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Global stability of traveling waves for Nagumo equations with degenerate diffusion

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Abstract

This paper is concerned with the global nonlinear stability with possibly large perturbations of the unique sharp / smooth traveling waves for the degenerate diffusion equations with Nagumo (bistable) reaction. Two technical issues arise in this study. One is the shortage of weak regularity of sharp traveling waves, the other difficulty is the non-absorbing initial-perturbation around the smooth traveling waves at the far field $x = +\infty$. For the sharp traveling wave case, we technically construct weak sub- and super-solutions with semi-compact supports via translation and scaling of the unique sharp traveling wave to characterize the motion of the steep moving edges and avoid the weak regularity of the solution near the steep edges. For the smooth traveling wave case, we artfully combine both the translation and scaling type sub- and super-solutions and the translation and superposition type sub- and super-solutions in a systematical manner.

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1. Introduction and main results

We consider the global stability of the unique traveling wave for the following reaction diffusion equation with degenerate porous medium diffusion and Nagumo (or bistable) type reaction

$$\begin{cases} u_t = (u^m)_{xx} + f(u), \\ u(x,0) = u_0(x), \end{cases} \tag{1.1}$$

where $u(x,t)$ is the unknown population density, the degenerate diffusion $(u^m)_{xx}$ with $m > 1$ represents the density dependent diffusion phenomena, and the reaction term $f(u)$ satisfies (the two stable equilibrium points are normalized to 0 and 1 for convenience):

$$\begin{cases} \text{there exists } \theta \in (0,1), \text{ such that } f(0) = f(\theta) = f(1) = 0, \\ f(u) > 0, \forall u \in (\theta, 1),\ f(u) < 0, \forall u \in (0,\theta) \cup (1, +\infty), \\ f \in C^2([0,+\infty)), \text{ and } f'(0) < 0,\ f'(1) < 0,\ f''(0) \neq 0. \end{cases} \tag{1.2}$$

Background of studies. The density dependent diffusion phenomena have attracted considerable attention in various fields, such as the density dependent dispersal in biological dynamics (Gurney and Nisbet [13], Gurtin and MacCamy [12], Aronson [2]; Shiguesada et al. [33]; Nagai and Minura [24], Newman [26], Murray [23]), and the temperature dependent thermal conductivity (Kersner [15], Mendez and Fort [22]). The Nagumo (or bistable) type reaction often arises in genetics or population dynamics, describing that kinetics exhibit positive growth rate for population densities within a threshold value $u \in (\theta, 1)$, and decay for densities below such value $u \in (0, \theta)$, see [1,4,5,9,20,21,25,28]. A typical example of the Nagumo (or bistable) type reaction is the cubic nonlinearity $f(u) = u(1-u)(u-\theta)$.

Compared with the linear diffusion case ($m = 1$), the porous medium diffusion (1.1) with $m > 1$ degenerates at where $u(x,t) = 0$ such that the initial perturbation propagates with finite speed, which is a remarkable distinguish feature from the non-degenerate diffusion equations. For the degenerate porous medium diffusion equation ($m > 1$) and the Fisher-KPP type reaction $f(u) = u(1-u)$, it was Aronson [2] who first observed the existence of the critical (minimal) wave speed $c^*(m)$ corresponding to a sharp traveling wave and the asymptotic propagation speed. Here, the sharp traveling wave means that the wave $u(x,t) = \phi(x - c^*t)$ has a regional degeneracy such that $\phi(x - c^*t) = \phi(\xi) \equiv 0$ for $\xi \in (\xi_0, \infty)$, and $\phi(x - c^*t) = \phi(\xi) > 0$ for $\xi \in (-\infty, \xi_0)$, where c^* is the wave speed, and ξ_0 is the edge of sharp traveling wave. Meanwhile for the degenerate porous medium diffusion equation ($m > 1$) and the Nagumo (or bistable) type reaction $f(u) = u(1-u)(u-\theta)$ with $\theta \in (0, 1/2)$, Aronson [2] also proved the existence of a unique sharp traveling wave solution with positive propagation speed. For more general Nagumo (or bistable) type reaction, the existence of the unique (up to translation) sharp / smooth traveling wave solution $u(x,t) = \phi(x - c^*t)$ connecting the two stable equilibrium points was proved by Hosono [14], where the propagation speed is positive, zero, or negative, depending on the sign of $s(f,m) := \int_0^1 f(u)u^{m-1}\mathrm{d}u$. The local stability result with small perturbation around the unique sharp / smooth traveling wave solution was also established by Hosono [14] based on the artful construction of weak sub- or super-solutions. Recently, the spectral stability of the linearized operator around the traveling waves for degenerate diffusion equations with Nagumo reaction was formulated by Leyva, Lṕez Ríos, and Plaza [16]. For the study of the existence and asymptotic behavior of traveling waves for degenerate diffusion equations with Fisher-KPP (monostable) type reaction or Nagumo (bistable) type reaction, we refer to [3,6,8,10,11,18,19,27,29–32,34–36] and the references therein.

Regarding the local stability of sharp / smooth traveling wave solution with small initial perturbation showed by Hosono [14], there is another more interesting but challenging question: are these sharp / smooth traveling waves globally stable with arbitrarily large initial perturbations? To answer this question is the main issue of this paper.

For the sake of convenience, we recall the existence result of the unique (up to translation) traveling wave solutions for the degenerate diffusion equation (1.1) with Nagumo (bistable) type reaction satisfying (1.2). Throughout this paper, we always assume that $m > 1$ and $f(u)$ satisfies (1.2). The traveling wave solutions of (1.1) connecting the two equilibria 1 and 0 with the form $u(x,t) = \phi(\xi) = \phi(x - c^*t)$ satisfy the following degenerate elliptic equation

$$\begin{cases} c^*\phi' + (\phi^m)'' + f(\phi) = 0, \\ \phi(-\infty) = 1, \ \phi(+\infty) = 0. \end{cases} \tag{1.3}$$

Proposition 1.1 *([14]). There exist a unique speed $c^* = c^*(f,m)$ with the same sign as $s(f,m) := \int_0^1 f(u)u^{m-1}\mathrm{d}u$ and a unique (up to translation) traveling wave solution $u(x,t) = \phi(\xi) = \phi(x - c^*t)$ with $\xi = x - c^*t$ connecting the two stable equilibrium points ($\phi(-\infty) = 1$ and $\phi(+\infty) = 0$) for the degenerate diffusion equation (1.1) with Nagumo (bistable) type reaction. Moreover, the solution $\phi(\xi)$ has the following properties:*

(i) $1 - \phi(\xi)$ decays exponentially with its derivatives up to order two as $\xi \to -\infty$.

*(iia) If $s(f,m) > 0$, then $c^* > 0$, and $\phi(\xi)$ is of **sharp** type: there exists a $\omega \in \mathbb{R}$ (called the edge of the support) such that $\phi(\xi) > 0$ and $\phi'(\xi) < 0$ for all $\xi \in (-\infty, \omega)$, $\phi(\xi) \equiv 0$ for all $\xi \in [\omega, +\infty)$, and*

$$\lim_{\xi \to \omega^-} (\phi^{m-1}(\xi))' = -\frac{m-1}{m}c^*.$$

*(iib) If $s(f,m) = 0$, then $c^* = 0$, and $\phi(\xi)$ is of **sharp** type as in (iia) and*

$$\lim_{\xi \to \omega^-} (\phi^{m-1}(\xi))' = -\frac{m-1}{m}c^* = 0, \quad \lim_{\xi \to \omega^-} (\phi^{m-1}(\xi))'' = -\frac{(m-1)^2}{m(m+1)} f'(0).$$

*(iic) If $s(f,m) < 0$, then $c^* < 0$, and $\phi(\xi)$ is of **smooth** type such that $\phi(\xi) > 0$ and $\phi'(\xi) < 0$ for all $\xi \in (-\infty, +\infty)$, and $\phi(\xi)$ decays exponentially with its derivatives up to order two as $\xi \to +\infty$.*

The above existence result of the unique sharp / smooth traveling wave solution for the degenerate diffusion equation (1.1) with Nagumo (bistable) type reaction reveals an interesting phenomenon: the weighted energy $s(f,m) := \int_0^1 f(u)u^{m-1}\mathrm{d}u$ can be regarded as the total growth effect of the Nagumo type reaction, such that if $s(f,m)$ is positive then the population propagates with positive speed (equivalently, the traveling wave solution $u(x,t) = \phi(x - c^*t)$ is increasing with respect to t for any fixed location x). The shape of the traveling wave also indicates that the sharp type traveling wave with semi-compact support occurs only if the population is invading with non-negative speed $c^* \geq 0$ and non-negative growth effect $s(f,m) \geq 0$.

The unique sharp / smooth traveling wave solution is proved to be locally L^∞ stable by Hosono [14] if the perturbation is small in the following sense: (i) $\sup_{x\in\mathbb{R}} |u_0(x) - \phi(x - x_0)| < \delta$ for some shift x_0 and sufficiently small δ; (ii) for the sharp type traveling wave, i.e., cases (iia) and (iib) in Proposition 1.1, additionally it assumes that $\operatorname{supp} u_0(x) \subset (-\infty, \omega + x_0 + \delta)$, where ω is the edge of the support of the sharp type traveling wave. In other words, if the initial data $u_0(x)$ is close to some shifted traveling wave $\phi(x - x_0)$ and the edge of the support of the initial data is also close to the edge of the support of the sharp type traveling wave, then the solution stays close to the shifted traveling wave.

The traveling wave solutions for degenerate diffusion equations with various reactions, especially those with critical (or minimal) wave speed or unique wave speed if possible, usually lose the regularity due to the degeneracy of the viscosity. Hence the local or global stability results of traveling waves with weak regularity for degenerate diffusion equations are quite limited. For degenerate diffusion equations with Fisher-KPP reaction, Biró [7] first proved the nonlinear stability by sub- and super-solutions method for the sharp fronts with critical wave speed $c = c^*$. In [17], Leyva and Plaza established the spectral stability in exponentially weighted spaces of non-critical traveling waves for degenerate diffusion equations of Fisher-KPP type. Very recently, we proved the convergence to sharp traveling waves of solutions for Burgers-Fisher-KPP equations with degenerate diffusion in [37], and for the degenerate diffusion equations with combustion reaction in [38], respectively.

As for the degenerate diffusion equations with Nagumo (bistable) reaction considered in this paper, up to our knowledge, the only existing rigorous works on the stability of traveling waves are the above mentioned local stability result with small perturbations by Hosono [14] and the spectral stability of the linearized operator around the traveling waves in [16], but the global stability of these waves with large initial data is still open as we know. Different from the previous studies [7,37,38], there are two technical issues for the Nagumo (bistable) degenerate diffusion equations. One is that, for the case of sharp traveling waves, the weak regularity of the solutions causes some essential difficulties for establishing the estimates for the wave stability. The other is that, for the smooth traveling waves, the carried out estimates on the initial perturbations cannot be absorbed, due to the exponential decay of the smooth traveling waves at the far field $x = +\infty$.

Main results. In this paper, we shall prove the global nonlinear stability results with large perturbations. According to the existence results of the unique sharp / smooth traveling wave in Proposition 1.1, the unique traveling wave $\phi(\xi)$ in cases (iia) and (iib) corresponding to $c^* \geq 0$ (equivalently $s(f,m) \geq 0$) is of sharp type with semi-compact support; while the unique traveling wave $\phi(\xi)$ in case (iic) corresponding to $c^* < 0$ (equivalently $s(f,m) < 0$) is of smooth type. Since the sharp type traveling wave behaves differently from the smooth type one near the zero equilibrium point, we state and prove their global stability results separately.

For the case $s(f,m) = \int_0^1 f(u)u^{m-1}\mathrm{d}u \geq 0$, the unique traveling wave is of sharp type, we show that the solution to the Cauchy problem (1.1) of Nagumo type reaction diffusion equation with degenerate diffusion propagates in the form of sharp traveling wave $\phi(\xi)$ and also converges to the sharp traveling wave with a shift.

Theorem 1.1 *(Stability of sharp traveling waves). Suppose that*

$$s(f,m) := \int_0^1 f(u)u^{m-1}\mathrm{d}u \geq 0.$$

For any solution $u(x,t)$ of the Cauchy problem (1.1) with initial data u_0 such that

$$u_0 \in L^\infty(\mathbb{R}), u_0(x) \geq 0, \operatorname{supp} u_0 \subset (-\infty, x_*] \text{ for some } x_* \in \mathbb{R}, \liminf_{x\to-\infty} u_0(x) > \theta, \tag{1.4}$$

there exists a $x_0 \in \mathbb{R}$ such that $u(x,t)$ converges to $\phi(x - c^*t - x_0)$:

$$\lim_{t\to+\infty} \sup_{x\in\mathbb{R}} |u(x,t) - \phi(x - c^*t - x_0)| = 0, \tag{1.5}$$

where $\phi(\xi) = \phi(x - c^*t)$ is the sharp traveling wave with wave speed $c^* \geq 0$ in cases (iia) and (iib) in Proposition 1.1 (with the edge of the support $\omega = 0$ by shifting).

Remark. The condition $\liminf_{x\to-\infty} u_0(x) > \theta$ of the initial data in (1.4) is almost necessary. Otherwise for some initial data $u_0(x)$ such that $u_0(x) \leq \theta$ for all $x \in \mathbb{R}$, we can see that $u(x,t) \leq \theta$ for all $x \in \mathbb{R}$ and $t > 0$ according to the comparison principle with supper-solution $\bar{u}(x,t) \equiv \theta$. It follows that the solution does not converge to any traveling wave connecting 1 and 0.

Theorem 1.1 shows the evolution of the free boundary of the solution to Cauchy problem (1.1) for initial data u_0 with semi-compact support. It follows that the edge of the support propagates asymptotically at the same speed as the unique sharp type traveling wave.

Corollary 1.1. *Let $u(x,t)$ be the solution of the Cauchy problem (1.1) with initial data u_0 satisfying (1.4) for the case $s(f,m) \geq 0$, and let $\zeta(t)$ be the free boundary of the semi-compact support of the solution $u(x,t)$, i.e., $\zeta(t) := \sup\{x \in \mathbb{R}; u(x,t) > 0\}$. Then there exists a $x_0 \in \mathbb{R}$ such that*

$$\lim_{t\to+\infty} (\zeta(t) - c^*t) = x_0.$$

For the case $s(f,m) = \int_0^1 f(u)u^{m-1}\mathrm{d}u < 0$, the unique traveling wave is of smooth type, we show that the solution to the Cauchy problem (1.1) of Nagumo type reaction diffusion equation with degenerate diffusion propagates in the form of smooth traveling wave $\phi(\xi)$ and also converges to the smooth traveling wave with a shift.

Theorem 1.2 *(Stability of smooth traveling waves). Suppose that*

$$s(f,m) := \int_0^1 f(u)u^{m-1}\mathrm{d}u < 0.$$

For any solution $u(x,t)$ of the Cauchy problem (1.1) with initial data u_0 such that

$$u_0 \in L^\infty(\mathbb{R}), u_0(x) > 0, \limsup_{x\to+\infty} |\ln u_0(x) - \ln\phi(x)| < +\infty, \liminf_{x\to-\infty} u_0(x) > \theta, \tag{1.6}$$

there exists a $x_0 \in \mathbb{R}$ such that $u(x,t)$ converges to $\phi(x - c^*t - x_0)$:

$$\lim_{t\to+\infty} \sup_{x\in\mathbb{R}} |u(x,t) - \phi(x - c^*t - x_0)| = 0, \tag{1.7}$$

where $\phi(\xi) = \phi(x - c^*t)$ is the smooth traveling wave with wave speed $c^* < 0$ in case (iic) in Proposition 1.1.

Remark. The condition $\liminf_{x\to-\infty} u_0(x) > \theta$ of the initial data in (1.6) is almost necessary as explained in the Remark for the sharp traveling wave case. Meanwhile, the condition $\limsup_{x\to+\infty} |\ln u_0(x) - \ln\phi(x)| < +\infty$ means that the initial value $u_0(x)$ is asymptotically proportional (in a bounded range) to the smooth traveling wave.

Features, difficulties and strategies. We explain the main features and the idea of the proofs for the global stability results with large perturbations of the Nagumo (bistable) type reaction diffusion equation (1.1) with degenerate diffusion.

• To gain the global stability result Theorem 1.1 of sharp traveling waves, the main difficulty is the solutions with too weak regularities such that we would not be able to establish the suitable estimates in the stability proof. Inspired by the previous studies [7,37,38], we shall technically construct the weak sub- and super-solutions with semi-compact supports via translation and scaling of the unique sharp traveling wave profile. Here, we observe an advantage that we can successfully establish the estimates on the evolution of the free boundary (moving steep edge), at where the solutions are at most Hölder continuous, such that we can avoid the difficulty caused by the weak regularity of the solution and allows for large perturbations in the meantime.

• To show the global stability result Theorem 1.2 of smooth traveling wave, we shall propose the sub- and super-solutions constructed via translation, scaling and superposition, as a new development from the previous studies in [7,14]. We recognize an arising difficulty that, for exponentially decaying smooth traveling wave $\phi(\xi)$, the perturbation $\phi(\xi) + \delta$, no matter how small it is, cannot be absorbed by the translation and scaling $F \cdot \phi(\xi - x_0)$, where F is a function showed in Lemma 2.1 later. We solve this difficulty by artfully combining the translation and scaling type sub- and super-solutions and the translation and superposition type sub- and super-solutions in a systematical manner.

• Numerical simulations will be carried out at the last section of the paper, and these numerical results significantly show the global stability for sharp traveling waves with $c^* > 0$ and $c^* = 0$, and the smooth traveling waves with $c^* < 0$, respectively. Note that, the numerical computations on sharp waves at the edges are quite difficult and usually unstable. In order to overcome such a numerical difficulty, we adopt a new numerical method, the so-called sharp-profile-based difference scheme as we originally proposed in [39].

This paper is organized as follows. The global stability of sharp type traveling wave with possibly large perturbations for the degenerate diffusion equations (1.1) with Nagumo (bistable) reaction is proved in Section 2, and the global stability of smooth type traveling wave with possibly large perturbations is left to Section 3. Numerical simulations are carried out in Section 4.

2. Global stability of sharp traveling wave

In this section, we prove the global stability of sharp type traveling wave with possibly large perturbations for the degenerate diffusion equations (1.1) with Nagumo (bistable) reaction under the condition that

$$s(f,m)=\int_0^1 f(u)u^{m-1}\mathrm{d}u\geq 0.$$

The key step of the global L^∞ stability of sharp traveling wave is to estimate the evolution of the free boundary since both the unique sharp traveling wave $\phi(x-c^*t)$ and the solution $u(x,t)$ to the Cauchy problem are at most Hölder continuous near the free boundary. We employ and generalize the construction of weak sub- and super-solutions with semi-compact supports via translation and scaling developed by Z. Biró [7] for degenerate diffusion equations of porous medium type with Fisher-KPP type reaction $f(u)=u^p-u^q$ such that $1\leq p<\min\{m,q\}$. The above method was extended to the construction of weak sub- and super-solutions for degenerate diffusion equations with advection and general Fisher-KPP type reaction satisfying some growth condition in [37] and also for degenerate diffusion equations with combustion type reaction in [38].

Lemma 2.1 *([7,37]). Let $F(t)$ and $G(t)$ be the solutions to the following ordinary differential system*

$$\begin{cases} F'(t)=\varepsilon_1 F(t)(1-F(t)),\\ G'(t)=c^*F^{m-1}(t)-\varepsilon_2(1-F(t)),\\ F(0)=F_0>0, \qquad G(0)=G_0, \end{cases} \tag{2.1}$$

where ε_1 and ε_2 are positive constants. Then

(i) $\lim_{t\to+\infty}F(t)=1$, and the convergence rate is exponential;

(ii) if $F_0<1$, then $F(t)$ is strictly increasing, while if $F_0>1$, then $F(t)$ is strictly decreasing, such that $F(t)\in[\min\{F_0,1\},\max\{F_0,1\}]$;

(iii) $\lim_{t\to+\infty}G'(t)=c^*$;

(iv) there exists $x_0\in\mathbb{R}$ such that $\lim_{t\to+\infty}(G(t)-c^*t)=x_0$, and the convergence rate is exponential.

Let $\phi(\xi)=\phi(x-c^*t)$ be the unique sharp traveling wave with wave speed $c^*\geq 0$ in cases (iia) and (iib) in Proposition 1.1 with the edge of the support shifted to $\omega=0$. Define

$$W(x,t):=F(t)\cdot\phi(x-G(t)),$$

where $F(t)$ denotes the amplitude of the weak solution, and $G(t)$ characterize the evolution of the free boundary. For the sake of convenience, we fix two constants

$$F_1\in(\theta,\min\{\liminf_{x\to-\infty}u_0(x),1\}), \qquad F_2>\max\{\sup_{x\in\mathbb{R}}u_0(x),1\}, \tag{2.2}$$

for any given initial data u_0 satisfying (1.4) in Theorem 1.1 since $\liminf_{x\to-\infty} u_0(x) > \theta$. Clearly, we see that $F_1 < 1$ and $F_2 > 1$. Further, we define the following auxiliary continuous function

$$H(F,\phi) := \begin{cases} \dfrac{F^m f(\phi) - f(F\phi)}{F-1}, & F \in [F_1, F_2]\backslash\{1\},\ \phi \in [0,1], \\ mf(\phi) - \phi f'(\phi), & F = 1,\ \phi \in [0,1]. \end{cases} \tag{2.3}$$

Lemma 2.2. *For Nagumo (bistable) reaction $f(u)$ satisfying (1.2), and for any given $F_1 \in (\theta, 1)$ and $F_2 > 1$, the auxiliary function $H(F,\phi)$ defined in (2.3) is continuous on $[F_1, F_2] \times [0,1]$.*

Proof. We rewrite the definition (2.3) of $H(F,\phi)$ into the following equality for any $F \in [F_1, F_2]\backslash\{1\}$ and $\phi \in [0,1]$

$$\begin{aligned} H(F,\phi) =& \frac{F^m f(\phi) - f(F\phi)}{F-1} \\ =& \frac{F^m - 1}{F-1} f(\phi) - \frac{f(F\phi) - f(\phi)}{F-1} \\ =& mf(\phi) \cdot \int_0^1 (1 + s(F-1))^{m-1} \mathrm{d}s - \phi \cdot \int_0^1 f'(\phi + s(F-1)\phi) \mathrm{d}s. \end{aligned} \tag{2.4}$$

It follows that $H(F,\phi)$ is continuous on $[F_1, F_2] \times [0,1]$. □

Lemma 2.3. *For Nagumo (bistable) reaction $f(u)$ satisfying (1.2), and for any given $F_1 \in (\theta, 1)$ and $F_2 > 1$, there exists a positive constant $M_0 > 0$ such that*

$$\inf_{F\in[F_1,F_2],\phi\in[0,1]} H(F,\phi) \geq -M_0, \qquad \inf_{F\in[F_1,F_2],\phi\in(0,1]} \frac{H(F,\phi)}{\phi} \geq -M_0.$$

Proof. According to Lemma 2.2, $H(F,\phi)$ is continuous on $[F_1, F_2] \times [0,1]$, thus attains its minimum. Employing the expression (2.4), we see that

$$\begin{aligned} \frac{H(F,\phi)}{\phi} =& m\frac{f(\phi)}{\phi} \cdot \int_0^1 (1 + s(F-1))^{m-1} \mathrm{d}s - \int_0^1 f'(\phi + s(F-1)\phi) \mathrm{d}s \\ \geq& -m \cdot \|f'\|_{L^\infty(0,1)} \cdot F_2^{m-1} - \|f'\|_{L^\infty(0,F_2)} \geq -M_0. \end{aligned}$$

The proof is complete. □

Lemma 2.4. *For Nagumo (bistable) reaction $f(u)$ satisfying (1.2), and for any given $F_1 \in (\theta, 1)$ and $F_2 > 1$, there exist positive constants $\mu_1 \in (0, 1-\theta)$, $\lambda_1 \in (0, 1-\theta)$, and $\delta_1 > 0$, such that*

$$\inf_{F\in[1-\lambda_1,1+\lambda_1],\phi\in[1-\mu_1,1]} H(F,\phi) \geq \delta_1 > 0.$$

Proof. According to the assumption (1.2), $f \in C^2$ and $f'(1) < 0$. There exists a positive constant $\delta_1^* > 0$ (it suffices to take $\delta_1^* = |f'(1)|/2$), and a neighborhood $[1-\mu_1^*, 1+\mu_1^*]$ with $\mu_1^* \in (0, 1-\theta)$ such that $f'(u) \le -\delta_1^* < 0$ for all $u \in [1-\mu_1^*, 1+\mu_1^*]$.

We choose positive constants $\mu_1 \in (0, 1-\theta)$, $\lambda_1 \in (0, 1-\theta)$ such that both ϕ and $F\phi$ lie in the neighborhood $[1-\mu_1^*, 1+\mu_1^*]$ for all $F \in [1-\lambda_1, 1+\lambda_1]$ and $\phi \in [1-\mu_1, 1]$. For example, we take $\mu_1 = \mu_1^*/2$ and $\lambda_1 = \mu_1^*/2$, then $F\phi \le 1+\lambda_1 < 1+\mu_1^*$ and $F\phi \ge (1-\lambda_1)(1-\mu_1) \ge (1-\mu_1^*/2)^2 > 1-\mu_1^*$.

Note that $\mu_1 \in (0, 1-\theta)$ then $f(\phi) > 0$ for all $\phi \in (1-\mu_1, 1) \subset (\theta, 1)$. According to (2.4), we have

$$
\begin{aligned}
H(F,\phi) =& m f(\phi) \cdot \int_0^1 (1+s(F-1))^{m-1} \mathrm{d}s - \phi \cdot \int_0^1 f'(\phi + s(F-1)\phi) \mathrm{d}s \\
\ge& -\phi \cdot \int_0^1 f'(\phi + s(F-1)\phi) \mathrm{d}s \\
\ge& -\phi \cdot \sup_{u \in [\min\{\phi, F\phi\}, \max\{\phi, F\phi\}]} f'(u) \\
\ge& (1-\mu_1)\delta_1^* =: \delta_1 > 0,
\end{aligned} \tag{2.5}
$$

for all $F \in [1-\lambda_1, 1+\lambda_1]$ and $\phi \in [1-\mu_1, 1]$, since both ϕ and $F\phi$ lie in the neighborhood $[1-\mu_1^*, 1+\mu_1^*]$ such that $f'(u) \le -\delta_1^* < 0$. $\square$

Lemma 2.5. *For Nagumo (bistable) reaction $f(u)$ satisfying (1.2), and for any given $F_1 \in (\theta, 1)$ and $F_2 > 1$, let $\mu_1 \in (0, 1-\theta)$ and $\lambda_1 \in (0, 1-\theta)$ be the constants in Lemma 2.4, then there exists a positive constant $\delta_2 > 0$, such that*

$$\inf_{F \in [1+\lambda_1, F_2], \phi \in [1-\mu_1, 1]} H(F,\phi) \ge \delta_2 > 0.$$

Proof. According to the assumption (1.2), $f \in C^2$ and $f'(1) < 0$. It follows that $f'(u)$ is negative near $u = 1$, but $f'(u)$ can be sign-changing for $u \in [1, F_2]$. Fortunately, we can modify $f(u)$ to deduce the same conclusions. Since $f \in C^2$, $f'(1) < 0$ and $f(u) < 0$ for all $u > 1$. There exists a constant $\delta_2^* > 0$ (we may assume that $\delta_2^* \in (0, \delta_1^*]$) such that $f(u) \le -\delta_2^*(u-1)$ for all $u \in [1, F_2]$. Define an alternative function

$$\tilde{f}(u) := \begin{cases} f(u), & u \in [0, 1), \\ -\delta_2^*(u-1), & u \in [1, F_2]. \end{cases}$$

Then $\tilde{f}$ is Lipschitz continuous and $f(u) \le \tilde{f}(u)$ for all $u \in [0, F_2]$. They only differ for $u > 1$. We use $\tilde{f}'(u)$ to denote the generalized derivative of $\tilde{f}(u)$, which is consistent with its almost everywhere derivative. Further we replace one of the function f in the definition of $H(F,\phi)$ by $\tilde{f}$ for $F \in (1, F_2]$ and $\phi \in [0, 1]$, that is,

$$\tilde{H}(F,\phi) := \frac{F^m f(\phi) - \tilde{f}(F\phi)}{F-1}.$$

Therefore, similar to the expression (2.4), we also have (note that $f(u) \equiv \tilde{f}(u)$ for $u \in [0, 1]$)

$$\begin{aligned}\tilde{H}(F,\phi) =& \frac{F^m f(\phi) - \tilde{f}(F\phi)}{F-1} \\ =& \frac{F^m - 1}{F-1} f(\phi) - \frac{\tilde{f}(F\phi) - \tilde{f}(\phi)}{F-1} \\ =& m f(\phi) \cdot \int_0^1 (1 + s(F-1))^{m-1} \mathrm{d}s - \phi \cdot \int_0^1 \tilde{f}'(\phi + s(F-1)\phi) \mathrm{d}s. \end{aligned} \tag{2.6}$$

The advantage of the above modification is that

$$\tilde{H}(F,\phi) = \frac{F^m f(\phi) - \tilde{f}(F\phi)}{F-1} \le \frac{F^m f(\phi) - f(F\phi)}{F-1} = H(F,\phi), \quad F \in (1, F_2], \phi \in [0, 1],$$

and the Lipschitz continuous function $\tilde{f}$ satisfies that $\tilde{f}'(u) \le -\delta_2^* < 0$ for all $u \in [1, F_2]$. We then conclude the desired estimate on $H(F,\phi)$ via the estimate on $\tilde{H}(F,\phi)$.

According to the proof of Lemma 2.4, $f'(u) \le -\delta_1^* < 0$ for all $u \in [1-\mu_1^*, 1+\mu_1^*]$. Then for all $F \in [1+\lambda_1, F_2]$ and $\phi \in [1-\mu_1, 1]$, both ϕ and $F\phi$ reside in the interval $[1-\mu_1^*, F_2]$. We see that $\tilde{f}'(u) \le -\delta_1^* < 0$ for all $u \in [1-\mu_1^*, 1)$ and $\tilde{f}'(u) \le -\delta_2^* < 0$ for all $u \in [1, F_2]$. Similar to (2.5) in the proof of Lemma 2.4, we have

$$\begin{aligned}\tilde{H}(F,\phi) &\ge -\phi \cdot \int_0^1 \tilde{f}'(\phi + s(F-1)\phi)\mathrm{d}s \\ &\ge -\phi \cdot \sup_{u \in [\min\{\phi, F\phi\}, \max\{\phi, F\phi\}]} \tilde{f}'(u) \\ &\ge (1-\mu_1)\delta_2^* =: \delta_2 > 0.\end{aligned}$$

The proof is complete since $H(F,\phi) \ge \tilde{H}(F,\phi)$. □

Lemma 2.6. *For Nagumo (bistable) reaction $f(u)$ satisfying (1.2), and for any given $F_1 \in (\theta, 1)$ and $F_2 > 1$, let $\lambda_1 \in (0, 1-\theta)$ be the constant in Lemma 2.4, then there exist positive constants $\mu_2 \in (0, 1-\theta)$ and $\delta_3 > 0$, such that*

$$\inf_{F \in [F_1, 1-\lambda_1], \phi \in [1-\mu_2, 1]} H(F,\phi) \ge \delta_3 > 0.$$

Proof. This is proved in a similar way as Lemma 2.5 in [38]. Note that $F_1 \in (\theta, 1)$ and $\lambda_1 > 0$, then $[F_1, 1-\lambda_1] \subset (\theta, 1)$. Take $F_1^* \in (\theta, F_1)$, such as $F_1^* = (\theta + F_1)/2$, then $[F_1, 1-\lambda_1] \subset [F_1^*, 1-\lambda_1] \subset (\theta, 1)$. According to (1.2), there exists a positive constant $\delta_3^* > 0$ such that

$$\inf_{u \in [F_1^*, 1-\lambda_1]} f(u) \ge \delta_3^* > 0,$$

since f is positive and continuous on $(\theta, 1)$. Noticing that $f(1) = 0$ and $f'(1) < 0$, we see that there exists a positive constant $\mu_2^* > 0$ and $\mu_2^* < \lambda_1$, such that

$$\sup_{u \in [1-\mu_2^*, 1]} f(u) \leq \frac{\delta_3^*}{2}.$$

The last step is to take $\mu_2 \in (0, \mu_2^*)$ such that $F\phi$ resides in the interval $[F_1^*, 1 - \lambda_1]$ for all $F \in [F_1, 1 - \lambda_1]$ and $\phi \in [1 - \mu_2, 1]$. It suffices to take $\mu_2 > 0$ sufficiently small such that $(1 - \mu_2)F_1 > F_1^*$, which is possible since $F_1 > F_1^*$.

According to (2.4) and the above selection, we have

$$\begin{aligned} H(F, \phi) &= \frac{F^m - 1}{F - 1} f(\phi) - \frac{f(F\phi) - f(\phi)}{F - 1} \\ &\geq \frac{f(F\phi) - f(\phi)}{1 - F} \geq f(F\phi) - f(\phi) \\ &\geq \inf_{u \in [F_1^*, 1-\lambda_1]} f(u) - \sup_{u \in [1-\mu_2^*, 1]} f(u) \geq \frac{\delta_3^*}{2} =: \delta_3 > 0, \end{aligned}$$

for any $F \in [F_1, 1 - \lambda_1]$ and $\phi \in [1 - \mu_2, 1]$. □

Lemma 2.7. *For Nagumo (bistable) reaction $f(u)$ satisfying (1.2), and for any given $F_1 \in (\theta, 1)$ and $F_2 > 1$, there exist positive constants $\mu_0 \in (0, 1 - \theta)$ and $\delta_0 > 0$ such that*

$$\inf_{F \in [F_1, F_2], \phi \in [1-\mu_0, 1]} H(F, \phi) \geq \delta_0 > 0.$$

Proof. Taking $\mu_0 = \min\{\mu_1, \mu_2\} > 0$ and $\delta_0 = \min\{\delta_1, \delta_2, \delta_3\} > 0$, we arrive at the positive infimum of $H(F, \phi)$ over $[F_1, F_2] \times [1 - \mu_0, 1]$ according to Lemma 2.4, Lemma 2.5, and Lemma 2.6. □

Lemma 2.8. *Let $\phi(\xi) = \phi(x - c^* t)$ be the unique sharp traveling wave in Proposition 1.1 in cases (iia) and (iib) under the condition $s(f, m) = \int_0^1 f(u) u^{m-1} \mathrm{d}u \geq 0$. For Nagumo (bistable) reaction $f(u)$ satisfying (1.2), and for any given $F_1 \in (\theta, 1)$ and $F_2 > 1$, there exist positive constants $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$, such that*

$$\varepsilon_1 F\phi(\xi) + \varepsilon_2 F\phi'(\xi) - H(F, \phi(\xi)) \leq 0, \tag{2.7}$$

for any $F \in [F_1, F_2]$ and any $\xi \in \mathbb{R}$. The above inequality (2.7) is valid for any smaller ε_1 and any larger ε_2.

Proof. Note that the edge of the support is shifted to zero (i.e., $\omega = 0$) and the sharp traveling wave $\phi(\xi) \equiv 0$, for all $\xi \geq 0$, we only need to consider the case $\xi < 0$. According to properties in Proposition 1.1, $\phi(-\infty) = 1$, $\phi'(\xi) < 0$ for all $\xi \in (-\infty, 0)$. There exists a unique $\xi_* \in (-\infty, 0)$ such that $\phi(\xi_*) = 1 - \mu_0$, where μ_0 is the positive constant in Lemma 2.7.

For $\xi \in [\xi^*, 0)$, we see that $\phi(\xi) \in (0, \phi(\xi_*)] = (0, 1-\mu_0]$. According to the asymptotic behavior of the unique sharp traveling wave in Proposition 1.1 in cases (iia) and (iib), we see that:

Case (iia). We have

$$\lim_{\xi \to 0^-} (\phi^{m-1}(\xi))' = -\frac{m-1}{m} c^*.$$

Then

$$\lim_{\xi \to 0^-} \frac{|\phi'(\xi)|}{\phi(\xi)} = \lim_{\xi \to 0^-} \frac{\frac{c^*}{m}(-\frac{m-1}{m} c^* \xi)^{\frac{1}{m-1}-1}}{(-\frac{m-1}{m} c^* \xi)^{\frac{1}{m-1}}} = +\infty.$$

Case (iib). We have

$$\lim_{\xi \to \omega^-} (\phi^{m-1}(\xi))' = -\frac{m-1}{m} c^* = 0, \qquad \lim_{\xi \to \omega^-} (\phi^{m-1}(\xi))'' = -\frac{(m-1)^2}{m(m+1)} f'(0) =: 2\beta_0 > 0.$$

Then similarly

$$\lim_{\xi \to 0^-} \frac{|\phi'(\xi)|}{\phi(\xi)} = \lim_{\xi \to 0^-} \frac{\frac{2}{m-1} \beta_0^{\frac{1}{m-1}} |\xi|^{\frac{2}{m-1}-1}}{\beta_0^{\frac{1}{m-1}} |\xi|^{\frac{2}{m-1}}} = +\infty.$$

Note that $\phi'(\xi) < 0$ and $\phi(\xi) > 0$ for all $\xi \in [\xi^*, 0)$, $\frac{|\phi'(\xi)|}{\phi(\xi)}$ is positive and continuous on $[\xi^*, 0)$ with positive infinity limit when approaching 0. We conclude that there exists a positive constant $\delta_4 > 0$ such that

$$\min_{\xi \in [\xi_*, 0)} \frac{|\phi'(\xi)|}{\phi(\xi)} = \delta_4 > 0.$$

Consider the following function

$$J(F, \xi) := -\varepsilon_1 F - \varepsilon_2 F \frac{\phi'(\xi)}{\phi(\xi)} + \frac{H(F, \phi(\xi))}{\phi(\xi)}, \qquad F \in [F_1, F_2], \ \ \xi \in [\xi^*, 0),$$

for positive constants ε_1 and ε_2 to be determined. According to Lemma 2.3, we have

$$\begin{aligned} J(F, \xi) &= -\varepsilon_1 F + \varepsilon_2 F \frac{|\phi'(\xi)|}{\phi(\xi)} + \frac{H(F, \phi(\xi))}{\phi(\xi)} \\ &\geq -\varepsilon_1 F_2 + \varepsilon_2 F_1 \delta_4 - M_0 \geq 0, \end{aligned}$$

provided that ε_2 is sufficiently large and ε_1 is sufficiently small. Therefore,

$$\varepsilon_1 F \phi(\xi) + \varepsilon_2 F \phi'(\xi) - H(F, \phi(\xi)) = -J(F, \xi) \cdot \phi(\xi) \leq 0,$$

for $\xi \in [\xi^*, 0)$ and $F \in [F_1, F_2]$.

For $\xi \in (-\infty, \xi^*)$, we see that $\phi(\xi) \in (\phi(\xi_*), 1) = (1 - \mu_0, 1)$. In this case, $H(F, \phi(\xi))$ admits positive infimum $\delta_0 > 0$ according to Lemma 2.7. Hence,

$$\varepsilon_1 F\phi(\xi) + \varepsilon_2 F\phi'(\xi) - H(F, \phi(\xi)) \leq \varepsilon_1 F_2 - \delta_0 \leq 0,$$

provided that $\varepsilon_1 < \delta_0/F_2$. The proof is complete. $\square$

Now we are ready to construct weak sub- and super-solutions with semi-compact supports.

Lemma 2.9. *Define*

$$W(x,t) := F(t)\phi(x - G(t)), \tag{2.8}$$

where $F(t)$ and $G(t)$ are the solutions in Lemma 2.1 with initial data (F_0, G_0), ε_1 and ε_2 are the constants in Lemma 2.8, $\phi(\xi) = \phi(x - c^*t)$ is the unique sharp traveling wave in Proposition 1.1 in cases (iia) and (iib) under the condition $s(f,m) = \int_0^1 f(u)u^{m-1}\mathrm{d}u \geq 0$. Then $W(x,t)$ is a sub-solution to (1.1) if $F_0 \in [F_1, 1)$; $W(x,t)$ is an super-solution to (1.1) if $F_0 \in (1, F_2]$. Moreover there exists $x_0 \in \mathbb{R}$ such that

$$\lim_{t\to+\infty} \sup_{\xi\in\mathbb{R}} |W(\xi + c^*t, t) - \phi(\xi - x_0)| = 0.$$

Here, the sub- and super-solutions refer only to the parabolic equation without comparison of the initial data.

Proof. Lemma 2.1 shows that $F(t) \in [F_0, 1) \subset [F_1, 1)$ if $F_0 \in [F_1, 1)$ and $F(t) \in (1, F_2]$ if $F_0 \in (1, F_2]$. Note that $\phi(\xi) \equiv 0$ for all $\xi \geq 0$, and $\phi(\xi) > 0$ for all $\xi < 0$. It suffices for sub-solutions to prove that for any $t > 0$ and $x < G(t)$,

$$F'(t)\phi(\eta) - F(t)\phi'(\eta)G'(t) - F^m(t)(\phi^m)''(\eta) - f(F(t)\phi(\eta)) \leq 0, \tag{2.9}$$

where $\eta := x - G(t) < 0$. Since $\phi(\xi)$ is a sharp traveling wave, satisfying (1.3), substituting (1.3) into (2.9), we see (2.9) is equivalent to

$$F'(t)\phi(\eta) - (F(t)G'(t) - c^*F^m(t))\phi'(\eta) + F^m(t)f(\phi) - f(F(t)\phi(\eta)) \leq 0.$$

According to the differential system (2.1), (2.9) is further transformed into

$$\varepsilon_1 F(t)(1 - F(t))\phi(\eta) + \varepsilon_2 F(t)(1 - F(t))\phi'(\eta) + F^m(t)f(\phi) - f(F(t)\phi(\eta)) \leq 0,$$

which is valid according to Lemma 2.8 since

$$\begin{aligned}
&\varepsilon_1 F(t)(1 - F(t))\phi(\eta) + \varepsilon_2 F(t)(1 - F(t))\phi'(\eta) + F^m(t)f(\phi) - f(F(t)\phi(\eta)) \\
=&(1 - F(t)) \cdot (\varepsilon_1 F(t)\phi(\eta) + \varepsilon_2 F(t)\phi'(\eta) - H(F(t), \phi(\eta))) \leq 0,
\end{aligned}$$

for $F(t) \in [F_1, 1)$ with $F_0 \in [F_1, 1)$. It follows that $W(x,t)$ is a sub-solution if $F_0 \in [F_1, 1)$. The above inequality is reversed for $F(t) \in (1, F_2]$ with $F_0 \in (1, F_2]$ such that $W(x,t)$ corresponds to super-solutions.

The convergence follows from Lemma 2.1 such that

$$\begin{aligned}
&\lim_{t\to+\infty}\sup_{\xi\in\mathbb{R}} |W(\xi + c^* t, t) - \phi(\xi - x_0)| \\
=&\lim_{t\to+\infty}\sup_{\xi\in\mathbb{R}} |F(t)\phi(\xi + c^* t - G(t)) - \phi(\xi - x_0)| \\
\le&\lim_{t\to+\infty}\sup_{\xi\in\mathbb{R}} \big(|(F(t)-1)\phi(\xi + c^* t - G(t))| + |\phi(\xi + c^* t - G(t)) - \phi(\xi - x_0)|\big) \\
\le&\lim_{t\to+\infty} |F(t)-1| + \lim_{t\to+\infty}\sup_{\xi\in\mathbb{R}} |\phi(\xi + c^* t - G(t)) - \phi(\xi - x_0)| \\
=&0,
\end{aligned}$$

since $\lim_{t\to+\infty} F(t) = 1$ and $\lim_{t\to+\infty}(G(t) - c^* t) = x_0$. The proof is complete. □

Lemma 2.10. *Let $F(t)$ and $G(t)$ be the solutions in Lemma 2.1 with initial data (F_0, G_0). Then there exists a function $\omega(\cdot) \ge 0$ such that*

$$|G(t) - c^* t - G_0| \le \omega(|F_0 - 1|), \qquad \lim_{F_0\to 1} \omega(|F_0 - 1|) = 0.$$

Proof. This is proved in the proof of Lemma 2.5 in [37], only related to the ordinary differential system (2.1). □

Now we can show the global L^∞ stability result for the sharp traveling wave for the case $s(f,m) = \int_0^1 f(u)u^{m-1}\mathrm{d}u \ge 0$. The proof is based on the compact analysis under the construction of sub- and super-solutions with semi-compact supports. We follow the same line as the proof in [7], the proof of Lemma 2.6 in [37], and also the proof of Lemma 2.10 in [38].

Lemma 2.11. *Let $u(x,t)$ be the solution of the Cauchy problem (1.1) with initial data u_0 satisfying (1.4) under the condition $s(f,m) = \int_0^1 f(u)u^{m-1}\mathrm{d}u \ge 0$. Then there exists a $x_0 \in \mathbb{R}$ such that $u(x,t)$ converges to $\phi(x - c^* t - x_0)$ in the sense that for any $0 < \varepsilon < 1$, there exists $T > 0$ such that*

$$(1-\varepsilon)\phi(\xi - x_0 + \varepsilon) \le u(\xi + c^* t, t) \le (1+\varepsilon)\phi(\xi - x_0 - \varepsilon), \tag{2.10}$$

for all $t \ge T$ and $\xi \in \mathbb{R}$.

Proof. For any u_0 satisfying (1.4), let F_1 and F_2 be the constants as defined in (2.2). By appropriately shifting x_1 and x_2, we have

$$F_1\phi(x - x_1) < u_0(x) < F_2\phi(x - x_2).$$

There exist sub- and super-solutions in the form

$$\overline{W}(x,t) := \overline{F}(t)\phi(x - \overline{G}(t)), \quad \underline{W}(x,t) := \underline{F}(t)\phi(x - \underline{G}(t)),$$

where $\overline{F}(t)$ and $\overline{G}(t)$ are solutions to problem (2.1) in Lemma 2.1 with initial data $\overline{F}(0) = F_2$ and $\overline{G}(0) = x_2$, similarly, $\underline{F}(t)$ and $\underline{G}(t)$ are solutions to problem (2.1) with initial data $\underline{F}(0) = F_1$ and $\underline{G}(0) = x_1$, such that

$$\underline{W}(x,0) < u_0(x) < \overline{W}(x,0). \tag{2.11}$$

Therefore, Lemma 2.9 shows that $\underline{W}(x,t)$ and $\overline{W}(x,t)$ are sub- and super-solutions to the equation (1.1), which together with the comparison of initial data (2.11) imply that $\underline{W}(x,t)$ and $\overline{W}(x,t)$ are sub- and super-solutions to the Cauchy problem with initial data $u_0(x)$.

According to Lemma 2.1, both $\underline{G}(t) - c^*t$ and $\overline{G}(t) - c^*t$ converge to some points ξ_1^* and ξ_2^* and reside in a bounded interval $[\xi_1, \xi_2]$. It follows from the comparison with the sub- and super-solutions $\underline{W}(x,t)$ and $\overline{W}(x,t)$ that

$$\underline{F}(t)\phi(\xi - \xi_1) \le u(\xi + c^*t, t) \le \overline{F}(t)\phi(\xi - \xi_2), \quad t > 0,\ \xi \in \mathbb{R}, \tag{2.12}$$

which provides bounded estimates of the free boundary in the moving coordinates.

Denote $z(\xi,t) := u(\xi + c^*t, t)$ in the moving coordinates. For any sequence $\{t_n\}$ with $\lim_{n\to\infty} t_n = +\infty$, denote $z_n(\xi) := u(\xi + c^*t_n, t_n)$. The compact analysis shows the existence of a function $z(\xi) \in C^\alpha(\mathbb{R})$ and a convergent subsequence of $\{z_n\}$, denoted by $\{z_n\}$ itself for simplicity, such that

$$\lim_{n\to\infty} \sup_{\xi\in\mathbb{R}} |z_n(\xi) - z(\xi)| = 0. \tag{2.13}$$

There exists $\theta_n \in (t_n, t_{n+1})$ such that $\{z(\xi, \theta_n)\}$ converges to the unique sharp traveling wave $\phi(\xi - x_0)$ with some shift x_0, and the free boundary of $z(\xi, \theta_n)$, denoted by ζ_n, converges to x_0,

$$\lim_{n\to\infty} \zeta_n = x_0, \qquad \lim_{n\to\infty} \sup_{\xi\in\mathbb{R}} |z(\xi,\theta_n) - \phi(\xi - x_0)| = 0. \tag{2.14}$$

Lemma 4 of [7] shows that the closeness of boundary and the L^∞ convergence of function sequences imply the uniform similarity in shape, i.e., for any $\varepsilon_* > 0$, there exists $\delta > 0$ such that

$$(1 - \varepsilon_*)\phi(\xi - x_0 + \varepsilon_*) \le z(\xi, \theta_n) \le (1 + \varepsilon_*)\phi(\xi - x_0 - \varepsilon_*), \tag{2.15}$$

provided that $|\zeta_n - x_0| \le \delta$ and $|z(\xi,\theta_n) - \phi(\xi - x_0)| \le \delta$. Here we note that the above inequality (2.15) is not valid for smooth type traveling wave, it is true for sharp type traveling wave with sharp edge. Starting from time $T = \theta_n$, we construct weak sub- and super-solutions in the same form as in Lemma 2.9 but with more accurate amplitude $F(t)$ and free boundary $G(t)$. We may take $\varepsilon_* > 0$ smaller, such that $\varepsilon_* + \omega(\varepsilon_*) < \varepsilon$, where $\omega(\cdot)$ is the function in Lemma 2.10. Lemma 2.10 shows that if the amplitude $F(t)$ is sufficiently close to 1, then the bound of the evolution of the free boundary $|G(t) - c^*t - G_0|$ can be sufficiently small. Therefore,

$$(1 - \varepsilon)\phi(\xi - x_0 + \varepsilon) < u(\xi + c^*t, t) < (1 + \varepsilon)\phi(\xi - x_0 - \varepsilon),$$

for all $t \ge T$. The proof is completed. □

Proof of Theorem 1.1. This is proved in Lemma 2.11. □

3. Global stability of smooth traveling wave

In this section, we prove the global stability of smooth type traveling wave with possibly large perturbations for the degenerate diffusion equations (1.1) with Nagumo (bistable) reaction under the condition

$$s(f,m) = \int_0^1 f(u)u^{m-1}\mathrm{d}u < 0.$$

The auxiliary function $H(F,\phi)$ defined in (2.3) and its properties in Lemma 2.3 and Lemma 2.7 in Section 2 are also needed for the case $s(f,m) < 0$ in this section. However, Lemma 2.8 for the unique sharp type traveling wave with semi-compact support should be modified to the following lemma for the unique smooth type traveling wave without semi-compact support.

Lemma 3.1. *Let $\phi(\xi) = \phi(x - c^*t)$ be the unique smooth traveling wave in Proposition 1.1 in case (iic) under the condition $s(f,m) = \int_0^1 f(u)u^{m-1}\mathrm{d}u < 0$. For Nagumo (bistable) reaction $f(u)$ satisfying (1.2), and for any given $F_1 \in (\theta, 1)$ and $F_2 > 1$, there exist positive constants $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$, such that*

$$\varepsilon_1 F\phi(\xi) + \varepsilon_2 F\phi'(\xi) - H(F,\phi(\xi)) \le 0, \tag{3.1}$$

for any $F \in [F_1, F_2]$ and any $\xi \in \mathbb{R}$.

Proof. According to properties of the unique smooth traveling wave in Proposition 1.1, we know that $\phi(-\infty) = 1$, $\phi(+\infty) = 0$, and $\phi'(\xi) < 0$ for all $\xi \in (-\infty, 0)$. There exists a unique $\xi_* \in (-\infty, +\infty)$ such that $\phi(\xi_*) = 1 - \mu_0$, where μ_0 is the positive constant in Lemma 2.7.

For $\xi \in [\xi^*, +\infty)$, we have $\phi(\xi) \in (0, \phi(\xi_*)] = (0, 1 - \mu_0]$. According to the asymptotic behavior of the unique sharp traveling wave in Proposition 1.1 in case (iic), we see that $\phi(\xi)$ decays exponentially with its derivatives up to order two as $\xi \to +\infty$. Then

$$\lim_{\xi\to+\infty} \frac{|\phi'(\xi)|}{\phi(\xi)} = \lim_{\xi\to+\infty} \frac{A\lambda_0 \mathrm{e}^{-\lambda_0\xi}}{A\mathrm{e}^{-\lambda_0\xi}} = \lambda_0 > 0,$$

where λ_0 is the exponential decay rate. It is shown by asymptotic analysis that $\lambda_0 = \frac{f'(0)}{c^*} > 0$ for the case $s(f,m) = \int_0^1 f(u)u^{m-1}\mathrm{d}u < 0$ such that $c^* < 0$ according to the proof of Theorem 1 in Hosono [14].

Note that $\phi'(\xi) < 0$ and $\phi(\xi) > 0$ for all $\xi \in [\xi^*, +\infty)$, $\frac{|\phi'(\xi)|}{\phi(\xi)}$ is positive and continuous on $[\xi^*, +\infty)$ with positive limit when approaching $+\infty$. It follows the existence of a positive constant $\delta_5 > 0$ such that

$$\inf_{\xi\in[\xi_*,+\infty)} \frac{|\phi'(\xi)|}{\phi(\xi)} = \delta_5 > 0.$$

The remaining part of this proof is similar to Lemma 2.8 since the property of $H(F,\phi)$ is also valid for this case, and hence we omit the details. □

We construct sub- and super-solutions via translation and scaling of the unique traveling wave formally in the same way as those in Lemma 2.9 in Section 2. The difference lies in the shapes of the traveling waves and meanwhile the sub- and super-solutions, they have semi-compact supports in Section 2 and they are all positive in this section.

Lemma 3.2. *Define*

$$W(x,t) := F(t)\phi(x - G(t)), \tag{3.2}$$

where $F(t)$ and $G(t)$ are the solutions in Lemma 2.1 with initial data (F_0, G_0), ε_1 and ε_2 are the constants in Lemma 3.1, $\phi(\xi) = \phi(x - c^* t)$ is the unique smooth traveling wave in Proposition 1.1 in case (iic) under the condition $s(f,m) = \int_0^1 f(u)u^{m-1}\mathrm{d}u < 0$. Then $W(x,t)$ is a sub-solution to (1.1) if $F_0 \in [F_1, 1)$; $W(x,t)$ is an super-solution to (1.1) if $F_0 \in (1, F_2]$. Moreover there exists $x_0 \in \mathbb{R}$ such that

$$\lim_{t\to+\infty} \sup_{\xi\in\mathbb{R}} |W(\xi + c^* t, t) - \phi(\xi - x_0)| = 0.$$

Here, the sub- and super-solutions refer only to the parabolic equation without comparison of the initial data.

Proof. Based on the property of $F(t)$ and $G(t)$ in Lemma 2.1 and the inequality (3.1) in Lemma 3.1, the assertions are valid in a similar way as the proof of Lemma 2.9. □

Different from the sharp type case in the proof of Lemma 2.11, here for the smooth traveling wave, the closeness of the functions $|z(\xi,\theta_n) - \phi(\xi - x_0)| \le \delta$ does not imply (2.15) since $\phi(\xi - x_0) + \delta$ cannot be controlled by the exponentially decaying function $(1+\varepsilon_*)\phi(\xi - x_0 - \varepsilon_*)$. Therefore, the argument in Lemma 2.11 of construction sub- and super-solutions with more accurate amplitude $F(t)$ starting from a new time $T = \theta_n$ is inapplicable. We employ an alternative construction scheme by combining the method of Z. Biró [7] and the method of Hosono [14] together to show the global stability of smooth traveling wave.

Lemma 3.3. *Let $u(x,t)$ be the solution of the Cauchy problem (1.1) with initial data u_0 satisfying (1.6) under the condition $s(f,m) = \int_0^1 f(u)u^{m-1}\mathrm{d}u < 0$. Then there exists a $x_0 \in \mathbb{R}$ such that $u(x,t)$ converges to $\phi(x - c^* t - x_0)$ in the sense that for any $0 < \varepsilon < 1$, there exists $T > 0$ such that*

$$\sup_{\xi\in\mathbb{R}} |u(\xi + c^* t, t) - \phi(\xi - x_0)| < \varepsilon, \quad t \ge T. \tag{3.3}$$

Proof. The first step is to construct sub- and super-solutions in the form of (3.2) in Lemma 3.2 with additional comparison of the initial data. For any u_0 satisfying (1.6), let F_1 and F_2 be the constants as defined in (2.2). According to the condition (1.6) of the initial data

$$\limsup_{x\to+\infty} |\ln u_0(x) - \ln \phi(x)| < +\infty,$$

we see that there exist shifts $x_1, x_2 \in \mathbb{R}$ such that

$$F_1\phi(x-x_1) \le u_0(x) \le F_2\phi(x-x_2). \tag{3.4}$$

This is achievable since $\phi(x)$ decays exponentially for the case $s(f,m) < 0$ with asymptotic behavior $\phi(x) \sim \mathrm{e}^{-\lambda_0 x}$ as $x \to +\infty$, where $\lambda_0 = \frac{f'(0)}{c^*} > 0$ according to the proof of Theorem 1 in [14], and the difference in the amplitude can be absorbed by the translation of the smooth traveling wave.

We construct sub- and super-solutions in the form

$$\overline{W}(x,t) := \overline{F}(t)\phi(x-\overline{G}(t)), \quad \underline{W}(x,t) := \underline{F}(t)\phi(x-\underline{G}(t)),$$

where $\overline{F}(t)$ and $\overline{G}(t)$ are solutions to problem (2.1) in Lemma 2.1 with initial data $\overline{F}(0) = F_2$ and $\overline{G}(0) = x_2$, and $\underline{F}(t)$ and $\underline{G}(t)$ are solutions to problem (2.1) with initial data $\underline{F}(0) = F_1$ and $\underline{G}(0) = x_1$. According to Lemma 3.2, $\underline{W}(x,t)$ and $\overline{W}(x,t)$ are sub- and super-solutions to the equation (1.1). Noticing the comparison of initial data (3.4), we see that $\underline{W}(x,t)$ and $\overline{W}(x,t)$ are sub- and super-solutions to the Cauchy problem with initial data $u_0(x)$.

Similar to the proof of Lemma 2.11, compact analysis shows that there exists $\theta_n \in (t_n, t_{n+1})$ such that $\{z(\xi,\theta_n)\}$ converges to the unique smooth traveling wave $\phi(\xi - x_0)$ with some shift x_0,

$$\lim_{n\to\infty} \sup_{\xi\in\mathbb{R}} |z(\xi,\theta_n) - \phi(\xi - x_0)| = 0. \tag{3.5}$$

At this stage, we cannot construct sub- and super-solutions with more accurate amplitude $F(t)$ starting from the new time $T = \theta_n$ as the proof of Lemma 2.11 since the inequality (2.15) is invalid for smooth type traveling wave. Fortunately, we combine an alternative construction scheme of Hosono [14] by seeking sub- and super-solutions in the following form

$$\underline{u}(x,t) := [\max(0, \phi(x - c^*t - z_1)^{m-1} - r_1\mathrm{e}^{-\mu t})]^{\frac{1}{m-1}},$$
$$\overline{u}(x,t) := [\phi(x - c^*t - z_2)^m + r_2\mathrm{e}^{-\mu t}]^{\frac{1}{m}},$$

with appropriately chosen parameters z_1, z_2, r_1, r_2, μ. Detailed proof can be found in the proof of Theorem 2 in [14]. Then for any $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$\sup_{\xi\in\mathbb{R}} |u(\xi + c^*t, t) - \phi(\xi - x_0)| < \varepsilon, \quad t > t_n,$$

follows from the small perturbation at time t_n:

$$\sup_{\xi\in\mathbb{R}} |z(\xi,\theta_n) - \phi(\xi - x_0)| < \delta,$$

which is true according to (3.5) for large n. The proof is completed. □

Proof of Theorem 1.2. This is proved in Lemma 3.3. □

4. Numerical simulations

This section is devoted to the numerical simulations for the global nonlinear stability with possibly large perturbations of the unique sharp / smooth traveling waves for the degenerate diffusion equations (1.1) with Nagumo (bistable) reaction.

According to the existence and asymptotic properties of sharp / smooth traveling waves in Proposition 1.1, the unique (up to translation) traveling waves $\phi(\xi) = \phi(x - c^* t)$ connecting the two stable equilibrium points ($\phi(-\infty) = 1$ and $\phi(+\infty) = 0$) are sharp with positive speed $c^* = c^*(f, m) > 0$ for the case $s(f, m) = \int_0^1 f(u)u^{m-1}\mathrm{d}u > 0$, and with zero speed $c^* = 0$ for the case $s(f, m) = 0$; while the unique traveling wave is smooth with negative speed $c^* < 0$ for the case $s(f, m) < 0$.

Difficulty in numerical issues. We mention that for the cases of $s(f, m) = \int_0^1 f(u)u^{m-1}\mathrm{d}u \geq 0$, the sharp traveling waves $\phi(\xi) = \phi(x - c^* t)$ and the solutions $u(x, t)$ to the Cauchy problem (1.1) for initial data with semi-compact support, both propagate with distinct sharp edges of the support. They are at most Hölder continuous near the edges of the support. Therefore, the classical second order difference scheme of $(u^m)_{xx}$ does not work well near the moving boundary (edge of the support). The numerical computations with the cases of steep corners are quite unstable and difficult, as we know. Inspired by our previous study in [39], in order to numerically demonstrate the sharp traveling waves (with sharp edge), we [39] first proposed a so-called sharp-profile-based difference scheme for the degenerate diffusion Fisher-KPP equation with time delay. Here without time delay, based on the different types of asymptotic profiles (traveling waves), we shall adopt different approaches. When $s(f, m) = \int_0^1 f(u)u^{m-1}\mathrm{d}u \geq 0$, we shall expect the asymptotic profiles as the sharp traveling waves with steep corners, and we shall employ the sharp-profile-based difference scheme for the convergence to sharp traveling waves for the degenerate diffusion Nagumo (bistable) equations (1.1). On the other hand, when $s(f, m) = \int_0^1 f(u)u^{m-1}\mathrm{d}u < 0$, we shall expect the asymptotic profiles as the regular smooth traveling waves, and we shall employ the classical second order difference scheme for the convergence to smooth traveling waves.

Sharp-profile-based difference scheme. Here we only show how to treat the challenging cases for the solutions with sharp edges in numerical computations. We take the typical case $m = 2$ in the degenerate porous medium diffusion $(u^m)_{xx}$ and $f(u) = f_\theta(u) = u(1 - u)(u - \theta)$ in the Nagumo (bistable) reaction for numerical simulations:

$$\begin{cases} u_t = (u^2)_{xx} + u(1 - u)(u - \theta), \\ u|_{t=0} = u_0(x), \end{cases}$$

where the initial data is taken as

$$u_0(x) = (1 - \mathrm{e}^{\frac{x}{2}})_+ - 0.6 \sin(\frac{\pi(x - 2)}{10}) \cdot \chi_{[-32,-12]}(x),$$

such that the support is $(-\infty, 0]$ and oscillates near the edge of the support with large perturbations. And $0 < \theta < 1$ is set with different values as follows: we take $\theta = 0.4$, 0.6 and 0.8, respectively, such that $s(f_\theta, m) = \int_0^1 f_\theta(u)u^{m-1}\mathrm{d}u > 0$ for $\theta = 0.4$; $s(f_\theta, m) = \int_0^1 f_\theta(u)u^{m-1}\mathrm{d}u = 0$ for $\theta = 0.6$; and $s(f_\theta, m) = \int_0^1 f_\theta(u)u^{m-1}\mathrm{d}u < 0$ for $\theta = 0.8$.

For the convergence to sharp traveling wave, we shall apply for the sharp-profile-based difference scheme. The detailed procedures are carried out as follows. Proposition 1.1 shows that the sharp traveling wave behaves as (we take the case $m = 2$ and $c^* > 0$ for example, other cases can be handled similarly)

$$\phi(\xi) = \Big(-\frac{(m-1)c^*}{m}\xi \Big)_+ + o(|\xi|), \quad \xi \to 0. \tag{4.1}$$

(i) Besides the partition points $x_{-N} < x_{-N+1} < \cdots < x_0 < \cdots < x_{N-1} < x_N$, we additionally introduce an edge point $\hat{x} \in (x_{k-1}, x_k]$ for some $-N < k \le N$ depending on time t such that $u(t, x_j) = 0$ for $j > k$ and $u(t, x_j) > 0$ for $j < k$;

(ii) The second order derivatives $(u^2)_{xx}\big|_{x_j}$ away from the sharp edge are calculated by the classical second order difference scheme and $(u^2)_{xx}\big|_{x_j} = 0$ for $j > k$ since locally $u(t, x) \equiv 0$ near x_j;

(iii) For $(u^2)_{xx}$ near the sharp edge, i.e., for $(u^2)_{xx}\big|_{x_k}$, we use the profile ansatz according to (4.1)

$$u(t, x) = c_1(-x + \hat{x})_+ + c_2(-x + \hat{x})_+^2 + o((-x + \hat{x})_+^2), \quad x \to \hat{x},$$

and the values $u(t, x_k)$, $u(t, x_{k+1})$, $u(t, x_{k+2})$ to fit the coefficients c_1 and c_2, then

$$u^2(t, x) = c_1^2(-x + \hat{x})_+^2 + 2c_1c_2(-x + \hat{x})_+^3 + o((-x + \hat{x})_+^3), \quad x \to \hat{x},$$

such that we calculate $(u^2)_{xx}\big|_{x_k} = 2c_1^2 + 12c_1c_2(-x_k + \hat{x})_+$.

(iv) We compute the values $u(t + \Delta t, x_j)$ according to the equation (1.1), and special attention should be paid to the values $u(t + \Delta t, x_j)$ near the edge. We use the ansatz

$$u(t + \Delta t, x) = c_1'(-x + \hat{x}')_+ + c_2'(-x + \hat{x}')_+^2 + o((-x + \hat{x}')_+^2), \quad x \to \hat{x}',$$

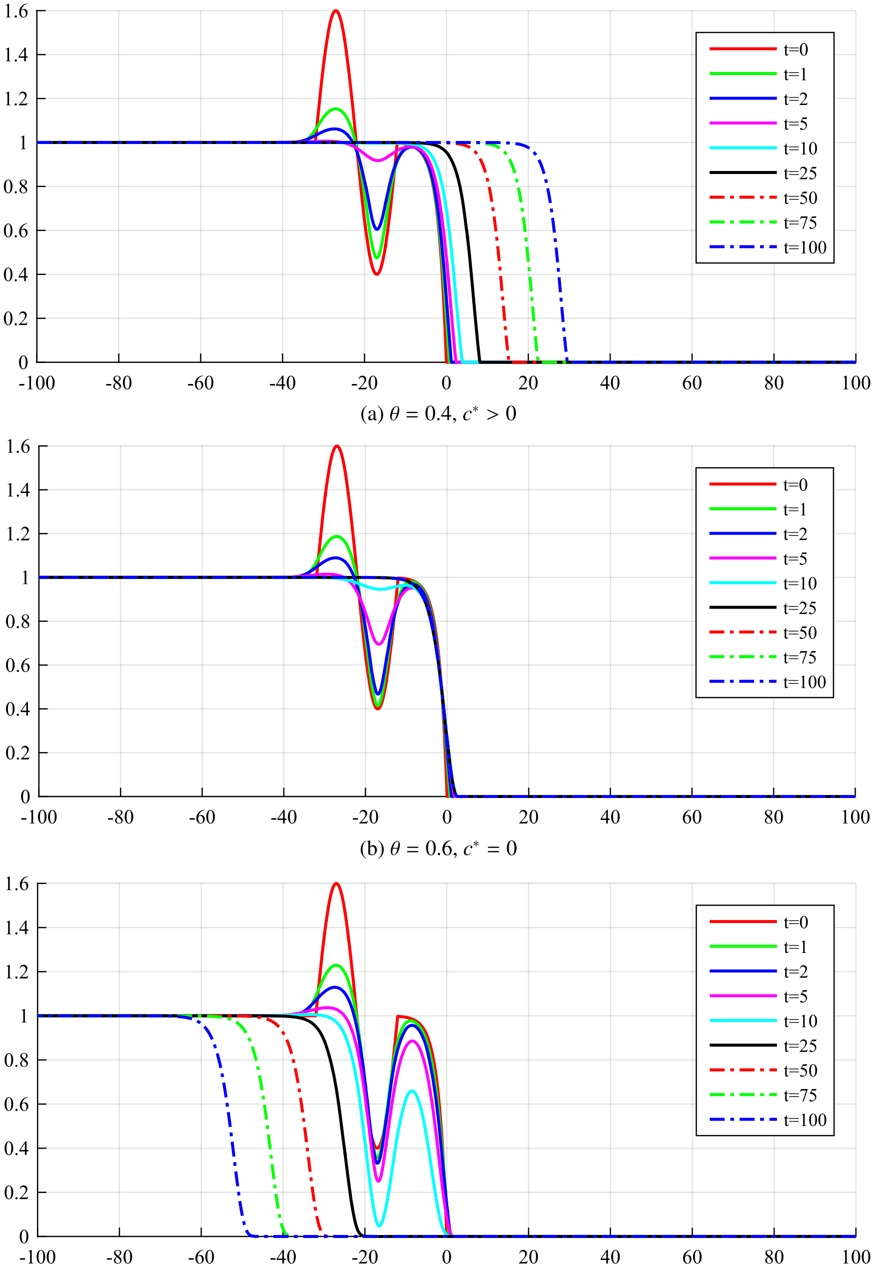
where $\hat{x}'$ is the new edge point, and the new values $u(t+\Delta t, x_k)$, $u(t+\Delta t, x_{k+1})$, $u(t+\Delta t, x_{k+2})$ to fit the new coefficients c_1', c_2', and the new edge $\hat{x}'$. If the distance $x_k - \hat{x}'$ is larger than or equal to Δx, we modify

$$u(t + \Delta t, x_{k-1}) = c_1'(-x_{k-1} + \hat{x}')_+ + c_2'(-x_{k-1} + \hat{x}')_+^2,$$

which means the solution propagates across x_{k-1} and the values are calculated based on the edge profile.

Numerical results. We are going to carry out numerical computations in three cases with $\theta = 0.4$, 0.6 and 0.8, respectively. Our numerical simulations show that: (i) for $\theta = 0.4$, the solution $u(x, t)$ propagates with positive speed and converges asymptotically to the unique traveling wave with sharp edge corresponding to speed $c^* > 0$, see the illustration Fig. 1a; (ii) for $\theta = 0.6$, the solution $u(x, t)$ is standing with zero speed and converges asymptotically to the unique traveling wave with sharp edge and speed $c^* = 0$, see the illustration Fig. 1b; (iii) for $\theta = 0.8$, the solution $u(x, t)$ propagates with negative speed and converges asymptotically to the unique smooth traveling wave with speed $c^* < 0$, see the illustration Fig. 1c.

Additionally, according to the simulations presented in Fig. 1, we see that the wave crests and troughs for large perturbations with oscillations decay rapidly with the lapse of time; it is shown that the perturbation seems to disappear and the solution behaves as the traveling wave at $t = 10$ for the case $\theta = 0.4$ with positive speed in Fig. 1a, while similar phenomena happen at $t = 25$



(a) $\theta = 0.4$, $c^* > 0$

(b) $\theta = 0.6$, $c^* = 0$

(c) $\theta = 0.8$, $c^* < 0$

Fig. 1. Convergence to sharp traveling wave for $c^* \geq 0$ and to smooth traveling wave for $c^* < 0$. (For interpretation of the colors in the figure(s), the reader is referred to the web version of this article.)

for the case $\theta = 0.6$ with zero speed in Fig. 1b and for the case $\theta = 0.8$ with negative speed in Fig. 1c.

In all cases, the numerical simulations coincide with the theoretical stability results proved in Theorem 1.1 for sharp traveling wave with $c^* \geq 0$ and Theorem 1.2 for smooth traveling wave with $c^* < 0$, such that, even with large initial data, the solutions to the Cauchy problem time-asymptotically behave like some sharp / smooth traveling waves.

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Data availability

Data will be made available on request.

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