

Research Article

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Sharp viscous shock waves for relaxation model with degeneracy

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Abstract: This article is concerned with a relaxation model with degeneracy. Different from the existing studies on the regular viscous shock waves, we recognize that when the sub-characteristic condition is degenerate, the dynamical system possesses sharp viscous shock waves, a new type of viscous shock waves with sharp corners and being regionally degenerate in a half space. The regularities of these sharp viscous shock waves are further derived. Some of them can be smooth due to different degree of degeneracy, but most of them are Hölder continuous with sharp corners. Finally, the exact sharp viscous shock waves are constructed for three typical equations, and the corresponding numerical simulations with given initial data are also presented, which perfectly match the exact sharp viscous shock waves and also numerically demonstrate the stability of sharp shock waves.

Keywords: relaxation model, sharp viscous shock waves, regularity, degeneracy

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1 Introduction

The phenomenon of relaxation arises from many physical situations such as traffic flows, river flows, chromatography, and kinetic theory [5,8,21]. It can be described by the following hyperbolic equations of conservation laws:

$$\begin{cases} u_t + \tilde{f}(u, v)_x = 0, \\ v_t + \tilde{g}(u, v)_x = \tilde{h}(u, v), \end{cases} \quad x \in \mathbb{R}^1, \quad t \in \mathbb{R}_+. \quad (1.1)$$

With the essential character of relaxation, in this article, we consider the Jin-Xin relaxation model [13] by setting $\tilde{f}(u, v) = v$, $\tilde{g}(u, v) = g(u)$, and $\tilde{h}(u, v) = \frac{f(u) - v}{\varepsilon}$ in (1.1) as follows:

$$\begin{cases} u_t + v_x = 0, \\ v_t + g(u)_x = \frac{f(u) - v}{\varepsilon}, \end{cases} \quad x \in \mathbb{R}^1, \quad t \in \mathbb{R}_+, \quad (1.2)$$

where the term of $-\frac{v}{\varepsilon}$ presents the effect of relaxation, $\varepsilon > 0$ is the so-called relaxation time, and the flux function $f(u)$ is non-convex but $g(u)$ is convex with

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$$g''(u) > 0, \text{ for } u \text{ under consideration.} \quad (1.3)$$

Under the scaling $(x, t) \rightarrow (\varepsilon x, \varepsilon t)$, system (1.2) is reduced to

$$\begin{cases} u_t + v_x = 0, \\ v_t + g(u)_x = f(u) - v, \end{cases} \quad x \in \mathbb{R}^1, \quad t \in \mathbb{R}_+. \quad (1.4)$$

In this article, we mainly consider the aforementioned Jin-Xin relaxation model (1.4) subject to the initial data

$$(u, v)(0, x) = (u_0(x), v_0(x)) \rightarrow (u_{\pm}, v_{\pm}), \quad \text{as } x \rightarrow \pm\infty, x \in \mathbb{R}^1, \quad (1.5)$$

where $(u_{\pm}, v_{\pm})$ are the state constants. Without loss of generality, we assume

$$u_- > u_+.$$

The relaxation model was first investigated by Liu [18] for the nonlinear stability criteria for diffusion waves, expansion waves, and the regular smooth traveling waves (called also the regular viscous shock waves) by considering $\tilde{f}(u, v) = v$, $\tilde{g}(u, v) = au$, and $\tilde{h}(u, v) = \frac{f(u) - v}{\varepsilon}$ in (1.1). In [18], to guarantee the existence of smooth regular traveling waves $(U, V)(x - st)$ connected with the constant states $(u_{\pm}, v_{\pm})$, the so-called strict sub-characteristic condition must hold:

$$g'(u) > s^2, \quad \text{for } u \text{ under consideration,}$$

where s is the wave speed determined by the Rankine-Hugoniot condition

$$s = \frac{v_+ - v_-}{u_+ - u_-} = \frac{f(u_+) - f(u_-)}{u_+ - u_-}.$$

The relaxation time limits as $\varepsilon \rightarrow 0$ were deeply studied by Chen and Liu [7] and Chen-Levermore-Liu [6], and the numerical analysis and computations for relaxation time limits were significantly carried out by Jin and Xin [13], respectively. The stability of traveling wave with non-convex flux for the same model was studied by Liu et al. [16,17]. The optimal algebraic time convergence rates and the exponential time convergence rates to the traveling waves were further derived by Mei and Yang [20] (see also the boundary cases by Mei and Rubino [19], Hsiao et al. [12], and references therein). These traveling waves mentioned in the aforementioned works are the regular (smooth) viscous shock waves.

In this article, when the sub-characteristic condition is degenerate:

$$g'(u_+) = s^2, \quad (1.6)$$

we recognize that, such a degenerate sub-characteristic condition will cause all viscous shock waves for the dynamic system to be sharp with corner edges (we call them the sharp viscous shock waves), namely, there exists a number ξ_0 , such that the viscous shock waves $(U, V)(x - st) = (U, V)(\xi)$ satisfy $(U, V)(\xi) > (u_+, v_+)$ for $\xi \in (-\infty, \xi_0)$, and $(U, V)(\xi) \equiv (u_+, v_+)$ for $\xi \in [\xi_0, +\infty)$. Usually, the viscous shock wave $(U, V)(\xi)$ forms a corner at $\xi = \xi_0$. For a clearer visualization of regular viscous shock waves and sharp viscous shock waves, we present them in Figure 1.

Different from the regular viscous shock waves for model (1.1), the sharp viscous shock waves lose their regularity and become non-differentiable. Particularly, we expect that the sharp viscous shock waves have different regularities at the corner ξ_0 based on the different degeneracies of $f(u)$ and $g(u)$ near u_+ as $\xi \rightarrow \xi_0^-$,

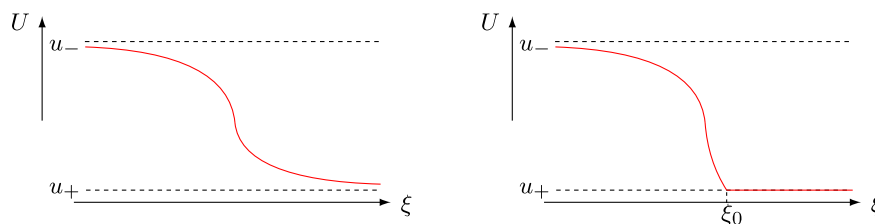


Figure 1: (left) Regular viscous shock wave; (right) sharp viscous shock wave.

and different decay rates at the far field $\xi = -\infty$. The main purpose of this article is to show the existence and uniqueness (up to shift) of the sharp viscous shock waves, and their regularities, as well as their decay rates at the far fields.

Regarding sharp traveling wave, it was first observed by Aronson [1] in 1980 for Fisher-KPP equation with degenerate diffusion

$$u_t - (u^m)_{xx} = u(1 - u), \quad m > 1,$$

when the wave speed is the minimum speed $s = s^*$. In particular, when $m = 2$, an exact sharp traveling wave was constructed by Gilding and Kersner [10]. The study was then generalized by De Pablo and Vázquez [9] for

$$u_t - (u^m)_{xx} = u^n(1 - u), \quad m > 1, \quad n > 1.$$

As shown in [1,9,10], the sharp traveling waves occur only for the critical traveling waves with the minimum wave speed $s = s^*$, and the non-critical traveling waves with speed $s > s^*$ are the regular smooth traveling waves.

For the Nagomo equation with degenerate diffusion

$$u_t - (u^m)_{xx} = u(1 - u)(u - \theta) = f(\bar{u}), \quad m > 1, \quad \theta \in (0, 1/2),$$

Hosono [11] proved the existence of the traveling waves. The waves are sharp when the speed of waves is non-negative: $s = s(\bar{f}, m) = \int_0^1 f(\bar{u})u^{m-1} du \geq 0$, and the waves are regular traveling waves when the speed of waves is negative: $s = s(\bar{f}, m) = \int_0^1 f(\bar{u})u^{m-1} du < 0$.

The sharp traveling waves are also investigated by Xu et al. [24] for the combustion equation with degenerate diffusion and by Audrito and Vázquez [2,3] and Xu et al. [23] for the double nonlinear degenerate diffusion equations, recently (see also the significant works in [4,14,15,22] and references therein).

From the aforementioned studies, we know that all sharp waves exist for these types of reaction-diffusion equations with degenerate diffusion. The degeneracy of diffusion essentially causes the traveling waves to be sharp.

Regarding hyperbolic system of conservation laws from fluid dynamics, including the relaxation model (1.1), usually the shock waves are either the discontinuous solutions (inviscid shock waves) when the hyperbolic system losses its viscosity, such as Euler equations, or the smooth viscous shock waves when the hyperbolic system possesses the viscosity, such as Navier-Stokes equations. But the sharp viscous shock waves have not been observed for the hyperbolic system of conservation laws, so far. From the degenerate sub-characteristic condition $s^2 = g'(u_+)$, we realize that the hyperbolic system (1.4) becomes degenerate at the constant state u_+ . Such a degeneracy will weaken the smooth viscous shock waves and lead them to be sharp with corner and regionally degenerate in the half space. This new discovery opens a window for the study of sharp viscous shock waves to hyperbolic conservation laws and could be applied to a large amount of hyperbolic equations from fluid dynamics. On the other hand, in this article, we also construct three exact sharp viscous shock waves for the relaxation model (1.4) with specially chosen functions $f(u)$ and $g(u)$. For the given initial data, we carry out some numerical simulations, which perfectly demonstrate that the solutions behave like sharp viscous shock waves. Namely, we numerically verify the stability of sharp viscous shock waves. For its theoretical proof, we leave it for future.

The article is organized as follows. In Section 2, we give the main existence theorem of sharp viscous shock wave solutions. In Section 3, the existence and regularity of sharp viscous shock waves are investigated in detail. In Section 4, three typical examples are given, and the exact sharp viscous shock waves are obtained. The corresponding numerical simulations are carried out, too.

Notation. In what follows, c denotes a generic constant that can change from one line to another. We denote $f(x) \sim g(x)$ as $x \rightarrow x_0$ when $c^{-1}g \leq f \leq cg$ in a neighborhood of x_0 .

2 Main results

Before stating our main theorems, we make the following definitions.

Definition 2.1. (Regular viscous shock waves) The solution $(U, V)(\xi)$, where $\xi = x - st$, to (1.4) is said to be a pair of regular viscous shock waves if $(U, V)(\xi) \in C^1(\mathbb{R})$ with $g(U) \in C^1(\mathbb{R})$, and it holds

$$\begin{cases} -sU_\xi + V_\xi = 0, \\ -sV_\xi + g(U)_\xi = f(U) - V, \\ (U, V)(\pm\infty) = (u_\pm, v_\pm), \end{cases} \quad (2.1)$$

where $\partial_\xi = \frac{d}{d\xi}$ and s is the propagation speed of shock waves.

Definition 2.2. (Sharp viscous shock waves) The viscous shock wave solution $(U, V)(\xi)$ to (1.4) is said to be sharp if $(U, V)(\xi) \in W_{\text{loc}}^{1,1}(\mathbb{R}) \cap L^\infty(\mathbb{R})$ with $g(U) \in W_{\text{loc}}^{1,1}(\mathbb{R})$, and there exists ξ_0 such that $(U, V)(\xi) \equiv (u_+, v_+)$ when $\xi \in [\xi_0, +\infty)$; meanwhile, it holds that when $\xi \in (-\infty, \xi_0)$,

$$\begin{cases} -sU_\xi + V_\xi = 0, \\ -sV_\xi + g(U)_\xi = f(U) - V, \\ (U, V)(-\infty) = (u_-, v_-), \\ (U, V)(\xi) \rightarrow (u_+, v_+), \quad \text{as } \xi \rightarrow \xi_0^-. \end{cases} \quad (2.2)$$

Remark 1. The sharp viscous shock wave solution $(U, V)(\xi)$ can also take the form of $(U, V)(\xi) \equiv (u_-, v_-)$ when $\xi \in (-\infty, \xi_0]$. Meanwhile, it holds that when $\xi \in (\xi_0, +\infty)$,

$$\begin{cases} -sU_\xi + V_\xi = 0, \\ -sV_\xi + g(U)_\xi = f(U) - V, \\ (U, V)(+\infty) = (u_+, v_+), \\ (U, V)(\xi) \rightarrow (u_-, v_-), \quad \text{as } \xi \rightarrow \xi_0^+. \end{cases} \quad (2.3)$$

We now state the main existence theorem as follows.

Theorem 2.3. (Existence of sharp viscous shock waves) Assume that $u_- > u_+$ and the degenerate sub-characteristic condition (1.6): $s^2 = g'(u_+)$ holds.

- Necessary condition: if system (1.4) admits a sharp viscous shock wave $(U, V)(\xi)$, where $\xi = x - st$, connecting (u_-, v_-) and (u_+, v_+) with the corner at ξ_0 , namely, $(U, V)(\xi)$ is monotone when $\xi \in (-\infty, \xi_0)$ and $(U, V) \equiv (u_+, v_+)$ when $\xi \in [\xi_0, +\infty)$, then $u_\pm, v_\pm$ and s must satisfy the Rankine-Hugoniot condition:

$$s = \frac{v_+ - v_-}{u_+ - u_-} = \frac{f(u_+) - f(u_-)}{u_+ - u_-}, \quad (2.4)$$

and the generalized entropy condition:

$$q(U) = f(U) - f(u_\pm) - s(U - u_\pm) < 0, \quad \text{for } U \in (u_+, u_-). \quad (2.5)$$

- Sufficient condition: suppose that the Rankine-Hugoniot condition (2.4) and the generalized shock condition (2.5) hold, then for a given number ξ_0 , there exists a sharp viscous shock wave solution $(U, V)(\xi)$ of (1.4) with the corner at ξ_0 satisfying Definition 2.2, which is unique up to shift in ξ .

Of even greater significance is the fact that the regularity of the sharp viscous shock wave $(U, V)(\xi)$ at the corner ξ_0 depends on the degeneracies of $f(u)$ and $g(u)$ at u_+ . The degeneracy of sub-characteristic condition $s^2 = g'(u_+)$ may essentially affect the regularity of the sharp viscous shock waves. Here, we give the definition of degeneracy degree for the sharp viscous shock waves.

Definition 2.4. (Degeneracy degree of sharp viscous shock waves) Let $(U, V)(x - st)$ be the sharp viscous shock waves for system (1.4).

- 1-degree of degeneracy: if $s^2 = g'(u_+)$ but $g''(u_+) \neq 0$, then we call the sharp viscous shock waves to be 1-degree of degeneracy.
- n -degree of degeneracy: if $s^2 = g'(u_+)$ but

$$g''(u_+) = \dots = g^{(n)}(u_+) = 0 \text{ and } g^{(n+1)}(u_+) \neq 0, \text{ for some integer } n \geq 2, \quad (2.6)$$

then we call the sharp viscous shock waves to be the n -degree of degeneracy.

- Singular 1-degree of degeneracy: if

$$|g'(u) - s^2| = O(1)|u - u_+|^\alpha, \text{ as } u \rightarrow u_+, \text{ for } 0 < \alpha < 1, \quad (2.7)$$

which implies

$$s^2 = g'(u_+), \quad g''(u_+) = \infty,$$

then we call the sharp viscous shock waves to be the singular 1-degree of degeneracy.

- Singular n -degree of degeneracy: if

$$|g'(u) - s^2| = O(1)|u - u_+|^{n+\alpha-1}, \text{ as } u \rightarrow u_+, \text{ for } 0 < \alpha < 1, \quad (2.8)$$

which implies

$$s^2 = g'(u_+), \quad g''(u_+) = \dots = g^{(n)}(u_+) = 0, \quad g^{(n+1)}(u_+) = \infty,$$

then we call the sharp viscous shock waves to be the singular n -degree of degeneracy.

Regarding the generalized entropy condition (2.5), it is easy to see that when $f(u)$ is convex, then the generalized entropy condition (2.5) is equivalent to Lax's entropy condition:

$$f'(u_+) < s < f'(u_-). \quad (2.9)$$

When $f(u)$ is non-convex, the generalized shock condition (2.5) implies

$$f'(u_+) \leq s \leq f'(u_-). \quad (2.10)$$

In the case of $s = f'(u_+)$ or $s = f'(u_-)$, we call the entropy condition (2.5) to be *degenerate at $U = u_+$ or u_-* . As shown in the following, the degeneracy of the entropy condition will also essentially affect the regularity of the sharp viscous shock waves. We denote the degeneracy of entropy condition (2.5) as follows.

Definition 2.5. (Degeneracy of entropy condition) When the Lax's entropy condition (2.9) holds, namely, $f'(u_+) < s < f'(u_-)$, the entropy condition is said to be non-degenerate.

When $f'(u_+) = s < f'(u_-)$, the entropy condition is said to be degenerate at u_+ . In particular, if it holds

$$f''(u_+) = \dots = f^{(k_+)}(u_+) = 0 \text{ and } f^{(k_++1)}(u_+) \neq 0, \text{ for some integer } k_+ \geq 2, \quad (2.11)$$

then the entropy condition is said to be degenerate k_+ -times at u_+ .

Similarly, when $f'(u_+) < s = f'(u_-)$, the entropy condition is said to be degenerate at u_- . In particular, if it holds

$$f''(u_-) = \dots = f^{(k_-)}(u_-) = 0 \text{ and } f^{(k_-+1)}(u_-) \neq 0, \text{ for some integer } k_- \geq 2. \quad (2.12)$$

then the entropy condition is said to be degenerate k_- -times at u_- .

Now, we are ready to state the second main results on the regularity of sharp viscous shock waves.

Theorem 2.6. (Regularity of sharp viscous shock waves) Assume that $u_- > u_+$ and the degenerate sub-characteristic condition (1.6) holds. Let $(U, V)(\xi)$ be a sharp viscous shock wave satisfying the Rankine-Hugoniot condition (2.4) and the generalized shock condition (2.5), where $\xi = x - st$.

- For the 1-degree of degeneracy of sharp viscous shock waves, namely, $s^2 = g'(u_+)$ but $g''(u_+) \neq 0$, then the regularity of the sharp viscous shock waves is:

- if $f'(u_+) < s \leq f'(u_-)$, then

$$(U, V)(\xi) \in C^0(\mathbb{R}). \quad (2.13)$$

- if $f'(u_+) = s \leq f'(u_-)$, $f''(u_+) \neq 0$, then

$$(U, V)(\xi) \in C^\infty(\mathbb{R}). \quad (2.14)$$

- if $f'(u_+) = s \leq f'(u_-)$, $f''(u_+) = \dots = f^{(k_+)}(u_+) = 0$, and $f^{(k_++1)}(u_+) \neq 0$, then

$$(U, V)(\xi) \in C^\infty(\mathbb{R}). \quad (2.15)$$

- For the n -degree of degeneracy of sharp viscous shock waves, namely, $s^2 = g'(u_+)$, $g''(u_+) = \dots = g^{(n)}(u_+) = 0$, but $g^{(n+1)}(u_+) \neq 0$, then the regularity of sharp viscous shock waves is

- if $f'(u_+) < s \leq f'(u_-)$, then

$$(U, V)(\xi) \in C^{\frac{1}{n}}(\mathbb{R}), \quad (2.16)$$

where $C^{\frac{1}{n}}(\mathbb{R})$ is the Hölder continuous space with the exponent $0 < \frac{1}{n} < 1$ for $n \geq 2$.

- if $f'(u_+) = s \leq f'(u_-)$, $f''(u_+) \neq 0$, then

$$(U, V)(\xi) \in \begin{cases} C^0(\mathbb{R}), & \text{if } n = 2, \\ C^{\frac{1}{n-1}}(\mathbb{R}), & \text{if } n > 2, \end{cases} \quad (2.17)$$

where $C^{\frac{1}{n-1}}(\mathbb{R})$ is the Hölder continuous space with the exponent $0 < \frac{1}{n-1} < 1$ for $n > 2$.

- if $f'(u_+) = s \leq f'(u_-)$, $f''(u_+) = \dots = f^{(k_+)}(u_+) = 0$, and $f^{(k_++1)}(u_+) \neq 0$, then

$$(U, V)(\xi) \in \begin{cases} C^{\frac{1}{n-k_+}}(\mathbb{R}), & \text{if } k_+ < n - 1, \\ C^0(\mathbb{R}), & \text{if } k_+ = n - 1, \\ C^\infty(\mathbb{R}), & \text{if } k_+ \geq n, \end{cases} \quad (2.18)$$

where $C^{\frac{1}{n-k_+}}(\mathbb{R})$ is the Hölder continuous space with the exponent $0 < \frac{1}{n-k_+} < 1$ for $k_+ < n - 1$.

- For the singular 1-degree of degeneracy of sharp viscous shock waves, namely,

$$|g'(u) - s^2| = O(1)|u - u_+|^\alpha, \text{ as } u \rightarrow u_+, \text{ for } 0 < \alpha < 1, \quad (2.19)$$

which implies

$$s^2 = g'(u_+), \quad g''(u_+) = \infty,$$

then the regularity of sharp viscous shock waves is:

- if $f'(u_+) < s \leq f'(u_-)$, then

$$(U, V)(\xi) \in C^{m,\beta}(\mathbb{R}), \quad (2.20)$$

where $m = \left\lfloor \frac{1}{\alpha} \right\rfloor$, the maximum integer less than or equal to $\frac{1}{\alpha}$, and $0 \leq \beta = \frac{1}{\alpha} - \left\lfloor \frac{1}{\alpha} \right\rfloor < 1$, and $C^{m,\beta}(\mathbb{R})$ is the m th differentiable space with β -Hölder continuity.

- if $f'(u_+) = s \leq f'(u_-)$, $f''(u_+) \neq 0$, then

$$(U, V)(\xi) \in C^\infty(\mathbb{R}); \quad (2.21)$$

- if $f'(u_+) = s \leq f'(u_-)$, with $f''(u_+) = \dots = f^{(k_+)}(u_+) = 0$ and $f^{(k_++1)}(u_+) \neq 0$, then

$$(U, V)(\xi) \in C^\infty(\mathbb{R}); \quad (2.22)$$

- For the singular n -degree of degeneracy of sharp viscous shock waves, namely,

$$|g'(u) - s^2| = O(1)|u - u_+|^{n+\alpha-1}, \text{ as } u \rightarrow u_+, \text{ for } 0 < \alpha < 1, \quad (2.23)$$

which implies

$$s^2 = g'(u_+), \quad g''(u_+) = \dots = g^{(n)}(u_+) = 0, \quad g^{(n+1)}(u_+) = \infty,$$

then the regularity of sharp viscous shock waves is:

★ if $f'(u_+) < s \leq f'(u_-)$, then

$$(U, V)(\xi) \in C^{\frac{1}{n+a-1}}(\mathbb{R}). \quad (2.24)$$

★ if $f'(u_+) = s \leq f'(u_-)$, $f''(u_+) \neq 0$, then

$$(U, V)(\xi) \in \begin{cases} C^{m,\beta}(\mathbb{R}), & \text{if } n = 2, \\ C^{\frac{1}{n+a-2}}(\mathbb{R}), & \text{if } n > 2. \end{cases} \quad (2.25)$$

★ if $f'(u_+) = s \leq f'(u_-)$, with $f''(u_+) = \dots = f^{(k_+)}(u_+) = 0$ and $f^{(k_++1)}(u_+) \neq 0$, then

$$(U, V)(\xi) \in \begin{cases} C^{\frac{1}{n+a-k_+-1}}(\mathbb{R}), & \text{if } n \geq k_+ + 2, \\ C^{m,\beta}(\mathbb{R}), & \text{if } n = k_+ + 1, \\ C^\infty(\mathbb{R}), & \text{if } n < k_+ + 1. \end{cases} \quad (2.26)$$

Theorem 2.7. (Decay properties) Assume that $u_- > u_+$ and the degenerate sub-characteristic condition (1.6) holds. Let $(U, V)(\xi)$ be a sharp viscous shock wave satisfying the Rankine-Hugoniot condition (2.4) and the generalized shock condition (2.5), where $\xi = x - st$. Then, the decay properties of the sharp viscous shock waves $(U, V)(\xi)$ at far field $\xi = -\infty$ are

$$\begin{aligned} |(U, V)(\xi)| &\sim |(U - u_-, V - v_-)(\xi)| \sim \exp(-\hat{\lambda}|\xi|), \quad \text{for } f'(u_+) \leq s < f'(u_-), \\ |(U, V)(\xi)|^{\frac{1}{k_-}} &\sim |(U - u_-, V - v_-)(\xi)| \sim |\xi|^{-1}, \quad \text{for } f'(u_+) < s = f'(u_-), f''(u_-) \neq 0, \\ |(U, V)(\xi)|^{\frac{1}{1+k_-}} &\sim |(U - u_-, V - v_-)(\xi)| \sim |\xi|^{-\frac{1}{k_-}}, \quad \text{for } f'(u_+) < s = f'(u_-), f''(u_-) = 0, \end{aligned} \quad (2.27)$$

where $k_- \geq 2$ and $\hat{\lambda} := \frac{f'(u_-) - s}{g'(u_-) - g'(u_+)} > 0$.

3 Existence and regularity analysis of sharp viscous shock waves

This section is devoted to the proof of the sufficient and necessary conditions for the existence of sharp viscous shock waves, their regularities, and their decay properties at $\xi = -\infty$.

3.1 Existence of sharp viscous shock waves

Proof of Theorem 2.3. First, we derive the necessary conditions (2.4) and (2.5), namely, for any given sharp viscous shock waves, they must satisfy the Rankine-Hugoniot condition (2.4) and the generalized shock condition (2.5). In fact, let $(U, V)(x - st) = (U, V)(\xi)$ be a pair of sharp viscous shock waves. From Definition 2.2, there exists a number ξ_0 such that

$$(U, V)(\xi) \rightarrow \begin{cases} (u_-, v_-), & \text{as } \xi \rightarrow -\infty, \\ (u_+, v_+), & \text{as } \xi \rightarrow \xi_0^-, \end{cases} \quad \text{and } (U, V)(\xi) \text{ is monotonic for } \xi \in (-\infty, \xi_0), \quad (3.1)$$

and

$$(U, V)(\xi) \equiv (u_+, v_+), \quad \text{for } \xi \in [\xi_0, +\infty), \quad (3.2)$$

where $u_+ < u_-$ and s is the propagation speed of the sharp viscous shock wave. From (1.4), we have that when $\xi \in (-\infty, \xi_0)$, the sharp viscous shock wave satisfies

$$\begin{cases} -sU_\xi + V_\xi = 0, \\ -sV_\xi + g(U)_\xi = f(U) - V, \end{cases} \quad \xi \in (-\infty, \xi_0). \quad (3.3)$$

Substituting (3.3)₁ into (3.3)₂, we have

$$(g'(U) - s^2)U_\xi = f(U) - V, \quad \text{for } \xi \in (-\infty, \xi_0). \quad (3.4)$$

Remarkably, (3.4) is degenerate if $g'(U) - s^2 = 0$. In virtue of the convexity of $g(u)$, we have $g'(u_-) > g'(U) > g'(u_+)$ for $U \in [u_+, u_-]$. Thus, we realize that equation (3.4) is degenerate at $U = u_+$ with

$$g'(u_+) = s^2. \quad (3.5)$$

This is the so-called degenerate sub-characteristic condition. Note the degenerate sub-characteristic condition, and the degenerate differential equation (3.4) is reduced to

$$(g'(U) - g'(u_+))U_\xi = f(U) - V, \quad \text{for } \xi \in (-\infty, \xi_0). \quad (3.6)$$

Noting that $(U, V)(-\infty) = (u_-, v_-)$ and $(U, V)(\xi_0) = (u_+, v_+)$ and integrating (3.3)₁ over $(-\infty, \xi)$ and (ξ, ξ_0) , respectively, we obtain

$$-sU + V = -su_+ + v_+ = -su_- + v_-. \quad (3.7)$$

Since $(U_\xi, V_\xi)(-\infty) = (0, 0)$, by taking ξ to $-\infty$ in (3.3)₂, we obtain

$$v_- = f(u_-). \quad (3.8)$$

On the other hand, considering that $(U_\xi, V_\xi) \equiv (0, 0)$ when $\xi \in (\xi_0, +\infty)$ and by virtue of the degenerate sub-characteristic condition (3.5), we have

$$v_+ = f(u_+). \quad (3.9)$$

Applying (3.8) and (3.9) to (3.7), it is then rewritten as

$$-sU + V = -su_\pm + v_\pm = -su_\pm + f(u_\pm), \quad \text{for } \xi \in (-\infty, \xi_0). \quad (3.10)$$

Clearly, (3.10) implies the Rankine-Hugoniot condition:

$$s = \frac{v_+ - v_-}{u_+ - u_-} = \frac{f(u_+) - f(u_-)}{u_+ - u_-}. \quad (3.11)$$

Substituting (3.10) into (3.6), we have

$$(g'(U) - g'(u_+))U_\xi = f(U) - f(u_\pm) - s(U - u_\pm) = q(U), \quad \text{for } \xi \in (-\infty, \xi_0). \quad (3.12)$$

Denote

$$U_\xi = \frac{f(U) - f(u_\pm) - s(U - u_\pm)}{g'(U) - g'(u_+)} = h(U), \quad \text{for } \xi \in (-\infty, \xi_0). \quad (3.13)$$

Since the sharp viscous shock wave $U(\xi)$ is monotonically decreasing for $\xi \in (-\infty, \xi_0)$, namely, $U_\xi < 0$ for $\xi \in (-\infty, \xi_0)$, and $g(U)$ is convex, which implies $g'(U) - g'(u_+) > 0$ for $U > u_+$, then from (3.13), we immediately obtain the generalized entropy condition:

$$q(U) = f(U) - f(u_\pm) - s(U - u_\pm) < 0, \quad \text{for } U \in (u_+, u_-).$$

This completes the proof of necessary conditions.

Next, we are going to prove the sufficient conditions. Let the Rankine-Hugoniot condition (2.4) and the generalized entropy condition (2.5) hold. We are now going to constructively prove the existence of sharp viscous shock waves, which are unique up to a shift. Given a number ξ_0 , and let

$$(U, V)(\xi) \equiv (u_+, v_+) \quad \text{for } \xi \in [\xi_0, \infty),$$

and $(U, V)(\xi)$ satisfy the following equations in $(-\infty, \xi_0)$:

$$\begin{cases} -sU_\xi + V_\xi = 0, \\ -sV_\xi + g(U)_\xi = f(U) - V, \end{cases} \quad \xi \in (-\infty, \xi_0),$$

which, by substituting the first equation to the second equation and applying the Rankine-Hugoniot condition (2.4), can be reduced to

$$U_\xi = \frac{f(U) - f(u_\pm) - s(U - u_\pm)}{g'(U) - g'(u_\pm)} = h(U), \text{ for } \xi \in (-\infty, \xi_0).$$

For $\xi \in (-\infty, \xi_0)$, the aforementioned ordinary differential equation is equivalent to

$$\begin{cases} \frac{dU}{h(U)} = d\xi, \\ U(\xi_*) = u_*, \end{cases} \quad (3.14)$$

where we set $u_* = \frac{u_+ + u_-}{2}$ and $U(\xi_*) = u_* \in (u_+, u_-)$, where $\xi_* \in (-\infty, \xi_0)$. Integrating the aforementioned equation yields

$$H(U) = \int_{u_*}^U \frac{1}{h(\tau)} d\tau = \xi - \xi_*, \quad \xi \in (-\infty, \xi_0). \quad (3.15)$$

Note that, by the generalized entropy condition (2.5), we have

$$H'(U) = \frac{1}{h(U)} < 0, \text{ for } U \in (u_+, u_-), \quad (3.16)$$

which means that $H(U)$ is invertible. From $H(U) = \xi - \xi_*$, we obtain the existence of sharp viscous shock wave $U(\xi) = H^{-1}(\xi)$ for $\xi \in (-\infty, \xi_0)$. Furthermore, from (3.7), we obtain the other component of sharp viscous shock wave $V(\xi)$ by

$$V(\xi) = sH^{-1}(\xi) - su_\pm + v_\pm, \text{ for } \xi \in (-\infty, \xi_0). \quad (3.17)$$

Thus, we have proved the existence of sharp viscous shock wave $(U, V)(\xi)$, which are monotonic for $\xi \in \mathbb{R}$, and unique up to a shift. $\square$

3.2 Regularity of sharp viscous shock waves

With the existence of sharp viscous shock waves, we are going to derive the asymptotic behavior of the sharp viscous shock waves at far fields as $\xi \rightarrow -\infty$ and at the corner ξ_0 in detail.

Proof of Theorem 2.6. Let $(U, V)(\xi)$ be the sharp viscous shock wave with a corner at ξ_0 of (1.4) with (1.5). For $\xi \in [\xi_0, +\infty)$, it holds $(U, V)(\xi) \equiv (u_+, v_+)$. Then, we have $(U_\xi, V_\xi) \equiv (0, 0)$ when $\xi \in (\xi_0, +\infty)$.

Case 1. The 1-degree degeneracy: $g'(u_+) = s^2$ but $g''(u_+) \neq 0$.

When $f'(u_+) < s \leq f'(u_-)$, by virtue of (3.13), it holds

$$\begin{aligned} \lim_{\xi \rightarrow \xi_0^-} U_\xi &= \lim_{\xi \rightarrow \xi_0^-} \frac{f(U(\xi)) - f(u_+) - s(U(\xi) - u_+)}{g'(U(\xi)) - g'(u_+)} \\ &= \lim_{\xi \rightarrow \xi_0^-} \frac{\frac{f(U) - f(u_+)}{U - u_+} - s}{\frac{g'(U) - g'(u_+)}{U - u_+}} \\ &= \frac{f'(u_+) - s}{g''(u_+)} \neq 0. \end{aligned} \quad (3.18)$$

Thus, we have

$$\lim_{\xi \rightarrow \xi_0^-} U_\xi \neq \lim_{\xi \rightarrow \xi_0^+} U_\xi,$$

which implies

$$(U, V)(\xi) \in C^0(\mathbb{R}), \text{ for 1-degree of degeneracy with } f'(u_+) < s \leq f'(u_-). \quad (3.19)$$

On the other hand, when $s = f'(u_+)$ with $f''(u_+) \neq 0$, by Taylor's formula, we have, for $\xi \in (-\infty, \xi_0)$,

$$\begin{aligned} f(U) - f(u_+) - s(U - u_+) &= f(U) - f(u_+) - f'(u_+)(U - u_+) \\ &= \frac{1}{2}f''(u_+)(U - u_+)^2 + o((U - u_+)^2). \end{aligned} \quad (3.20)$$

(3.13) and (3.20) give

$$\lim_{\xi \rightarrow \xi_0^-} U_\xi = \lim_{\xi \rightarrow \xi_0^-} \left[\frac{f''(u_+)(U - u_+)}{2g''(u_+)} + o(U - u_+) \right] = 0. \quad (3.21)$$

This implies

$$\lim_{\xi \rightarrow \xi_0^-} U_\xi = \lim_{\xi \rightarrow \xi_0^+} U_\xi = 0.$$

Furthermore, noting that

$$U_{\xi\xi} = \left[\frac{f'(U) - s}{g'(U) - g'(u_+)} - \frac{[f(U) - f(u_+) - s(U - u_+)]g''(U)}{[g'(U) - g'(u_+)]^2} \right] U_\xi, \quad (3.22)$$

then we obtain

$$\lim_{\xi \rightarrow \xi_0^-} U_{\xi\xi} = \frac{f''(u_+)}{2g''(u_+)} \lim_{\xi \rightarrow \xi_0^-} U_\xi = 0.$$

This guarantees

$$\lim_{\xi \rightarrow \xi_0^-} U_{\xi\xi} = \lim_{\xi \rightarrow \xi_0^+} U_{\xi\xi} = 0.$$

Repeating the same procedure, we can prove

$$\lim_{\xi \rightarrow \xi_0^-} \frac{d^k U}{d\xi^k} = \lim_{\xi \rightarrow \xi_0^+} \frac{d^k U}{d\xi^k} = 0, \quad \text{for any integer } k \geq 1.$$

We have proved

$$(U, V)(\xi) \in C^\infty(\mathbb{R}), \text{ for 1-degree of degeneracy with } f'(u_+) = s \leq f'(u_-), f''(u_+) \neq 0. \quad (3.23)$$

When $s = f'(u_+)$ with $f''(u_+) = \dots = f^{(k_+)}(u_+) = 0$ and $f^{(k_++1)}(u_+) \neq 0$, by Taylor's formula, we have, for $\xi \in (-\infty, \xi_0)$,

$$\begin{aligned} f(U) - f(u_+) - s(U - u_+) &= f(U) - f(u_+) - f'(u_+)(U - u_+) \\ &= \frac{1}{(k_+ + 1)!} f^{(k_++1)}(u_+)(U - u_+)^{k_++1} + o((U - u_+)^{k_++1}). \end{aligned} \quad (3.24)$$

Thus, (3.13) and (3.24) give

$$\lim_{\xi \rightarrow \xi_0^-} U_\xi = \lim_{\xi \rightarrow \xi_0^-} \left[\frac{f^{(k_++1)}(u_+)}{(k_+ + 1)!g''(u_+)} (U - u_+)^{k_+} + o((U - u_+)^{k_+}) \right] = 0. \quad (3.25)$$

By virtue of (3.22), we can similarly obtain

$$\lim_{\xi \rightarrow \xi_0^-} U_{\xi\xi} = 0.$$

Furthermore, in this case, we can also prove

$$\lim_{\xi \rightarrow \xi_0^-} \frac{d^k U}{d\xi^k} = \lim_{\xi \rightarrow \xi_0^+} \frac{d^k U}{d\xi^k} = 0, \quad \text{for any integer } k \geq 1.$$

Therefore, we have proved

$$(U, V)(\xi) \in C^\infty(\mathbb{R}), \quad (3.26)$$

for 1-degree of degeneracy with $f'(u_+) = s \leq f'(u_-)$, $f''(u_+) = \dots = f^{(k_+)}(u_+) = 0$, $f^{(k_++1)}(u_+) \neq 0$, where $k_+ \geq 2$.

Case 2. n -degree of degeneracy: $g'(u_+) = s^2$, $g''(u_+) = \dots = g^{(n)}(u_+) = 0$, $g^{(n+1)}(u_+) \neq 0$.

When $f'(u_+) < s \leq f'(u_-)$, it holds

$$f(U) - f(u_+) - s(U - u_+) \sim [f'(u_+) - s](U - u_+), \quad \text{as } \xi \rightarrow \xi_0^-,$$

and

$$g'(U) - g'(u_+) \sim \frac{g^{(n+1)}(u_+)}{n!}(U - u_+)^n, \quad \text{as } \xi \rightarrow \xi_0^-,$$

which imply

$$U_\xi = \frac{f(U) - f(u_+) - s(U - u_+)}{g'(U) - g'(u_+)} \sim \frac{[f'(u_+) - s]n!}{g^{(n+1)}(u_+)}(U - u_+)^{1-n} \rightarrow -\infty, \quad \text{as } \xi \rightarrow \xi_0^-,$$

and

$$U(\xi) - u_+ \sim |\xi - \xi_0|^{\frac{1}{n}} \rightarrow 0, \quad \text{as } \xi \rightarrow \xi_0^-.$$

This indicates the regularity of sharp viscous shock waves as Hölder continuity:

$$U(\xi) - u_+ \in C^{\frac{1}{n}}(\mathbb{R}).$$

Therefore, we have proved the regularity:

$$(U, V)(\xi) \in C^{\frac{1}{n}}(\mathbb{R}), \quad \text{for } n\text{-degree of degeneracy with } f'(u_+) < s \leq f'(u_-), \quad (3.27)$$

where $n \geq 2$.

When $f'(u_+) = s \leq f'(u_-)$ with $f'(u_+) \neq 0$, we can show

$$\begin{aligned} |U_\xi| &= \left| \frac{f(U) - f(u_+) - s(U - u_+)}{g'(U) - g'(u_+)} \right| \\ &\sim \left| \frac{n!}{2} \frac{f''(u_+)}{g^{(n+1)}(u_+)} \right| |U - u_+|^{2-n} \\ &= \begin{cases} \tilde{\lambda} \neq 0, & \text{if } n = 2, \\ \tilde{\lambda} |U - u_+|^{-(n-2)} \rightarrow \infty, & \text{if } n > 2, \end{cases} \quad \text{as } \xi \rightarrow \xi_0^-, \end{aligned} \quad (3.28)$$

where $\tilde{\lambda} = \left| \frac{n!}{2} \frac{f''(u_+)}{g^{(n+1)}(u_+)} \right|$. We can conclude the regularities of the sharp viscous shock waves as follows:

$$(U, V)(\xi) \in \begin{cases} C^0(\mathbb{R}), & \text{if } n = 2, \\ C^{\frac{1}{n-1}}(\mathbb{R}), & \text{if } n > 2, \end{cases} \quad \text{for } n\text{-degree of degeneracy with } f'(u_+) = s \leq f'(u_-), f''(u_+) \neq 0.$$

When $f'(u_+) = s \leq f'(u_-)$ with $f''(u_+) = \dots = f^{(k_+)}(u_+) = 0$ and $f^{(k_++1)}(u_+) \neq 0$, we can similarly show, as $\xi \rightarrow \xi_0^-$, that

$$\begin{aligned} |U_\xi| &= \left| \frac{f(U) - f(u_+) - s(U - u_+)}{g'(U) - g'(u_+)} \right| \\ &\sim \left| \frac{n!}{(k_+ + 1)!} \frac{f^{(k_++1)}(u_+)}{g^{(n+1)}(u_+)} \right| |U - u_+|^{k_++1-n} \\ &= \begin{cases} \bar{\lambda} |U - u_+|^{-(n-k_+-1)} \rightarrow \infty, & \text{if } k_+ < n - 1, \\ \bar{\lambda} \neq 0, & \text{if } k_+ = n - 1, \\ \bar{\lambda} |U - u_+|^{k_++1-n} \rightarrow 0, & \text{if } k_+ > n - 1, \end{cases} \quad \text{as } \xi \rightarrow \xi_0^-, \end{aligned} \quad (3.29)$$

where $\bar{\lambda} := \left| \frac{n!}{(k_+ + 1)!} \frac{f^{(k_++1)}(u_+)}{g^{(n+1)}(u_+)} \right|$. Therefore, similar to (3.26) and (3.27), we can conclude the regularities of the sharp viscous shock waves as follows:

$$(U, V)(\xi) \in \begin{cases} C^{\frac{1}{n-k_+}}(\mathbb{R}), & \text{if } k_+ < n-1, \\ C^0(\mathbb{R}), & \text{if } k_+ = n-1, \\ C^\infty(\mathbb{R}), & \text{if } k_+ \geq n, \end{cases}$$

for n -degree of degeneracy with $f'(u_+) = s \leq f'(u_-)$, $f''(u_+) = \dots = f^{(k_+)}(u_+) = 0$, $f^{(k_++1)}(u_+) \neq 0$.

Case 3. Singular 1-degree of degeneracy: $|g'(u) - s^2| = O(1)|u - u_+|^\alpha$ as $u \rightarrow u_+$ for $0 < \alpha < 1$.

When $f'(u_+) < s \leq f'(u_-)$, we have

$$|f(U) - f(u_+) - s(U - u_+)| = |f'(u_+) - s||U - u_+|, \text{ as } U(\xi) \rightarrow u_+, \text{ i.e., } \xi \rightarrow \xi_0^-.$$

As shown (3.29), we similarly have

$$\begin{aligned} |U_\xi| &= \left| \frac{f(U) - f(u_+) - s(U - u_+)}{g'(U) - g'(u_+)} \right| \\ &\sim \frac{|f'(u_+) - s||U - u_+|}{O(1)|U - u_+|^\alpha} \\ &\sim O(1)|U - u_+|^{1-\alpha} \\ &\rightarrow 0, \text{ as } \xi \rightarrow \xi_0^-, \end{aligned} \quad (3.30)$$

which implies

$$(U, V)(\xi) \in C^{m,\beta}(\mathbb{R}), \text{ for singular 1-degree of degeneracy with } f'(u_+) < s \leq f'(u_-), \quad (3.31)$$

where $m = \left\lfloor \frac{1}{\alpha} \right\rfloor > 1$, and $0 \leq \beta = \frac{1}{\alpha} - \left\lfloor \frac{1}{\alpha} \right\rfloor < 1$.

When $f'(u_+) = s \leq f'(u_-)$ with $f''(u_+) \neq 0$, we have

$$|f(U) - f(u_+) - s(U - u_+)| = O(1)|U - u_+|^2, \text{ as } U(\xi) \rightarrow u_+, \text{ i.e., } \xi \rightarrow \xi_0^-.$$

We similarly have

$$\begin{aligned} |U_\xi| &= \left| \frac{f(U) - f(u_+) - s(U - u_+)}{g'(U) - g'(u_+)} \right| \\ &\sim \frac{O(1)|U - u_+|^2}{O(1)|U - u_+|^\alpha} \\ &\sim O(1)|U - u_+|^{2-\alpha} \\ &\rightarrow 0, \text{ as } \xi \rightarrow \xi_0^-, \end{aligned} \quad (3.32)$$

By virtue of (3.22), we obtain

$$\lim_{\xi \rightarrow \xi_0^-} U_{\xi\xi} = [O(1)(u - u_+)^{1-\alpha}] \lim_{\xi \rightarrow \xi_0^-} U_\xi = 0.$$

This guarantees

$$\lim_{\xi \rightarrow \xi_0^-} U_{\xi\xi} = \lim_{\xi \rightarrow \xi_0^+} U_{\xi\xi} = 0.$$

Repeating the same procedure, we can prove

$$\lim_{\xi \rightarrow \xi_0^-} \frac{d^k U}{d\xi^k} = \lim_{\xi \rightarrow \xi_0^+} \frac{d^k U}{d\xi^k} = 0, \text{ for any integer } k \geq 1,$$

which implies

$$(U, V)(\xi) \in C^\infty(\mathbb{R}), \text{ for singular 1-degree of degeneracy with } f'(u_+) = s \leq f'(u_-), f'(u_+) \neq 0. \quad (3.33)$$

When $f'(u_+) = s \leq f'(u_-)$ with $f''(u_+) = \dots = f^{(k_+)}(u_+) = 0$ and $f^{(k_++1)}(u_+) \neq 0$, we have

$$|f(U) - f(u_+) - s(U - u_+)| = O(1)|U - u_+|^{k_++1}, \text{ as } U(\xi) \rightarrow u_+, \text{ i.e., } \xi \rightarrow \xi_0^-.$$

As shown in (3.29), we similarly have

$$\begin{aligned} |U_\xi| &= \left| \frac{f(U) - f(u_+) - s(U - u_+)}{g'(U) - g'(u_+)} \right| \\ &\sim \frac{O(1)|U - u_+|^{k_++1}}{O(1)|U - u_+|^\alpha} \\ &\sim O(1)|U - u_+|^{k_++1-\alpha} \\ &\rightarrow 0, \text{ as } \xi \rightarrow \xi_0^-, \end{aligned} \quad (3.34)$$

Similar to (3.33), we can derive

$$\lim_{\xi \rightarrow \xi_0^-} \frac{d^k U}{d\xi^k} = \lim_{\xi \rightarrow \xi_0^-} \frac{d^k U}{d\xi^k} = 0, \text{ for any integer } k \geq 1,$$

which implies

$$(U, V)(\xi) \in C^\infty(\mathbb{R}), \quad (3.35)$$

for singular 1-degree of degeneracy with $f'(u_+) = s \leq f'(u_-)$, $f''(u_+) = \dots = f^{(k_+)}(u_+) = 0$, $f^{(k_++1)}(u_+) \neq 0$.

Case 4. Singular n -degree of degeneracy: $|g'(u) - s^2| = O(1)|u - u_+|^{n+\alpha-1}$ as $u \rightarrow u_+$ for $0 < \alpha < 1$.

When $f'(u_+) < s \leq f'(u_-)$, we have

$$|f(U) - f(u_+) - s(U - u_+)| = |f'(u_+) - s||U - u_+|, \text{ as } U(\xi) \rightarrow u_+, \text{ i.e., } \xi \rightarrow \xi_0^-.$$

As shown in (3.29), we similarly have

$$\begin{aligned} |U_\xi| &= \left| \frac{f(U) - f(u_+) - s(U - u_+)}{g'(U) - g'(u_+)} \right| \\ &\sim \frac{|f'(u_+) - s||U - u_+|}{O(1)|U - u_+|^{n+\alpha-1}} \\ &\sim O(1)|U - u_+|^{-(\alpha+n-2)} \\ &\rightarrow \infty, \text{ as } \xi \rightarrow \xi_0^-, \end{aligned} \quad (3.36)$$

which implies

$$(U, V)(\xi) \in C^{\frac{1}{n+\alpha-1}}(\mathbb{R}), \text{ for singular } n\text{-degree of degeneracy with } f'(u_+) < s \leq f'(u_-). \quad (3.37)$$

When $f'(u_+) = s \leq f'(u_-)$ with $f''(u_+) \neq 0$, we have

$$|f(U) - f(u_+) - s(U - u_+)| = O(1)|U - u_+|^2, \text{ as } U(\xi) \rightarrow u_+, \text{ i.e., } \xi \rightarrow \xi_0^-.$$

As shown in (3.28), we similarly have

$$\begin{aligned} |U_\xi| &= \left| \frac{f(U) - f(u_+) - s(U - u_+)}{g'(U) - g'(u_+)} \right| \\ &\sim \frac{O(1)|U - u_+|^2}{O(1)|U - u_+|^{n+\alpha-1}} \\ &\sim O(1)|U - u_+|^{3-n-\alpha} \\ &\sim \begin{cases} O(1)|U - u_+|^{1-\alpha} \rightarrow 0, & \text{if } n = 2, \\ O(1)|U - u_+|^{-(n+\alpha-3)} \rightarrow \infty, & \text{if } n > 2, \end{cases} \text{ as } \xi \rightarrow \xi_0^-, \end{aligned} \quad (3.38)$$

which implies

$$(U, V)(\xi) \in \begin{cases} C^{m,\beta}(\mathbb{R}), & \text{if } n = 2, \\ C^{\frac{1}{n+\alpha-2}}(\mathbb{R}), & \text{if } n > 2, \end{cases} \quad (3.39)$$

for singular n -degree of degeneracy with $f'(u_+) = s \leq f'(u_-)$, $f''(u_+) \neq 0$, where $m = \left\lceil \frac{1}{\alpha} \right\rceil$, $0 \leq \beta = \frac{1}{\alpha} - \left\lceil \frac{1}{\alpha} \right\rceil < 1$.

When $f'(u_+) = s \leq f'(u_-)$ with $f''(u_+) = \dots = f^{(k_+)}(u_+) = 0$ and $f^{(k_++1)}(u_+) \neq 0$, we have

$$|f(U) - f(u_+) - s(U - u_+)| = O(1)|U - u_+|^{k_++1}, \quad \text{as } U(\xi) \rightarrow u_+, \quad \text{i.e., } \xi \rightarrow \xi_0^-.$$

As shown in (3.29), we similarly have

$$\begin{aligned} |U_\xi| &= \left| \frac{f(U) - f(u_+) - s(U - u_+)}{g'(U) - g'(u_+)} \right| \\ &\sim \frac{O(1)|U - u_+|^{k_++1}}{O(1)|U - u_+|^{n+a-1}} \\ &\sim O(1)|U - u_+|^{k_++2-n-a} \\ &\sim \begin{cases} O(1)|U - u_+|^{-(n+a-k_+-2)} \rightarrow \infty, & \text{if } n \geq k_+ + 2, \\ O(1)|U - u_+|^{k_++2-n-a} \rightarrow 0, & \text{if } n \leq k_+ + 1, \end{cases} \quad \text{as } \xi \rightarrow \xi_0^-, \end{aligned} \quad (3.40)$$

which implies

$$(U, V)(\xi) \in \begin{cases} C^{\frac{1}{n+a-k_+-1}}(\mathbb{R}), & \text{if } n \geq k_+ + 2, \\ C^{m,\beta}(\mathbb{R}), & \text{if } n = k_+ + 1, \\ C^\infty(\mathbb{R}), & \text{if } n < k_+ + 1. \end{cases} \quad (3.41)$$

The proof is complete. $\square$

3.3 Asymptotic behavior of sharp viscous shock waves at far field

Proof of Theorem 2.7. When $s < f'(u_-)$, it holds that as $\xi \rightarrow -\infty$,

$$U_\xi = \frac{\frac{f(U) - f(u_-)}{U - u_-} - s}{g'(U) - g'(u_+)}(U - u_-) \sim \frac{f'(u_-) - s}{g'(u_-) - g'(u_+)}(U - u_-), \quad \text{when } f'(u_+) \leq s < f'(u_-), \quad (3.42)$$

which implies

$$|U(\xi) - u_-| \sim \exp(-\hat{\lambda}|\xi|), \quad \text{as } \xi \rightarrow -\infty, \quad \text{when } f'(u_+) \leq s < f'(u_-), \quad (3.43)$$

with $\hat{\lambda} = \frac{f'(u_-) - s}{g'(u_-) - g'(u_+)} > 0$.

When $s = f'(u_-)$, it holds that as $\xi \rightarrow -\infty$,

$$U_\xi \sim \begin{cases} \frac{f''(u_-)}{2(g'(u_-) - g'(u_+))}(U - u_-)^2, & \text{if } f''(u_-) \neq 0, \\ \frac{f^{(k_+)}(u_-)}{(k_+ + 1)!(g'(u_-) - g'(u_+))}(U - u_-)^{k_++1}, & \text{if } f''(u_-) = 0. \end{cases} \quad (3.44)$$

We then obtain

$$|U(\xi) - u_-| \sim \begin{cases} |\xi|^{-1}, & \text{if } f''(u_-) \neq 0, \\ |\xi|^{-\frac{1}{k_+}}, & \text{if } f''(u_-) = 0, \end{cases} \quad \text{as } \xi \rightarrow -\infty, \quad \text{when } s = f'(u_-), \quad (3.45)$$

where $k_+ \geq 2$.

Thus, the proof of Theorem 2.7 is complete. $\square$

4 Examples of exact sharp viscous shock waves and numerical simulations

In this section, we give three examples and construct the exact sharp viscous shock waves. We also carry out some numerical simulations that perfectly match the exact sharp viscous shock waves and numerically demonstrate the stability of sharp viscous shock waves. The theoretical proof of stability of sharp viscous shock waves is challenging and still open so far. This will be our next target.

Example 1. Let $(u_+, v_+) = (0, 0)$, $(u_-, v_-) = (1, \frac{1}{2})$, $f(u) = u^3 - \frac{1}{2}u^2$, and $g(u) = u^2 + \frac{1}{4}u$. We consider the Cauchy problem for

$$\begin{cases} u_t + v_x = 0, \\ v_t + \left(2u + \frac{1}{4}\right) \cdot u_x = u^3 - \frac{1}{2}u^2 - v, \end{cases} \quad x \in \mathbb{R}^1, \quad t \in \mathbb{R}_+, \quad (4.1)$$

with the initial datum satisfying

$$(u, v)(x, 0) = (u_0, v_0)(x) \rightarrow (u_{\pm}, v_{\pm}), \quad \text{as } x \rightarrow \pm\infty. \quad (4.2)$$

The flux function $f(u)$ is non-convex:

$$f''(u) \begin{cases} < 0, & u \in \left[0, \frac{1}{6}\right), \\ > 0, & u \in \left[\frac{1}{6}, 1\right]. \end{cases} \quad (4.3)$$

Based on Theorems 2.3 and 2.6, system (4.1) admits sharp viscous shock wave solutions $(U, V)(\xi) \in C^0$ with an edge at some fixed number ξ_0 , for instance, we consider $\xi_0 = 0$, where $\xi = x - st$. The shock speed satisfies $s = \frac{f(u_+) - f(u_-)}{u_+ - u_-} = \frac{1}{2}$ and $f'(0) < s < f'(1)$.

We are going to derive the exact sharp viscous shock waves. Substituting the sharp viscous shock wave solutions $(U, V)(\xi)$ into (4.1), we find that $U(\xi)$ satisfies

$$U(\xi) \equiv 0, \quad \text{for } \xi \in [0, +\infty), \quad (4.4)$$

and

$$U_\xi = \frac{U^2}{2} - \frac{U}{4} - \frac{1}{4}, \quad \text{for } \xi \in (-\infty, 0). \quad (4.5)$$

The ordinary equation (4.5) with the initial value $U(\xi_0) = 0$ admits a unique solution

$$U(\xi) = \frac{3}{2 + 4e^{\frac{3}{4}\xi}} - \frac{1}{2}, \quad \text{for } \xi \in (-\infty, 0). \quad (4.6)$$

The exact viscous shock waves for (4.1) with (4.10) for any $t > 0$ are given by

$$(U, V)(x - st) = \begin{cases} \left(\frac{3}{2 + 4 \exp\left(\frac{3}{4}(x - \frac{1}{2}t)\right)} - \frac{1}{2}, \frac{3}{4 + 8 \exp\left(\frac{3}{4}(x - \frac{1}{2}t)\right)} - \frac{1}{4} \right), & \text{for } x - \frac{t}{2} < 0, \\ (0, 0), & \text{for } x - \frac{t}{2} \geq 0. \end{cases} \quad (4.7)$$

At the far field as $\xi \rightarrow -\infty$, the exact solutions keep exponential convergence rates. Meanwhile, at the corner point $\xi_0 = 0$, it becomes evident that the exact solutions keep sharp type. The exact sharp viscous shock waves are shown in Figure 2 as follows.

On the other hand, we set an initial data to system (4.1), and carry out the numerical simulations. We will compare the numerical results with the exact sharp shock waves in (4.7). From (4.7), the sharp shock wave solutions $(U, V)(\xi)$ behave like

$$|(U - u_-, V - v_-)(\xi)| \sim e^{-\frac{3}{4}|\xi|}, \quad \text{as } \xi \rightarrow -\infty, \quad (4.8)$$

and

$$(U_\xi, V_\xi) = \left(-\frac{1}{4}, -\frac{1}{8}\right), \quad \text{as } \xi \rightarrow 0^-. \quad (4.9)$$

To ensure that the initial data $(u_0, v_0)(x)$ decay similarly to the sharp shock waves $(U, V)(\xi)$ at both the far field as $\xi \rightarrow -\infty$ and at the corner $\xi_0 = 0$, we choose the initial value as

$$u_0(x) = \begin{cases} 0, & x \geq 0, \\ -\frac{1}{4}x, & -\frac{8}{3} \leq x < 0, \\ 1 - \frac{1}{3}e^{\frac{3}{4}x+2}, & x < -\frac{8}{3}, \end{cases} \quad v_0(x) = \begin{cases} 0, & x \geq 0, \\ -\frac{1}{8}x, & -\frac{8}{3} \leq x < 0, \\ \frac{1}{2} - \frac{1}{6}e^{\frac{3}{4}x+2}, & x < -\frac{8}{3}. \end{cases} \quad (4.10)$$

Numerical results for the aforementioned Cauchy problem are shown in Figure 3. The 2-D plots (left) in Figure 3 illustrate the sharp shock wave solutions $u(x, t)$ at $t = 0, t = 5, t = 10, t = 15$ and $t = 20$, respectively, while the 3-D plots (right) show the solution $u(x, t)$ spanning from $t = 0$ to $t = 20$. The solution $v(x, t)$ exhibits a similar pattern; we omit it. The simulation solutions $u(x, t)$ depicted in Figure 3 are sharp and loss their regularities at corner points. Moreover, we can observe that the corner moves over time t at the speed $s = \frac{1}{2}$.

Comparing the numerical solution $u(x, t)$ presented in Figure 3 with the exact sharp viscous shock wave $U(x - st)$ presented in Figure 2, we see that they look almost identically. This numerically demonstrates the stability of sharp viscous shock wave for equation (4.1) with the initial data (4.10).

Example 2. Let $(u_+, v_+) = (-\frac{1}{2}, \frac{3}{8})$, $(u_-, v_-) = (1, 0)$, $f(u) = u^3 - u$, and $g(u) = \left(u + \frac{1}{2}\right)^4 + \frac{1}{16}\left(u + \frac{1}{2}\right)$. We consider the Cauchy problem for

$$\begin{cases} u_t + v_x = 0, \\ v_t + \left(4u^3 + 6u^2 + 3u + \frac{9}{16}\right) \cdot u_x = u^3 - u - v, \end{cases} \quad x \in \mathbb{R}^1, \quad t \in \mathbb{R}_+, \quad (4.11)$$

with the initial datum satisfying

$$(u, v)(x, 0) = (u_0, v_0)(x) \rightarrow (u_\pm, v_\pm), \quad \text{as } x \rightarrow \pm\infty. \quad (4.12)$$

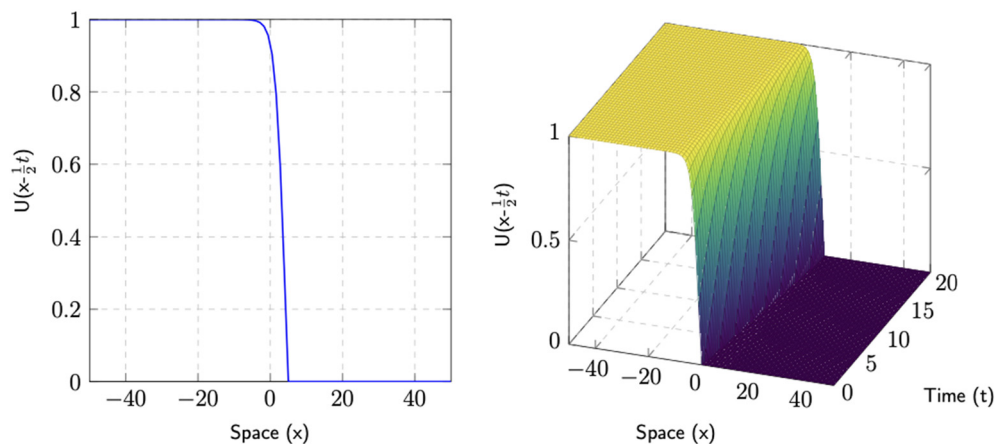


Figure 2: Exact sharp viscous shock wave for Example 1.

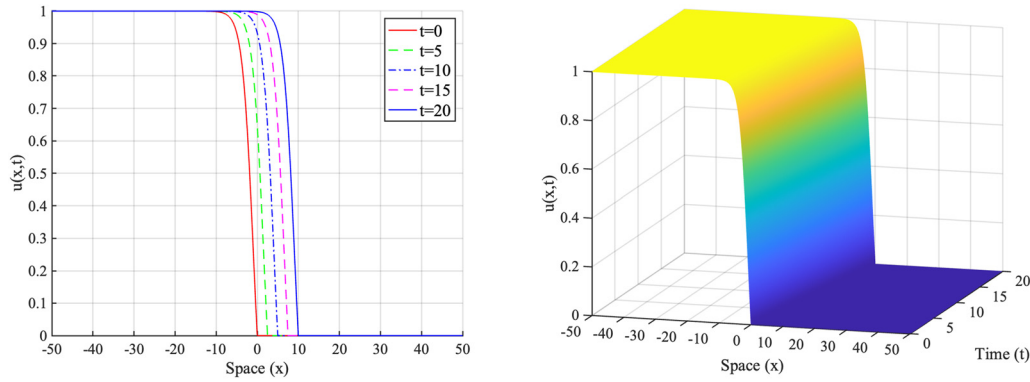


Figure 3: Numerical solution $u(x, t)$ for Example 1, which behaves like the exact sharp viscous shock waves in Figure 2, with the corner point moving to the right over time.

The flux function $f(u)$ is non-convex:

$$f''(u) \begin{cases} < 0, & u \in \left[-\frac{1}{2}, 0\right], \\ > 0, & u \in (0, 1]. \end{cases}$$

Based on Theorem 2.3 and 2.6, system (4.11) admits sharp shock wave solutions $(U, V)(\xi) \in C^{\frac{1}{2}}$ with an edge at ξ_0 (without loss of generality, we take $\xi_0 = 0$), where $\xi = x - st$. The shock speed satisfies $s = \frac{f(u_+) - f(u_-)}{u_+ - u_-} = -\frac{1}{4}$ and $f'(-\frac{1}{2}) = s < f'(1)$.

Now, we are going to derive the exact sharp viscous shock waves. Substituting the sharp shock wave solutions $(U, V)(\xi)$ into (4.11), we find that $U(\xi)$ satisfies

$$U(\xi) \equiv -\frac{1}{2}, \quad \text{for } \xi \in [0, +\infty), \quad (4.13)$$

and

$$U_\xi = \frac{U - 1}{4U + 2}, \quad \text{for } \xi \in (-\infty, 0). \quad (4.14)$$

Solving the ordinary equation (4.14) with the initial value $U(0) = -\frac{1}{2}$ and by virtue of inverse function theorem, we obtain the unique solution in terms of its inverse function

$$U(\xi) = \Xi^{-1}(\xi), \quad \text{for } \xi \in (-\infty, 0), \quad (4.15)$$

where

$$\xi = 4U + 6 \ln \frac{2(1 - U)}{3} + 2 = \Xi(U), \quad \text{for } U \in \left[-\frac{1}{2}, 1\right]. \quad (4.16)$$

The exact solutions for (4.11) with (4.20) for any $t > 0$ are given by

$$(U, V)(x, t) = \begin{cases} \left(\Xi^{-1}\left(x + \frac{t}{4}\right), -\frac{\Xi^{-1}\left(x + \frac{t}{4}\right) + \frac{1}{4}}{4} \right), & \text{for } x + \frac{t}{4} < 0, \\ \left(-\frac{1}{2}, \frac{3}{8} \right), & \text{for } x + \frac{t}{4} \geq 0. \end{cases} \quad (4.17)$$

It is easily observed from (4.17) that the solutions exhibit sharp at the corner points.

From (4.17), the sharp shock wave solutions $(U, V)(\xi)$ behave like

$$|(U - u_-, V - v_-)(\xi)| \sim e^{-\frac{|\xi|}{6}}, \quad \text{as } \xi \rightarrow -\infty, \quad (4.18)$$

and

$$(U_\xi, V_\xi) \rightarrow (-\infty, +\infty), \quad \text{as } \xi \rightarrow 0^-, \quad (4.19)$$

Now, we are going to carry out numerical simulations for a given initial data and then compare these numerical solutions with the exact sharp shock waves. We choose the initial value as

$$u_0(x) = \begin{cases} -\frac{1}{2}, & x \geq 0, \\ \sqrt{\frac{|x|}{6}} - \frac{1}{2}, & -\frac{3}{2} \leq x < 0, \\ 1 - e^{\frac{x}{6} + \frac{1}{4}}, & x < -\frac{3}{2}, \end{cases} \quad v_0(x) = \begin{cases} \frac{3}{8}, & x \geq 0, \\ \frac{3}{8} - \sqrt{\frac{|x|}{96}}, & -\frac{3}{2} \leq x < 0, \\ \frac{1}{4}e^{\frac{x}{6} + \frac{1}{4}}, & x < -\frac{3}{2}. \end{cases} \quad (4.20)$$

Figure 5 shows our numerical simulation result $u(x, t)$ for the Cauchy problem for system (4.11) with the initial data (4.20), and we have also omitted the graph of $v(x, t)$ to avoid redundancy. A sequence of two-dimensional images (left) illustrates the sharp shock wave solutions $u(x, t)$ at $t = 0, t = 5, t = 10, t = 15$, and $t = 20$, respectively, while the three-dimensional image (right) depicts the solution $u(x, t)$ spanning from $t = 0$ to $t = 20$. We observe that the solutions behave like sharp shock waves, distinctly showcasing their sharpness at the corner points. The numerical simulation for $u(x, t)$ in Figure 5 is almost identical to the exact sharp shock wave presented in Figure 4. This also numerically verifies the stability of sharp viscous shock wave.

Example 3. Let $(u_+, v_+) = \left(\frac{1}{4}, -\frac{3}{64}\right)$, $(u_-, v_-) = (1, 0)$, $f(u) = u^3 - u^2$, and $g(u) = \frac{1}{128}u^2$. We consider the Cauchy problem for

$$\begin{cases} u_t + v_x = 0, \\ v_t + \frac{1}{64}uu_x = u^3 - u^2 - v, \end{cases} \quad x \in \mathbb{R}^1, \quad t \in \mathbb{R}_+, \quad (4.21)$$

with the initial datum satisfying

$$(u, v)(x, 0) = (u_0, v_0)(x) \rightarrow (u_\pm, v_\pm), \quad \text{as } x \rightarrow \pm\infty. \quad (4.22)$$

The flux function $f(u)$ is non-convex:

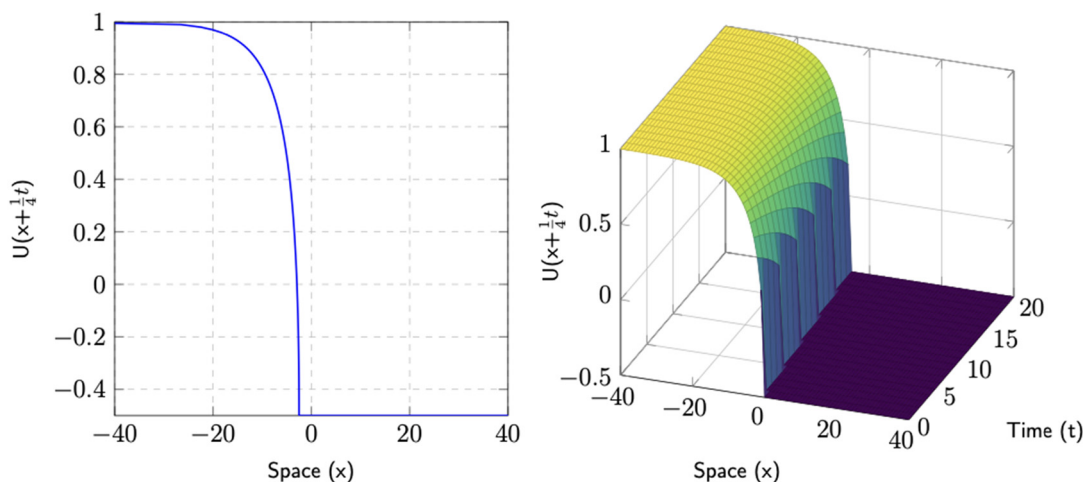


Figure 4: Exact sharp viscous shock wave $U(x - st)$ for Example 2.

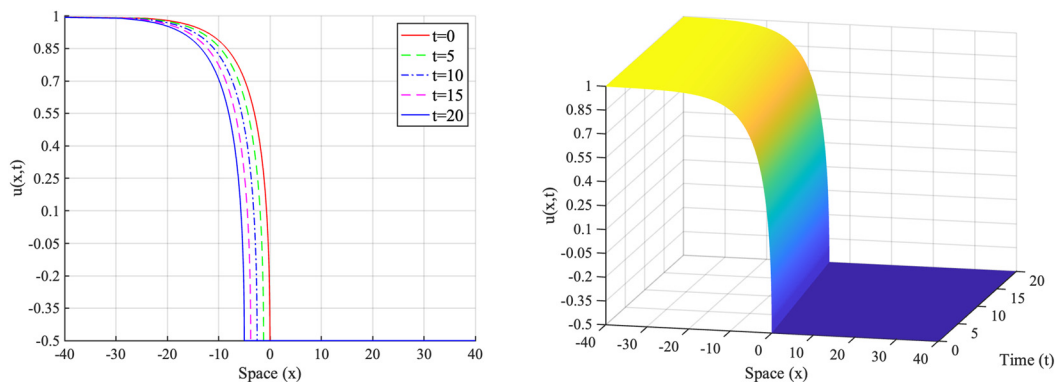


Figure 5: Numerical solution $u(x, t)$ for Example 2, which behaves like the exact sharp viscous shock waves in Figure 4, with the corner point moving to the right over time.

$$f''(u) \begin{cases} < 0, & u \in \left[\frac{1}{4}, \frac{1}{3}\right), \\ > 0, & u \in \left[\frac{1}{3}, 1\right]. \end{cases} \quad (4.23)$$

Based on Theorems 2.3 and 2.6, system (4.21) admits sharp shock wave solutions $(U, V)(\xi) \in C^0$ with an corner at ξ_0 , where $\xi = x - st$. The shock speed satisfies $s = \frac{f(u_+) - f(u_-)}{u_+ - u_-} = \frac{1}{16}$ and $f'(\frac{1}{4}) < s < f'(1)$.

Now, we derive the exact sharp viscous shock waves. Substituting the sharp shock wave solutions $(U, V)(\xi)$ into (4.21), we find that $U(\xi)$ satisfies

$$U(\xi) \equiv \frac{1}{4}, \quad \text{for } \xi \in [0, +\infty), \quad (4.24)$$

and

$$U_\xi = 64(U - 1)\left(U + \frac{1}{4}\right), \quad \text{for } \xi \in (-\infty, 0). \quad (4.25)$$

The ordinary equation (4.25) with the initial value $U(\xi_0) = \frac{1}{4}$ admits a unique solution

$$U(\xi) = \frac{5}{4 + 6e^{80\xi}} - \frac{1}{4}, \quad \text{for } \xi \in (-\infty, 0). \quad (4.26)$$

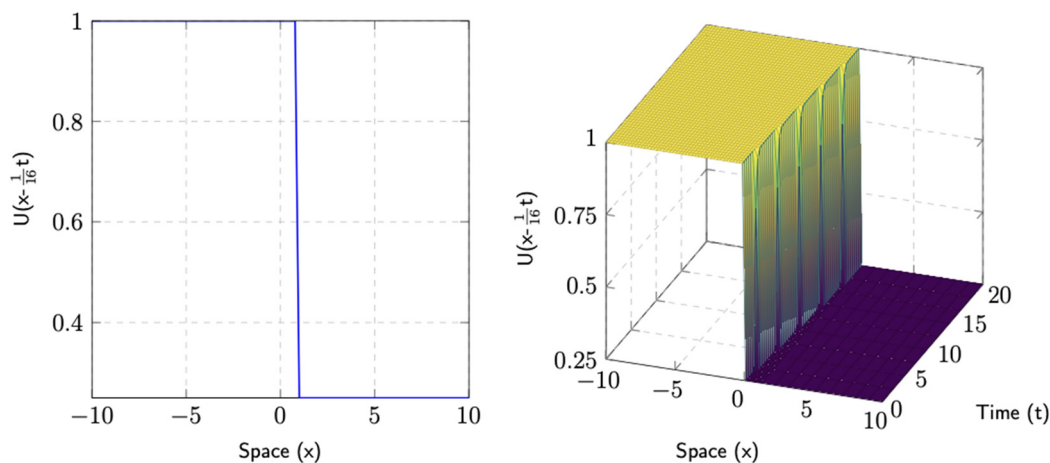


Figure 6: Exact sharp viscous shock wave for Example 3.

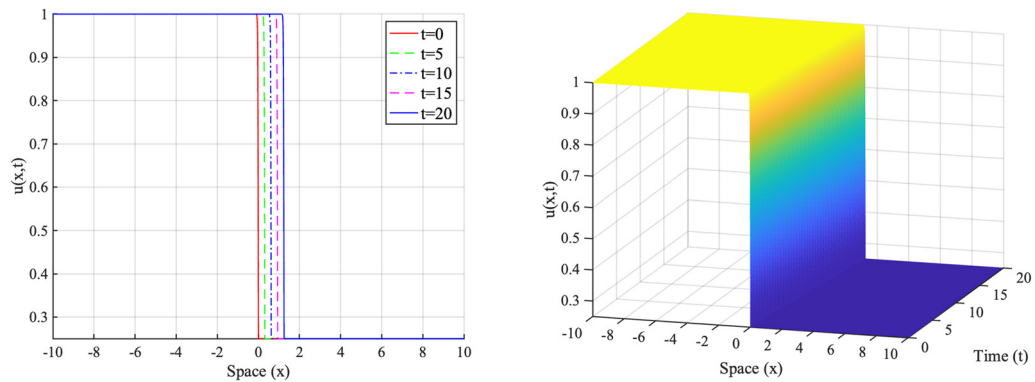


Figure 7: Numerical solution $u(x, t)$ for Example 3, which behaves like the exact sharp viscous shock waves in Figure 6, with the corner point moving to the right over time.

The exact solutions for (4.21) with (4.30) for any $t > 0$ are given by

$$(U, V)(x, t) = \begin{cases} \left(\frac{5}{4 + 6 \exp(80x - 5t)} - \frac{1}{4}, \frac{5}{4 + 6 \exp(80x - 5t)} - \frac{1}{4} \right), & \text{for } x - \frac{t}{16} < 0, \\ \left(\frac{1}{4}, -\frac{3}{64} \right), & \text{for } x - \frac{t}{16} \geq 0. \end{cases} \quad (4.27)$$

At the far field $\xi \rightarrow -\infty$, the exact solutions keep exponential convergence speeds. Meanwhile, at the corner point $\xi_0 = 0$, we easily observe that the exact solutions exhibit sharp type.

Next, we are going to carry out some numerical simulations in this case. From (4.25), the sharp shock $(U, V)(\xi)$ behave like

$$|(U - u_-, V - v_-)(\xi)| \sim e^{-80|\xi|}, \quad \text{as } \xi \rightarrow -\infty, \quad (4.28)$$

and

$$(U_\xi, V_\xi) = \left(-24, -\frac{3}{2} \right), \quad \text{as } \xi \rightarrow 0^-. \quad (4.29)$$

Thus, we choose the initial value as

$$u_0(x) = \begin{cases} \frac{1}{4}, & x \geq 0, \\ \frac{1}{4} - 24x, & -\frac{3}{160} \leq x < 0, \\ 1 - \frac{3}{10}e^{80x + \frac{2}{3}}, & x < -\frac{3}{160}, \end{cases} \quad v_0(x) = \begin{cases} -\frac{3}{64}, & x \geq 0, \\ -\frac{3}{64} - \frac{3}{2}x, & -\frac{3}{160} \leq x < 0, \\ -\frac{3}{160}e^{80x + \frac{2}{3}}, & x < -\frac{3}{160}. \end{cases} \quad (4.30)$$

Figure 7 presents our numerical simulation result $u(x, t)$ for the Cauchy problem for system (4.21) with (4.30). We can also note a significant loss of regularity in $u(x, t)$ at the corner points. Moreover, the shapes of $u(x, t)$ have become much steeper compared to the previous two examples, due to the effect of higher exponential convergence speed as $x \rightarrow -\infty$. The shape of the numerical simulation $u(x, t)$ presented in Figure 7 is almost identical to the exact sharp viscous shock wave $U(x - st)$ presented in Figure 6. This numerically demonstrates the stability of sharp viscous shock wave.

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