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Variational approach of critical sharp front speeds in degenerate diffusion model with time delay

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Abstract

For the classical reaction diffusion equation, the *a priori* speed of fronts is determined exactly in the pioneering paper (Benguria and Depassier 1996 *Commun. Math. Phys.* **175** 221–227) by variational characterization method. In this paper, we study the age-structured population dynamics using a degenerate diffusion equation with time delay. We show the existence and uniqueness of sharp critical fronts, where the sharp critical front is C^1-smooth when the diffusion degeneracy is weaker with $1 < m < 2$, and the sharp critical front is non-C^1-smooth (piecewise smooth) when the diffusion degeneracy is stronger with $m \geqslant 2$, and the non-critical waves are C^2-smooth. We give a new variational approach for the critical wave speed and investigate how the time delay affects the propagation mechanism of fronts. It is shown that the time delay slows down the critical wave speed.

Keywords: variational approach, time delay, degenerate diffusion, critical wave speed, sharp type wave

Mathematics Subject Classification numbers: 35K65, 35K57, 35C07.

(Some figures may appear in colour only in the online journal)

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1. Introduction and preliminaries

In order to describe the population dynamics of single species with age-structure and density-dependent diffusion, we consider a class of degenerate diffusion equation with time delay

$$\frac{\partial u}{\partial t} = D\Delta u^m - d(u) + b(u(t-r,x)), \quad x \in \mathbb{R}, \quad t > 0, \tag{1.1}$$

where u denotes the total mature population of the species at location x and time $t > 0$, $r \geqslant 0$ is the maturation time, $b(u(t-r,x))$ is the birth function, $D > 0$ represents the diffusivity of the population, and $d(u)$ is the death rate function. The diffusion term $D\Delta u^m$ (with $m > 1$) is considered to be in the form of porous medium type, which is dependent on the population density due to the population pressure [11, 21, 24]. Ecologically, many species exhibit positive density-dependent dispersal. In this case, large dispersal takes place in high-populated regions, but low mobility occurs in less crowded situations [2].

The main purpose of this paper is to study the critical sharp traveling waves for (1.1). A traveling wave solution is a special solution in the form of $u(t,x) = \phi(x+ct)$ with wave speed c. We show the existence and uniqueness of a critical traveling wave for (1.1), and investigate the geometric shape of this wave, which is partially compactly supported (called 'sharp wave'). Inspired by the work of Benguria and Depassier [5] for the regular diffusion equation, we give a new variational characterization of this sharp critical wave for the density-dependent degenerate diffusion equation with time delay. We demonstrate that, the diffusion degeneracy and the time delay both act as two important mechanisms for the propagation of wave front. The speed of this sharp critical front is reduced due to the time delay effect. The critical wave speed $c^*(m,D,r)$ (defined in (2.2)) of model (1.1) with time delay is smaller than $c^*(m,D,0)$ of the case without time delay as shown in theorem 2.1, and when the diffusion degeneracy is weaker with $1 < m < 2$, the wave front is still C^1-smooth, while, when the diffusion degeneracy is stronger with $m \geqslant 2$, then the wave front is non-C^1-smooth. Theorem 2.2 shows that all the non-critical waves are C^2-smooth.

Throughout this paper, we assume that the functions $d(s)$ and $b(s)$ satisfy the following hypotheses:

$$\begin{aligned} &\text{There exist } u_- = 0, \quad u_+ > 0 \text{ such that} \quad d, b \in C^2([0,u_+]), d(0) = b(0) = 0, \\ &d(u_+) = b(u_+), \quad b'(0) > d'(0) \geqslant 0, d'(u_+) \geqslant b'(u_+) \geqslant 0, \quad d'(s) \geqslant 0, \quad b'(s) \geqslant 0. \end{aligned} \tag{1.2}$$

Here, both $u_- = 0$ and $u_+ > 0$ are constant equilibria of (1.1), and the functions $b(u)$, $d(u)$ are both nondecreasing. The assumption (1.2) is summarized from the classical Fisher–KPP equation [8], which is also satisfied in a large number of evolution equations in ecology, for example, the well-studied Nicholson's blowflies equation [10] with the death functions $d_1(u) = \delta u$ or $d_2(u) = \delta u^2$, and the birth function

$$b_1(u) = pue^{-au^q}, \quad p > 0, \quad q > 0, \quad a > 0;$$

and the Mackey–Glass equation [15] with the growth function

$$b_2(u) = \frac{pu}{1+au^q} \quad p > 0, \quad q > 0, \quad a > 0.$$

Since the pioneering work of Schaaf [23] on traveling fronts in delayed reaction–diffusion equations, this field has been extensively investigated [1, 9, 25]. So *et al* [25] studied traveling front solutions of the Nicholson's blowflies equation. They showed that a monotone wave front exists connecting the equilibria for monotone birth function. In [7] the existence of non-monotone traveling fronts of delayed reaction–diffusion equation is proved, which includes several specific classes of birth rate functions. For the stability of critical and non-critical traveling waves, we refer the readers to [6, 14, 17–20] and the references therein.

Density-dependent dispersal has been observed in many biological populations [11, 21, 24]. Individuals migrate from densely populated areas to sparsely areas to avoid overcrowding [22]. The probability of leaving the current site increases with the local population density. This density-dependent dispersal mechanism arises from competition between conspecifics or deteriorating environmental conditions [16].

One of the most interesting features of the systems with degenerate diffusion is the existence of sharp type traveling wave. Sharp type traveling waves are zero on a half-plane and decay to zero in a continuous but non-smooth way. The interest in sharp traveling waves is related to the important property of finite speed of propagation, as showed in [26]. The sharp traveling wave plays an important role in the analysis of the propagation properties of degenerate diffusion equations since the equation generates compactly supported solutions if the initial value is compactly supported. The appearance of sharp profiles was first discussed in the pioneering work of Aronson [2], which is within the framework of models for degenerate biological invasion. In the case without time delay, the sharp type traveling wave is unique, and the corresponding speed is the minimal admissible traveling wave speed of all types, and further the speed is also the spreading speed, see the very recent paper by Audrito and Vázquez [3] for a doubly nonlinear diffusion equation.

When the system is with a time delay $r > 0$, the situation changes dramatically. Time delay and degenerate diffusion lead to essential difficulties in proving the similar results as in [3] concerning the existence, uniqueness and critical properties of sharp traveling waves for delayed system. To the best of our knowledge, the study of traveling waves for the monostable delayed model (1.1) with degenerate diffusion was initiated in [12, 27]. But the existence of the critical traveling waves $\phi(x + c^*t)$ still remains open. It is proved in [27] that the admissible traveling wave speeds for (2.3) are greater than or equal to $c^*(m, D, r) > 0$ if the time delay is small (see theorem 2.4 in [27]), but nothing is known about the type of the critical traveling waves. If the birth rate function is not restricted to be monotone, the authors in [12] proved that (2.3) admits traveling waves for some $c(r)$ if the time delay is small but those speeds are not critical (see theorem 1.2 in [12]). In our recent work [28], we prove the existence of sharp type traveling wave (wavefront or semi-wavefront). However, the uniqueness of such waves and whether it is corresponding to the minimal admissible wave speed still remain open. In this paper, we will answer those unsettled questions in [12, 27, 28] under the hypotheses (1.2).

Our main results provide insights into the dynamics of critical traveling wave solutions for (1.1). Theorem 2.1 stated below shows that model (1.1) admits a unique sharp traveling wave and it is monotonically increasing. Here, we develop a phase transform approach to show more precise behavior about traveling waves. Generally speaking, since the trajectories may intersect with each other due to the time delay, traditional phase plane analysis method is incapable of showing the existence of solutions of time delayed model. Surprisingly, we find that it provides a blueprint to draw more precise information about the solutions.

Critical traveling wave plays an important role in biological invasion. For this degenerate diffusion equation with time delay, the critical traveling wave is of sharp type. This sharp traveling wave $\phi(x + c^*t)$ separates the existence or the nonexistence of wavefronts. When $c < c^*$, there is no traveling wave for (1.1). When $c > c^*$, all the traveling waves are smooth and positive.

It is worth mentioning that the time delay leads to speed reducing mechanism of critical wave speed. For the case without time delay and with linear diffusion (i.e. $m = 1$ and $r = 0$), it is proved by Benguria and Depassier [5] that

$$c^*(1, D, 0) = \max\left\{2\sqrt{D(b'(0) - d'(0))}, \sup_{g\in\mathcal{D}} J_1(g)\right\},$$

where J_1 and $\mathcal{D}$ are defined in (2.3). As far as we know, this is the first reference that provides an effective method to calculate the velocity of the fronts. A well-known result for the classical Fisher–KPP equation $u_t - D\Delta u = u(1 - u/K)$ shows that $c^*(1, D, 0) = 2\sqrt{D}$ for this special types of birth and death functions. In this paper, we give a new variational characterization of the critical wave speed for this degenerate diffusion model with time delay. We show that there is a reducing mechanism of critical wave speed caused by time delay as shown in theorem 2.1.

The rest of this paper is organized as follows. In section 2, we present the main results on the existence, uniqueness and critical properties of sharp traveling wave solutions. Section 3 is devoted to the proof of the existence and properties of critical traveling wave solutions. We show that the critical wave for model (1.1) is sharp type and it is unique. Then we prove the effect of time delay to the critical wave speed and the regularity of the sharp traveling wave.

2. Main results

We consider the degenerate diffusion equation with time delay (1.1). We are looking for the traveling wave solutions that connect the two equilibria $u_- = 0$ and $u_+ =: K$. Under the hypotheses (1.2), the birth function $b(u)$ is monotonically increasing on $[u_-, u_+] = [0, K]$. Let $\phi(\xi)$, where $\xi = x + ct$ and $c > 0$, be the traveling wave solution of (1.1), we get (we write ξ as t for the sake of simplicity)

$$\begin{cases} c\phi'(t) = D(\phi^m(t))'' - d(\phi(t)) + b(\phi(t - cr)), \quad t \in \mathbb{R}, \\ \phi(-\infty) = 0, \quad \phi(+\infty) = K. \end{cases} \tag{2.1}$$

The traveling wave with partially compact support (we will call this ‘sharp type’) plays a crucial role in the analysis of the propagation properties of degenerate diffusion equations. Here, we try to show the uniqueness of sharp type traveling wave and the critical property of the corresponding wave speed for the time delayed case. Furthermore, we will prove that the time delay slows down the critical wave speed.

We are looking for traveling waves of sharp type and try to show the relation between the corresponding wave speed and all the admissible traveling wave speeds. We present the following definition of sharp and smooth traveling waves, see figure 1 for illustration. Here are some notations used throughout the paper: $L^1_{\mathrm{loc}}(\mathbb{R})$ is the set of locally Lebesgue integrable functions, $C^1 = C^1(\overline{\mathbb{R}})$,

$$C^b_{\mathrm{unif}}(\mathbb{R}) := \{\phi \in C(\mathbb{R}) \cap L^\infty(\mathbb{R}); \quad \phi \text{ is uniformly continuous on } \mathbb{R}\},$$

and

$$W^{1,2}_{\mathrm{loc}}(\mathbb{R}) := \{\phi; \phi \in W^{1,2}(\Omega) \quad \text{for any compact subset } \Omega \subset \mathbb{R}\},$$

where $W^{1,2}(\Omega) := \{\phi \in L^2(\Omega); \phi' \in L^2(\Omega)\}$.

Definition 2.1. A profile function $\phi(t)$ is said to be a traveling wave solution (TW) of (2.3) if $\phi \in C^b_{\mathrm{unif}}(\mathbb{R})$, $0 \leqslant \phi(t) \leqslant K := u_+$, $\phi(-\infty) = 0$, $\phi(+\infty) = K$, $\phi^m \in W^{1,2}_{\mathrm{loc}}(\mathbb{R})$, and $\phi(t)$ satisfies (2.3) in the sense of distributions. The TW $\phi(t)$ is said to be of sharp type if the support of $\phi(t)$

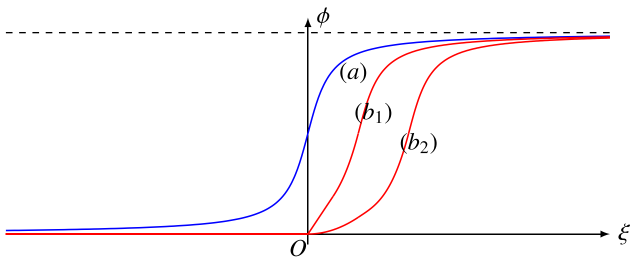

Figure 1. Traveling waves: (a) smooth type; (b_1) non-C^1 sharp type; (b_2) C^1 sharp type.

is semi-compact, i.e., supp$\phi = [t_0, +\infty)$ for some $t_0 \in \mathbb{R}$, $\phi(t) > 0$ for $t > t_0$. On the contrary, the TW $\phi(t)$ is said to be of smooth type if $\phi(t) > 0$ for all $t \in \mathbb{R}$.

Furthermore, for the sharp TW $\phi(t)$, if $\phi'' \notin L^1_{\text{loc}}(\mathbb{R})$, we say that $\phi(t)$ is a non-C^1 type sharp TW; otherwise, if $\phi'' \in L^1_{\text{loc}}(\mathbb{R})$, we say that $\phi(t)$ is a C^1 type sharp TW.

Without loss of generality, we may always shift t_0 to 0 for the sharp type TW. Therefore, a sharp type TW $\phi(t)$ is a special TW such that $\phi(t) = 0$ for $t \leqslant 0$, and $\phi(t) > 0$ for $t > 0$.

For any given $m > 1$, $D > 0$, and $r \geqslant 0$, we define the critical (or minimal) wave speed $c^*(m, D, r)$ for the degenerate diffusion equation (2.3) as follows

$$c^*(m, D, r) := \inf\{c > 0; (2.3) \quad \text{admits increasing TWs with speed } c\}. \tag{2.2}$$

For the case without time delay and with degenerate diffusion (i.e. $m > 1$ and $r = 0$), it is proved in [4, 12] that

$$c^*(m, D, 0) = \sup_{g\in\mathcal{D}} \left\{ 2\sqrt{D} \int_0^K \sqrt{-ms^{m-1}g(s)g'(s)(b(s) - d(s))}\mathrm{d}s \right\} =: \sup_{g\in\mathcal{D}} J_m(g), \tag{2.3}$$

where $\mathcal{D} = \left\{ g \in C^1([0, K]);\ g(K) = 0,\ \int_0^K g(s)\mathrm{d}s = 1,\ g'(s) < 0,\ \forall s \in (0, K) \right\}$. For the case without time delay and with linear diffusion (i.e. $m = 1$ and $r = 0$), it is proved by Benguria and Depassier [5] that

$$c^*(1, D, 0) = \max \left\{ 2\sqrt{D(b'(0) - d'(0))}, \sup_{g\in\mathcal{D}} J_1(g) \right\}.$$

A well-known result for the classical Fisher–KPP equation $u_t - D\Delta u = u(1 - u/K)$ shows that $c^*(1, D, 0) = 2\sqrt{D}$ for this special type of birth and death functions.

In this paper we show that (2.3) admits a unique sharp type TW, and the sharp TW is monotonically increasing and corresponding to the minimal wave speed $c^*(m, D, r)$, and further $c^*(m, D, r) < c^*(m, D, 0)$ for any time delay $r > 0$.

Our main results are as follows.

Theorem 2.1 (Critical sharp traveling wave). *Assume that $d(s)$ and $b(s)$ satisfy (1.2), and $m > 1$, $r \geqslant 0$. There exists a unique $c^* = c^*(m, D, r) > 0$ defined in (2.2) satisfying $c^*(m, D, r) < c^*(m, D, 0)$ for any time delay $r > 0$, such that (2.3) admits a unique (up to shift) sharp traveling wave $\phi(x + c^*t)$ with speed c^*, which is the critical traveling wave of (2.3) and is monotonically increasing.*

Theorem 2.2 (Smooth traveling waves). *Assume that the conditions in theorem 2.1 hold. For any $c > c^* = c^*(m, D, r) > 0$ defined in (2.2), the traveling wave $\phi(x + ct)$ of (2.3) with speed c is smooth type for all $m > 1$ such that $\phi \in C^2(\mathbb{R})$; while for any $c \leqslant c^* = c^*(m, D, r)$, there is no smooth type traveling waves of (2.3) with speed c.*

Remark 2.1. Theorems 2.1 and 2.2 show that the sharp traveling wave $\phi(x + c^*t)$ is unique and critical in the following senses:

(a) the speed c^* is the unique wave speed such that (2.3) admits sharp type traveling waves (no other wave speed for sharp traveling waves);
(b) the speed c^* is critical such that no traveling waves exist for $c < c^*$ and $\phi(x + c^*t)$ is the wave with minimal wave speed;
(c) $\phi(x + c^*t)$ is the unique traveling wave with speed c^*, i.e., no smooth traveling waves with the critical wave speed exist.

Therefore, a traveling wave of (2.3) is sharp if and only if the corresponding wave speed is critical.

Remark 2.2. The existence of non-critical traveling waves was proved in [27], where no regularity results were given as the types of traveling waves, sharp or smooth, were not settled. We also note that the C^2 regularity and existence of traveling waves were shown in [12] for some wave speeds $c > c^*(m, D, 0)$, which according to our results here (theorem 2.1) are non-critical since the critical wave speed $c^*(m, D, r) < c^*(m, D, 0)$. Theorem 2.2 presents the C^2 regularity of non-critical traveling waves with any speed $c > c^*(m, D, r)$.

The sharp traveling wave is classified into C^1 type and non-C^1 type according to the degeneracy index m.

Theorem 2.3 (Regularity of sharp wave). *Assume that the conditions in theorem 2.1 hold. If $m \geqslant 2$, then the sharp traveling wave is of non-C^1 type; while if $1 < m < 2$, then the sharp traveling wave is of C^1 type.*

Remark 2.3. Regarding the regularity of sharp waves, roughly speaking the degeneracy strengthens as $m > 1$ increases and the regularity of the case $m \geqslant 2$ is weaker than that of $1 < m < 2$. For the case $1 < m < 2$, the sharp traveling wave remains C^1 regularity but not analytic.

3. Proof of the main results

In this section, we first solve (2.3) locally for sharp type solutions, and then we develop a phase transform approach to show the monotone dependence of the wave speed. The sharp traveling wave is the unique local solution that exists globally, monotonically increases on the whole real line, and has the least upper bound K. The corresponding wave speed is characterized via a variational approach inspired by Benguria and Depassier [5] (see also Huang *et al* [12]), and is also compared with the smooth type traveling wave speeds.

The existence of sharp TW for the case without time delay is proved in [12] for the Nicholson's blowfly model. It is also valid for the general birth rate and death rate functions without time delay and here we only focus on the case with time delay $r > 0$.

For any given $D > 0$ and $r > 0$, we solve (2.3) step by step. First, noticing that the sharp wave solution $\phi(t) = 0$ for $t \leqslant 0$ and then $\phi(t - cr) = 0$ for $t \in [0, cr)$, (2.3) is locally reduced to

$$\begin{cases} c\phi'(t) = D(\phi^m(t))'' - d(\phi(t)), \\ \phi(0) = 0, \quad (\phi^m)'(0) = 0, \quad t \in (0, cr), \end{cases} \tag{3.1}$$

whose solutions are not unique and we choose the maximal one such that $\phi(t) > 0$ for $t \in (0, cr)$ as shown in the following lemma. Here, $(\phi^m)'(0) = 0$ is a necessary and sufficient condition such that the zero extension of $\phi(t)$ to the left satisfies (2.3) locally near 0 in the sense of distributions.

The following three lemmas are formulated and proved in our related paper [28] in order to show the existence of sharp TWs in a more general setting. We omit the proofs here for the sake of simplicity.

Lemma 3.1. ([28]) *For any $c > 0$, the degenerate ODE (3.6) admits a unique maximal solution $\phi_c^1(t)$ on $(0, cr)$ such that $\phi_c^1(t) > 0$ on $(0, cr)$ and*

$$\phi_c^1(t) = \left(\frac{(m-1)c}{Dm}t\right)^{\frac{1}{m-1}} + o\left(t^{\frac{1}{m-1}}\right), \quad t \to 0^+.$$

Next, let $\phi_c^2(t)$ be the solution of the following initial value ODE problem

$$\begin{cases} c\phi'(t) = D(\phi^m(t))'' - d(\phi(t)) + b(\phi_c^1(t - cr)), \\ \phi(r) = \phi_c^1(r), \quad \phi'(r) = (\phi_c^1)'(r), \quad t \in (cr, 2cr). \end{cases} \tag{3.2}$$

The above steps can be continued unless $\phi_c^k(t)$ blows up or decays to zero in finite time for some $k \in \mathbb{N}$. Let $\phi_c(t)$ be the connecting function of those functions on each step, i.e.,

$$\phi_c(t) = \begin{cases} \phi_c^1(t), & t \in [0, cr), \\ \phi_c^2(t), & t \in [cr, 2cr), \\ \vdots & \\ \phi_c^k(t), & t \in [(k-1)cr, kcr), \\ \vdots & \end{cases} \tag{3.3}$$

for some finite steps such that $\phi_c(t)$ blows up or decays to zero, or for infinite steps such that $\phi_c(t)$ is defined on $(0, +\infty)$ and zero extended to $(-\infty, 0)$ for convenience.

Lemma 3.2. ([28]) *For any given m, D and $r > 0$, there exists a constant $\underline{c} > 0$ such that if $c \leqslant \underline{c}$, then $\phi_c(t)$ decays to zero in finite time.*

Lemma 3.3. ([28]) *For any given m, D and $r > 0$, there exists a constant $\overline{c} > 0$ such that if $c \geqslant \overline{c}$, then $\phi_c(t)$ grows up to $+\infty$ as t tends to $+\infty$.*

In the paper [28], the birth rate function is not restricted to be monotone in $[0, K]$, and the time delay together with the non-monotone structure of birth rate function gains the possibility of the existence of oscillating sharp TWs. As a result, the monotone dependence of $\phi_c(t)$ with respect to c is generally not true, and the uniqueness of the wave speed for sharp type TWs remains open under these general settings.

Here in this paper, under the assumption (1.2), the birth rate function is monotonically increasing in $[0, K]$ and we can present a positive answer to the above questions.

Lemma 3.4 (Continuous dependence)**.** *For any given m, D and $r > 0$, the solution $\phi_c(t)$ is locally continuously dependent on c. That is, for any $c > 0$ and any given $T > 0$ and $\varepsilon > 0$,* *there exists a* $\delta > 0$ *such that for any* $|c_1 - c| < \delta$ *and* $c_1 > 0$ *we have*

$$|\phi_{c_1}(t) - \phi_c(t)| < \varepsilon, \quad \forall t \in (0, T_1 - \varepsilon),$$

where $T_1 = \min\{T, T_c\}$ *with* $(0, T_c)$ *being the existence interval of* $\phi_c(t)$.

Proof. Without loss of generality, we may assume that $T_1 > cr + \varepsilon$. The proof is divided into two parts: the continuous dependence of the singular ODE (3.6) within $(0, cr)$ and the continuous dependence of a regular ODE within $(cr, T_1 - \varepsilon)$.

Step 1. We prove that $\phi_c(t)$ together with $\phi_c{}'(cr)$ is continuously dependent on c for $t \in (0, cr)$. Since $T_c > cr$, we see that $\phi_c(t)$ is positive for $t \in (0, cr]$. We note that the maximal solution $\phi_c(t)$ is the unique solution such that $\phi_c(t) > 0$ in a right neighbor of 0 and the asymptotic analysis lemma 3.1 shows that $\phi_c(t)$ is locally monotonically and continuously dependent on c within some interval $(0, t_1) \subset (0, cr)$. In (t_1, cr), $\phi_c(t)$ is bounded away from zero and (3.6) is a regular ODE, and the continuously dependence follows from the classical theory.

Step 2. We prove that $\phi_c(t)$ is continuously dependent on c for $t \in (cr, T_1 - \varepsilon)$. As $(0, T_c)$ is the existence interval of $\phi_c(t)$, $\phi_c(t)$ is bounded from above and below for $t \in [cr, T_c - \varepsilon]$ such that $\phi_c(t) \in [M_1, M_2] \subset (0, +\infty)$ for some $M_2 > M_1 > 0$. According to the construction of $\phi_c(t)$, we see that $\phi_c(t)$ satisfies

$$\begin{cases} c\phi'(t) = D(\phi^m(t))'' - d(\phi(t)) + b(\phi(t - cr)), & t \in (cr, T_1 - \varepsilon), \\ \phi(cr) = \phi_c(cr), \quad \phi'(cr) = \phi_c'(cr), \end{cases}$$

which is a regular ODE without singularity on bounded interval. This completes the proof. □

Lemma 3.5. *For any given* $c > 0$, $\phi_c(t)$ *is locally strictly increasing on* $(0, t_*)$ *for some* $t_* \in (0, +\infty]$. *We may slightly abuse the notation and denote the maximal interval in which* $\phi_c(t)$ *is strictly increasing and* $\phi_c(t) < K$ *also by* $(0, t_*)$. *If* $t_* < +\infty$ *and* $\phi_c(t_*) < K$, *then* $\phi_c(t)$ *decreases to zero and will never grows up to* K *after* t_*.

Proof. The local monotonicity follows from lemma 3.1. We first show that $\phi_c(t)$ can not be constant after t_*. Otherwise,

$$b(\phi(t - cr)) = c\phi'(t) - D(\phi^m(t))'' + d(\phi(t)) = d(\phi(t))$$

is a constant after t_*, which contradicts the strictly monotone increasing of $\phi_c(t)$ before t_* since $t - cr < t_*$ for $t \in (t_*, t_* + cr)$.

Now we argue by contradiction, assume that $\phi_c(t)$ is not always decreasing after t_*, then there exists a $t_1 > t_*$ such that $\phi_c(t)$ is strictly decreasing in (t_*, t_1) and $\phi_c{}'(t_1) = 0$. Since $\phi_c(t)$ and $\phi_c^m(t)$ attain their local maximums at t_* and local minimums at t_1, then

$$\phi_c'(t_*) = 0, \quad (\phi_c^m)''(t_*) \leqslant 0, \quad d(\phi_c(t_*)) \leqslant b(\phi_c(t_* - cr)),$$
$$\phi_c'(t_1) = 0, \quad (\phi_c^m)''(t_1) \geqslant 0, \quad d(\phi_c(t_1)) \geqslant b(\phi_c(t_1 - cr)),$$

and

$$b(\phi_c(t_1 - cr)) \leqslant d(\phi_c(t_1)) < d(\phi_c(t_*)) \leqslant b(\phi_c(t_* - cr)),$$

which implies that $\phi_c(t_1 - cr) < \phi_c(t_* - cr)$ according to the monotonicity of $b(\cdot)$. Case 1: if $t_1 - t_* \leqslant cr$, then $t_* - cr < t_1 - cr \leqslant t_*$, and $\phi_c(t_1 - cr) > \phi_c(t_* - cr)$ as $\phi_c(t)$ is strictly increasing before t_*. We arrive at a contradiction. Case 2: if $t_1 - t_* > cr$, then $t_1 - cr > t_*$, $\phi_c(t_1 - cr) > \phi_c(t_1)$ as $\phi_c(t)$ is strictly decreasing in (t_*, t_1), and then

$$b(\phi_c(t_1)) < b(\phi_c(t_1 - cr)) \leqslant d(\phi_c(t_1)).$$

Noticing that $\phi_c(t_1) < \phi_c(t_*) < K$, we have another contradiction to the fact $b(s) > d(s)$ for all $s \in (0, K)$. Therefore, $\phi_c(t)$ is always decreasing after t_* before reaching zero, and zero is the only equilibrium that smaller than K. The proof is completed. □

The monotonicity of birth function $b(\cdot)$ plays a crucial role in the proof of the monotonicity of the sharp type local solution $\phi_c(t)$ in lemma 3.5 and the monotonicity of $\phi_c(t)$ is the foundation of the proof of the following monotone dependence of ϕ_c with respect to c. Here we develop a phase transform approach to show more precise behavior about $\phi_c(t)$. We note that generally speaking this method is incapable of showing the existence of traveling waves with time delay due to the nonlocal property of time delay. However, it could provide more precise information about the solutions if we have already known or assumed the existence.

For any $c > 0$, let $(0, t_*)$ be the maximal interval such that $\phi_c(t)$ is strictly increasing and $\phi_c(t) < K$ on $(0, t_*)$ as defined in lemma 3.5. Then three cases can happen:

(a) $t_* < +\infty$ and $\phi_c(t_*) = K$, then $\phi_c{}'(t_*) \geqslant 0$, $\phi_c(t)$ may grow or decay after t_*;
(b) $t_* < +\infty$ and $\phi_c(t_*) < K$, then $\phi_c{}'(t_*) = 0$ and $\phi_c(t)$ decays to zero and never grows up to K after t_*;
(c) $t_* = +\infty$, then $\phi_c(t)$ is strictly increasing on $(0, +\infty)$ and $\phi_c(+\infty) \leqslant K$, $\phi_c{}'(+\infty) = 0$.

The assertion about case (b) follows from Lemma 3.5. Hereafter, we only consider $\phi_c(t)$ within $(0, t_*)$.

Lemma 3.6 (Monotone dependence). *The function $\phi_c(t)$ is strictly monotonically increasing with respect to c in their joint interval.*

Proof. To prove the monotone dependence of ϕ_c with respect to c, we develop a phase transform approach with time delay inspired by [12, 13], see also [27, 28] for the proof of nonexistence of traveling waves by using this phase transform approach. We note that lemma 3.1 implies the locally monotone dependence for t near 0, but it is not enough as we need the globally monotone dependence.

From the construction of $\phi_c(t)$, we see that $\phi_c(t)$ is strictly increasing on $(0, t_*)$, either $\phi_c(t_*) = K$ or $\phi_c{}'(t_*) = 0$, and $\phi_c(t)$ satisfies the following equation as we solve (2.3) locally step by step

$$\begin{cases} c\phi'(t) = D(\phi^m(t))'' - d(\phi(t)) + b(\phi(t - cr)), \quad t \in (0, t_*), \\ \phi(0) = 0, \quad \phi(t)\,\text{is strictly increasing on}\,(0, t_*). \end{cases} \tag{3.4}$$

Let

$$\psi_c(t) = D(\phi_c^m(t))' = Dm\phi_c^{m-1}(t)\phi_c'(t).$$

Since $\phi_c(t)$ is strictly increasing on $(0, t_*)$, (3.9) is transformed into

$$\begin{cases} \phi'(t) = \dfrac{\psi(t)}{Dm\phi^{m-1}(t)}, \\ \psi'(t) = \dfrac{c\psi(t)}{Dm\phi^{m-1}(t)} - \left(b(\hat{\phi}_{cr}(t)) - d(\phi(t))\right), \end{cases} \tag{3.5}$$

where $\hat{\phi}_{cr}(t) := \phi_c(t - cr)$.

The key ingredient of the phase transform approach is to write $t \in (0, t_*)$ as an inverse function of $\phi \in (0, \phi_c(t_*))$ according to $\phi = \phi_c(t)$ and interpret $\phi_c(t - cr)$ as a function of ϕ depending on ψ_c. That is, $\tilde{t}(\phi) := \phi_c^{-1}(\phi)$ for $\phi \in (0, \phi_c(t_*))$, and $\tilde{\psi}(\phi) := \psi_c(\tilde{t}(\phi))$, and further

$$\tilde{\phi}_{cr}(\phi) := \hat{\phi}_{cr}(\tilde{t}(\phi)) = \phi_c(\tilde{t}(\phi) - cr) = \inf_{\theta \geqslant 0} \left\{ \int_\theta^\phi \frac{Dms^{m-1}}{\tilde{\psi}(s)} \mathrm{d}s \leqslant cr \right\}. \tag{3.6}$$

Since $\phi_c(t)$ is strictly increasing, we see that $\tilde{\phi}_{cr}(\phi) < \phi$ according to (3.10) (or (3.11)). The local solution $\phi_c(t)$ in $(0, t_*)$ corresponds to a trajectory in the phase plane (ϕ, ψ) of (3.10) and $\tilde{\psi}(\phi) > 0$ except for some possible isolated points t_1 such that ${\phi_c}'(t_1) = 0$ and then $\tilde{\psi}(\phi_c(t_1)) = 0$. Those points are isolated according to the strictly increasing property of $\phi_c(t)$. As we have made change of the variables, we can write (3.10) into

$$\frac{d\tilde{\psi}}{d\phi} = c - \frac{Dm\phi^{m-1}(b(\tilde{\phi}_{cr}(\phi)) - d(\phi))}{\tilde{\psi}}, \quad \phi \in (0, \phi_c(t_*)). \tag{3.7}$$

From lemma 3.1, we know that $(\phi_c^m)'(0^+) = 0$, i.e., $\psi_c(0) = 0$ and $\tilde{\psi}(0) = 0$.

Let Γ_c be the curve of

$$\bar{\psi}(\phi) := \frac{Dm\phi^{m-1}(b(\phi) - d(\phi))}{c}.$$

Then $\bar{\psi}(0) = \bar{\psi}(K) = 0$, $\bar{\psi}(\phi) > 0$ for $\phi \in (0, K)$, $\bar{\psi}(\phi)$ is decreasing near K and Γ_c divides $(0, K) \times (0, +\infty)$ into two parts, $E_1 := \left\{(\phi, \psi); \quad \phi \in (0, K), \quad 0 < \psi < \bar{\psi}(\phi)\right\}$ and $E_2 := ((0, K) \times (0, +\infty)) \setminus E_1$. Since $\tilde{\phi}_{cr}(\phi) < \phi$, we have

$$\frac{d\tilde{\psi}}{d\phi} = c - \frac{Dm\phi^{m-1}(b(\tilde{\phi}_{cr}(\phi)) - d(\phi))}{\tilde{\psi}} > c - \frac{Dm\phi^{m-1}(b(\phi) - d(\phi))}{\tilde{\psi}} \geqslant 0,$$

for $(\phi, \tilde{\psi}) \in E_2$. In cases (b) and (c), $\psi_c(t_*) = 0$ and then $\tilde{\psi}(\phi_c(t_*)) = 0$. That is, the trajectory lies below Γ_c when t nears t_* and then

$$\sup_{\phi \in (0, \phi_c(t_*))} \tilde{\psi}(\phi) \leqslant \sup_{\phi \in (0, K)} \bar{\psi}(\phi).$$

Otherwise, $\tilde{\psi}(\phi_c(t_*))$ will be positive, which contradicts cases (b) and (c).

After settling down the generalized phase plane, we now divide the proof of the monotonically increasing dependence of ϕ_c with respect to c into two steps. Let $c_1 > c_2 > 0$ and $\phi_{c_1}(t)$, $\phi_{c_2}(t)$ be the local solutions on the interval $(0, t_*(c_1))$ and $(0, t_*(c_2))$, respectively. The symbols $t_*(c_1)$ and $t_*(c_2)$ are t_* corresponding to c_1 and c_2 separately defined in lemma 3.5. The functions $\tilde{t}(\phi)$, $\tilde{\psi}(\phi)$ and $\tilde{\phi}_{cr}(\phi)$ defined above corresponding to c_1 are denoted by $\tilde{t}_1(\phi)$, $\tilde{\psi}_1(\phi)$, $\tilde{\phi}_{c_1 r}(\phi; \tilde{\psi}_1)$. Then the functions $\tilde{t}_2(\phi)$, $\tilde{\psi}_2(\phi)$, $\tilde{\phi}_{c_2 r}(\phi; \tilde{\psi}_2)$ follow similarly corresponding to c_2.

Step 1. We assert that $\phi_{c_1}(t) > \phi_{c_2}(t)$ for all $t \in (0, c_1 r]$ and $\tilde{\psi}_1(\phi) > \tilde{\psi}_2(\phi)$ for $\phi \in (0, \phi_{c_1}(c_1 r)]$. For the phase plane corresponding to c_1, (3.12) reads

$$\frac{d\tilde{\psi}_1}{d\phi} = c_1 - \frac{Dm\phi^{m-1}(b(\tilde{\phi}_{c_1 r}(\phi; \tilde{\psi}_1)) - d(\phi))}{\tilde{\psi}_1}, \quad \phi \in (0, \phi_{c_1}(t_*(c_1))), \tag{3.8}$$

and the phase plane corresponding to c_2 follows similarly. Within $(0, c_1r)$, we have $\tilde{\phi}_{c_1r}(\phi;\tilde{\psi}_1) = \phi_{c_1}(\tilde{t}(\phi) - cr) = 0$ and (3.13) is actually

$$\frac{d\tilde{\psi}_1}{d\phi} = c_1 + \frac{Dm\phi^{m-1}d(\phi)}{\tilde{\psi}_1}, \quad \phi \in (0, \phi_{c_1}(c_1r)). \tag{3.9}$$

Therefore, for $\phi \in (0, \phi_{c_1}(c_1r))$ we have

$$\frac{d\tilde{\psi}_1}{d\phi} = c_1 + \frac{Dm\phi^{m-1}d(\phi)}{\tilde{\psi}_1} > c_2 + \frac{Dm\phi^{m-1}d(\phi)}{\tilde{\psi}_1}$$

and

$$\frac{d\tilde{\psi}_2}{d\phi} = c_2 - \frac{Dm\phi^{m-1}(b(\tilde{\phi}_{c_2r}(\phi;\tilde{\psi}_2)) - d(\phi))}{\tilde{\psi}_2} \leqslant c_2 + \frac{Dm\phi^{m-1}d(\phi)}{\tilde{\psi}_2}.$$

The comparison principle for the above singular ODE implies that $\tilde{\psi}_1(\phi) > \tilde{\psi}_2(\phi)$ for $\phi \in (0, \phi_{c_1}(c_1r))$, see also [12, 13] for details by calculating $d(\tilde{\psi}_1 - \tilde{\psi}_2)/d\phi$.

Here it should be noted that $\phi_{c_1}(t) > \phi_{c_2}(t)$ for all $t \in (0, c_1r]$ is not a simple conclusion of $\tilde{\psi}_1(\phi) > \tilde{\psi}_2(\phi)$ for $\phi \in (0, \phi_{c_1}(c_1r)]$ as the phase planes corresponding to c_1 and c_2 are different (the changes of variables are different, i.e., $\tilde{t}_1(\phi) \not\equiv \tilde{t}_2(\phi)$). We need to argue by contradiction. Suppose that $\phi_{c_1}(t) > \phi_{c_2}(t)$ is not true for all $t \in (0, c_1r]$, then there exists a $t_0 \in (0, c_1r]$ such that $\phi_{c_1}(t_0) = \phi_{c_2}(t_0)$. The choice of t_0 may not be unique, we choose the smallest one as $\phi_{c_1}(t) > \phi_{c_2}(t)$ for t near 0 according to lemma 3.1. Then $\phi'_{c_1}(t_0) \leqslant \phi'_{c_2}(t_0)$, $\phi_{c_1}(t_0) = \phi_{c_2}(t_0) =: \phi_0 \in (0, \phi_{c_1}(c_1r)]$ and $\tilde{t}_1(\phi_0) = t_0 = \tilde{t}_2(\phi_0)$,

$$\tilde{\psi}_1(\phi_0) = m\phi_0^{m-1}\phi'_{c_1}(t_0) \leqslant m\phi_0^{m-1}\phi'_{c_2}(t_0) = \tilde{\psi}_2(\phi_0),$$

which contradicts the fact that $\tilde{\psi}_1(\phi) > \tilde{\psi}_2(\phi)$ for all $\phi \in (0, \phi_{c_1}(c_1r)]$.

Step 2. We prove that $\phi_{c_1}(t) > \phi_{c_2}(t)$ for all $t \in (c_1r, \min\{t_*(c_1), t_*(c_2)\})$ and $\tilde{\psi}_1(\phi) > \tilde{\psi}_2(\phi)$ for $\phi \in (\phi_{c_1}(c_1r), \min\{\phi_{c_1}(t_*(c_1)), \phi_{c_2}(t_*(c_2))\})$. According to step 1, $\tilde{\psi}_1(\phi) > \tilde{\psi}_2(\phi)$ at $\phi = \phi_{c_1}(c_1r)$, let $\phi_* > \phi_{c_1}(c_1r)$ be the first point such that $\tilde{\psi}_1(\phi) > \tilde{\psi}_2(\phi)$ is not true as we are arguing by contradiction. Then $\tilde{\psi}_1(\phi) > \tilde{\psi}_2(\phi)$ for $\phi \in (0, \phi_*)$ and $\tilde{\psi}_1(\phi_*) = \tilde{\psi}_2(\phi_*)$, $\tilde{\psi}'_1(\phi_*) \leqslant \tilde{\psi}'_2(\phi_*)$. For $\phi \in (\phi_{c_1}(c_1r), \phi_*)$, we have $\phi > \phi_{c_1}(c_1r) > \phi_{c_2}(c_1r)$, $\tilde{t}_1(\phi) > c_1r > c_2r$, $\tilde{t}_2(\phi) > c_1r > c_2r$, and then (3.11) is simplified to

$$\int_{\tilde{\phi}_{c_1r}(\phi;\tilde{\psi}_1)}^{\phi} \frac{Dms^{m-1}}{\tilde{\psi}_1(s)}\mathrm{d}s = c_1r, \qquad \int_{\tilde{\phi}_{c_2r}(\phi;\tilde{\psi}_2)}^{\phi} \frac{Dms^{m-1}}{\tilde{\psi}_2(s)}\mathrm{d}s = c_2r. \tag{3.10}$$

Since $c_1r > c_2r$ and $\tilde{\psi}_1(\phi) > \tilde{\psi}_2(\phi)$ for $\phi \in (0, \phi_*)$, (3.15) tells us that

$$\tilde{\phi}_{c_1r}(\phi;\tilde{\psi}_1) < \tilde{\phi}_{c_2r}(\phi;\tilde{\psi}_2)$$

for $\phi \in (\phi_{c_1}(c_1r), \phi_*)$.

Now we use (3.13) to deduce that

$$\begin{aligned}\frac{d\tilde{\psi}_1}{d\phi} =& c_1 - \frac{Dm\phi^{m-1}(b(\tilde{\phi}_{c_1r}(\phi;\tilde{\psi}_1)) - d(\phi))}{\tilde{\psi}_1}\\ >& c_1 + \frac{Dm\phi^{m-1}d(\phi)}{\tilde{\psi}_1} - \frac{Dm\phi^{m-1}b(\tilde{\phi}_{c_2r}(\phi;\tilde{\psi}_2))}{\tilde{\psi}_1}\\ >& c_1 + \frac{Dm\phi^{m-1}d(\phi)}{\tilde{\psi}_1} - \frac{Dm\phi^{m-1}b(\tilde{\phi}_{c_2r}(\phi;\tilde{\psi}_2))}{\tilde{\psi}_2}, \quad \phi \in (0, \phi_*),\end{aligned}$$

and similarly for the phase plane of c_2 we have

$$\begin{aligned}\frac{d\tilde{\psi}_2}{d\phi} =& c_2 - \frac{Dm\phi^{m-1}(b(\tilde{\phi}_{c_2r}(\phi;\tilde{\psi}_2)) - d(\phi))}{\tilde{\psi}_2}\\ =& c_2 + \frac{Dm\phi^{m-1}d(\phi)}{\tilde{\psi}_2} - \frac{Dm\phi^{m-1}b(\tilde{\phi}_{c_2r}(\phi;\tilde{\psi}_2))}{\tilde{\psi}_2}, \quad \phi \in (0, \phi_*).\end{aligned}$$

It follows that at the point ϕ_*,

$$\frac{d\tilde{\psi}_1}{d\phi} - \frac{d\tilde{\psi}_2}{d\phi} > c_1 - c_2 > 0,$$

which is a contradiction to $\tilde{\psi}_1'(\phi_*) \leqslant \tilde{\psi}_2'(\phi_*)$. This argument by contradiction and step 1 show that $\tilde{\psi}_1(\phi) > \tilde{\psi}_2(\phi)$ for $\phi \in (0, \min\{\phi_{c_1}(t_*(c_1)), \phi_{c_2}(t_*(c_2))\})$. Using this fact, we can show that $\phi_{c_1}(t) > \phi_{c_2}(t)$ for all $t \in (0, \min\{t_*(c_1), t_*(c_2)\})$ in a similar procedure as in step 1. The proof is completed. □

Lemma 3.7. *There exists a unique $\hat{c} > 0$ such that $\phi_{\hat{c}}(t)$ strictly increasing on $(0, +\infty)$, $\phi_{\hat{c}}(+\infty) = K$ and the function $\phi_{\hat{c}}(t)$ is also unique.*

Proof. According to lemma 3.6, ϕ_c is strictly increasingly depending on $c > 0$. Lemmas 3.2 and 3.3 show that ϕ_c grows up to K at finite time if c is large and decays to 0 at finite time if c is small. Applying the continuous dependence lemma 3.4 and the monotone dependence lemma 3.6 with respect to c, we can define

$$\hat{c} = \{c > 0; \quad \phi_c(t)\,\text{grows up to}\,K\,\text{in finite time}\}. \tag{3.11}$$

Then $\hat{c} > 0$ according to lemma 3.2, $\phi_{\hat{c}}$ is unique as we solve (2.3) step by step, and $\hat{c}$ is the speed that satisfies the conditions in this lemma.

We show that $\hat{c}$ is the unique speed that has the properties in this lemma. Lemma 3.6 implies the strictly monotone dependence of $\phi_c(t)$ with respect to c. More precisely, from the proof of lemma 3.6 we know that $\tilde{\psi}(\phi)$ is also strictly monotone dependence with respect to c, where $\tilde{\psi}(\phi)$ is the trajectory in the generalized phase plane as in the proof of lemma 3.6. The strictly monotone dependence implies the uniqueness of $\hat{c}$. In fact, if there are $c_1 > c_2$ that have the properties in this lemma, let $\tilde{\psi}_1(\phi)$ and $\tilde{\psi}_2(\phi)$ be the functions defined as in the proof of lemma 3.6. Then $\tilde{\psi}_1(K) = \tilde{\psi}_2(K)$, which contradicts the strictly monotone dependence of $\tilde{\psi}(\phi)$ with respect to c. This completes the proof. □

Lemma 3.7 shows that $\phi_{\hat{c}}{}'(t) \geqslant 0$ and $\phi_{\hat{c}}(t)$ is strictly increasing in $(0, +\infty)$. We need to prove a strong version as follows.

Lemma 3.8. *The sharp traveling wave $\phi_{\hat{c}}(t)$ in lemma 3.7 satisfies $\phi_{\hat{c}}{}'(t) > 0$ for all $t \in (0, +\infty)$.*

Proof. It is obvious that $\phi_{\hat{c}}{}'(t) \geqslant 0$ and we argue by contradiction and assume that there exists a $t_0 \in (0, +\infty)$ such that $\phi_{\hat{c}}{}'(t_0) = 0$. If $\phi''_{\hat{c}}(t_0) \neq 0$, then $\phi_{\hat{c}}$ attains its local strictly extreme value at t_0, which cannot happen since $\phi_{\hat{c}}(t)$ is strictly increasing on $(0, +\infty)$. Therefore, $\phi''_{\hat{c}}(t_0) = 0$ and

$$\phi_{\hat{c}}(t) = \phi_{\hat{c}}(t_0) + A(t - t_0)^3 + o(|t - t_0|^3), \quad t \to t_0, \tag{3.12}$$

where $A \geqslant 0$ as $\phi_{\hat{c}}(t)$ is strictly increasing. The above expansion is valid as $\phi_{\hat{c}}$ is smooth away from the boundary of its support. We first assume that $A > 0$ and we write $\phi_{\hat{c}}(t)$ as $\phi(t)$ for simplicity. Now (2.3) reads

$$\begin{aligned} b(\phi(t - cr)) =& c\phi'(t) - D(\phi^m(t))'' + d(\phi(t)) \\ =& 3cA(t - t_0)^2 - 6DAm\phi^{m-1}(t_0)(t - t_0) + d(\phi(t_0)) \\ &+ 3Ad'(\phi(t_0))(t - t_0)^2 + o(|t - t_0|), \quad t \to t_0, \end{aligned}$$

where the right hand side is monotonically decreasing near t_0. By noticing that $\phi(t)$, $\phi(t - cr)$ and $b(\phi(t - cr))$ are strictly increasing, we arrive at a contradiction. If $A = 0$, we can expand (3.12) to higher odd order and proceed the above argument similarly. The proof is completed. □

In order to compare $\hat{c}$ with $c^*(m, D, r)$ defined by (2.2), we need to compare the sharp type traveling wave $\phi_{\hat{c}}(t)$ with other smooth type traveling waves.

Lemma 3.9. *There holds $\hat{c} = c^*(m, D, r)$. That is, $\hat{c}$ is the minimal admissible traveling wave speed. Furthermore, the wave speed for the smooth type traveling wave is greater than the unique sharp type traveling wave speed $\hat{c}$.*

Proof. Lemma 3.7 shows that the sharp type traveling wave is unique. We argue by contradiction in the following. Let $\hat{\phi}(t)$ be a smooth type traveling wave with speed $c_1 < \hat{c}$, and let $\phi_{c_1}(t)$ and $\phi_{\hat{c}}(t)$ be the local sharp type solutions defined by (3.8) corresponding to c_1 and $\hat{c}$ respectively. Since $c_1 < \hat{c}$ and $\phi_{\hat{c}}(t)$ increases in $(0, +\infty)$ with $\phi_{\hat{c}}(+\infty) = K$, we see that $\phi_{c_1}(t_*(c_1)) < K$ according to the strictly monotone dependence and the uniqueness of $\hat{c}$ in lemma 3.7, where $t_*(c_1)$ is defined in the proof of lemma 3.6.

Let $\tilde{\psi}(\phi)$ be the trajectory in the generalized phase plane corresponding to the sharp type $\phi_{c_1}(t)$ defined in the proof of lemma 3.6. Then $\tilde{\psi}(\phi_{c_1}(t_*(c_1))) = 0$ as $\phi'_{c_1}(t_*(c_1)) = 0$. For the monotonically increasing smooth type traveling wave solution $\hat{\phi}(t)$ we can also define the generalized phase plane and let $\hat{\psi}(\phi)$ be the trajectory corresponding to the smooth type traveling wave $\hat{\phi}(t)$. The local asymptotic analysis lemma 3.1 implies that

$$\tilde{\psi}(\phi) \sim c_1\phi, \quad \phi \to 0^+,$$

and similar analysis shows that

$$\hat{\psi}(\phi) \sim \frac{Dm(b'(0)\mathrm{e}^{-\lambda c_1 r} - d'(0))}{c_1}\phi^m, \quad \phi \to 0^+,$$

where $\lambda > 0$ is the unique solution of the equation $\lambda c_1 + d'(0) = b'(0)\mathrm{e}^{-\lambda c_1 r}$. The above local asymptotic behavior near zero shows that $\hat{\psi}(\phi) < \tilde{\psi}(\phi)$ for $\phi \in (0, \phi_*)$ with some $\phi_* > 0$. Similar to the proof of lemma 3.6, we can show that $\hat{\psi}(\phi) < \tilde{\psi}(\phi)$ for all $\phi \in (0, \phi_{c_1}(t_*(c_1))]$ and then $\hat{\psi}(\phi_0) = 0$ for some $\phi_0 \in (0, \phi_{c_1}(t_*(c_1))) \subset (0, K)$. Similar to the proof of lemma 3.8, we can prove that $\hat{\phi}'(t) > 0$ for all $t \in \mathbb{R}$ if $\hat{\phi}$ is a monotonically increasing smooth type traveling wave solution. That is, $\hat{\psi}(\phi) > 0$ for all $\phi \in (0, K)$, which is a contradiction to $\hat{\psi}(\phi_0) = 0$ for some $\phi_0 \in (0, K)$.

The above argument shows that the wave speed for the smooth type traveling wave is greater than or equal to the unique sharp type traveling wave speed $\hat{c}$. Next, we only need to show that there exist no smooth traveling waves with speed $\hat{c}$. The argument by contradiction is similar to the proof above with the modifications such that $\phi_{c_1}(t_*(c_1)) = K$ and $\hat{\psi}(\phi) < \tilde{\psi}(\phi)$ for all $\phi \in (0, K]$ and then $\hat{\psi}(\phi_0) = 0$ for some $\phi_0 \in (0, K)$. This completes the proof. □

The dependence of $c^*(m, D, r)$ with respect to the time delay r is formulated via a variational characterization inspired by Benguria and Depassier [5] and see also Huang *et al* [12].

Lemma 3.10. *The minimal traveling wave speed $c^*(m, D, r)$ for the time delay $r > 0$ is strictly smaller than that without time delay, i.e., $c^*(m, D, r) < c^*(m, D, 0)$.*

Proof. Let $\phi(t)$ be the unique sharp type traveling wave corresponding to the speed $c^*(m, D, r) = \hat{c}$ according to lemma 3.9. In lemma 3.6 we list three possible cases of the functions $\phi_c(t)$ corresponding to all $c > 0$, and combining the strictly monotone dependence (lemma 3.6) and continuous dependence (lemma 3.4) of ϕ_c with respect to c and the uniqueness of the sharp type traveling wave, we see that $\phi(t)$ is a special function in case (iii) and $\phi(t)$ is strictly increasing on $(0, +\infty)$, $\phi(+\infty) = K$ and $\phi'(+\infty) = 0$. Lemma 3.8 shows that $\phi'(t) > 0$ for all $t \in (0, +\infty)$.

In the proof of lemma 3.6, we develop the generalized phase plane (3.10) and (3.12) as $\tilde{\phi}_{cr}(\phi)$ is defined by (3.11). Additionally, $\tilde{\psi}(0) = 0$, $\tilde{\psi}(K) = 0$ since $\phi'(+\infty) = 0$, and $\tilde{\psi}(\phi) > 0$ for all $\phi \in (0, K)$ since $\phi'(t) > 0$ for all $t \in (0, \infty)$. Now, we rewrite (3.12) into

$$\frac{d\tilde{\psi}}{d\phi} = c - \frac{Dm\phi^{m-1}(b(\phi) - d(\phi))}{\tilde{\psi}} + \frac{Dm\phi^{m-1}(b(\phi) - b(\tilde{\phi}_{cr}(\phi)))}{\tilde{\psi}}, \quad \phi \in (0, K). \tag{3.13}$$

For any $g \in \mathcal{D} = \left\{ g \in C^1([0, K]);\ g(K) = 0,\ \int_0^K g(s)\mathrm{d}s = 1,\ g'(s) < 0,\ \forall s \in (0, K) \right\}$, we multiply (3.13) by $g(s)$ and integrate over $(0, K)$ to find

$$\begin{aligned} c =& \int_0^K g(\phi)\frac{d\tilde{\psi}}{d\phi}d\phi + \int_0^K g(\phi)\frac{Dm\phi^{m-1}(b(\phi) - d(\phi))}{\tilde{\psi}}d\phi \\ &- \int_0^K g(\phi)\frac{Dm\phi^{m-1}(b(\phi) - b(\tilde{\phi}_{cr}(\phi)))}{\tilde{\psi}}d\phi \\ =& \int_0^K - g'(\phi)\tilde{\psi}(\phi)d\phi + \int_0^K g(\phi)\frac{Dm\phi^{m-1}(b(\phi) - d(\phi))}{\tilde{\psi}}d\phi \\ &+ \left[g(\phi)\tilde{\psi}(\phi)\right]\Big|_{\phi=0}^{\phi=K} - \int_0^K g(\phi)\frac{Dm\phi^{m-1}(b(\phi) - b(\tilde{\phi}_{cr}(\phi)))}{\tilde{\psi}}d\phi \\ \geqslant& 2\sqrt{D}\int_0^K \sqrt{-ms^{m-1}g(s)g'(s)(b(s) - d(s))}\mathrm{d}s \\ &- \int_0^K g(\phi)\frac{Dm\phi^{m-1}(b(\phi) - b(\tilde{\phi}_{cr}(\phi)))}{\tilde{\psi}}d\phi, \end{aligned} \tag{3.14}$$

by Cauchy inequality and according to $\int_0^K g(s)\mathrm{d}s = 1$ and $\left[g(\phi)\tilde{\psi}(\phi)\right]|_{\phi=0}^{\phi=K} = 0$ as $\tilde{\psi}(0) = 0 = g(K)$.

It should be noted that the equality in (3.14) is attainable at some function $\hat{g}$ such that

$$-\hat{g}'(\phi)\tilde{\psi}(\phi) = \hat{g}(\phi)\frac{Dm\phi^{m-1}(b(\phi) - d(\phi))}{\tilde{\psi}}, \quad \phi \in (0, K), \tag{3.15}$$

with $\hat{g}(K) = 0$ and $\hat{g}'(\phi) < 0$ for all $\phi \in (0, K)$. In fact, such kind of solution $\hat{g}$ to (3.15) is solvable since $\tilde{\psi}(K) = 0$ and $\tilde{\psi}(\phi) \sim \kappa(K - \phi)$ as $\phi \to K^-$ for some $\kappa > 0$ and $\tilde{\psi}(\phi) > 0$ for all $\phi \in (0, K)$ according to lemma 3.8, see for example [12] for the phase plane without time delay. Then

$$(\ln\, \hat{g})' = \frac{\hat{g}'}{\hat{g}} = -\frac{Dm\phi^{m-1}(b(\phi) - d(\phi))}{\tilde{\psi}^2(\phi)} \sim -\frac{DmK^{m-1}(d'(K) - b'(K))}{\kappa^2(K - \phi)}, \, \phi \to K^-,$$

which has infinitely many solutions with $\hat{g}(K) = 0$. (Otherwise, if $\tilde{\psi}(K) \neq 0$, then $\hat{g}(\phi) \equiv 0$ is the unique solution of (3.15) with $\hat{g}(K) = 0$.) On the other hand, in the proof of lemma 3.9, we show that $\tilde{\psi}(\phi) \sim c\phi$ as $\phi \to 0^+$, and hence

$$(\ln\, \hat{g})' = \frac{\hat{g}'}{\hat{g}} = -\frac{Dm\phi^{m-1}(b(\phi) - d(\phi))}{\tilde{\psi}^2(\phi)} \sim -\frac{Dm(b'(0) - d'(0))\phi^{m-2}}{c^2}, \quad \phi \to 0^+.$$

It follows that $\hat{g}(0) < +\infty$ as $m > 1$ and $\hat{g} \in \mathcal{D}$ such that the equality in (3.14) is attainable. (Here we point out that for the smooth type traveling waves, $\hat{\psi}(\phi) \sim \mu\phi^m$ as $\phi \to 0^+$ for some $\mu > 0$ as in the proof of lemma 3.9, and then $\hat{g}(0) = +\infty$ in this case and the equality in (3.15) is not attainable.)

Now that we showed the attainable of the equality in (3.14) at $\hat{g}$, we have

$$\begin{aligned}
c =& 2\sqrt{D}\int_0^K \sqrt{-ms^{m-1}\hat{g}(s)\hat{g}'(s)(b(s) - d(s))}\mathrm{d}s \\
&- \int_0^K \hat{g}(\phi)\frac{Dm\phi^{m-1}(b(\phi) - b(\tilde{\phi}_{cr}(\phi)))}{\tilde{\psi}}d\phi \\
<& 2\sqrt{D}\int_0^K \sqrt{-ms^{m-1}\hat{g}(s)\hat{g}'(s)(b(s) - d(s))}\mathrm{d}s \\
\leqslant& \sup_{g\in\mathcal{D}} 2\sqrt{D}\int_0^K \sqrt{-ms^{m-1}\hat{g}(s)\hat{g}'(s)(b(s) - d(s))}\mathrm{d}s,
\end{aligned} \tag{3.16}$$

where the ‘ $<$ ‘ in (3.16) follows from the strictly monotone increasing of $\phi(c)$ such that $\tilde{\phi}_{cr} < \phi$ as in the proof of lemma 3.6. It should be noted that the variational principle for the speed of propagation of fronts for the density-dependent reaction–diffusion equation without time delay (i.e., the equality case of (3.16) for $r = 0$) was first obtained by Benguria and Depassier [4] in 1995. The variational approach is applied here in order to show that the time delay slows down the critical wave speed. The proof is completed. □

Proof of Theorem 2.1. The uniqueness of the wave speed of the sharp type traveling wave is proved in lemma 3.7. We see that the corresponding sharp wave is also unique (up to shift) since the maximal solution $\phi_c^1(t)$ in lemma 3.1 is the unique solution such that $\phi_c^1(t) > 0$ in a right neighbor of 0. The monotonicity follows from lemmas 3.5, 3.6 and 3.7. Lemma 3.9

implies that $c^*(m, D, r) = \hat{c}$. The positiveness of $c^*(m, D, r)$ and the uniqueness of $c^*(m, D, r)$ and the sharp type traveling wave follow from lemma 3.7. The dependence of $c^*(m, D, r)$ with respect to r is proved in lemma 3.10. The proof is completed by combining these lemmas. □

Proof of Theorem 2.2. This is proved in lemma 3.9. The regularity is trivial since $\phi(t) > 0$ for all $t \in \mathbb{R}$, where (2.3) is non-degenerate. □

Proof of Theorem 2.3. The asymptotic behavior near 0 in lemma 3.1 completes the proof, see also [28] for details. □

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