutting out a square from each of the four corners and bending up the sides. What is the maximum volume of such a box?

x	f′(x)	f″(x)	f(x)
$(-\infty, -2)$	$+$	$-$	$\nearrow \cap \Longrightarrow$
$x = -2$	0	$-$	$f(-2) = 3$ relative maximum
$(-2, -1)$	$-$	$-$	$\searrow \cap \Longrightarrow$
$x = -1$	$-$	0	inflection point
$(-1, 0)$	$-$	$+$	$\searrow \cup \Longrightarrow$
$x = 0$	0	$+$	$f(0) = 1$ relative minimum
$(0, 1)$	$+$	$+$	$\nearrow \cup \Longrightarrow$
$(1, \infty)$	$+$	$-$	$\nearrow \cap \Longrightarrow$

Table 1: Table for Question 1

Solutions to Review Questions

Q.1. Since $\lim_{x\to 1^-} f(x) = +\infty$, $\lim_{x\to 1^+} f(x) = -\infty$, and $\lim_{x\to\infty} f(x) = 0$, we know that $x = 1$ is the vertical asymptote and $y = 0$ is the horizontal asymptote. From the given conditions, we collect the information as a table (see Table 1: Table for Question 1), and sketch the graph of $f(x)$ as follows.



Q.2. **(a)**. The domain of $f(x)$ is

$$D = \{x|x + 1 \neq 0\} = \{x|x \neq -1\} = (-\infty, -1) \cup (-1, \infty).$$

(b). The vertical asymptote is $x = -1$, because it is the zero of the denominator such that $\lim_{x\to -1^-} f(x) = \infty$ and $\lim_{x\to -1^+} f(x) = -\infty$). On the other hand,

$$\lim_{x\to\pm\infty} \frac{2x}{x+1} = \lim_{x\to\pm\infty} \frac{2x/x}{(x+1)/x} = \lim_{x\to\pm\infty} \frac{2}{1+\frac{1}{x}} = 2,$$

x	f'(x)	f''(x)	f(x)
$(-\infty, -1)$	$+$	$+$	$\nearrow \cup \Longrightarrow$
$(-1, \infty)$	$+$	$-$	$\nearrow \cap \Longrightarrow$

Table 2: Table for Question 2

then the horizontal asymptote is $y = 2$.

(c). About the critical numbers, since

$$f'(x) = \left(\frac{2x}{x+1}\right)' = \frac{(2x)'(x+1) - 2x(x+1)'}{(x+1)^2} = \frac{2}{(x+1)^2} > 0,$$

there is no critical number.

(d). As shown before, $f'(x) > 0$ in the domain $(-\infty, -1) \cup (-1, \infty)$, so the graph of f is increasing in $(-\infty, -1) \cup (-1, \infty)$.

(e). No relative extrema, because there is no critical numbers.

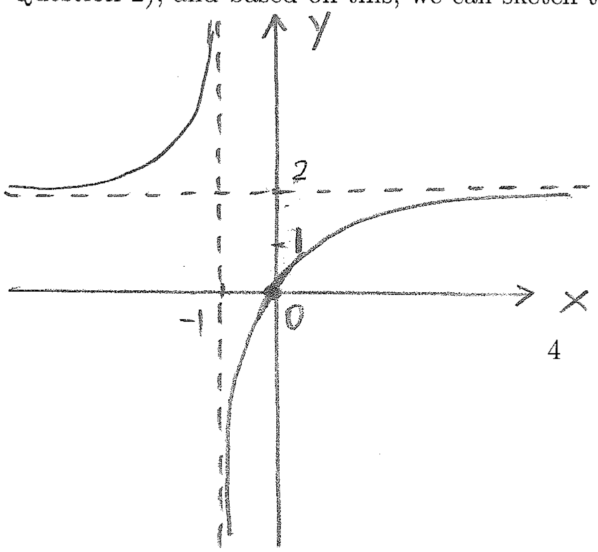
(f). Notice that,

$$f''(x) = \left(\frac{2}{(x+1)^2}\right)' = \left(2(x+1)^{-2}\right)' = -4(x+1)^{-3} = -\frac{4}{(x+1)^3}.$$

When $x > -1$, i.e., $x + 1 > 0$, $(x+1)^3 > 0$, we have $f''(x) = -\frac{4}{(x+1)^3} < 0$. So, f is concave downward in $(-1, \infty)$; While, when $x < -1$, i.e., $x + 1 < 0$, $(x+1)^3 < 0$, we have $f''(x) = -\frac{4}{(x+1)^3} > 0$. So, f is concave upward in $(-\infty, -1)$;

(g). Since $f''(x) = -\frac{4}{(x+1)^3} \neq 0$ in the domain $(-\infty, -1) \cup (-1, \infty)$, there is no inflection point.

(h). Collect all information as shown above, we first have a table (see Table 2: Table for Question 2), and based on this, we can sketch the graph of f as follows.



Q.3. From the profit function

$$P(x) = -0.04x^3 + 1200x - 10,000, \qquad \text{for } x \geq 0,$$

we have

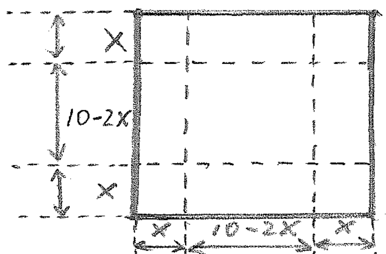
$$P'(x) = -0.12x^2 + 1200 = -0.12(x^2 - 10,000) = -0.12(x - 100)(x + 100),$$

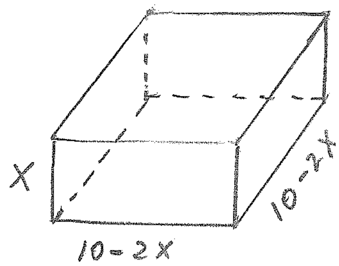
and

$$P''(x) = (-0.12x^2 + 1200)' = -0.24x.$$

So, $x = 100$ and $x = -100$ both are the critical numbers of $P(x)$ (i.e., $P'(x) = 0$). Since we need to restrict $x \geq 0$, there is only one critical number $x = 100$. Calculating $P''(100) = -0.24 \times 100 = -24 < 0$, by the second derivative test, we can verify that $P(100) = -0.04 \cdot (100)^3 + 1200 \cdot (100) - 10,000 = \$70,000$ is the maximum profit.

Q.4. Let x be the width of the side of the four corner squares, then the left side from the big square piece of cardboard is $10 - 2x$:





then the volume of the box is

$$V(x) = x(10 - 2x)(10 - 2x) = 4x^3 - 40x^2 + 100x$$

for $x \geq 0$ and $10 - 2x \geq 0$, namely, $0 \leq x \leq 5$. From

$$V'(x) = (4x^3 - 40x^2 + 100x)' = 12x^2 - 80x + 100 = 4(3x^2 - 20x + 25) = 4(3x - 5)(x - 5) = 0$$

we find two critical numbers: $x_1 = \frac{5}{3}$ and $x_2 = 5$. Since $V''(x) = (12x^2 - 80x + 100)' = 24x - 80$ and $V''(\frac{5}{3}) = -40 < 0$ and $V''(5) = 40 > 0$, by the second derivative test, $V(\frac{5}{3}) = \frac{5}{3}(10 - \frac{10}{3})^2 = \frac{2000}{27}$ is the relative maximum and $V(5) = 0$ is the relative minimum. On the other hand, $V(0) = 0$, comparing with $V(\frac{5}{3}) = \frac{2000}{27}$, the maximum volume of the box is $V(\frac{5}{3}) = \frac{2000}{27}$ by cutting out the four small squares with the side $x = \frac{5}{3}$ inches.