Global solutions for bistable degenerate reaction–diffusion equation with time-delay and nonlocal effect

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\textbf{ABSTRACT}

In this paper, we prove the existence, uniqueness and regularity of global solution to the bistable reaction–diffusion equation with degenerate diffusion and nonlocal time-delay. The adopted approach is the Holmgren’s approximation scheme combining the compactness analysis.

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1. Introduction

This paper concerns the nonlocal degenerate diffusion equation with time-delay, a model of population dynamics for single species [1–3]

\begin{equation}
\begin{aligned}
 & u_t(x,t) - \Delta u^m(x,t) = g(u(x,t), J * S(u(x,t-r))), \quad t \geq 0, \ x \in \mathbb{R}, \\
 & u(x,s) = u_0(x,s), \quad s \in [-r, 0], \ x \in \mathbb{R},
\end{aligned}
\end{equation}

with the convolution

\begin{equation}
J * S(u(x,t-r)) = \int_{\mathbb{R}} J(y) S(u(x-y,t-r)) dy.
\end{equation}

Here, $u(t,x)$ denotes the population density of single species at time $t$ and location $x$, $\Delta u^m(x,t)$ with $m > 1$ is the degenerate diffusion, and $g(\cdot, \cdot)$ is the nonlinear function involving the death rate function and the birth rate function, where $r > 0$ is the matured age of the single species, the so-called time-delay. Throughout the

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Here, we assume that the kernel $J(\cdot)$ is positive and unit, and Eq. (1.1) satisfies the bistable conditions, namely,

(H1) $J(y) \geq 0$ for $y \in \mathbb{R}$, and $\int_{\mathbb{R}} J(y) dy = 1$;

(H2) $g(u,v) \in C^2([0,1] \times [S(0),S(1)],\mathbb{R})$, $g_2(u,v) \geq 0$ for $(u,v) \in [0,1] \times [S(0),S(1)]$, $S(u) \in C^2([0,1],\mathbb{R})$ and $S'(u) \geq 0$ for $u \in [0,1]$;

(H3) $g(0,S(0)) = g(1,S(1)) = 0$, $g_1(1,S(1)) + g_2(1,S(1))S'(1) < 0$ and $g_1(0,S(0)) + g_2(0,S(0))S'(0) < 0$.

Here, $g_1(u,v) = \frac{\partial}{\partial u} g(u,v)$, $g_2(u,v) = \frac{\partial}{\partial v} g(u,v)$, and $S'(w) = \frac{d}{dw} S(w)$.

The typical models of $g(u,v)$ satisfying (H2) – (H3) are as follows:

- **Neural network** [4–6] with $g(u,v) = -\alpha u + v$ for $\alpha > 1$:
  \[
  u_t(x,t) - \Delta u^m(x,t) = -\alpha u + \int_{-\infty}^{\infty} J(x-y)S(u(y,t-r))dy, (t,x) \in \mathbb{R}^+ \times \mathbb{R}, r > 0.
  \]

- **Hodgkin–Huxley** [7,8] with $g(u,v) = u(1-u)v$, $S(u) = u - a$ for $a \in (0,1)$:
  \[
  u_t(x,t) - \Delta u^m(x,t) = u(1-u) \int_{-\infty}^{\infty} J(x-y)(u(y,t-r) - a)dy, (t,x) \in \mathbb{R}^+ \times \mathbb{R}, r > 0.
  \]

Eq. (1.1) possesses some challenging features caused by the degenerate diffusion, time-delay and the nonlocal effect, because the degenerate diffusion usually causes the solutions to lose their regularity, the large time-delay usually makes the solutions to be oscillating, and the nonlocal effect enhances the complexity of calculation. So it is worth to investigate the well-posedness of the solutions and the structure of the solutions, particularly, the traveling wave solutions. In this short note, we study the existence, uniqueness and regularity of the global solutions to (1.1), and leave the structure of sharp traveling waves for future.

For the nonlinear porous media equations without time-delay, the properties of the solutions have been extensively studied in [3,7–13]. The new phenomenon for sharp traveling waves with critical wave speed was first observed by Aronson [14] for Fisher–KPP equations, then further studied by De Pablo and Vazquez [15] and Malaguti and Marcelli [16] for the generalized Fisher–KPP equations with degeneracy of diffusion. The exact form of sharp traveling waves was obtained by Gilding and Kersner [17] in the case of $m = 2$. When these Fisher–KPP (mono-stable type) equations are affected by the time-delay, the sharp waves with or without oscillations were classified by Xu et al. [3,18,19] based on the sizes of time-delay and the wave speed, and the global stability of the non-sharp traveling waves were studied by Huang et al. [11] and Liu-Mei-Yang [12], recently. The asymptotic behavior of solutions for the Dirichlet boundary case were studied in [20–22].

For the regular Huxley (bistable type) reaction–diffusion equations, the smooth and monotone traveling waves were intensively investigated in [4–6,23–26]. When these bistable reaction–diffusion equations with degeneracy of diffusion and time-delay, the existence of sharp traveling waves was proved by Jin et al. [7,8].

However, for the case of the bistable reaction–diffusion equations with the effect of degenerate diffusion, time-delay and nonlocality, the relevant study is almost nothing as we know. As the first step, we are going to prove the existence, uniqueness and regularity of global solution for (1.1). This is the main issue of the present note. The adopted approach is the Holmgren’s iteration scheme with the help of the compactness estimates and the monotonic technique. The crucial steps of the Holmgren’s iteration scheme are to denote a monotonic function for constructing an auxiliary equation and establish the estimates for the solutions of the adjoint equation. Then by upper and lower solutions to iterate.
2. Main result and proof

Definition 2.1. A function \( u \in L^2_{loc}(\mathbb{R}_+ \times \mathbb{R}) \) is called a weak solution of (1.1) with (1.2) if 0 \( \leq \) \( u \leq \) 1, \( \nabla u^m \in L^2_{loc}(\mathbb{R}_+ \times \mathbb{R}) \), and for any \( T > 0 \) and \( \psi \in C^\infty_0((-r,T) \times \mathbb{R}) \),

\[
-\int_0^T \int_\mathbb{R} u(x,t)\psi_t dxdt + \int_0^T \int_\mathbb{R} \nabla u^m(x,t) \cdot \nabla \psi dxdt \\
= \int_\mathbb{R} u_0(x,0)\psi(0,x)dx + \int_0^T \int_\mathbb{R} g(u(x,t), J \ast S(u(x,t-r)))\psi dxdt.
\]

Definition 2.2. A function \( u \) is called an upper (lower) solution of (1.1) if the following inequality holds in the sense of distributions,

\[
u_t(t,x) - (u^m)_{xx}(t,x) - g(u(t,x), J \ast S(u(t-r,x))) \geq (\leq) 0.
\]

Theorem 2.1. Assume that \((H_1)-(H_3)\) hold. Let the initial data \( u_0(s,x) \in L^\infty([-r,0] \times \mathbb{R}) \) satisfy

\[
0 < u_0(s,x) < 1, (s,x) \in [-r,0] \times \mathbb{R},
\]

and

\[
\lim_{x \to -\infty} u_0(s,x) = 0, \quad \lim_{x \to +\infty} u_0(s,x) = 1, \text{ uniformly in } s \in [-r,0].
\]

Then there exists a uniquely global solution of (1.1) with Hölder continuity \( C^{\frac{1}{2m}}(\mathbb{R}_+ \times \mathbb{R}) \) such that \( u \in \mathcal{D} \), and

\[
0 < u(t,x) < 1, (t,x) \in \mathbb{R}_+ \times \mathbb{R},
\]

and

\[
\lim_{x \to -\infty} u(t,x) = 0, \quad \lim_{x \to +\infty} u(t,x) = 1, \text{ uniformly in } t \in \mathbb{R}_+.
\]

where \( \mathcal{D} = \{ u | u \in L^\infty(\mathbb{R}_+ \times \mathbb{R}) \cap C^{\frac{1}{2m}}(\mathbb{R}_+ \times \mathbb{R}); (u^m)_x \in L^\infty(\mathbb{R}_+; L^2_{loc}(\mathbb{R})); (u^m)_t \in L^2_{loc}(\mathbb{R}_+ \times \mathbb{R}) \} \).

For the local equation

\[
\begin{aligned}
u_t(t,x) - \Delta u^m(x,t) &= g(u(x,t), S(u(x,t-r))), & t \geq 0, & x \in \mathbb{R}, \\
u(x,s) &= u_0(x,s), & s \in [-r,0], & x \in \mathbb{R},
\end{aligned}
\]

we have the similar result.

Theorem 2.2. Assume that \((H_2)-(H_3)\) hold. Let the initial data \( u_0(s,x) \in L^\infty([-r,0] \times \mathbb{R}) \) satisfies (2.2) and (2.3). Then there exists a uniquely global solution of (2.5) with Hölder continuity \( C^{\frac{1}{2m}}(\mathbb{R}_+ \times \mathbb{R}) \) such that \( u \in \mathcal{D} \) and

\[
0 < u(t,x) < 1, (t,x) \in \mathbb{R}_+ \times \mathbb{R},
\]

and

\[
\lim_{x \to -\infty} u(t,x) = 0, \quad \lim_{x \to +\infty} u(t,x) = 1, \text{ uniformly in } t \in \mathbb{R}_+.
\]
Since the proof of Theorem 2.2 is pretty similar to Theorem 2.1, therefore we just need to prove Theorem 2.1. Note that Eq. (1.1) is degenerate at $u = 0$, so we will adopt the Holmgren’s approximation scheme and compactness method in this section to treat it.

Let us define a sufficiently smooth sequence $u'_0(s, x)$ with $l \gg 1$ such that $0 < u'_0(s, x) < 1$ for $(s, x) \in [-r, 0] \times [-l, l]$ and $\lim_{t \to \infty} u'_0(s, x) = u_0(s, x)$ uniformly in $x$ and $s$. Now we consider the initial–boundary-value problem as follows:

$$\begin{cases}
u_t(x, t) - \Delta u^m(x, t) = g(u(x, t), J * S(u(x, t - r))), & t \geq 0, \ x \in [-l, l], \\
u(-l, t) = \frac{1}{t}, u(l, t) = 1 - \frac{1}{t}, & t \geq 0, \\
u(x, s) = u'_0(s, x), & (s, x) \in [-r, 0] \times [-l, l] .
\end{cases} \quad (2.6)$$

We state the definitions of the upper and lower solutions of (2.6) and some lemmas before we prove the existence of the approximate solutions to (2.6).

**Definition 2.3.** A function $u \in C(\mathbb{R}_+ \times [-l, l])$ with $u^m \in W^{1,2}_{loc}(\mathbb{R}_+ \times [-l, l])$ is called an upper (lower) solution of (2.6), if it satisfies

$$\begin{cases}
u_t(x, t) - \Delta u^m(x, t) \geq (\leq) g(u(x, t), J * S(u(x, t - r))), & t \geq 0, \ x \in [-l, l], \\
u(-l, t) \geq (\leq) \frac{1}{t}, u(l, t) \geq (\leq) 1 - \frac{1}{t}, & t \geq 0, \\
u(x, s) \geq (\leq) u'_0(s, x), & (s, x) \in [-r, 0] \times [-l, l].
\end{cases}$$

Let $T > 0$ be a given positive constant and $Q_T := [0, T] \times [-l, l]$, define a space

$$C_+([0, T]; H^1([-l, l])) = \{v(t, x)|v(t, x) > 0 \text{ for } (t, x) \in Q_T \text{ and } v \in C([0, T]; H^1([-l, l]))\}. \quad (2.7)$$

For $v(t, x) \in C_+([0, T]; H^1([-l, l]))$, denote

$$F(v) := K_1 v(t, x) + g(v, J * S(v_r)), \ v_r := v(t - r, x), \quad (2.8)$$

where

$$K_1 = \sup_{(u, v) \in [0, 1] \times [S(0), S(1)]} \ |g_1(u, v)|. \quad (2.9)$$

For $0 < v_1(t, x) \leq v_2(t, x), \ (t, x) \in [-r, T] \times [-l, l]$, we have

$$F(v_2) - F(v_1) = K_1 (v_2 - v_1) + g(v_2, J * S(v_2 - v_1)) - g(v_1, J * S(v_2 - v_1)) + g(v_1, J * S(v_2 - v_1)) - g(v_1, J * S(v_1 - r))$$

$$= K_1 (v_2 - v_1) + g_1(\theta_1, J * S(v_2 - v_1))(v_2 - v_1) + g_2(v_1, J * S(\theta_2))S'(\theta_3)J * (v_2 - v_1 - v_1 - r)$$

$$= (K_1 + g_1(\theta_1, J * S(v_2))) (v_2 - v_1) + g_2(v_1, J * S(\theta_2))S'(\theta_3)J * (v_2 - v_1 - r)$$

$$\geq 0, \ (t, x) \in Q_T, \ \theta_i \in [0, 1], \ i = 1, 2, 3,$$ which follows from the definition of $K_1$ and $(H_2)$. Let $v_1 = 0$, then $F(v) \geq 0$ for $(t, x) \in Q_T$. We first pay attention to the following problem

$$\begin{cases}
u_t(x, t) - \Delta u^m(x, t) + K_1 u(t, x) = F(v), & t \geq 0, \ x \in [-l, l], \\
u(-l, t) = \frac{1}{t}, u(l, t) = 1 - \frac{1}{t}, & t \geq 0, \\
u(x, s) = u'_0(s, x), & (s, x) \in [-r, 0] \times [-l, l] .
\end{cases} \quad (2.10)$$

From [12], the following lemmas hold.
Lemma 2.3 ([12]). Let \( u(t,x) \) be the solution of (2.10) with \( 0 < v(t,x) < 1 \) for \((t,x) \in Q_T \) and \( 0 < u_0'(s,x) < 1 \) for \((s,x) \in [-r,0] \times [-l,l] \). Then

\[
0 < u(t,x) < 1, \quad (t,x) \in Q_T.
\]  

(2.11)

Lemma 2.4 ([12]). Let \( 0 < v_1(t,x) \leq v_2(t,x) < 1 \) for \((t,x) \in [-r,T] \times [-l,l] \) and \( u_1(t,x) \) and \( u_2(t,x) \) be the solution of (2.10) with respect to \( v_1(t,x) \) and \( v_2(t,x) \), respectively. Then

\[
0 < u_1(t,x) \leq u_2(t,x) < 1, \quad (t,x) \in Q_T.
\]  

(2.12)

Lemma 2.5. For any given \( f(t,x) \in C_0^\infty(Q_T) \), where \( Q_T := [0,T] \times [-l,l] \) with \( l > 1, \epsilon \in (0,1), \eta \in (0,1) \), and \( 0 \leq A(t,x) \in L^\infty(\mathbb{R}) \cap C(\mathbb{R}), B(t,x) \in C^1(\mathbb{R}) \) is bounded. Let \( \psi_{\epsilon, \eta}(t,x) \) be the solution of the following parabolic problem

\[
\begin{aligned}
\psi_t + (A_\epsilon(t,x) + \eta)\psi_{xx} + B_\epsilon \psi &= f(t,x), & (t,x) \in Q_T, \\
\psi(t,-A) &= 0, & \psi(t,A) = 0, & t \in [0,T], \\
\psi(T,x) &= 0, & x \in [-l,l], 
\end{aligned}
\]  

(2.13)

where \( A_\epsilon(t,x) \) and \( B_\epsilon(t,x) \) are the smooth approximations of \( A(t,x) \) and \( B(t,x) \) respectively, such that

\[
A(t,x) \leq A_\epsilon(t,x) \leq A(t,x) + \epsilon, \quad \lim_{\epsilon \to 0} B_\epsilon(t,x) = B(t,x).
\]  

(2.14)

Then \( \psi_{\epsilon, \eta}(t,x) \) satisfies

\[
\sup_{(t,x) \in Q_T} |\psi_{\epsilon, \eta}(t,x)| \leq C, \quad \int_{Q_T} (A_\epsilon(t,x) + \eta) \left( \frac{\partial^2 \psi_{\epsilon, \eta}(t,x)}{\partial x^2} \right)^2 \, dx \, dt \leq C\eta^{-1},
\]  

(2.15)

and

\[
\int_{Q_T} \left( \frac{\partial \psi_{\epsilon, \eta}(t,x)}{\partial x} \right)^2 \, dx \, dt \leq C\eta^{-1}, \quad \sup_{t \in [0,T]} \int_{-l}^{l} \left| \frac{\partial \psi_{\epsilon, \eta}(t,x)}{\partial x} \right| \, dx \leq C,
\]  

(2.16)

where \( C \) denotes a generic constant which is independent of \( \eta \) and \( \epsilon \), and may take different values in different cases.

Proof. The proof can be similarly done as shown in Lemma 3.2.1 and Lemma 3.2.2 of [13], so we omit its detail.

Now, applying Lemmas 2.3–2.5 to Eq. (2.10), we can prove the existence of the solution for (2.6). \( \square \)

Lemma 2.6. Let \( u_0'(s,t) \in C_+([-r,0];H^1([-l,l])) \) satisfy \( 0 < u_0'(s,t) < 1 \). Then the problem (2.6) admits a unique smooth solution \( u(t,x) \in C_+([0,T];H^1([-l,l])) \) for any \( T > 0 \), satisfying

\[
0 < u(t,x) < 1, \quad (t,x) \in [0,T] \times [-l,l].
\]  

(2.17)

Proof. From Definition 2.3, \( \bar{u} := 1 \) and \( v := 0 \) are a pair of upper and lower solutions of (2.6). Note that, for given function \( v \), the IBVP (2.10) defines an operator \( \mathcal{P} \) such that the solution of (2.10) can be written in the form of \( u = \mathcal{P}(v) \). Thus, by initially setting \( v = u_0' \) and \( v = \bar{v} \), respectively, we can construct two sequences \( \{u^n\} \) and \( \{\bar{v}^n\} \) by \( u^n = \mathcal{P}(u^{n-1}) \) and \( \bar{v}^n = \mathcal{P}(\bar{v}^{n-1}) \), for \( n = 1,2,\ldots \), respectively.

Thanks to Lemmas 2.3 and 2.4, we can prove

\[
0 = u^0 \leq u^1 \leq \cdots \leq u^n \leq \cdots \leq \bar{v}^0 \leq \cdots \leq \bar{v}^n \leq \cdots \leq \bar{v}^1 = 1.
\]
Thus there exist two limit functions $\hat{u}(t, x)$ and $\tilde{u}(t, x)$ with respect to $\{u^n\}$ and $\{\overline{u}^n\}$ as $n \to \infty$, respectively, satisfying

$$0 < \hat{u}(t, x) \leq \tilde{u}(t, x) < 1, \quad (t, x) \in [0, T] \times [-l, l].$$

(2.18)

Next we need to prove $\hat{u}(t, x) = \tilde{u}(t, x)$ as the unique solution of (2.6). Let $w = \tilde{u}(t, x) - \hat{u}(t, x)$, then

$$\begin{cases}
w_t - \partial_x^2(A(t, x)w) - B(t, x)w = 0, & (t, x) \in [0, T] \times [-l, l], \\
w(t, -l) = 0, w(t, l) = 0, & t > 0, \\
w(s, x) = 0, & (s, x) \in [-r, 0] \times [-l, l],
\end{cases}$$

(2.19)

where

$$A(t, x) = m \int_0^1 (\theta \tilde{u} + (1 - \theta)\hat{u})^{m-1} d\theta.$$

We note that $0 < A(t, x) \in C([0, T] \times [-l, l])$, due to (2.18).

For $t \in [0, r], t - r \in [-r, 0],

$$g(\tilde{u}, J * S(\tilde{u}_r)) - g(\hat{u}, J * S(\hat{u}_r)) = g(\tilde{u}, J * S(u_0^0)) - g(\hat{u}, J * S(u_0^0)) =: B(t, x)w,$$

where $B(t, x) := \int_0^1 g_1(\theta \tilde{u} + (1 - \theta)\hat{u}, J * S(u_0^0))d\theta$. Since $|g_1(u, v)| < K_1$ for $(u, v) \in [0, 1] \times [S(0), S(1)]$, $|B(t, x)| < K_1$, then $w$ satisfies

$$\begin{cases}
w_t - \partial_x^2(A(t, x)w) - B(t, x)w = 0, & (t, x) \in [0, T] \times [-l, l], \\
w(t, -l) = 0, w(t, l) = 0, & t > 0, \\
w(s, x) = 0, & (s, x) \in [-r, 0] \times [-l, l].
\end{cases}$$

(2.20)

By the definition of weak solutions, for any test function $\psi \in C^\infty([0, r] \times [-l, l])$, the solution of (2.20) satisfies

$$\int_{Q_r} w(\psi_t + A(t, x)\psi_{xx} + B(t, x)\psi)dxdt = 0,$$

(2.21)

where $Q_r := [0, r] \times [-l, l]$.

Since $A(t, x)$ and $B(t, x)$ are merely bounded and measurable, it is difficult to analyze the solvability of the adjoint problem directly. Based on this reason, for any given $f(t, x) \in C^\infty_0(Q_r)$, we focus on the following approximation adjoint problem

$$\begin{cases}
\psi_t + (A_\varepsilon(t, x) + \eta)\psi_{xx} + B^\delta_{\varepsilon, \eta}\psi = f(t, x), & (t, x) \in Q_r, \\
\psi(t, -l) = 0, \psi(t, l) = 0, \quad & t \in [0, r], \\
\psi(r, x) = 0, \quad & x \in [-l, l].
\end{cases}$$

(2.22)

Here, $\delta > 0$ and $\eta > 0$ are sufficiently small constants, for $|\tilde{u} - \hat{u}| \geq \delta, A(t, x) \geq L(\delta) > 0$, let

$$b^\delta_{\eta} = \begin{cases}
(A(t, x) + \eta)^{-\frac{1}{2}}B, & |\tilde{u} - \hat{u}| \geq \delta, \\
0, & |\tilde{u} - \hat{u}| < \delta,
\end{cases}$$

thus $A_\varepsilon(t, x)$ and $b^\delta_{\varepsilon, \eta}$ are the smooth approximations of $A(t, x)$ and $b^\delta_{\eta}$, and $B^\delta_{\varepsilon, \eta} = b^\delta_{\varepsilon, \eta}(A_\varepsilon + \eta)^{\frac{1}{2}}$.

We take the solution $\psi_{\varepsilon, \eta}$ of (2.22) as the test function in (2.21), then

$$\int_{Q_r} wfdxdt = \int_{Q_r} w(\psi_t + (A_\varepsilon + \eta)\psi_{xx} + B^\delta_{\varepsilon, \eta}\psi)dxdt = \int_{Q_r} w\eta\psi_{xx}dxdt + \int_{Q_r} w(A_\varepsilon - A)\psi_{xx}dxdt + \int_{Q_r} w(B^\delta_{\varepsilon, \eta} - B)\psi dxdt.$$

From [13], let $\varepsilon \to 0, \eta \to 0, \delta \to 0$, then the right-hand side of the above equation tends to zero. By the arbitrariness of $f$, then $w = 0$ for $(t, x) \in Q_r$, namely, $\hat{u} = \tilde{u}$ for $t \in [0, r]$.

Repeating the above procedure to $[r, 2r], [2r, 3r], \ldots$, we obtain $\hat{u} = \tilde{u}$ for $t \in [0, T] \times [-l, l]$. The proof is complete. $\square$
Proof of Theorem 2.1. Let $a \in (-l + 2, l - 2)$, and $\eta(x) \in C_0^\infty((a - 2, a + 2))$ be a function satisfying

$$0 \leq \eta(x) \leq 1; \ |\eta'(x)| \leq 1; \ \eta(x) = 1, \ x \in (a - 1, a + 1).$$

Similar to [12], by Young’s inequality, the mean value theorem of integrals and $0 < u < 1$, it is easy to deduce

$$\int_t^{t+\sigma} u^{m-1}|u_s|^2 \eta^4 dx ds + \sup_t \int |(u^m)_x|^2 \eta^4 dx \leq C,$$

where $C$ is independent of $l$. We denote the weak limit of $u^l$ by $u$ as $l \to \infty$. The solutions of (1.1) satisfy $0 \leq u \leq 1$ and $u^m \in L^\infty(\mathbb{R}^+_+; H^1_{loc}(\mathbb{R}))$. The solutions also satisfy $(u^m)_t \in L^2((t, t+\delta) \times (a - 1, a + 1))$. For any $t_2 \leq t_1 \in \mathbb{R}^+_+, x \in \mathbb{R}$, take a ball $B_r$ of radius $r$ centered at $x$, where $r = |t_1 - t_2|^{\frac{1}{2}}$. By the Poincaré inequality and the mean value theorem, there exists $x^* \in B_r$ such that

$$|u^m(t_1, x^*) - u^m(t_2, x^*)| \leq C|t_1 - t_2|^{\frac{1}{2}} r^{-\frac{1}{2}}.$$

Then

$$|u^m(t_1, x) - u^m(t_2, x)| \leq C|t_1 - t_2|^{\frac{1}{4}},$$

which implies that $u^m \in C^{\frac{1}{4}, \frac{1}{2}}(\mathbb{R}^+_+ \times \mathbb{R})$. Furthermore, $u \in C^{\frac{1}{2}, \frac{1}{2}}(\mathbb{R}^+_+ \times \mathbb{R})$.

For the uniqueness of original solution, let $u_1$ and $u_2$ be two solutions of (1.1), and denote $v = u_1 - u_2$. When $t \in [0, r]$, $u_1(t-r,x) = u_2(t-r,x) = u_0(t-r,x)$, we note that

$$\left\{\begin{array}{l}
u_t - \partial_x^2(A(t)x) v - B(t,x)v = 0, \quad (t,x) \in \mathbb{R}^+_+ \times \mathbb{R}, \\
\lim_{x \to \pm \infty} v(t,x) = 0, \quad t \in [0,r], \\
v(s,x) = 0, \quad (s,x) \in [-r,0] \times \mathbb{R},
\end{array}\right.$$\n
where

$$A(t,x) := m \int_0^1 (\theta u_1 + (1-\theta) u_2)^{m-1} d\theta,$$

and

$$B(t,x) := \int_0^1 g_1(\theta u_1 + (1-\theta) u_2, J * S(u_0)) d\theta.$$\n
From the definition of weak solution, the weak form for the above equation is reduced to

$$\int_{t_0}^{t_1} \int_{\mathbb{R}} u(t,x) \left[ \frac{\partial \psi}{\partial t} + DA(t,x) \frac{\partial^2 \psi}{\partial x^2} + B(t,x) \psi \right] dx dt = 0,$$

(2.24)

with any test function $\psi \in C_0^\infty([-r,T] \times \mathbb{R})$. For any smooth functions $g(x,t) \in C_0^\infty(R \times (0, r))$, let $B_{R_0}$ be the ball with radius $R_0 > 0$ such that supp $g \subset B_{R_0}$. For any $R > R_0 + 1$, $\eta > 0$, and $\varepsilon > 0$, $A_\varepsilon(t,x)$ and $B_\varepsilon(t,x)$ are the smooth approximations of $A(t,x)$ and $B(t,x)$ respectively, we consider the following adjoint problem in $C_0^\infty([0,r] \times \mathbb{R})$:

$$\left\{\begin{array}{l}
\frac{\partial \psi}{\partial t} + (DA_\varepsilon(t,x) + \eta) \frac{\partial^2 \psi}{\partial x^2} + B_\varepsilon(t,x) \psi = -g(x,t), \quad |x| < R, \ t \in (0,r], \\
\frac{\partial \psi}{\partial \nu} = 0, \quad |x| = R, \ t \in (0,r], \\
\psi(x,r) = 0, \quad |x| < R.
\end{array}\right.$$\n
(2.25)

Let $\psi(t,x) = \psi_{\varepsilon,\eta}(t,x)$ be the solution of (2.25) with zero extension to the whole space of $\mathbb{R}$, and let $\xi_R(x) \in C_0^\infty(\mathbb{R})$ be a cut-off function such that $0 \leq \xi_R(x) \leq 1$, $\xi_R(x) = 1$ for $|x| \leq R - 1$, and $\xi_R(x) = 0$ for $|x| > R - \frac{1}{2}$. Particularly, by taking $\phi(t,x) = \xi_R(x) \psi_{\varepsilon,\eta}(t,x)$ as the test function in (2.24), using Lemma 2.5, by the same fashion of Proposition 3.2 in [12], we derive

$$\int_0^r \int g(x,t) dx dt = 0.$$

Repeating the same procedure and using the arbitrariness of $g(x,t)$, we can prove the uniqueness. The proof of Theorem 2.1 is complete. \qed
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