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Novel convergence to steady-state for Nicholson's blowflies equation with Dirichlet boundary

Jiaqi Yang [a,b], Ming Mei [b,c,*], Yang Wang [d,b]

[a] School of Mathematics, Jilin University, Changchun, Jilin, 130012, China
[b] Department of Mathematics and Statistics, McGill University, Montreal, Quebec, H3A 2K6, Canada
[c] Department of Mathematics, Champlain College Saint-Lambert, Saint-Lambert, Quebec, J4P 3P2, Canada
[d] School of Mathematical Sciences, Shanxi University, Taiyuan, Shanxi, 030006, China

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ABSTRACT

This short note is concerned with Nicholson's blowflies equation with Dirichlet boundary. For the challenging non-monotone case, we show the exponential convergence rate by the energy method, where the key observation is that the energy integrations for the perturbations on the inner boundary will be disappeared automatically, even the solutions at the boundary are non-zero.

1. Introduction

In this paper, we consider the diffusive Nicholson's blowflies equation with Dirichlet boundary

$$\begin{cases} u_t - D\triangle u + \delta u = f(u(t-r,x)), & (t,x) \in \mathbb{R}_+ \times \Omega, \\ u(s,x) = u_0(s,x), & (s,x) \in [-r,0] \times \Omega, \\ u|_{\partial\Omega} = 0, & t > 0. \end{cases} \tag{1.1}$$

where $u(t,x)$ represents the matured population of Australian blowflies at time t and location x. $D > 0$ is the spatial diffusion coefficient, $\delta > 0$ is the death rate. $f(u(t-r,x)) := pu(t-r,x)e^{-au(t-r,x)}$ is the birth rate function with the birth rate coefficient $p > 0$ (the maximum per capita daily egg production rate), and the number $a > 0$, where $\frac{1}{a}$ stands for the size at which the flies population reproduces at the

* Corresponding author at: Department of Mathematics, Champlain College Saint-Lambert, Saint-Lambert, Quebec, J4P 3P2, Canada.

E-mail addresses: Yangjq18@mails.jlu.edu.cn (J. Yang), ming.mei@mcgill.ca (M. Mei), ywang2005@sxu.edu.cn (Y. Wang).

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maximum rate. $r > 0$ is the time-delay, the time required for a newborn to become matured. Ω is a bounded open domain with smooth boundary $\partial\Omega$ in the space $\mathbb{R}^n$ with the dimension $n > 1$. $u_0(s,x) \geq 0$ for $(s,x) \in [-r,0] \times \Omega$ with $u_0(0,x) > 0$ for $x \in \Omega$ is the given initial data, which satisfies the compatibility conditions for the Dirichlet boundary as $u_0|_{\partial\Omega} = 0$ for $s \in [-r,0]$.

Clearly, Eq. (1.1) has two constant equilibria $u_- = 0$ and $u_+ = \frac{1}{a}\ln\frac{p}{\delta}$. When $0 < \frac{p}{\delta} \leq 1$, then $u_+ \leq u_- = 0$; and when $\frac{p}{\delta} > 1$, then $u_+ > u_- = 0$. Eq. (1.1) may also possess non-trivial steady-states when $\frac{p}{\delta} > 1$.

For the Dirichlet boundary problem, So and Yang [1] first obtained the threshold convergence results when the physical quantities δ, p and D satisfy certain conditions. Let $\phi(x)$ be a steady-state solution to (1.1), namely,

$$\begin{cases} -D\triangle\phi + \delta\phi = f(\phi(x)), & x \in \Omega, \\ \phi|_{\partial\Omega} = 0. \end{cases} \tag{1.2}$$

Clearly, $\phi = 0$ is a trivial steady-state of (1.2). As showed in [1], when $D\lambda_1 + \delta > p$, (1.2) possesses a unique trivial steady-state $\phi(x) \equiv 0$, and when $D\lambda_1 + \delta < p$, (1.2) possesses a unique positive (non-trivial) steady-state $0 < \phi(x) \leq \frac{1}{a}\ln\frac{p}{\delta}$ for $x \in \Omega$, where λ_1 is the principal eigenvalue to the following eigenvalue problem

$$\begin{cases} -\triangle\psi = \lambda\psi, & x \in \Omega, \\ \psi|_{\partial\Omega} = 0. \end{cases} \tag{1.3}$$

When $D\lambda_1 + \delta > p$, So and Yang [1] proved the global convergence of the solution of (1.1) to the trivial steady-state $\phi = 0$ by the energy method. When $D\lambda_1 + \delta < p$, So and Yang [1] further obtained the convergence of the solution $u(t,x)$ to the non-trivial steady-state by the technical but complex domain-decomposition method. The nonlocal equations with Dirichlet boundary were further generalized in [2–5], and the degenerate diffusion case was studied in [6]. But, no convergence rates were showed in all mentioned cases due the technical reasons by their adopted methods.

For the studies of the Nicholson's blowflies equations with Neumann boundary, the convergence of the solutions to their steady-states were extensively investigated in [7–10]. For the Cauchy problem, Nicholson's blowflies equation (1.1) possesses traveling waves $u = \phi(x + ct)$ connecting two constant states $u_- = 0$ and $u_+ = \frac{1}{a}\ln\frac{p}{\delta}$ in [11–16], with the wave speed $c \geq c_* > 0$, where c_* is the minimal wave speed. These waves are proved to be globally stable in [17–23]. For the recent developments on Nicholson's blowflies equations involving patch structure, we refer to [24–26].

For $e < \frac{p}{\delta} < e^2$, the equation of (1.1) is non-monotone, and the positive steady-state $\phi(x)$ will be non-monotone, and in some part of Ω it satisfies $0 < \phi(x) \leq \frac{1}{a}$ and in the other part it holds $\phi(x) > \frac{1}{a}$. Let us decompose the domain Ω in to two parts:

$$\Omega_A = \{x \in \Omega | 0 \leq \phi(x) \leq \frac{1}{a}\}, \text{ and } \Omega_B = \{x \in \Omega | \phi(x) > \frac{1}{a}\}.$$

As we show below, since $\phi(x)$ is a C^2-differentiable cone for x in Ω, cutting-off $\phi(x)$ by the plane $\phi = \frac{1}{a}$ yields that the level set

$$\partial\Omega_B = \{x \in \Omega | \phi(x) = \frac{1}{a}\} \tag{1.4}$$

is a C^2-closed-curve for $n = 2$ or a C^2-closed-surface for $n \geq 3$. See the example Fig. 1 for the structure of Ω_B.

Here, we are interested in investigating the convergence to the non-trivial steady-state in the region Ω_B, because in this region the solution may be oscillating when the time-delay is big. We observe that, after taking perturbation of u for (1.1) around the steady-state $\phi(x)$, the integration for the energy estimates on the boundary $\partial\Omega_B$ will be vanishing automatically. This guarantees us to derive a new exponential

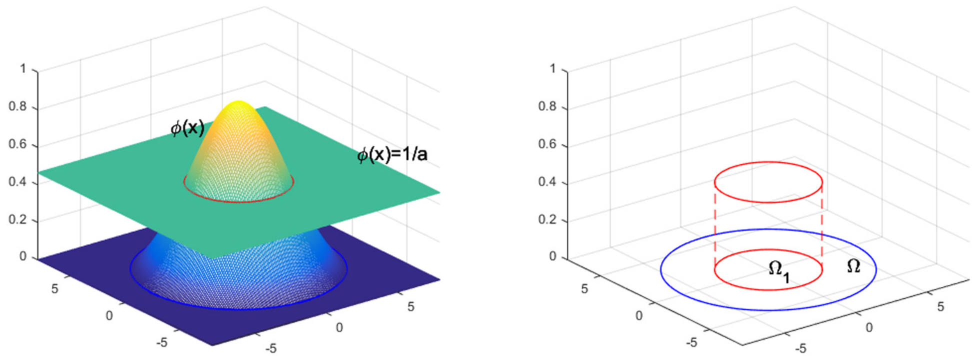

Fig. 1. The image of $\phi(x)$ and the structure of the region Ω_B.

convergence by the basic but efficient energy method as follows:

$$\|u(t,\cdot)-\phi(\cdot)\|_{L^2(\Omega_B)} \le Ce^{-\mu_1 t}, \quad \text{for some } \mu_1 > 0.$$

This is the main target in the short note.

2. Main result and proof

We first recall the existence, uniqueness and boundedness of the non-trivial steady-state $0 < \phi(x) \le u_+ = \frac{1}{a}\ln\frac{p}{\delta}$ (see [1] for details). For the existence and uniqueness of the solution (1.1), it has been proved in [1] that $u \in C_+^{1,1+\alpha}([0,+\infty)\times\Omega)$. Here, we give a better regularity of the solution, once the initial data is with a better regularity.

Lemma 2.1. *When $0 < \frac{p}{\delta} < e^2$ with $r > 0$, once the initial data satisfies $u_0 \in C_+^{1,2}([-r,0]\times\Omega)$, then the IBVP (1.1) possesses a unique smooth solution $u \in C_+^{1,2}([0,+\infty)\times\Omega)$, and $0 \le u(t,x) \le \frac{p}{ea\delta}$ for all $x \in \Omega$ and $t > 0$.*

Proof. The proof is obvious. In fact, when $t \in [0,r]$, namely $t-r \in [-r,0]$, which implies $f(u(t-r,x)) = f(u_0(t-r,x))$, so (1.1) is a linear parabolic equation with the initial–boundary value conditions in $[0,t]\times\Omega$, hence it possesses a unique smooth solution $u \in C_+^{1,2}([0,r]\times\Omega)$ once the initial data satisfies $u_0 \in C_+^{1,2}([-r,0]\times\Omega)$. Similarly, when $t \in [r,2r]$, we may prove that $u \in C_+^{1,2}([r,2r]\times\Omega)$. Repeating the same procedure to each interval $t \in [mr,(m+1)r]$, we can finally prove $u \in C_+^{1,2}([0,+\infty)\times\Omega)$. □

Next we are going to prove the boundary $\partial\Omega_B$ as a closed curve in the 2-D case and a closed surface in the higher dimensional case of $n \ge 3$.

Lemma 2.2. *Let $D\lambda_1 + \delta < p$. Then the non-trivial steady-state $\phi(x)$ of the stationary problem (1.2) is a C^2 convex cone on Ω, and the boundary $\partial\Omega_B = \{x \in \Omega | \phi(x) = \frac{1}{a}\}$ is a unique C^2-closed-curve for $n = 2$, or a unique C^2-closed-surface for $n \ge 3$.*

Proof. It has been proved in [1] that the stationary problem (1.2) exists a unique steady-state $\phi \in C^2(\Omega)$ satisfying $0 < \phi(x) \le u_+ = \frac{1}{a}\ln\frac{p}{\delta}$ for $x \in \Omega$. We now prove that $\phi(x)$ is a C^2 convex cone on Ω. In fact, note that $0 < \phi(x) \le u_+ = \frac{1}{a}\ln\frac{p}{\delta}$ for $x \in \Omega$, one can verify

$$\frac{p}{\delta} - e^{a\phi} \ge \frac{p}{\delta} - e^{au_+} = 0.$$

which can guarantee (1.2) to yield

$$-D\triangle\phi = f(\phi(x)) - \delta\phi = p\phi e^{-a\phi} - \delta\phi = \delta\phi e^{-a\phi}\Big(\frac{p}{\delta} - e^{a\phi}\Big) \ge 0, \text{ for } x \in \Omega. \tag{2.1}$$

Namely, it proves that all intersection curves $\phi(x)$ are concave downward in any vertical sliding plans (space) of $\langle x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n\rangle = \langle b_1, \dots, b_{i-1}, b_{i+1}, \dots, b_n\rangle \in \mathbb{R}^{n-1} \cap \Omega$ with a given constant vector $\langle b_1, \dots, b_{i-1}, b_{i+1}, \dots, b_n\rangle$, for $i = 1, \dots, n$. Therefore, $\phi(x)$ is a C^2 convex cone on Ω (see Fig. 1). This immediately guarantees that the level set $\partial\Omega_B = \{x \in \Omega | \phi(x) = \frac{1}{a}\}$ is a closed-curve for $n = 2$, or a closed-surface for $n \ge 3$. □

Now we state our main theorem of the exponential convergence of the solution to the steady-state for the Dirichlet boundary problem (1.1).

Theorem 2.3. *Let the initial data satisfy $u_0 \in C_+([-r, 0] \times \Omega)$. In the case of $e < \frac{p}{\delta} < e^2$, it holds:*

$$\|u(t,\cdot) - \phi(\cdot)\|_{L^2(\Omega_B)} \le C_1 e^{-\mu_1 t} \tag{2.2}$$

where $C_1 = C_1(u_0, \phi) > 0$ is a constant dependent on the initial perturbation $u_0 - \phi$, and $\mu_1 > 0$ is a small number such that

$$2\mu_1 + \delta(e^{2\mu_1 r} - 1) \le \delta(2 - \ln\frac{p}{\delta}). \tag{2.3}$$

Proof. When $e < \frac{p}{\delta} < e^2$ and $x \in \Omega_B$, namely $\phi(x) \ge \frac{1}{a}$, we use the energy method to prove the exponential convergence (2.2) in Ω_B. The key observation is that, the integration on the boundary of $\partial\Omega_B$ can be disappeared.

Let us define $v(t,x) := u(t,x) - \phi(x)$, which satisfies, from (1.1) and (1.2), that

$$\begin{cases} v_t - D\triangle v + \delta v = F(t-r,x), & (t,x) \in R_+ \times \Omega_B, \\ v|_{\partial\Omega_B} = [u(t,x) - \phi(x)]|_{\partial\Omega_B} =: g(t,x) & t > 0, x \in \partial\Omega_B, \\ v|_{t=s} = v_0(s,x) := u_0(s,x) - \phi(x), & (s,x) \in [-r,0] \times \Omega_B, \end{cases} \tag{2.4}$$

where

$$F(t-r,x) := f(v(t-r,x) + \phi(x)) - f(\phi(x)) = f'(\phi(x))v(t-r,x) + f''(\hat\phi(x))v^2(t-r,x)$$

for some function $\hat\phi(t,x)$ between $\phi(x)$ and $u(t,x)$, due to the Taylor's formula.

Multiplying (2.4) by $e^{2\mu_1 t}v(t,x)$, we have

$$\begin{aligned} &\Big\{\frac{1}{2}e^{2\mu_1 t}v^2\Big\}_t - De^{2\mu_1 t}\nabla(v\nabla v) + De^{2\mu_1 t}|\nabla v|^2 + (\delta - \mu_1)v^2 \\ &= e^{2\mu_1 t}f'(\phi(x))v(t,x)v(t-r,x) + e^{2\mu_1 t}f''(\tilde\phi)v(t,x)v^2(t-r,x). \end{aligned} \tag{2.5}$$

Integrating the above equation over Ω_B with respect to x, we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\{e^{2\mu_1 t}\|v(t)\|^2_{L^2(\Omega_B)}\} - De^{2\mu_1 t}\int_{\Omega_B}\nabla(v\nabla v)dx \\ &\quad + De^{2\mu_1 t}\|\nabla v(t)\|^2_{L^2(\Omega_B)} + (\delta - \mu_1)e^{2\mu_1 t}\|v(t)\|^2_{L^2(\Omega_B)} \\ &= e^{2\mu_1 t}\int_{\Omega_B} f'(\phi(x))v(t,x)v(t-r,x)dx \\ &\quad + e^{2\mu_1 t}\int_{\Omega_B} f''(\hat\phi)v(t,x)v^2(t-r,x)dx. \end{aligned} \tag{2.6}$$

Noting from Lemma 2.2 that the boundary $\partial\Omega_B$ is close and smooth, which makes the boundary integral to be disappeared (even the boundary value is non-zero):

$$-D\int_{\Omega_B}\nabla(v\nabla v)dx=-D\int_{\partial\Omega_B}v\nabla vdx=-\frac{D}{2}\int_{\partial\Omega_B}\nabla(v^2)dx=0. \tag{2.7}$$

Substituting (2.7) to (2.6), we further get

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\{e^{2\mu_1 t}\|v(t)\|^2_{L^2(\Omega_B)}\}+De^{2\mu_1 t}\|\nabla v(t)\|^2_{L^2(\Omega_B)}+(\delta-\mu_1)e^{2\mu_1 t}\|v(t)\|^2_{L^2(\Omega_B)}\\&=e^{2\mu_1 t}\int_{\Omega_B}f'(\phi(x))v(t,x)v(t-r,x)dx+e^{2\mu_1 t}\int_{\Omega_B}f''(\hat{\phi})v(t,x)v^2(t-r,x)dx.\end{aligned} \tag{2.8}$$

Notice that, it has been showed in [1] that $\lim_{t\to\infty}u(t,\cdot)|_{\Omega_B}=\phi|_{\Omega_B}$. Namely, for any given small number $\varepsilon>0$, there exists a large number $t_*\gg 1$ such that, when $t\geq t_*$, then

$$\max_{x\in\Omega_B}|v(t,x)|=\max_{x\in\Omega_B}|u(t,x)-\phi(x)|\leq\varepsilon,\quad \text{for } t\geq t_*. \tag{2.9}$$

Integrating (2.8) over $[t_*,t]$ with respect to t yields

$$\begin{aligned}&e^{2\mu_1 t}\|v(t)\|^2_{L^2(\Omega_B)}+2D\int_{t_*}^t e^{2\mu_1 s}\|\nabla v(s)\|^2_{L^2(\Omega_B)}ds+2(\delta-\mu_1)\int_{t_*}^t e^{2\mu_1 s}\|v(s)\|^2_{L^2(\Omega_B)}ds\\&=e^{2\mu_1 t_*}\|v(t_*)\|^2_{L^2(\Omega_B)}+2\int_{t_*}^t e^{2\mu_1 s}\int_{\Omega_B}f'(\phi(x))v(s,x)v(s-r,x)dxds\\&\quad+2\int_{t_*}^t e^{2\mu_1 s}\int_{\Omega_B}f''(\hat{\phi})v(s,x)v^2(s-r,x)dxds.\end{aligned} \tag{2.10}$$

For the right hand side of (2.10), we first note the fact $|f'(\phi(x))|\leq|f'(u_+)|=\delta(\ln\frac{p}{\delta}-1)$, for all $x\in\Omega_B$, due to $\frac{1}{a}<\phi(x)\leq u_+$ for $x\in\Omega_B$ and $e<\frac{p}{\delta}<e^2$, which leads to, by using the Cauchy inequality and the variable change $s-r\to s$, that

$$\begin{aligned}&\left|2\int_{t_*}^t e^{2\mu_1 s}\int_{\Omega_B}f'(\phi(x))v(s,x)v(s-r,x)dxds\right|\\&\leq|f'(u_+)|\int_{t_*}^t e^{2\mu_1 s}[\|v(s)\|^2_{L^2(\Omega_B)}+\|v(s-r)\|^2_{L^2(\Omega_B)}]ds\\&=|f'(u_+)|\int_{t_*}^t e^{2\mu_1 s}\|v(s)\|^2_{L^2(\Omega_B)}ds+|f'(u_+)|\int_{t_*-r}^{t-r}e^{2\mu_1(s+r)}\|v(s)\|^2_{L^2(\Omega_B)}ds\\&\leq|f'(u_+)|(1+e^{2\mu_1 r})\int_{t_*}^t e^{2\mu_1 s}\|v(s)\|^2_{L^2(\Omega_B)}ds\\&\quad+|f'(u_+)|e^{2\mu_1 r}\int_{t_*-r}^{t_*}e^{2\mu_1 s}\|v(s)\|^2_{L^2(\Omega_B)}ds.\end{aligned} \tag{2.11}$$

For the nonlinear term of (2.10), from (2.9) it can be estimated as

$$\begin{aligned}&\left|2\int_{t_*}^t e^{2\mu_1 s}\int_{\Omega_B}f''(\tilde{\phi})v(s,x)v^2(s-r,x)dxds\right|\\&\leq C\int_{t_*}^t e^{2\mu_1 s}\max_{x\in\Omega_B}|v(s,x)|\int_{\Omega_B}v^2(s-r,x)dxds\\&\leq C\varepsilon\int_{t_*}^t e^{2\mu_1 s}\int_{\Omega_B}v^2(s-r,x)dxds\qquad[\text{ change of variable: } s-r\to s]\\&=C\varepsilon\int_{t_*-r}^{t-r}e^{2\mu_1(s+r)}\int_{\Omega_B}v^2(s,x)dxds\end{aligned}$$

$$
\begin{aligned}
&= C\varepsilon e^{2\mu_1 r}\int_{t_*-r}^{t-r} e^{2\mu_1 s}\|v(s)\|^2_{L^2(\Omega_B)}ds \\
&= C\varepsilon e^{2\mu_1 r}\int_{t_*}^{t-r} e^{2\mu_1 s}\|v(s)\|^2_{L^2(\Omega_B)}ds + C\varepsilon e^{2\mu_1 r}\int_{t_*-r}^{t_*} e^{2\mu_1 s}\|v(s)\|^2_{L^2(\Omega_B)}ds \\
&\le C\varepsilon e^{2\mu_1 r}\int_{t_*}^{t} e^{2\mu_1 s}\|v(s)\|^2_{L^2(\Omega_B)}ds + C\varepsilon e^{2\mu_1 r}\int_{t_*-r}^{t_*} e^{2\mu_1 s}\|v(s)\|^2_{L^2(\Omega_B)}ds.
\end{aligned}
\tag{2.12}
$$

Substituting (2.11) and (2.12) to (2.10), we further have

$$
\begin{aligned}
&e^{2\mu_1 t}\|v(t)\|^2_{L^2(\Omega_B)} + 2D\int_{t_*}^{t} e^{2\mu_1 s}\|\nabla v(s)\|^2_{L^2(\Omega_B)}ds + B_{\mu_1}\int_{t_*}^{t} e^{2\mu_1 s}\|v(s)\|^2_{L^2(\Omega_B)}ds \\
&= e^{2\mu_1 t_*}\|v(t_*)\|^2_{L^2(\Omega_B)} + [|f'(u_+)| + C\varepsilon]e^{2\mu_1 r}\int_{t_*-r}^{t_*} e^{2\mu_1 s}\|v(s)\|^2_{L^2\Omega_B}ds,
\end{aligned}
\tag{2.13}
$$

where

$$
\begin{aligned}
B_{\mu_1} : &= 2\delta - 2\mu_1 - |f'(u_+)|(1 + e^{2\mu_1 r}) - C\varepsilon e^{2\mu_1 r} \\
&= 2\delta - 2|f'(u_+)| - C\varepsilon - 2\mu_1 - (|f'(u_+)| + C\varepsilon)(e^{2\mu_1 r} - 1) \\
&= 2\delta\Big(2 - \ln\frac{p}{\delta}\Big) - C\varepsilon - 2\mu_1 - (|f'(u_+)| + C\varepsilon)(e^{2\mu_1 r} - 1)
\end{aligned}
\tag{2.14}
$$

Here, we used $|f'(u_+)| = \delta(\ln\frac{p}{\delta} - 1)$. Since $e < \frac{p}{\delta} < e^2$ (i.e., $2 - \ln\frac{p}{\delta} > 0$), and $0 < \varepsilon \ll 1$, then we may take ε as

$$
\varepsilon = \frac{1}{C}\delta(2 - \ln\frac{p}{\delta}),
$$

such that

$$
2\delta\Big(2 - \ln\frac{p}{\delta}\Big) - C\varepsilon = \delta\Big(2 - \ln\frac{p}{\delta}\Big) > 0.
$$

This ensures that, there exists a small number $0 < \mu_1 \ll 1$, such that

$$
\begin{aligned}
B_{\mu_1} &= 2\delta\Big(2 - \ln\frac{p}{\delta}\Big) - C\varepsilon - 2\mu_1 - (|f'(u_+)| + C\varepsilon)(e^{2\mu_1 r} - 1) \\
&= \delta\Big(2 - \ln\frac{p}{\delta}\Big) - 2\mu_1 - \delta(e^{2\mu_1 r} - 1) \\
&\ge 0.
\end{aligned}
$$

Since $t_* = t_*(\varepsilon)$ is a fixed number as ε is given, and $v(t, x)$ is bounded in the finite region $[t_* - r, t_*] \times \Omega_B$, then we finally prove

$$
e^{2\mu_1 t_*}\|v(t_*)\|^2_{L^2(\Omega_B)} + [|f'(u_+)| + C\varepsilon]e^{2\mu_1 r}\int_{t_*-r}^{t_*} e^{2\mu_1 s}\|v(s)\|^2_{L^2\Omega_B}ds \le C.
\tag{2.15}
$$

Substituting (2.15) to (2.13), and noting $B_{\mu_1} > 0$, we immediately obtain (2.2). This completes the proof. □

Remark 2.1. Here we proved the exponential convergence to the steady-state in the region of Ω_B. This is the first attempt to show the convergence rate to the steady-state in the bounded domain Ω_B. But for the convergence in the region Ω_A, we could not get any kind of convergence rate by the existing methods. Moreover, for the critical case $\frac{p}{\delta} = e^2$, it is quite not clear to show the convergence, so far. We expect that the linear convergence could be held when the time delay r is sufficiently small, and the dynamic system would be periodically oscillating when the time delay is bigger than the Hopf bifurcation point $r = r_0$. These will be somewhat interesting topics for future targets.

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