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Sharp oscillatory traveling waves of structured population dynamics model with degenerate diffusion

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Abstract

We consider a population dynamics model with degenerate diffusion and time delay. We discover sharp-oscillatory waves with sharp edges and non-decaying oscillations arising from density-dependent dispersal and reproduction with age structure. Degenerate diffusion and bad effect of time delay prevent the use of existing approaches. Here, we develop a new delayed iteration framework to show the existence of these peculiar waves. In particular, the estimate of the admissible wave speed is highly nontrivial. We employ a new phase transform method coupled with the detailed analysis of phase energy. Furthermore, we give a complete characterization of the dynamical behaviors of various kinds of waves. Our results indicate that simple invasion rules can generate complex wave patterns and provide some interesting insights into the ecological dynamics.

Keywords: Traveling waves; Time delay; Degenerate diffusion; Oscillatory waves; Sharp waves

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1. Introduction

This paper is concerned with the traveling wave solutions to the following population dynamics model for single species with age structure and degenerate diffusion

$$\begin{cases} \dfrac{\partial u}{\partial t} = D\Delta u^m - d(u) + b(u(t-r,x)), & x \in \mathbb{R},\ t > 0, \\ u(s,x) = u_0(s,x), & x \in \mathbb{R},\ s \in [-r,0], \end{cases} \tag{1.1}$$

Here, $D > 0$ denotes the diffusion coefficient, $u = u(t,x)$ represents the density of total mature population at location x and time t, $r > 0$ is the mature time, $D\Delta u^m$ with $m > 1$ is the density-dependent diffusion. Such a degenerate diffusion means that the spatial-diffusion rate increases with population density, particularly, zero density implies non-diffusion. This is with more ecological sense [3,28,29]. Two nonlinear functions $b(u)$ and $d(u)$ represent the birth rate and the death rate of the matured population respectively. In this paper we focus on the mono-stable case for the equation, which includes the classical Fisher-KPP equations, the Nicholson's blowflies equations and Mackey-Glass equations.

In a very recent article [31], Sullivan et al. explored the fluctuations in invasion speed in a set of discrete-time population models with density-dependent dispersal, and a generalized Nicholson's blowflies model with Allee effect and random dispersal. One of the open questions raised at the end of [31] is to identify the roles of several intrinsic density-dependent mechanisms, including the nonlinear dispersal, and structured population growth, in the spatial invasion dynamics. To answer such an open question, we study a continuous population dynamics model with time delay, wherein density-dependent dispersal and demography are coupled. On the one hand, we identify the different geometric wave patterns from wave speed c, degeneracy index m in density dependent dispersal and maturation time r in growth function (Fig. 2 – Fig. 4). We investigate the wave propagating dynamics in this ecological process: how fast the population propagates and what shape it forms. On the other hand, in contrast to classic models with random dispersal, there exists a peculiar form of sharp-oscillating traveling waves which possess the two distinct features, sharp profiles caused by degeneracy and oscillations due to large time delay. It is worthy of mentioning that these waves are unusual because their properties are controlled by both the leading edge and the bulk of the front. We show that delayed non-monotone growth generates the oscillations at high densities, and causes dynamical changes in the bulk of waves, while degenerate diffusion generates the sharp leading edge at low densities (Fig. 5 – Fig. 6).

Degenerate diffusion caused by competition between conspecifics or deteriorating environmental conditions is a common feature of population spreading modeling in ecology [3,9,23,27]. However, there are numerous gaps in our knowledge about the wave dynamical behaviors of degenerate diffusion equations compared with the well-studied linear diffusion case [1,6,8,16,21,26,30,32]. The degeneracy raises the possibility of sharp type traveling waves, where the population density u decreases to zero at a finite point, rather than decaying to zero asymptotically. For the study of monotone sharp waves in degenerate diffusion equations without time delay we refer to the works of Aronson [2], Pablo and Vàzquez [5], Gilding and Kersner [7], and the references therein. Recently, some detailed discussions of degenerate diffusion equations with time delay (1.1) are emerging. An additional complication appearing in the delayed and degenerate case is the possible non-monotonicity and non-smoothness of wavefronts. But even the existence of smooth monotone fronts for equations with small time delay was proved only very recently, in the fundamental contribution [13] by Huang et al. based on a perturbation approach. Later then, we [36] proved the existence of smooth monotone traveling wave solutions for any time delay only under the monotonicity assumption of the birth function $b(u)$ by the upper and lower solutions method combining with the viscosity vanishing method.

The main novelty of this paper lies in the peculiar form of waves with sharp edges and non-decaying oscillations, which is a brand new phenomenon in propagating waves. Due to the lack of monotonicity, the problem does not admit any comparison principle. Furthermore, the large time delay prevents the use of perturbation method, and the degeneracy makes the analysis rather complicated. Consequently, the existing methods are not applicable. Indeed, we solve the delayed degenerate diffusion equations with so much ingenuity by proposing a new delayed iteration approach. The method makes use of the delicate structure of time delay and sharp edge, thereby reducing the first iteration to the solvability of a degenerate elliptic problem with singularity. Thanks to this important insight, such a framework permits traveling waves with a partially compact support and oscillations. To the best of our knowledge, this is the first framework of showing the existence of sharp traveling waves for the degenerate diffusion equation with time delay.

Another challenging problem lies in the precise estimate of the lower bound of the admissible wave speeds. In the linear diffusion case, non-monotone growth function $b(u)$ and time delay r produce oscillations in population density. Despite these complicated fluctuation at high population density, the admissible wave speed is still determined by the dynamics at low densities, more precisely, the characteristic equation around equilibrium zero. In our model, the appearance of degenerate diffusion leads to the failure of this "linear determinacy principle" [17]. The propagating speed no longer depends on the pull of populations at low densities in front of the wave, but rather depends on the whole wave structure corresponding to the total population density range. This directly places limitations on the way we estimate the wave speed via classic characteristic theory. It is noted that the minimal admissible wave speed may decay to zero as time delay tends to infinity, and even for the case without time delay, the value of wave speed in degenerate equations is extremely difficult to compute or estimate. Instead of calculating the admissible wave speed directly, we develop a new phase transform approach coupled with the analysis of phase energy to estimate the corresponding wave speeds such that there is no sharp or smooth waves for arbitrary time delay with correspondingly small wave speed. Generally speaking, classical phase plane analysis does not work for delayed equation because of intersecting trajectories in the phase plane. Our phase transform method, however, could be a blueprint to draw the prospective contradiction for proving the wave properties and further give a speed estimate by analyzing the phase energy for a broad family of degenerate equations.

The rest of the paper is organized as follows. In Section 2, we present the main results on the existence of sharp oscillatory traveling waves. The proofs of the existence of sharp oscillatory waves are carried out in Section 3, Section 4 and Section 5, respectively. The classification of various kinds of waves for equation (1.1) is presented in Section 6.

2. Main results

We are looking for traveling wave solutions of (1.1) connecting two equilibria 0 and $\kappa > 0$ satisfying

$$d(0) = b(0), \quad d(\kappa) = b(\kappa),$$

in some sense that they may oscillate around the positive equilibrium κ. Let $\phi(\xi)$ with $\xi = x + ct$ and $c > 0$ be a traveling wave solution of (1.1), we get (we write ξ as t for the sake of simplicity)

$$c\phi'(t) = D(\phi^m(t))'' - d(\phi(t)) + b(\phi(t - cr)), \qquad t \in \mathbb{R}. \tag{2.1}$$

Our aim is to present a classification of various types of wave solutions with the admissible wave speeds depending on the time delay. Especially, we will show the existence of sharp oscillatory waves, which is never mentioned in the existing literature.

From biological experiments, a typical equation for (1.1) is the mono-stable case. Namely, (1.1) admits two constant equilibria $u_- = 0$ and $u_+ = \kappa > 0$, where $u_- = 0$ is unstable and $u_+ = \kappa$ is stable for the spatially homogeneous equation associated with (1.1). The important examples include the classical Fisher-KPP equation [10] and a large number of evolution equations in ecology, for example, the well-studied diffusive Nicholson's blowflies equation and Mackey-Glass equation [19,20,24,25] with the death function $d(u) = \delta u$, the birth function

$$b_1(u) = pue^{-au^q}, \text{ or } b_2(u) = \frac{pu}{1 + au^q}, \quad p > 0,\ q > 0,\ a > 0;$$

and the age-structured population model [4,11,15,18] with

$$d(u) = \delta u^2, \text{ and } b(u) = pe^{-\gamma r}u, \quad p > 0,\ \delta > 0,\ \gamma > 0.$$

Summarizing these examples, throughout the paper we assume that the death rate function $d(\cdot)$ satisfies

$$d \in C^2([0, +\infty)), \quad d(0) = 0,\ d'(s) > 0,\ d''(s) \geq 0 \text{ for } s > 0, \tag{2.2}$$

and the birth function $b(\cdot)$ satisfies the following unimodality condition:

$$\begin{aligned} &b \in C^1(\mathbb{R}_+; \mathbb{R}_+) \text{ has only one positive local extremum point } s = s_M \\ &\text{(global maximum point) and } b(0) = 0, b(\kappa) = d(\kappa), b'(0) > d'(0), \\ &b'(\kappa) < d'(\kappa), d(s) < b(s) \leq b'(0)s \text{ for } s \in (0, \kappa). \end{aligned} \tag{2.3}$$

We also need the following feedback condition when showing the divergent semi-wavefronts

$$(b(s) - \kappa)(s - \kappa) < 0, \quad s \in [d^{-1}(\theta), d^{-1}(M)] \backslash \{\kappa\}. \tag{2.4}$$

Here, we restrain ourselves to the case $s_M < \kappa$ in which b is non-monotone in $[0, \kappa]$ and $b(s_M) > b(\kappa) = d(\kappa)$. We set $M := b(s_M) = \max_{s \in [0,\kappa]} b(s)$, $\theta := b(d^{-1}(M))$ and according to the monotone increasing of the death function $d(u)$, it holds $s_M < \kappa < d^{-1}(M)$. An example for the Nicholson's blowflies model is as follows (Fig. 1).

Before stating our main results, let us give some definitions and notations. Since (1.1) is degenerate for $u = 0$, we employ the following definition of weak solutions for the initial-value problem (1.1), where the time delay $r \geq 0$, $m > 1$, $D > 0$, $u_0 \in L^2((-r, 0) \times \Omega)$ for any compact set $\Omega \subset \mathbb{R}$.

Definition 2.1 *(Weak solutions).* A function $u \in L^2_{\text{loc}}((0, +\infty) \times \mathbb{R})$ is called a weak solution of (1.1) if $0 \leq u \in L^\infty((0, +\infty) \times \mathbb{R})$, $\nabla u^m \in L^2_{\text{loc}}((0, +\infty) \times \mathbb{R})$, and for any $T > 0$ and $\psi \in C_0^\infty((-r, T) \times \mathbb{R})$

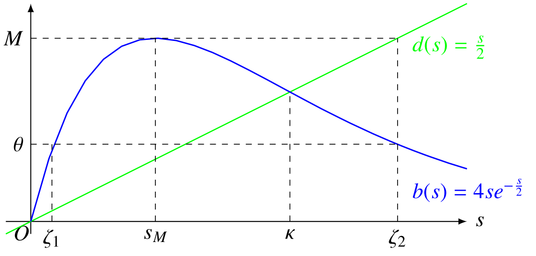

Fig. 1. The structure on functions $b(u)$ and $d(u)$.

$$
\begin{aligned}
&-\int_0^T\int_{\mathbb{R}} u(t,x)\frac{\partial\psi}{\partial t}dxdt + D\int_0^T\int_{\mathbb{R}} \nabla u^m\cdot\nabla\psi dxdt + \int_0^T\int_{\mathbb{R}} d(u(t,x))\psi dxdt\\
&=\int_{\mathbb{R}} u_0(0,x)\psi(0,x)dx + \int_r^{\max\{T,r\}}\int_{\mathbb{R}} b(u(t-r,x))\psi(t,x)dxdt\\
&+\int_0^{\min\{T,r\}}\int_{\mathbb{R}} b(u_0(t-r,x))\psi(t,x)dxdt.
\end{aligned}
$$

The initial-value problem (1.1) with degenerate diffusion and time delay can be solved step by step. For $t\in(0,r)$, the birth rate function $b(u(t-r,x))$ is determined by the initial value such that $b(u(t-r,x)) = b(u_0(t-r,x))$, and then the well-posedness of the degenerate diffusion equation (1.1) is shown in [34,35]. Moreover, if $0 \le u_0(s,x) \in L^\infty((-r,0)\times\mathbb{R})$, then $0 \le u(t,x) \le \max\{\|u_0\|_{L^\infty((-r,0)\times\mathbb{R})}, \kappa\}$ according to the comparison principle. Next for $t\in(r,2r)$, the well-posedness and the uniform boundedness follow similarly. This shows the global solvability of the initial-value problem (1.1).

The traveling wave solution $\phi(t)$ may be non-monotone and even non-decaying oscillating around the positive equilibrium κ since the birth function $b(u)$ is non-monotone. Meanwhile, it is also expected that the degenerate diffusion equation (1.1) may admit sharp type wave solution with semi-compact support. So we define the following various types of waves.

Definition 2.2 *(Sharp wavefronts).* A function $0 \le \phi(t) \in W^{1,1}_{\text{loc}}(\mathbb{R}) \cap L^\infty(\mathbb{R})$ with $\phi^m(t) \in W^{1,1}_{\text{loc}}(\mathbb{R})$ is said to be a **semi-wavefront** of (1.1) if

(i) the profile function ϕ satisfies (2.1) in the sense of distributions,

(ii) $\phi(-\infty) = 0$, and $0 < \liminf_{t\to+\infty}\phi(t) \le \limsup_{t\to+\infty}\phi(t) < +\infty$,

(iii) the leading edge of $\phi(t)$ near $-\infty$ is monotonically increasing in the sense that there exists a maximal interval $(-\infty, t_0)$ with $t_0 \in (-\infty, +\infty]$ such that $\phi(t)$ is monotonically increasing in it and if $t_0 < +\infty$ then $\phi(t_0) > \kappa$. We say that t_0 is the boundary of the leading edge of ϕ.

A semi-wavefront $\phi(t)$ is said to be a **wavefront** of (1.1) if ϕ converges to κ as t tends to $+\infty$, i.e., $\phi(+\infty) = \kappa$.

A semi-wavefront (including wavefront) is said to be **sharp** if there exists a $t_* \in \mathbb{R}$ such that $\phi(t) = 0$ for all $t \le t_*$ and $\phi(t) > 0$ for all $t > t_*$. Otherwise, it is said to be a **smooth** semi-wavefront (or smooth wavefront) if $\phi(t) > 0$ for all $t \in \mathbb{R}$.

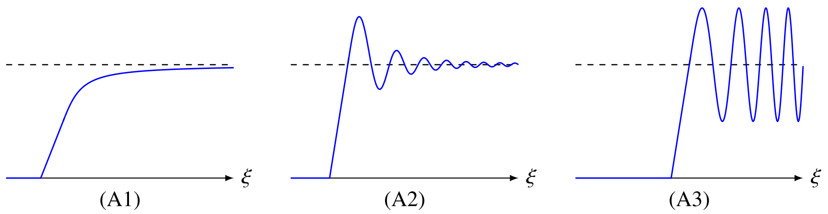

Fig. 2. Sharp waves — piecewise-C^1 type: (A1) monotone wavefront; (A2) non-monotone wavefront; (A3) divergent semi-wavefront.

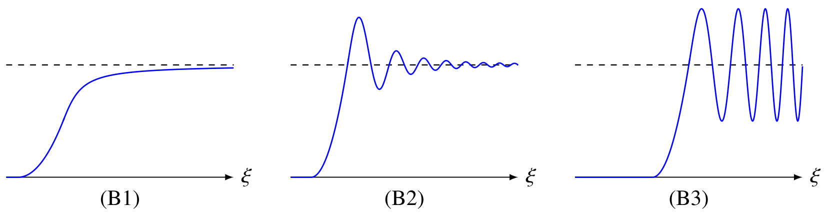

Fig. 3. Sharp waves — C^1 type: (B1) monotone wavefront; (B2) non-monotone wavefront; (B3) divergent semi-wavefront.

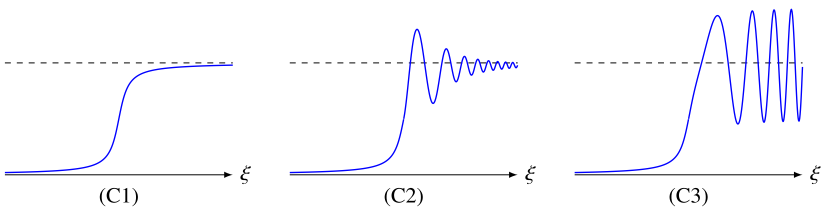

Fig. 4. Smooth waves: (C1) monotone wavefront; (C2) non-monotone wavefront; (C3) divergent semi-wavefront.

Furthermore, for the sharp semi-wavefronts (including wavefronts) $\phi(t)$, if $\phi'' \notin L^1_{\text{loc}}(\mathbb{R})$, we say that $\phi(t)$ is a **piecewise-C^1** type sharp wave; otherwise, if $\phi'' \in L^1_{\text{loc}}(\mathbb{R})$, we say that $\phi(t)$ is a C^1 type sharp wave.

According to the above definition, the possible waves are classified into **monotone wavefronts**, **non-monotone wavefronts**, and **divergent semi-wavefronts** considering the monotonicity or convergency near $+\infty$; and meanwhile these waves can also be classified into **piecewise-C^1 type sharp**, **C^1 type sharp** and **smooth** type concerned with the degeneracy and regularity near $-\infty$ or t_* such that $\phi(t)$ tends to zero. See Fig. 2, Fig. 3 and Fig. 4 for illustration. In the case of sharp type, we can always shift t_* to 0 for convenience.

The waves $\phi(t)$ may develop many possible features, the combinations of piecewise-C^1 type sharp, C^1 type sharp, or smooth near $\phi = 0$ and monotone wavefront, non-monotone wavefront or divergent semi-wavefront as t tends to infinity, see Fig. 2, Fig. 3 and Fig. 4 for illustration. Among all those waves, we note that the divergent semi-wavefront of sharp type is the new discovery in this paper.

For any given $m > 1$, $D > 0$ and $r \ge 0$, we define the critical wave speed $c_*(m, r, b, d)$ for the degenerate diffusion equation (2.1) as follows

$$c_*(m, r, b, d) := \inf\{c > 0; \text{(2.1) admits semi-wavefronts (including wavefronts)}\}. \tag{2.5}$$

Here we omit the dependence of the wave speed $c_*(m, r, b, d)$ on the parameter D for simplicity since the dependence is trivial via a re-scaling method such that the speed with $D > 0$ is the speed with $D = 1$ multiplied by $\sqrt{D}$. This note is applicable for all the wave speeds in this paper.

Our main results are as follows. First of all, we state the existence and nonexistence results of various sharp waves, including semi-wavefronts and wavefronts.

Theorem 2.1 *(Existence of sharp waves). For any $m > 1$, $D > 0$ and $r \geq 0$, there exists a constant $c_0(m, r, b, d) > 0$ depending on m, r and the structure of $b(\cdot), d(\cdot)$, such that for $c = c_0(m, r, b, d)$, (2.1) admits sharp wave $\phi(t)$ (semi-wavefronts or wavefronts, piecewise-C^1 or C^1) with $\phi(t) \equiv 0$ for $t \leq 0$,*

$$0 < \zeta_1 \leq \liminf_{t\to+\infty} \phi(t) \leq \limsup_{t\to+\infty} \phi(t) \leq \zeta_2,$$

and

$$|\phi(t) - C_1 t_+^{\lambda}| \leq C_2 t_+^{\Lambda}, \quad \text{for any } t \in (-\infty, 1),$$

where $t_+ = \max\{t, 0\}$, $\lambda = 1/(m - 1)$ and $\Lambda > \lambda$, $C_1, C_2 > 0$ are constants.

The sharp waves are classified into C^1 type and piecewise-C^1 type according to the degeneracy index m.

Theorem 2.2 *(Regularity of sharp waves). If $m \geq 2$, then the sharp waves in Theorem 2.1 are piecewise-C^1 (as illustrated in Fig. 2); while if $1 < m < 2$, the sharp waves in Theorem 2.1 are C^1 (as shown in Fig. 3).*

Remark 2.1. Roughly speaking, the degeneracy strengthens as $m > 1$ increases and the regularity of the case $m \geq 2$ is weaker than that of $1 < m < 2$. For the case $1 < m < 2$, the sharp waves remain C^1 regularity but not analytic.

The waves may be oscillatory or divergent according to the wave speed and the time delay.

Theorem 2.3 *(Oscillatory or divergent waves). Assume that $m > 1$, $r > 0$, $b'(\kappa) < 0$, and the birth rate function $b(\cdot)$ satisfies the feedback condition (2.4). Then there exist $c_\kappa = c_\kappa(m, r, b'(\kappa), d'(\kappa))$ and $c^* = c^*(m, r, b'(\kappa), d'(\kappa))$ with $0 < c_\kappa \leq c^* \leq +\infty$, such that the waves (wavefronts or semi-wavefronts, sharp or smooth, if exist) with speed $c > c_\kappa$ are oscillatory and these waves with speed $c > c^*$ are divergent. Moreover,*

$$c_\kappa(m, r, b'(\kappa), d'(\kappa)) = \frac{\mu_\kappa(m, b'(\kappa), d'(\kappa)) + o(1)}{r}, \quad r \to +\infty,$$

and if further $b'(\kappa) < -d'(\kappa)$, then

$$c^*(m, r, b'(\kappa), d'(\kappa)) = \frac{\mu^*(m, b'(\kappa), d'(\kappa)) + o(1)}{r}, \quad r \to +\infty,$$

where $\mu_\kappa(m, b'(\kappa), d'(\kappa)) := \sqrt{\frac{2Dm\kappa^{m-1}\omega_\kappa}{b'(\kappa)}} e^{\frac{\omega_\kappa}{2}}$, $\omega_\kappa < -2$ *is the unique negative root of the equation* $2d'(\kappa) = b'(\kappa)e^{-\omega_\kappa}(2 + \omega_\kappa)$, *and* $\mu^*(m, b'(\kappa), d'(\kappa)) := \pi\sqrt{\frac{Dm\kappa^{m-1}}{-b'(\kappa)-d'(\kappa)}}$.

Remark 2.2. We note that it may hold that $c_\kappa = c^* = +\infty$ under some conditions, especially for small time delay, where no non-monotone waves are guaranteed by Theorem 2.3. In fact, the existence of monotone waves is proved in [13] for small time delay. Although the existence of monotone waves does not directly preclude the existence of non-monotone waves, there may exist a critical value $r_{\min}$ of time delay such that only monotone waves exist for $r \leq r_{\min}$, as indicated by Kwong and Ou in [16] that oscillatory or periodic behavior of the waves may be generated by the Hopf bifurcation near the positive equilibrium.

Remark 2.3. If the number $c_\kappa(m, r, b'(\kappa), d'(\kappa))$ in Theorem 2.3 is less than the minimal admissible wave speed, which happens if the time delay r is suitable large under some conditions, then all the waves are oscillatory; while if $c^*(m, r, b'(\kappa), d'(\kappa))$ is less than the minimal admissible wave speed then all the waves are divergent (no wavefront exists).

We present a lower bound of all the admissible wave speeds as follows.

Theorem 2.4 *(Lower bound of wave speeds). For any* $m > 1$, $D > 0$ *and* $r \geq 0$, *there exists a constant* $\dot{c}(m, r, b, d) > 0$ *depending on* m, r *and the structure of* $b(\cdot), d(\cdot)$, *such that,* (2.1) *admits no wave solution* $\phi(t)$ *(semi-wavefronts or wavefronts, sharp or smooth) for any* $c < \dot{c}(m, r, b, d)$. *Moreover,*

$$\dot{c}(m, r, b, d) = \frac{\mu_0(m, b(\cdot), d(\cdot)) + o(1)}{r}, \quad r \to +\infty,$$

where $\mu_0(m, b(\cdot), d(\cdot)) > 0$.

Theorem 2.4 shows the nonexistence of all kinds of waves with smaller speeds. Here we present a sufficient condition for the existence of sharp divergent semi-wavefront.

Theorem 2.5 *(Existence of sharp divergent semi-wavefront). Assume that the function* $b(\cdot)$ *satisfies the feedback condition* (2.4) *and let* $\zeta_1 \in (0, \kappa)$ *be the constant such that* $b(\zeta_1) = b(d^{-1}(M))$, *where* $M = \max_{s\in[0,\kappa]} b(s)$. *Then, there exist positive constants* ε_0, K_0, r_0, *such that if* $b'(s) \geq \varepsilon_0$ *for* $s \in (0, \zeta_1)$, $b'(\kappa) \leq -K_0$ *and* $r \geq r_0$, *then the constants* $\mu_0(m, b(\cdot), d(\cdot)) > \mu^*(m, b'(\kappa), d'(\kappa))$ *in Theorem 2.3 and Theorem 2.4, and the sharp type wave in Theorem 2.1 is a divergent semi-wavefront.*

Remark 2.4. Theorem 2.5 shows the case that $\mu_0(m, b(\cdot), d(\cdot)) > \mu^*(m, b'(\kappa), d'(\kappa))$, which implies

$$\dot{c}(m, r, b, d) > c^*(m, r, b'(\kappa), d'(\kappa)) \geq c_\kappa(m, r, b'(\kappa), d'(\kappa))$$

for large time delay according to Theorem 2.3 and Theorem 2.4. The sharp waves in Theorem 2.1 corresponds to the wave speed $c_0(m, r, b, d)$, which is greater than or equal to $\dot{c}(m, r, b, d)$ according to Theorem 2.4. Therefore, $c_0(m, r, b, d)$ is greater than both $c^*(m, r, b'(\kappa), d'(\kappa))$ and $c_\kappa(m, r, b'(\kappa), d'(\kappa))$. It follows from Theorem 2.3 that the sharp waves are divergent. Actually, all the waves are divergent in this case.

Theorem 2.6 *(Existence of smooth waves). For any $m > 1$, $D > 0$ and $r \geq 0$, there exists a constant $\hat{c}(m, r, b, d) > 0$ depending on m, r and the structure of $b(\cdot), d(\cdot)$, such that for any $c > \hat{c}(m, r, b, d)$, (2.1) admits smooth wave solutions $\phi(t)$ (semi-wavefronts or wavefronts, see Fig. 4) with*

$$0 < \zeta_1 \leq \liminf_{t\to+\infty} \phi(t) \leq \limsup_{t\to+\infty} \phi(t) \leq \zeta_2,$$

and

$$|\phi(t) - C_1 e^{\lambda t}| \leq C_2 e^{\Lambda t}, \quad \text{for any } t < 0,$$

where $\lambda > 0$ is the unique root of $\chi_0(\lambda) = 0$ (χ_0 is defined by (5.1)) and $\Lambda > \lambda$, $C_1, C_2 > 0$ are constants.

Remark 2.5. In the above theorems, we have introduced constants $\hat{c}(m, r, b, d)$, $\dot{c}(m, r, b, d)$ and $c_0(m, r, b, d)$, with obviously

$$\dot{c}(m, r, b, d) \leq c_*(m, r, b, d) \leq \min\{\hat{c}(m, r, b, d), c_0(m, r, b, d)\},$$

where $c_*(m, r, b, d)$ is the minimal wave speed, or say critical wave speed, defined by (2.5). We conjecture that the sharp type traveling wave is unique, and the corresponding wave speed

$$c_*(m, r, b, d) = c_0(m, r, b, d) = \hat{c}(m, r, b, d).$$

In other words, the critical wave of the degenerate model is the unique sharp wave, and the speeds of smooth waves are greater than the speed of sharp wave. The difficulty lies in the fact that all the above thresholds of wave speeds are determined by the whole structure of the equation instead of the linearized characteristic function near zero equilibrium and the proofs of the existences of different types of waves are based on various approaches without consistency among the thresholds. Those conjectures are true for the case without time delay, see for example [13], and for the case with time delay and quasi-monotonicity, see our paper [37], where we develop a phase transform approach and formulate a variational characterization of the wave speed under the monotonicity condition.

Remark 2.6. The speeds $\hat{c}(m, r, b, d)$, $\dot{c}(m, r, b, d)$, $c_0(m, r, b, d)$ and $c_*(m, r, b, d)$ all are assumed to be dependent on the structure of functions $b(\cdot)$ and $d(\cdot)$. It is well known that for the linear diffusion equation without time delay, i.e., $m = 1, r = 0$, $c_*(1, 0, b, d) = 2\sqrt{D(b'(0) - d'(0))}$ provides that b, d satisfy some concave structure. Obviously, the critical wave speed of the linear diffusion equation is totally determined by the linearization near zero. However, the critical wave speed of the degenerate diffusion equation is nonlinearly determined. The wave front behaviors are controlled by the whole structure.

3. Existence of sharp waves and their regularity

In this section, we develop a new delayed iteration approach based on an observation of the delicate structure of time delay and sharp edge to solve the delayed degenerate equation. As far as we know, this is the first framework of showing the existence of sharp traveling wave solution for the degenerate diffusion equation with large time delay. A sharp wave solution $\phi(t)$ is a special solution such that $\phi(t) \equiv 0$ for $t \le 0$ and $\phi(t) > 0$ for $t > 0$. The result is also valid for the general birth rate and death rate functions without time delay and here we only focus on the case with time delay.

For any given $m > 1$, $D > 0$ and $r > 0$, we solve (2.1) step by step. First step, noticing that the sharp wave solution $\phi(t) = 0$ for $t \le 0$ and then $\phi(t - cr) = 0$ for $t \in [0, cr)$, (2.1) is locally reduced to

$$\begin{cases} c\phi'(t) = D(\phi^m(t))'' - d(\phi(t)), & t \in (0, cr), \\ \phi(0) = 0, \quad (\phi^m)'(0) = 0, \end{cases} \tag{3.1}$$

whose solutions are not unique and we choose the maximal one such that $\phi(t) > 0$ for $t \in (0, cr)$ as shown in the following lemma. Here, $(\phi^m)'(0) = 0$ is the necessary and sufficient condition such that the zero extension of $\phi(t)$ to the left satisfies (2.1) locally near 0 in the sense of distributions.

Lemma 3.1. *For any $c > 0$, the degenerate ODE (3.1) admits a maximal solution $\phi_c^1(t)$ on $(0, cr)$ such that $\phi_c^1(t) > 0$ on $(0, cr)$ and*

$$\phi_c^1(t) = \left(\frac{(m-1)c}{Dm} t\right)^{\frac{1}{m-1}} + o(t^{\frac{1}{m-1}}), \quad t \to 0^+.$$

Proof. Clearly, $\phi_0(t) \equiv 0$ is a solution of (3.1). But we are looking for the solution such that $\phi_c^1(t) > 0$ on $(0, cr)$. Consider the generalized phase plane related to (3.1) and define $\psi_c^1(t) = D[(\phi_c^1(t))^m]'$, then $(\phi_c^1(t), \psi_c^1(t))$ solve the following singular ODE system on $(0, cr)$

$$\begin{cases} \phi'(t) = \dfrac{\psi(t)}{Dm\phi^{m-1}(t)}, \\ \psi'(t) = \dfrac{c\psi(t)}{Dm\phi^{m-1}(t)} + d(\phi(t)). \end{cases} \tag{3.2}$$

We solve (3.2) with the condition $(\phi_{c,\epsilon}^1(0), \psi_{c,\epsilon}^1(0)) = (0, \epsilon)$ with $\epsilon > 0$, whose existence, continuous dependence and suitable regularity follow from the phase plane analysis. Let ϵ tend to zero and $(\phi_c^1(t), \psi_c^1(t))$ be the limiting function. Then $\phi_c^1(t)$ is the maximal solution of (3.1) and $\phi_c^1(t) > 0$ on $(0, cr)$. Asymptotic analysis shows that

$$\psi_c^1(t) = Dm(\phi_c^1(t))^{m-1}\phi_c^{1\prime}(t) = c\phi_c^1(t) + o(\phi_c^1(t)), \quad t \to 0^+,$$

which means that

$$\phi_c^1(t) = \left(\frac{(m-1)c}{Dm} t + o(t)\right)^{\frac{1}{m-1}}, \quad t \to 0^+. \quad \square$$

Second step, let $\phi_c^2(t)$ be the solution of the following initial value ODE problem

$$\begin{cases} c\phi'(t) = D(\phi^m(t))'' - d(\phi(t)) + b(\phi_c^1(t - cr)), & t \in (cr, 2cr), \\ \phi(cr) = \phi_c^1(cr), \quad \phi'(cr) = (\phi_c^1)'(cr). \end{cases} \tag{3.3}$$

Define $\psi_c^2(t) = D[(\phi_c^2(t))^m]'$, then $(\phi_c^2(t), \psi_c^2(t))$ solve the following system on $(cr, 2cr)$

$$\begin{cases} \phi'(t) = \dfrac{\psi(t)}{Dm\phi^{m-1}(t)}, \\ \psi'(t) = \dfrac{c\psi(t)}{Dm\phi^{m-1}(t)} + d(\phi(t)) - b(\phi_c^1(t - cr)). \end{cases} \tag{3.4}$$

The above steps can be continued unless $\phi_c^k(t)$ blows up or decays to zero in finite time for some $k \in \mathbb{N}^+$. Let $\phi_c(t)$ be the connecting function of those functions on each step, i.e.,

$$\phi_c(t) = \begin{cases} \phi_c^1(t), & t \in [0, cr), \\ \phi_c^2(t), & t \in [cr, 2cr), \\ \dots \\ \phi_c^k(t), & t \in [(k-1)cr, kcr), \\ \dots \end{cases} \tag{3.5}$$

for some finite steps such that $\phi_c(t)$ blows up or decays to zero, or for infinite steps such that $\phi_c(t)$ is defined on $(0, +\infty)$ and zero extended to $(-\infty, 0)$ for convenience.

Lemma 3.2. *For any given m, D and $r > 0$, there exists a constant $\underline{c} > 0$ such that if $c \le \underline{c}$, then $\phi_c(t)$ decays to zero in finite time.*

Proof. On the existence interval of $\phi_c(t)$, the pair $(\phi_c(t), \psi_c(t))$ with $\psi_c(t) := D[(\phi_c(t))^m]'$ is a trajectory in the phase plane

$$\begin{cases} \phi'(t) = \dfrac{\psi(t)}{Dm\phi^{m-1}(t)}, \\ \psi'(t) = \dfrac{c\psi(t)}{Dm\phi^{m-1}(t)} + d(\phi(t)) - b(\phi(t - cr)). \end{cases} \tag{3.6}$$

In Section 4, we will develop a phase transform approach and prove that all the admissible wave speeds have a lower bound. The result of Lemma 4.1 and its proof therein are independent of all the results about the existence of sharp waves in this Section. The proof of $\phi_c(t)$ decays to zero in finite time when c is sufficiently small is a consequence of Lemma 4.1. □

Lemma 3.3. *For any given m, D and $r > 0$, there exists a constant $\overline{c} > 0$ such that if $c \ge \overline{c}$, then $\phi_c(t)$ grows up to $+\infty$ as t tends to $+\infty$.*

Proof. On the existence interval of $\phi_c(t)$, the pair $(\phi_c(t), \psi_c(t))$ defined in the proof of Lemma 3.2 is a trajectory in the phase plane (3.6). Now, we utilize the phase plane analysis to show that when c is large enough, then $\phi_c(t)$ grows up to the positive infinity as t increases. For $t \in (0, cr)$, ϕ_c is strictly monotonically increasing according to (3.2). Let $(0, \zeta)$ be the maximal interval such that ϕ_c is strictly monotonically increasing and within this interval, we have $d\psi_c/d\phi_c$ satisfies

$$\begin{cases} \dfrac{d\psi}{d\phi} = c - \dfrac{Dm\phi^{m-1}(b(\phi_{cr}) - d(\phi))}{\psi} =: \dfrac{\Psi}{\Phi}, \\ \psi(0) = 0, \quad \psi(\phi) > 0, \qquad \phi \in (0, \zeta), \end{cases} \tag{3.7}$$

where ϕ_{cr} is the functional of ϕ_c and ψ_c (we regard ψ_c as a function of ϕ_c since ϕ_c is strictly increasing) defined by

$$\phi_{cr} = \inf_{\theta \in [0, \phi_c]} \left\{ \int_\theta^{\phi_c} \frac{Dms^{m-1}}{\psi_c(s)} ds \le cr \right\}.$$

Consider the following auxiliary problem

$$\begin{cases} \dfrac{d\psi}{d\phi} = c - \dfrac{Dm\phi^{m-1}(\tilde{b}(\phi) - d(\phi))}{\psi}, \\ \psi(0) = 0, \quad \psi(\phi) > 0, \qquad \phi \in (0, \zeta), \end{cases} \tag{3.8}$$

where $\tilde{b}(s) = \sup_{\theta \in (0,s)} b(\theta)$ is the quasi-monotone modification of $b(s)$ and the solution of (3.8) is denoted by $\underline{\psi}_c(\phi)$. Therefore, as $\phi_c(t)$ is strictly increasing (equivalently, $\psi_c(t) > 0$) we have

$$b(\phi_{cr}) \le \tilde{b}(\phi_{cr}) \le \tilde{b}(\phi),$$

and the comparison between (3.7) and (3.8) shows that

$$\psi_c(\phi) \ge \underline{\psi}_c(\phi), \qquad \phi \in (0, \zeta). \tag{3.9}$$

The phase plane analysis to (3.8) without time delay shows that there exists a $\overline{c} > 0$ such that if $c \ge \overline{c}$, then $\underline{\psi}_c(\phi)$ is positive for all $\phi \in (0, +\infty)$, which means according to (3.9) that $\psi_c(\phi) > 0$ for all $\phi \in (0, +\infty)$, $\phi_c(t)$ is always increasing for $t \in (0, +\infty)$. It follows that in fact $\zeta = +\infty$ and $\phi_c(t)$ grows up to $+\infty$ as t tends to $+\infty$. □

We also need the following continuous dependent property of $\phi_c(t)$ on c.

Lemma 3.4. *For any given m, D and $r > 0$, the solution $\phi_c(t)$ is locally continuously dependent on c. That is, for any $c > 0$ and any given $T > 0$ and $\varepsilon > 0$, there exists a $\delta > 0$ such that for any $|c_1 - c| < \delta$ and $c_1 > 0$ we have*

$$|\phi_{c_1}(t) - \phi_c(t)| < \varepsilon, \quad \forall t \in (0, T_1 - \varepsilon),$$

where $T_1 = \min\{T, T_c\}$ with T_c being the existence interval of $\phi_c(t)$.

Proof. Without loss of generality, we may assume that $T_1 > cr + \varepsilon$. The proof is divided into two parts: the continuous dependence of the singular ODE (3.1) within $(0, cr)$ and the continuous dependence of a regular ODE within $(cr, T_1 - \varepsilon)$.

Step I. We prove that $\phi_c(t)$ together with $\phi_c'(cr)$ is continuously dependent on c for $t \in (0, cr)$. Since $T_c > cr$, we see that $\phi_c(t)$ is positive for $t \in (0, cr]$. We note that the maximal solution

$\phi_c(t)$ is the unique solution such that $\phi_c(t) > 0$ in a right neighbor of 0 and the asymptotic analysis Lemma 3.1 shows that $\phi_c(t)$ is locally monotonically and continuously dependent on c within some interval $(0, t_1) \subset (0, cr)$. In (t_1, cr), $\phi_c(t)$ is bounded away from zero and (3.1) is a regular ODE, and the continuously dependence follows from the classical theory.

Step II. We prove that $\phi_c(t)$ is continuously dependent on c for $t \in (cr, T_1 - \varepsilon)$. As T_c is the existence interval of $\phi_c(t)$, $\phi_c(t)$ is bounded from above and below for $t \in [cr, T_c - \varepsilon]$ such that $\phi_c(t) \in [M_1, M_2] \subset (0, +\infty)$ for some $M_2 > M_1 > 0$. According to the construction of $\phi_c(t)$, we see that $\phi_c(t)$ satisfies

$$\begin{cases} c\phi'(t) = D(\phi^m(t))'' - d(\phi(t)) + b(\phi(t - cr)), & t \in (cr, T_1 - \varepsilon), \\ \phi(cr) = \phi_c(cr), \quad \phi'(cr) = \phi_c'(cr), \end{cases}$$

which is a regular ODE without singularity on a bounded interval. This completes the proof. □

According to the unimodality condition (2.3), if $s_M \geq \kappa$, then b is monotonically increasing on $[0, \kappa]$ and it is well known that the non-degenerate diffusion equation ($m = 1$) admits monotonically increasing wavefronts if and only if $c \geq c_*$ with $c_* > 0$ being the minimal wave speed determined by the characteristic equation near the equilibrium 0. It is also shown in [36] that the similar result holds for the degenerate diffusion equation ($m > 1$) except that the minimal wave speed is not determined by the corresponding characteristic equation, which indicates an essential difference between those two types of diffusion. Henceforth, we may restrain ourselves to the case $s_M < \kappa$ in which b is non-monotone in $[0, \kappa]$ and $b(s_M) > b(\kappa) = d(\kappa)$. We set $M := b(s_M) = \max b$, $\theta := b(d^{-1}(M))$ and according to the monotone increasing of the death function d, it holds $s_M < \kappa < d^{-1}(M)$.

The unimodality condition (2.3) is stronger than the following condition:

$$\begin{cases} b : \mathbb{R}_+ \to \mathbb{R}_+ \text{ is continuous and such that, for some } 0 < \zeta_1 < \zeta_2 : \\ \quad b([\zeta_1, \zeta_2]) \subseteq [d(\zeta_1), d(\zeta_2)] \text{ and } b([0, \zeta_1]) \subseteq [0, d(\zeta_2)]; \\ \quad \min_{s \in [\zeta_1, \zeta_2]} b(s) = b(\zeta_1); \\ \quad b(s) > d(s) \text{ for } s \in (0, \zeta_1] \text{ and } b \text{ is differentiable at } 0, \text{ with } b'(0) > d'(0); \\ \quad \text{in } [0, \zeta_2], \text{ the equation } b(s) = d(s) \text{ has exactly two solutions, } 0 \text{ and } \kappa. \end{cases} \tag{3.10}$$

Here we can take $\zeta_2 = d^{-1}(M) = d^{-1}(\max b)$, and $\zeta_1 \in (0, s_M)$ such that $b(\zeta_1) = \theta$, whose existence and uniqueness are ensured by the unimodality condition (2.3) as shown in the illustrative Fig. 1.

The following uniform permanence property is used to show that the above locally solved solution is actually a semi-wavefront if it exists globally. For the linear diffusion case ($m = 1$) in [32,33], the authors proved the uniform permanence property based on the variation of constants formula for semilinear differential equations. Their approach is not applicable to our case of degenerate diffusion equation. Here we adopt an alternative proof applicable for quasi-linear diffusion equations ($m > 1$).

Lemma 3.5 *(Uniform permanence property). Assume that the unimodality condition* (2.3) *or its consequence* (3.10) *holds with additionally* $\sup_{s \geq 0} b(s) \leq M$. *Then any non-trivial wave solution*

$\phi(t)$ of (2.1) satisfies

$$0 < \zeta_1 \leq \liminf_{t\to+\infty} \phi(t) \leq \limsup_{t\to+\infty} \phi(t) \leq \zeta_2 < +\infty.$$

Proof. We first prove that $\limsup_{t\to+\infty} \phi(t) \leq \zeta_2$. We proceed by contrary, supposing that there exists a $t_* \in \mathbb{R}$ such that $\phi(t_*) > \zeta_2$. Let (t_1, t_2) be the maximal interval such that $t_* \in (t_1, t_2)$ and $\phi(t) > \zeta_2$ in (t_1, t_2), i.e., (t_1, t_2) is the connected component containing t_* of the set that $\phi(t) > \zeta_2$. Since $\lim_{t\to-\infty} \phi(t) = 0 < \zeta_2$, we see that $t_1 \in (-\infty, t_*)$. If there is no local maximum point of $\phi(t)$ in (t_1, t_2), then $t_2 = +\infty$ and $\phi(t)$ is monotonically increasing and converges to some equilibrium greater than ζ_2 as t tends to positive infinity, which is impossible since the only positive equilibrium is $\kappa < \zeta_2$. Now let $t_0 \in (t_1, t_2)$ be one of the local maximum points. We have $\phi(t_0) \geq \phi(t_*) > \zeta_2$, $\phi'(t_0) = 0$, $(\phi^m(t))''|_{t=t_0} \leq 0$ as t_0 is also a maximum point of $\phi^m(t)$, and at this point t_0

$$\begin{aligned} c\phi'(t) - D(\phi^m(t))'' + d(\phi(t)) - b(\phi(t - cr)) \\ \geq d(\phi(t)) - b(\phi(t - cr)) > d(\zeta_2) - M = 0, \end{aligned}$$

which contradicts to the equation (2.1). Therefore, we proved that $\phi(t) \leq \zeta_2$ for all $t \in \mathbb{R}$.

We next prove that $\phi(t) > 0$ for $t \in \mathbb{R}$ unless $\phi(s) \equiv 0$ for all $s \leq t$, which is in fact possible for the sharp type wave solution. Suppose that there exists a t_* such that $\phi(t_*) = 0$ and $\phi(s) \not\equiv 0$ for $s \leq t_*$. Here at t_*, we have $\phi'(t) = 0$, $(\phi^m(t))'' \geq 0$ and

$$b(\phi(t - cr)) = c\phi'(t) - D(\phi^m(t))'' + d(\phi(t)) \leq 0,$$

which means $\phi(t_* - cr) = 0$ and $\phi(t_* - jcr) = 0$ for all $j \in \mathbb{Z}^+$ by induction. Supposing that t_0 is the boundary of the leading edge of $\phi(t)$ (see Definition 2.2) and in this case $t_0 < t_* < +\infty$, $\phi(t_0) > \kappa$, $\phi'(t_0) = 0$, $(\phi^m(t))''|_{t=t_0} \leq 0$, then we have at t_0

$$b(\phi(t - cr)) = c\phi'(t) - D(\phi^m(t))'' + d(\phi(t)) > d(\kappa) > 0.$$

It follows that $\phi(t_0 - cr) > 0$ and $\phi(t) > 0$ for $t \in (t_0 - cr, t_0)$ since ϕ is monotonically increasing in $(-\infty, t_0)$. That is, we find an interval in $(-\infty, t_*)$ longer than cr such that $\phi(t)$ has no zero point, which contradicts to $\phi(t_* - jcr) = 0$ for all $j \in \mathbb{Z}^+$. We conclude that $\phi(t) > 0$ for all $t > t_0$.

We finally prove that $\zeta_1 \leq \liminf_{t\to+\infty} \phi(t)$. Assuming that $\liminf_{t\to+\infty} \phi(t) < \zeta_1$, then there exists a sequence $\{t_n\}_{n=1}^{\infty}$ such that t_n tends to positive infinity and $\phi(t_n) < \zeta_1$. Let $A = \{t > t_1; \phi(t) < \zeta_1\}$. We denote the set of all the local minimum points of $\phi(t)$ in A by $A_{\min}$. We divide the following proof into two parts.

(i) If $A_{\min}$ is empty or bounded to the upwards, then $\phi(t)$ is eventually monotone and converges to some equilibrium in $[0, \zeta_1]$, which can only be 0. Therefore, $\phi(t)$ is monotonically decreasing on $[t_*, +\infty)$ and $\phi(t) \in [0, \varepsilon]$ for some sufficiently large t_*, where $\varepsilon \in (0, \zeta_1)$ such that

$$\frac{b(s) - d(s)}{s} \geq \frac{b'(0) - d'(0)}{2}, \quad \forall s \in (0, \varepsilon)$$

since the limit of the left hand side is $b'(0) - d'(0) > 0$ as s tends to zero. Now we have for $t > t_* + cr$, $\phi'(t) \le 0$, $\phi(t - cr) \ge \phi(t)$ and

$$b(\phi(t - cr)) \ge b(\phi(t)) \ge d(\phi(t)) + \frac{b'(0) - d'(0)}{2}\phi(t).$$

Here we have used the monotonicity of $b(s)$ on $[0, \varepsilon]$ since $b'(0) > 0$ and we may take ε even smaller if necessary. Then

$$D(\phi^m(t))'' = c\phi'(t) + d(\phi(t)) - b(\phi(t - cr)) \le -\frac{b'(0) - d'(0)}{2}\phi(t), \tag{3.11}$$

which must decay to zero at some finite point $\hat{t} \in (t_*, +\infty)$ and $\phi(t) > 0$ for $t < \hat{t}$ according to the phase plane analysis for this sublinear ordinary differential equation (3.11). At this point $\hat{t}$, we also have

$$b(\phi(t - cr)) = c\phi'(t) - D(\phi^m(t))'' + d(\phi(t)) = 0,$$

which contradicts to $\phi(\hat{t} - cr) > 0$.

(ii) If $A_{\min}$ is unbounded to the upwards. Suppose that t_0 is the boundary of the leading edge of $\phi(t)$ (see Definition 2.2) and in this case $t_0 < t_1 \le \inf A \le \inf A_{\min} < +\infty$, $\phi(t_0) > \kappa$, ($t_1 > t_0$ is trivial as we can modify the sequence $\{t_n\}_{n=1}^{\infty}$). We have already proved that $\phi(t) > 0$ for $t > t_0$, and the local regularity of non-degenerate diffusion equation (2.1) shows that there is no bounded accumulation point of $A_{\min}$. For any $t_* \in A_{\min}$, we find that $\phi'(t_*) = 0$, $(\phi^m(t))''|_{t=t_*} \ge 0$ and

$$b(\phi(t - cr)) = c\phi'(t) - D(\phi^m(t))'' + d(\phi(t)) \le d(\phi(t)) \quad \text{at } t_*. \tag{3.12}$$

According to the structure assumption on $b(s)$ and $d(s)$, we can choose positive constants $k_1 > 0$ and $k_2 \in (0, 1)$ such that $b(s) - d(s) \ge k_1 s$ and $b(\tilde{s}) \ge d(s)$ for all $s \in (0, \zeta_1)$ and $\tilde{s} \in [k_2 s, s]$. We deduce from (3.12) that $\phi(t_* - cr) < k_2\phi(t_*) < \phi(t_*)$. Noticing that t_* is a local minimum point, we see that $t_* - cr < t_1$ or there exists another local minimum point $\tilde{t}_* < t_*$ and $\tilde{t}_* \in A_{\min}$ such that $\phi(\tilde{t}_*) \le \phi(t_* - cr) < k_2\phi(t_*)$, which is denoted by $\tilde{t}_* = F(t_*)$ for convenience. Furthermore, if $F(t_*^1) = F(t_*^2)$ for two different minimum points $t_*^1, t_*^2 \in A_{\min}$ with $t_*^1 < t_*^2$, then $0 < t_*^2 - t_*^1 < cr$ as $t_*^2 - cr < t_*^1$, otherwise, $F(t_*^2) \ge t_*^1 > F(t_*^1)$, a contradiction. Therefore, we can choose a subsequence $\{s_n\}_{s=1}^{\infty} \subset A_{\min}$ such that $\phi(s_{n+1}) \ge \phi(s_n)/k_2$ for all $n \in \mathbb{Z}^+$, which contradicts to $k_2 \in (0, 1)$ and $\phi(t) < \zeta_1$ for all $t \in A$. The proof is completed. □

Now, we are able to prove the existence of sharp traveling waves.

Proof of Theorem 2.1. Let $(0, T_1)$ and $(0, T_2)$ be the maximal interval such that $\phi_{\underline{c}}(t)$ remains positive before decaying to zero and $\phi_{\overline{c}}(t) < \zeta_2$, respectively, where $\underline{c}$ and $\overline{c}$ are constants in Lemma 3.2 and Lemma 3.3. For any $T > \max\{T_1, T_2\}$, $\phi_c(T) \ge \zeta_2$ for some $c \ge \overline{c}$ and $\phi_c(T) \le 0$ for some $c \le \underline{c}$. The continuous dependence of $\phi_c(t)$ with respect to c on the compact interval $[0, T]$ (Lemma 3.4) implies that there exists a $c_T \in [\underline{c}, \overline{c}]$ such that $\phi_{c_T}(T) = \kappa$. Since the closed interval $[\underline{c}, \overline{c}]$ is compact, there exists a subsequence of $\{c_T\}$, i.e., $\{c_{T_i}\}_{i=1}^{\infty}$, and a $c_0 \in [\underline{c}, \overline{c}]$, such that $\lim_{i\to\infty} c_{T_i} = c_0$. Meanwhile, $\phi_{c_0}(t)$ exists on the whole $(0, +\infty)$, whose zero extension to the left is a sharp wave solution. The uniform permanence property Lemma 3.5 and the asymptotic expansion Lemma 3.1 indicate that the sharp wave solutions $\phi_{c_0}(t)$ satisfies

$$0 < \zeta_1 \le \liminf_{t\to+\infty} \phi_{c_0}(t) \le \limsup_{t\to+\infty} \phi_{c_0}(t) \le \zeta_2,$$

and

$$|\phi_{c_0}(t) - C_1 t_+^{\lambda}| \le C_2 t_+^{\Lambda}, \quad \text{for any } t \in (0, 1),$$

where $t_+ = \max\{t, 0\}$, $\lambda = 1/(m-1)$ and $\Lambda > \lambda$, $C_1, C_2 > 0$ are constants. □

Remark 3.1. The time delay together with the non-monotone structure of birth rate function $b(u)$ causes us essential difficulty in proving the monotonic dependence of $\phi_c(t)$ with respect to c. Actually, the possible existence of non-monotone semi-wavefront suggests that the monotonic dependence may be violated in general. Without this monotonic dependence, the uniqueness of the wave speed for wave solutions of sharp type remains open. For the case that the birth function is monotone, we show that the sharp type wave solution is unique in [37].

We show that the sharp waves are classified into C^1 type and piecewise-C^1 type according to the degeneracy index m as stated in Theorem 2.2.

Proof of Theorem 2.2. The asymptotic behavior near 0 in Lemma 3.1 completes the proof. □

4. Lower bounds of the wave speeds

This section is devoted to the formulation of a lower bound of the wave speeds and the nonexistence results for all kinds of waves with small speeds. It should be noted that the minimal admissible wave speed is generally dependent on the time delay and decays to zero if time delay tends to infinity (under some conditions on the birth function). Therefore, there is no existing method and result showing the lower bound of the wave speeds for degenerate diffusion equation with large time delay. For the case with monotone birth function and small time delay, we introduced a generalized phase plane analysis in [36] to show the nonexistence results.

Here, we develop a phase transform approach combined with the analysis of corresponding phase energy to show the nonexistence results of sharp or smooth waves with small speed depending on arbitrary time delay. We note that this method is generally incapable of showing the existence of traveling waves with time delay since the trajectories with time delay may intersect with each other. However, it can be a blueprint to draw the prospective contradiction for proving the nonexistence. As for as we know, this is the first approach dealing with degenerate diffusion equation with large time delay and non-monotone birth rate functions.

Lemma 4.1 *(Lower bound of wave speeds). For any $m > 1$, $D > 0$ and $r \ge 0$, there exists a constant $\dot{c}(m, r, b, d) > 0$ depending on m, r and the structure of $b(\cdot), d(\cdot)$, such that all the speeds of the wave solutions of* (2.1)*, no matter sharp or smooth, wavefronts or divergent semi-*

wavefronts, are great than or equal to $\dot{c}(m,r,b,d)$*. Moreover,*

$$\dot{c}(m,r,b,d) = \frac{\mu_0(m, b(\cdot), d(\cdot)) + o(1)}{r}, \quad r \to +\infty,$$

where $\mu_0(m, b(\cdot), d(\cdot)) > 0$.

Proof. For any given $m > 1$, $D > 0$ and $r \geq 0$, we need to find a constant $\dot{c}(m,r,b,d) > 0$, such that, (2.1) admits no wave solution $\phi(t)$ (semi-wavefronts or wavefronts, sharp or smooth) for any $c < \dot{c}(m,r,b,d)$. The nonexistence result is valid for a typical Nicholson's birth rate function and death rate function without time delay in [13]. We can verify that it is also true for the general type of b and d without time delay. Here we only prove the case with time delay $r > 0$.

We prove by contradiction and assume that ϕ_c is a wave solution corresponding to the speed c. Recall that ζ_1 and ζ_2 are the constants in (3.10). Since $b'(0) > d'(0)$, let $(0, \zeta_3)$ be the maximal interval such that

$$\psi_0(\phi) := \frac{Dm\phi^{m-1}(b(\phi) - d(\phi))}{c} \tag{4.1}$$

is increasing with respect to ϕ and denote $\zeta_0 = \min\{\zeta_1, \zeta_3\}$. It should be noted that ζ_0 is independent of c and r. Let $I_0 := (-\infty, t_0)$ be the maximal interval of the leading edge of ϕ_c and let $I_1 = (t_1, t_2)$ be the maximal subinterval of I_0 such that ϕ_c is positive, monotonically increasing and $\phi_c(t) < \zeta_0$. That is, there exists a unique $\hat{t}_0 < t_0$ such that $\phi_c(\hat{t}_0) = \zeta_0$ and we take $t_2 = \hat{t}_0$. If ϕ_c is of smooth type, then $t_1 = -\infty$, while if ϕ_c is sharp, we take $t_1 = 0$ instead. Within I_1, $\phi_c(t)$ is monotonically increasing and $b(\phi_c)$ is monotonically increasing with respect to ϕ_c as $\phi_c \leq \zeta_0 \leq \zeta_1$.

Now we introduce the phase transform approach, see for example [13,36]. Let

$$\psi_c(t) = D(\phi_c^m(t))'.$$

Since $\phi_c(t)$ is positive and monotonically increasing in I_1, we have the following singular phase plane where $(\phi_c(t), \psi_c(t))$ corresponds to a trajectory

$$\begin{cases} \phi'(t) = \dfrac{\psi(t)}{Dm\phi^{m-1}(t)} =: \Phi, \\ \psi'(t) = \dfrac{c\psi(t)}{Dm\phi^{m-1}(t)} + d(\phi(t)) - b(\phi_{cr}(t)) =: \Psi, \end{cases} \tag{4.2}$$

with $\phi_{cr}(t) = \phi(t - cr)$. We note that $\psi_c(t) \geq 0$ and the zero points of $\psi_c(t)$ is isolated since $\phi_c(t)$ is a given wave solution. According to the choice of I_1, we can regard $t \in I_1$ as an inverse function of ϕ_c and denote $\tilde{\psi}_c(\phi_c) = \psi_c(t(\phi_c)) \geq 0$. We redefine $\phi_{cr}(t)$ as a functional of ϕ_c and $\tilde{\psi}_c$ as follows

$$\phi_{cr} = \inf_{\theta \in [0, \phi_c]} \left\{ \int_{\theta}^{\phi_c} \frac{Dms^{m-1}}{\tilde{\psi}_c(s)} ds \leq cr \right\}. \tag{4.3}$$

Consider the following nonlocal problem

$$\begin{cases} \dfrac{d\psi}{d\phi} = c - \dfrac{Dm\phi^{m-1}(b(\phi_{cr}) - d(\phi))}{\psi} = \dfrac{\Psi}{\Phi}, \\ \psi(0) = 0, \quad \psi(\zeta_0) = Dm\zeta_0^{m-1}\phi_c'(t_2) > 0, \qquad \phi \in (0, \zeta_0). \end{cases} \tag{4.4}$$

Here, nonlocal means that ϕ_{cr} is a functional of ϕ and $\psi(\phi)$, which is caused by the time delay.

We draw a contradiction to the existence of solutions to (4.4) when c is sufficiently small with the help of the phase plane (4.2). The curve Γ_c corresponding to $\psi_0(\phi)$ defined in (4.1) divides $(0, \zeta_0) \times (0, +\infty)$ into two parts, $E_1 := \{(\phi, \psi); \phi \in (0, \zeta_0), 0 < \psi < \psi_0(\phi)\}$ and $E_2 := ((0, \zeta_0) \times (0, +\infty)) \backslash E_1$. For any $(\phi, \psi) \in \Gamma_c$, we have

$$\frac{\Psi}{\Phi} = c - \frac{Dm\phi^{m-1}(b(\phi_{cr}) - d(\phi))}{\psi} > c - \frac{Dm\phi^{m-1}(b(\phi) - d(\phi))}{\psi} = 0.$$

We can check that $\Psi/\Phi > 0$ for any $(\phi, \psi) \in E_2$. Let $\psi_1(\phi)$ be the solution of

$$\begin{cases} \dfrac{d\psi}{d\phi} = c + \dfrac{Dm\phi^{m-1}d(\phi)}{\psi}, \\ \psi(0) = 0, \psi(\phi) > 0, \phi \in (0, \zeta_0). \end{cases}$$

Asymptotic analysis shows that there exists a constant $C_1 > 0$ depending on the upper bound of c (independent of c if c is small) such that

$$\phi_1(\phi) \le C_1\phi, \quad \phi \in (0, \zeta_0).$$

The comparison principle of (4.4) shows that

$$\tilde{\psi}_c(\phi) \le \phi_1(\phi) \le C_1\phi, \quad \phi \in (0, \zeta_0). \tag{4.5}$$

Let $\epsilon \in (0, \zeta_0)$ be a constant such that

$$\int_0^\epsilon \phi^{m-1}d(\phi)d\phi < \frac{1}{4}\int_\epsilon^{\zeta_0} \phi^{m-1}(b(\phi) - d(\phi))d\phi, \tag{4.6}$$

and

$$\delta := \inf_{\phi \in (\epsilon, \zeta_0)} (b(\phi) - d(\phi)) > 0.$$

We note that ϵ and δ only depend on the structure of b and d. We assert that for any given $r > 0$, there exists a $c_1 > 0$ such that $b(\phi_{cr}) - d(\phi) > 0$ for all $\phi \in (\epsilon, \zeta_0)$ if $c \le c_1$. In fact, according to (4.5),

$$c_1 r \ge cr = \int_{\phi_{cr}}^{\phi} \frac{Dms^{m-1}}{\tilde{\psi}_c(s)} ds \ge \int_{\phi_{cr}}^{\phi} \frac{Dms^{m-1}}{C_1 s} ds \ge \frac{Dm}{C_1(m-1)}(\phi^{m-1} - \phi_{cr}^{m-1}),$$

and then using the uniform continuity of the function $f(s) := s^{1/(m-1)}$ on the interval $[\epsilon/2, \zeta_0]$ with the continuity modulus function being denoted by $\omega(\cdot)$, we have

$$
\begin{aligned}
0 < b(\phi) - b(\phi_{cr}) = b'(\theta)(\phi - \phi_{cr}) &\le \sup_{s\in(0,\zeta_0)} b'(s)\cdot(\phi - \phi_{cr}) \\
&\le \sup_{s\in(0,\zeta_0)} b'(s)\cdot \omega(|\phi^{m-1} - \phi_{cr}^{m-1}|) \\
&\le \sup_{s\in(0,\zeta_0)} b'(s)\cdot \omega(\frac{C_1(m-1)c_1 r}{Dm}) \le \frac{\delta}{2}
\end{aligned}
$$

for some $\theta \in (\phi_{cr}, \phi)$, provided that $c_1 r$ is sufficiently small such that $c_1 r = \mu_0 := \mu_0(m, b(\cdot), d(\cdot)) > 0$ (it suffices that c_1 is sufficiently small as r is given). Here we note that $\mu_0(m, b(\cdot), d(\cdot))$ is a constant depending on m, ϵ, ζ_0, δ, $\sup_{s\in(0,\zeta_0)} b'(s)$, which are all dependent on m and the structure of $b(\cdot)$ and $d(\cdot)$. The dependence of $\mu_0(m, b(\cdot), d(\cdot))$ on $b(\cdot)$ is basically on the structure of $b(\cdot)$ within $(0, \zeta_0)$ and $\zeta_0 \le \zeta_1$ with ζ_1 depending on the whole structure of $b(\cdot)$ on $(0, \zeta_2)$. Therefore,

$$
\begin{aligned}
b(\phi_{cr}) - d(\phi) &= (b(\phi) - d(\phi)) - (b(\phi) - b(\phi_{cr})) \\
&\ge (b(\phi) - d(\phi)) - \frac{\delta}{2} \ge \frac{b(\phi) - d(\phi)}{2}, \quad \phi \in (\epsilon, \zeta_0).
\end{aligned}
\tag{4.7}
$$

The first integral of (4.4) over $(0, \zeta_0)$ shows that

$$
\begin{aligned}
c\int_0^{\zeta_0} \tilde{\psi}_c(\phi)d\phi &= \frac{1}{2}\tilde{\psi}_c^2(\phi)\Big|_0^{\zeta_0} + \int_0^{\zeta_0} Dm\phi^{m-1}(b(\phi_{cr}) - d(\phi))d\phi \\
&\ge \int_0^{\epsilon} Dm\phi^{m-1}(b(\phi_{cr}) - d(\phi))d\phi + \int_\epsilon^{\zeta_0} Dm\phi^{m-1}(b(\phi_{cr}) - d(\phi))d\phi \\
&\ge -\int_0^{\epsilon} Dm\phi^{m-1}d(\phi)d\phi + \int_\epsilon^{\zeta_0} Dm\phi^{m-1}(b(\phi_{cr}) - d(\phi))d\phi \\
&\ge \Big(-\frac{1}{4} + \frac{1}{2}\Big)\int_\epsilon^{\zeta_0} Dm\phi^{m-1}(b(\phi) - d(\phi))d\phi,
\end{aligned}
$$

where we have used (4.6) and (4.7). On the other hand, we have

$$
c\int_0^{\zeta_0} \tilde{\psi}_c(\phi)d\phi \le c\int_0^{\zeta_0} C_1\phi d\phi \le c\frac{C_1}{2}\zeta_0^2.
$$

Now we arrive at a contradiction if we have chosen $c \le \dot{c}$ with

$$
\dot{c} = \min\{c_1, c_2\} = \min\{\frac{\mu_0(m, b(\cdot), d(\cdot))}{r}, c_2(m, b(\cdot), d(\cdot))\}
$$

such that

$$c_2 \frac{C_1}{2} \zeta_0^2 < \frac{1}{4} \int_{\epsilon}^{\zeta_0} Dm\phi^{m-1}(b(\phi) - d(\phi))d\phi.$$

The proof is completed. □

Proof of Theorem 2.4. This is equivalent to Lemma 4.1. □

5. Oscillatory property of waves

In this section, we show that the waves (sharp or smooth) are oscillatory convergent or divergent if the wave speed and the time delay is suitable large. More specifically, the sharp waves proved in Section 3 is oscillatory divergent if the time delay is large. We note that the lower bound demonstrated in Section 4 is needed in deriving the oscillatory properties.

Since the diffusion in (2.1) is degenerate for $\phi = 0$, and nonlinear with respect to ϕ near κ, we define the following characteristic functions for $c > 0$ near the two equilibria 0 and κ separately

$$\chi_0(\lambda) := b'(0)e^{-\lambda cr} - c\lambda - d'(0), \qquad \lambda > 0 \tag{5.1}$$

and

$$\chi_\kappa(\lambda) := Dm\kappa^{m-1}\lambda^2 + b'(\kappa)e^{-\lambda cr} - c\lambda - d'(\kappa), \qquad c > 0. \tag{5.2}$$

It should be noted that the characteristic function $\chi_0(\lambda)$ is totally deferent from that of linear diffusion case such that the quadratic $D\lambda^2$ for $m = 1$ is absent due to the degeneracy for $m > 1$. Therefore, the minimal admissible wave speed cannot be determined by the linearization of the equation near the zero equilibrium.

We see that $\chi_0(\lambda) = 0$ has a unique positive real root λ_0 for all $c > 0$. In fact, $\lambda_0 = \frac{\omega_r}{c}$ such that $\omega_r \in (0, b'(0) - d'(0))$ is the unique solution of $b'(0)e^{-r\omega_r} = \omega_r + d'(0)$ since $b'(0) > d'(0)$. However, the distribution of the roots of $\chi_\kappa(\lambda)$ is much more complicated and plays an essential role in determining the oscillatory property of the semi-wavefronts.

The characteristic equation near κ plays an essential role in the investigation of the monotonicity near κ. We have the following results concerned with the distribution of the roots of characteristic equation.

Lemma 5.1. *For $b'(\kappa) < 0$, there exists an extended real number $c_\kappa = c_\kappa(m, r, b'(\kappa), d'(\kappa)) \in (0, +\infty]$ such that the characteristic equation $\chi_\kappa(\lambda)$ defined in (5.2) has three real roots $\lambda_1 \leq \lambda_2 < 0 < \lambda_3$ if and only if $c \leq c_\kappa$. If c_κ is finite and $c = c_\kappa$, then $\chi_\kappa(\lambda)$ has a double root $\lambda_1 = \lambda_2 < 0$, while for $c > c_\kappa$ there does not exist any negative root to (5.2). Moreover, if $\lambda_j \in \mathbb{C}$ is a complex root of (5.2) for $c \in (0, c_\kappa]$, then $\Re\lambda_j < \lambda_2$. Furthermore, $c_\kappa(m, 0, b'(\kappa), d'(\kappa)) = +\infty$ and c_κ is strictly decreasing in its domain,*

$$c_\kappa(m, r, b'(\kappa), d'(\kappa)) = \frac{\mu_\kappa(m, b'(\kappa), d'(\kappa)) + o(1)}{r}, \quad r \to +\infty,$$

where $\mu_\kappa(m, b'(\kappa), d'(\kappa)) := \sqrt{\frac{2Dm\kappa^{m-1}\omega_\kappa}{b'(\kappa)}} e^{\frac{\omega_\kappa}{2}}$, and $\omega_\kappa < 0$ is the unique negative root of $2d'(\kappa) = b'(\kappa)e^{-\omega_\kappa}(2 + \omega_\kappa)$.

Proof. We note that (2.1) does not degenerate near κ and the linearization of the nonlinear diffusion equation (2.1) near κ is of the same type of the linear diffusion case. This lemma follows from a slight modification of Lemma 1.1 in [8]. Here we omit the details for the sake of simplicity. □

Now we prove that the waves are oscillatory if $c > c_\kappa$. For the case with linear diffusion ($m = 1$), it is proved in [8,32] that the waves are oscillatory if the wave speed are large enough based on the semigroup expression of differential equations with linear principle part. Here we provide a slightly different argument suitable for nonlinear diffusion.

Lemma 5.2. *Assume that $b'(\kappa) < 0$ and $c > c_\kappa$ as in Lemma 5.1, then (2.1) does not have any eventually monotone semi-wavefront.*

Proof. Lemma 5.1 implies that the characteristic function $\chi_\kappa(\lambda)$ around κ does not have any negative zeros. Arguing by contradiction, suppose that, there exists an eventually monotone traveling wavefront.

Set $w(t) = \phi(t) - \kappa$, then $w(t)$ is either decreasing and strictly positive or increasing and strictly negative on some interval $[T, +\infty)$ and satisfies

$$Dm(\phi(t)^{m-1}w'(t))' - cw'(t) = p(t)w(t) + k(t)w(t-h), \tag{5.3}$$

where $h = cr$ and

$$k(t) := -\frac{b(\phi(t-h)) - b(\kappa)}{\phi(t-h) - \kappa}, \quad p(t) := \frac{d(\phi(t)) - d(\kappa)}{\phi(t) - \kappa}.$$

Since $\phi(+\infty) = \kappa$, $0 < k(t) < -2b'(\kappa)$, and $0 < p(t) < 2d'(\kappa)$ for all sufficiently large t. We will show that for $c > c_\kappa$, $w(t)$ will oscillate about zero. As a consequence of Lemma 3.1.1 from [14], we can conclude that $w(t)$ cannot convey super-exponentially to 0. This fact and Corollary 24 in [32] imply the existence of a sequence $t_j \to +\infty$ and a real number $\delta > 0$ such that $|w(t_j)| = \max_{s \geq t_j} |w(s)|$ and $\max_{s \in [t_j - 3h, t_j]} |w(s)| \leq \delta |w(t_j)|$ for every j. Without loss of generality we assume that $w'(t_n) \leq 0$ and $0 < w(t) \leq w(t_n)$ for all $t \geq t_n$. Additionally, we can find a sequence $\{s_j\}$ with $\lim(s_j - t_j) = +\infty$ such that $|w'(s_j)| \leq w(t_j)$. Now, since $w(t)$ satisfies (5.3), we conclude that every $y_j(t) = w(t + t_j)/w(t_j) > 0$ is a solution of

$$Dm(\phi^{m-1}(t + t_j)y')' - cy' - p(t + t_j)y - k(t + t_j)y(t - h) = 0.$$

It is clear that $\lim_{j \to +\infty} k(t + t_j) = -b'(\kappa)$, $\lim_{j \to +\infty} p(t + t_j) = d'(\kappa)$, and $\lim_{j \to \infty} \phi(t + t_j) = \kappa$ uniformly on $\mathbb{R}_+$ and also that $0 < y_j(t) \leq \delta$ for all $t \geq -3h$, $j = 1, 2, 3, \ldots$.

We need to estimate $|y_j'(t)|$. Since $z_j(t) = m\phi^{m-1}(t + t_j)y_j'(t)$ solves the initial value problem $z_j(s_j - t_j) = w'(s_j)/w(t_j) \in [-1, 0]$ for equation

$$Dz'(t) - c\frac{1}{m\phi^{m-1}(t + t_j)}z(t) - p(t + t_j)y_j(t) - k(t + t_j)y_j(t - h) = 0,$$

we obtain that

$$z_j(t) = e^{\frac{1}{D}\int_{s_j-t_j}^{t} \frac{c}{m\phi^{m-1}(\tau+t_j)}d\tau} z_j(s_j - t_j)$$
$$+ \frac{1}{D}\int_{s_j-t_j}^{t} (p(t+t_j)y_j(s) + k(s+t_j)y_j(s-h))e^{\frac{1}{D}\int_s^t \frac{1}{m\phi^{m-1}(\tau+t_j)}d\tau} ds. \quad (5.4)$$

In consequence,

$$|y_j'(t)| \le C + C(2|g'(\kappa)|+1)d, \quad t \in [-2h, s_j - t_j], \quad j \in \mathbb{N}, \quad (5.5)$$

from which the uniform boundedness of the sequence $\{y_j'(t)\}$ on each compact interval $[-2h, \xi], \xi > -2h$, follows. Together with $0 < y_j(t) \le \delta, t \ge -3h$, inequality (5.5) implies the pre-compactness of the set $\{y_j(t), t \ge -2h, j \in \mathbb{N}\}$, in the compact open topology of $C([-2h, +\infty), \mathbb{R})$. Therefore, by the Arzela-Ascoli theorem combined with the diagonal method, we can indicate a subsequence $y_{j_k}(t)$ converging to a continuous function $y(t)$, $t \in [-2h, +\infty)$. This convergence is uniform on every bounded subset of $[-2h, +\infty)$. Additionally we may assume that $\lim_{k\to\infty} y_{j_k}'(0) = y_0'$ exists.

Next, putting $s_j - t_j = 0$ in (5.4), we find that

$$z_j(t) = m\phi^{m-1}(t+t_j)y_j'(t)$$
$$= e^{\frac{1}{D}\int_0^t \frac{c}{m\phi^{m-1}(\tau+t_j)}d\tau} z_j(0)$$
$$+ \frac{1}{D}\int_0^t (p(t+t_j)y_j(s) + k(s+t_j)y_j(s-h))e^{\frac{1}{D}\int_s^t \frac{1}{m\phi^{m-1}(\tau+t_j)}d\tau} ds, \ t \ge -h.$$

Integrating this relation between 0 and t and then taking the limit as $j \to \infty$ in the obtained expression, we obtain that

$$y(t) = 1 + \frac{Dm\kappa^{m-1}}{c}\left(e^{\frac{ct}{Dm\kappa^{m-1}}} - 1\right) y_0'$$
$$+ \int_0^t \frac{1}{Dm\kappa^{m-1}} \int_0^\sigma (d'(\kappa)y(s) - b'(\kappa)y(s-h))e^{\frac{c(t-s)}{Dm\kappa^{m-1}}} ds d\sigma, \ t \ge -h.$$

Therefore, $y(t)$ satisfies

$$Dm\kappa^{m-1}y''(t) - cy'(t) - d'(\kappa)y(t) + b'(\kappa)y(t-h) = 0, \quad t \ge -h. \quad (5.6)$$

Additionally, $y(0) = 1, y'(0) = y_0' \in [-1, 0]$ and $0 \le y(t) \le \delta, t \ge -2h$. Clearly, $y \in C^2(\mathbb{R}_+)$ and we claim that $y(t) > 0$ for all $t \ge 0$. Observe here that $y(t), t \ge -2h$, is non-increasing, and therefore $y(0) = 1, y(s) = 0$ imply $s > 0$. Let us suppose, for a moment, that $y(s) = 0$ and $y(\tau) > 0, \tau \in [-h, s)$. Then $y'(s) = 0, y(s-h) > 0$, so that (5.6) implies $y''(s) > 0$. Thus $y(t) > 0 = y(s)$ for all $t > s$ close to s which is not possible because y is non-increasing on $[-2h, +\infty)$.

We have proved that (5.6) has a bounded positive solution on $\mathbb{R}_+$. As it was established in [14] Lemma 3.1.1, this solution does not decay superexponentially. From Proposition 7.2 in [22] (see also Proposition 2.2 in [12]), we conclude that there are $b \le 0, \delta > 0$ and a nontrivial eigensolution $v(t)$ of (5.6) on the generalized eigenspace associated with the (nonempty) set Λ of eigenvalues with $\Re\lambda = b$, such that $y(t) = v(t) + O(\exp((b-\delta)t)), t \to +\infty$.

On the other hand, since $c > c^*$, we know from Lemma 5.1 that there are no real negative eigenvalues of (5.6) hence $\Im\lambda \neq 0$ for all $\lambda \in \Lambda$. From Lemma 2.3 in [12], we find that $y(t)$ is oscillatory, a contradiction. □

Proposition 5.1 *(Oscillatory waves). Assume that $m > 1$, $r > 0$, $b'(\kappa) < 0$, then there exists a number $c_\kappa = c_\kappa(m, r, b'(\kappa), d'(\kappa)) \in (0, +\infty]$ such that the semi-wavefronts with speed $c > c_\kappa$ cannot be eventual monotone (i.e., they must be oscillating around κ, convergent or divergent). Moreover,*

$$c_\kappa(m, r, b'(\kappa), d'(\kappa)) = \frac{\mu_\kappa(m, b'(\kappa), d'(\kappa)) + o(1)}{r}, \quad r \to +\infty,$$

where $\mu_\kappa(m, b'(\kappa), d'(\kappa)) := \sqrt{\frac{2Dm\kappa^{m-1}\omega_\kappa}{b'(\kappa)}}e^{\frac{\omega_\kappa}{2}}$, and $\omega_\kappa < -2$ is the unique negative root of $2d'(\kappa) = b'(\kappa)e^{-\omega_\kappa}(2 + \omega_\kappa)$.

Proof. This result follows from Lemma 5.1 and Lemma 5.2. □

Now we know that if $b'(\kappa) < 0$ and the birth rate function b satisfies the feedback condition (2.4), then for $c > c_\kappa$, the semi-wavefront $\phi(t)$ is slowly oscillating around the positive steady state. In the remaining part of this section, we show that these oscillations are non-decaying for the wave speeds greater than some constant c^*.

Before going further, it will be convenient to work with the scaled function $\varphi(s) = \phi(cs)$. Then φ is a positive solution of the delay differential equation

$$D\sigma(\varphi^m)''(t) - \varphi'(t) - d(\varphi(t)) + b(\varphi(t-r)) = 0, \quad t \in \mathbb{R},$$

where $\sigma = c^{-2}$. The characteristic equation around κ is

$$\chi^*(\lambda) = D\sigma m\kappa^{m-1}\lambda^2 - \lambda - d'(\kappa) + b'(\kappa)e^{-\lambda r}. \tag{5.7}$$

We recall the following definition in [32] concerned with the non-decaying oscillation around κ.

Definition 5.1. Suppose that $b'(\kappa) \le 0$. Let $c^* = c^*(m, r, b'(\kappa), d'(\kappa)) \in (0, +\infty]$ be the largest extended real number such that $\chi^*(\lambda)$ does not have roots in the half-plane $\{\Re z > 0\}$ other than a positive real root.

Lemma 5.3. *The inequality $c^*(m, r, b'(\kappa), d'(\kappa)) \ge c_\kappa(m, r, b'(\kappa), d'(\kappa))$ holds for all cases. If $b'(\kappa) \ge -d'(\kappa)$, then $c^*(m, r, b'(\kappa), d'(\kappa)) = +\infty$ for large time delay r; while if $b'(\kappa) < -d'(\kappa)$, then*

$$c^*(m, r, b'(\kappa), d'(\kappa)) = \frac{\mu^*(m, b'(\kappa), d'(\kappa)) + o(1)}{r}, \quad r \to +\infty,$$

where $\mu^*(m, b'(\kappa), d'(\kappa)) := \pi\sqrt{\frac{Dm\kappa^{m-1}}{-b'(\kappa) - d'(\kappa)}}$.

Proof. According to Lemma 5.1 and Lemma 1.1 in [8], for any $c \le c_\kappa$, any complex root λ_j of (5.2) has negative real part such that $\Re\lambda_j < \lambda_2 < 0$. It follows that $c^* \ge c_\kappa$ for all cases. If $c^* < +\infty$ and $c > c^*$, then (5.7) has a complex root with non-negative real part, denoted by $\alpha + i\beta$ with $\alpha \ge 0$ and $\beta > 0$. Then

$$Dm\kappa^{m-1}\sigma(\alpha+i\beta)^2 - (\alpha+i\beta) - d'(\kappa) + b'(\kappa)e^{-r(\alpha+i\beta)} = 0.$$

That is,

$$\begin{cases} Dm\kappa^{m-1}\sigma(\alpha^2-\beta^2) - \alpha - d'(\kappa) + b'(\kappa)e^{-r\alpha}\cos(r\beta) = 0, \\ 2Dm\kappa^{m-1}\sigma\alpha\beta - \beta - b'(\kappa)e^{-r\alpha}\sin(r\beta) = 0. \end{cases} \tag{5.8}$$

For large time delay r, we assert that $\alpha = o(1)$ as $r \to +\infty$. Otherwise, $|b'(\kappa)e^{-\lambda r}| < |D\sigma m\kappa^{m-1}\lambda^2 - \lambda - d'(\kappa)|$ for $\lambda \in \partial K$ for large time delay since the complex-valued function $D\sigma m\kappa^{m-1}\lambda^2 - \lambda - d'(\kappa)$ has at most one complex root within K, where $K := \{z; \Re z > \alpha/2\}$ in the complex plane. According to the Rouche's theorem, (5.7) admits at most one complex root (that is a positive real number), which is a contradiction. Now, we see that

$$b'(\kappa)e^{-r\alpha}\cos(r\beta) = d'(\kappa) + Dm\kappa^{m-1}\sigma\beta^2 + \alpha - Dm\kappa^{m-1}\sigma\alpha^2 > d'(\kappa) + Dm\kappa^{m-1}\sigma\beta^2,$$

which is impossible if $b'(\kappa) \in [-d'(\kappa), 0)$. For the case $b'(\kappa) < -d'(\kappa)$, we let c tend to c^*, then $\alpha + i\beta$ tends to a purely imaginary number iy, and the following limiting equation of (5.8) has a nonnegative solution

$$\begin{cases} -Dm\kappa^{m-1}\sigma y^2 - d'(\kappa) + b'(\kappa)\cos(ry) = 0, \\ -y - b'(\kappa)\sin(ry) = 0. \end{cases} \tag{5.9}$$

We note that according to the definition, c^* is smallest positive real number such that (5.7) has complex roots with non-negative real part except for the unique positive real root. That is, $\sigma = 1/(c^*)^2$ is the largest positive real number such that (5.9) has a solution. Asymptotic analysis as $r \to +\infty$ shows that $ry \to \pi$ and $Dm\kappa^{m-1}\pi^2/(c^*r)^2 \to -b'(\kappa) - d'(\kappa)$. The proof is completed. □

Finally, we present a sufficient condition for the existence of divergent semi-wavefronts.

Lemma 5.4. *Assume that $b'(\kappa) < 0$ and the birth rate function b satisfies the feedback condition* (2.4). *If $c > c^*$, then the semi-wavefront $\phi(t)$ does not converge to κ as $t \to +\infty$.*

Proof. We prove that the solution does not converge to κ using a similar argument in [32], which implies that the oscillation is non-decaying. By contradiction, we assume that $\phi(t) \to \kappa$ as $t \to +\infty$. Then $v(t) = \phi(t) - \kappa$ with $v(+\infty) = 0$, satisfies

$$Dm\sigma(\phi(t)^{m-1}v'(t))' - v'(t) - d_1(v(t)) + b_1(v(t-r)) = 0, \quad t \in \mathbb{R}, \tag{5.10}$$

where $b_1(s) := b(s+\kappa) - b(\kappa)$, $b_1(0) = 0$, $b'(0) = b'(\kappa)$, satisfies the feedback condition with respect to 0, and $d_1(s) := d(s+\kappa) - d(\kappa)$, $d_1(0) = 0$, $d_1'(0) = d'(\kappa)$.

Since $v(+\infty) = 0$, there exists a sequence $t_n \to +\infty$ with the property such that $|v(t_n)| = \max_{s\geq t_n} |v(s)|$. We can assume that v attains its local extremum at t_n so that $v'(t_n) = 0$, $v''(t)v(t_n) \leq 0$. These relations and (5.10) imply that $v(t_n)v(t_n - r) < 0$ and therefore $\mathrm{sc}(\overline{v}_{t_n})$ must be an odd integer. Since $\mathrm{sc}(\overline{v}_{t_n}) \leq 2$, $\mathrm{sc}(\overline{v}_{t_n}) = 1$. There are a unique $z_n \in (t_n - r, t_n)$ and a finite set F_n such that $v(s) < 0$ for $s \in [t_n - r, z_n)\backslash F_n$ and $v(s) \geq 0$ for $s \in [z_n, t_n]$. We can assume that $|v(t_n)| = \max\{|v(s)| : s \in [z_n, t_n]\}$, and that $\{r_n\}$, $r_n := t_n - z_n \in (0, r)$, is monotonically converging to $r^* \in [0, r]$. Set $y_n(t) = v(t + z_n)/v(t_n)$, $t \in \mathbb{R}$, then $y_n(t)$ satisfies

$$Dm\sigma(\phi(t)^{m-1}y'(t))' - y'(t) - q_n(t)y(t) + p_n(t-h)y(t-h) = 0, \quad t \in \mathbb{R},$$

where

$$p_n(t) = \begin{cases} b_1(v(t+z_n))/v(t+z_n), & \text{if } v(t+z_n) \neq 0, \\ b'(\kappa), & \text{if } v(t+z_n) = 0, \end{cases}$$

and

$$q_n(t) = \begin{cases} d_1(v(t+z_n))/v(t+z_n), & \text{if } v(t+z_n) \neq 0, \\ d'(\kappa), & \text{if } v(t+z_n) = 0. \end{cases}$$

Since $y_n(0) = 0$ and $|y_n(t)| \leq 1$, $t \geq 0$, and that $\lim_{n\to\infty} p_n(t) = b'(\kappa)$, $\lim_{n\to\infty} q_n(t) = d'(\kappa)$, $\lim_{n\to\infty} \phi(t) = \kappa$ uniformly in $t \in \mathbb{R}_+$. From (5.4), we get $|y_n(t)|$ is uniformly bounded in $C^1([-2r, \infty))$. Hence, using the similar arguments in Lemma 5.2, there exists a sub-sequence y_{n_j} converging to $y^*(t)$, which is the solution of the linear equation

$$Dm\sigma\kappa^{m-1}y''(t) - y'(t) - d'(\kappa)y(t) + b'(\kappa)y(t-h) = 0, \quad t \geq 2r. \tag{5.11}$$

From Proposition 7.2 in [22], for every sufficiently large $|\nu|$, $\nu < 0$, it holds that

$$y^*(t) = Y_0(t) + O(\exp(\nu t)), \quad t \to +\infty,$$

where $Y_0(t)$ is a nonempty finite sum of eigensolutions of the linear equation (5.11) associated to the eigenvalues in $\{\lambda \in \mathbb{C} : \Re(\lambda) \in (-\nu, 0]\}$. Thus, there exist $A > 0$, $\beta > 0$, $\alpha \geq 0$, $\zeta \in \mathbb{R}$, such that $y^*(t) = (A\cos(\beta t + \zeta) + o(1))e^{-\alpha t}$, $\quad t \geq 2r$. From [32] on the location of eigenvalues, we have $\beta > 2\pi/r$. Since y_{n_j} converges to y^* as $j \to \infty$, this ensures that y_{n_j} changes its sign at least three times for sufficient large j. It contradicts to $\mathrm{sc}(\overline{v}_{t_n}) = 1$ and completes the proof. □

Proposition 5.2 *(Non-decaying oscillating waves). Assume that the birth rate function $b(\cdot)$ satisfies the feedback condition (2.4) and $b'(\kappa) < 0$ and the time delay $r > 0$, then there exists a number $c^* = c^*(m, r, b'(\kappa), d'(\kappa)) \in (0, +\infty]$ such that the semi-wavefronts with speed $c > c^*$ have to develop non-decaying slow oscillations around κ. Moreover, if $b'(\kappa) \geq -d'(\kappa)$, then $c^*(m, r, b'(\kappa), d'(\kappa)) = +\infty$ for large time delay r; while if $b'(\kappa) < -d'(\kappa)$, then*

$$c^*(m, r, b'(\kappa), d'(\kappa)) = \frac{\mu^*(m, b'(\kappa), d'(\kappa)) + o(1)}{r}, \quad r \to +\infty,$$

where $\mu^*(m, b'(\kappa), d'(\kappa)) := \pi\sqrt{\frac{Dm\kappa^{m-1}}{-b'(\kappa)-d'(\kappa)}}$.

Proof. This result follows from Lemma 5.3 and Lemma 5.4. □

Proof of Theorem 2.3. This is proved in Proposition 5.1, Lemma 5.3 and Proposition 5.2. □

Remark 5.1. For $b'(\kappa) \in [-d'(\kappa), 0)$, we have $c^*(m, r, b'(\kappa), d'(\kappa)) = +\infty$ and then the speeds $c^*(m, r, b'(\kappa), d'(\kappa)) > c_\kappa(m, r, b'(\kappa), d'(\kappa))$ for large time delay. For $b'(\kappa) \in (-\infty, -d'(\kappa))$, we also have $c^*(m, r, b'(\kappa), d'(\kappa)) > c_\kappa(m, r, b'(\kappa), d'(\kappa))$ for large time delay since

$$\mu^*(m, b'(\kappa), d'(\kappa)) = \pi\sqrt{\frac{Dm\kappa^{m-1}}{-b'(\kappa) - d'(\kappa)}} > \sqrt{\frac{2Dm\kappa^{m-1}\omega_\kappa}{b'(\kappa)}}e^{\frac{\omega_\kappa}{2}} = \mu_\kappa(m, b'(\kappa), d'(\kappa)),$$

according to the fact that $2|\omega_\kappa|e^{-|\omega_\kappa|} \le 2/e < \pi^2$ for all ω_κ. In fact, we show that $c^*(m, r, b'(\kappa), d'(\kappa)) \ge c_\kappa(m, r, b'(\kappa), d'(\kappa))$ for all cases in Lemma 5.3.

Proof of Theorem 2.5. We show that for functions $b(\cdot)$ and $d(\cdot)$ with some structure condition, the curves $c_0(m, r, b, d)$ intersect with $c^*(m, r, b'(\kappa), d'(\kappa))$, and then the sharp divergent semi-wavefront exists. In fact, $\mu^*(m, b'(\kappa), d'(\kappa))$ is a constant only dependent on the local property of $b'(\kappa)$ and converge to zero as $b'(\kappa) \to -\infty$. From the proof of Theorem 2.4, we see that $\mu_0(m, b, d)$ depends on the structure of $b(\cdot)$ and $d(\cdot)$ within $(0, \zeta_1)$, where ζ_1 is determined by the whole structure of $b(\cdot)$ and $d(\cdot)$ as shown in Fig. 1. The local variation of $b'(\kappa)$ has minor effect on $\mu_0(m, b, d)$ (if the change of $b'(\kappa)$ has no effect on ζ_1, then $\mu_0(m, b, d)$ is fixed). Hence, for functions $b(\cdot)$ with appropriate structure near 0 and suitable large $-b'(\kappa)$, there holds

$$0 < \mu^*(m, b'(\kappa), d'(\kappa)) < \mu_0(m, b(\cdot), d(\cdot)),$$

and further

$$0 < c^*(m, r, b'(\kappa), d'(\kappa)) < \dot{c}(m, r, b, d) \le c_0(m, r, b, d),$$

for large time delay according to the asymptotic behavior in Theorem 2.4 proved in Section 4, Proposition 5.1 and Proposition 5.2. □

Remark 5.2. The condition in Theorem 2.5 holds for many kinds of birth functions. If fact, from the proof of Theorem 2.4, we see that $\mu_0(m, b, d)$ depends on the structure of $b(\cdot)$ and $d(\cdot)$ within $(0, \zeta_1)$, where ζ_1 is determined by the whole structure of $b(\cdot)$ and $d(\cdot)$ as shown in Fig. 1. If the change of $b'(\kappa)$ has no effect on ζ_1, then $\mu_0(m, b, d)$ is fixed. Hence, for functions $b(\cdot)$ with appropriate structure near 0 and suitable large $-b'(\kappa)$, the sharp divergent semi-wavefront exists.

6. Classification of various waves

This section is devoted to the classification of various waves defined in Definition 2.2. We have already shown the existence of sharp type waves in Section 3, especially the sharp divergent waves in Section 5. Here, we prove the existence of smooth waves and then present the classification of wave profiles according to the wave speed and time delay.

6.1. Existence of smooth waves

We employ the Schauder's Fixed Points Theorem to show the existence of monotone and non-monotone waves of smooth type. Compared with the linear diffusion case ($m = 1$), both the comparison principle and the solvability of degenerate elliptic problem ($m > 1$) are not obvious.

We recall the comparison principle of degenerate diffusion equation on an unbounded domain.

Lemma 6.1 *(Comparison Principle, [36]). Let $\phi_1, \phi_2 \in C(\mathbb{R}; \mathbb{R})$ such that for $i = 1, 2$, $0 \le \phi_i \in L^\infty(\mathbb{R})$, $\phi_i^m \in W^{1,2}_{loc}$, $\phi_1(t) > 0$ for all $t \in \mathbb{R}$, $\phi_i(t)$ is increasing for $t \le t_0$ with some fixed $t_0 \in \mathbb{R}$, $\liminf_{t\to\pm\infty}(\phi_1(t) - \phi_2(t)) \ge 0$, $\liminf_{t\to+\infty} \phi_1(t) > 0$ and ϕ_i satisfies the following inequality*

$$c\phi_1'(t) - D(\phi_1^m(t))'' + d(\phi_1(t)) \ge c\phi_2'(t) - D(\phi_2^m(t))'' + d(\phi_2(t))$$

in the sense of distributions. Then $\phi_1(t) \ge \phi_2(t)$ for all $t \in \mathbb{R}$.

We show the following solvability and monotonicity of degenerate equations on an unbounded domain. This is needed to verify the monotone leading edges of the waves, no matter monotone wavefronts, oscillatory wavefronts or divergent semi-wavefronts.

Lemma 6.2. *Assume that $0 \le \psi(t) \in L^\infty(\mathbb{R}) \cap C(\mathbb{R})$, ψ is monotonically increasing on $(-\infty, t_0]$ for some $t_0 \in \mathbb{R}$, and $\psi(t) \ge \psi(t_0) > 0$ for all $t > t_0$, then the following degenerate elliptic equation*

$$\begin{cases} c\phi'(t) - D(\phi^m(t))'' + d(\phi(t)) = \psi(t), \quad t \in \mathbb{R}, \\ \lim\limits_{t\to-\infty} \phi(t) = 0, \\ 0 < d^{-1}(\liminf\limits_{t\to+\infty} \psi(t)) \le \liminf\limits_{t\to+\infty} \phi(t) \le \limsup\limits_{t\to+\infty} \phi(t) \le d^{-1}(\limsup\limits_{t\to+\infty} \psi(t)) < +\infty, \end{cases} \tag{6.1}$$

admits at least one solution $\phi(t)$ such that $0 \le \phi(t) \in L^\infty(\mathbb{R})$, ϕ is monotonically increasing on $(-\infty, t_0]$, and $\phi(t) \ge \phi(t_0) > 0$ for all $t > t_0$.

Proof. Consider the following regularized problem for any $A > \max\{1, t_0\}$ with $-A < t_0$

$$\begin{cases} c\phi'(t) = D\big(m(|\phi(t)|^2 + 1/A)^{(m-1)/2}\phi'(t)\big)' - d(\phi(t)) + \psi(t), \quad t \in (-A, A), \\ \phi(-A) = d^{-1}(\psi(-A)), \quad \phi(A) = d^{-1}(\psi(A)). \end{cases} \tag{6.2}$$

The unique existence of solution to (6.2) is trivial. The solution is denoted by ϕ_A. We note that $d(s)$ is monotonically increasing and $\psi(t) \ge \psi(-A)$ for all $t \ge -A$ since $-A < t_0$ and $\psi(t)$ is increasing on $(-\infty, t_0)$. Comparison principle of elliptic equation shows that

$$0 < d^{-1}(\psi(-A)) \le \phi_A(t) \le d^{-1}(\sup \psi), \quad t \in (-A, A).$$

In fact, if this is not true, we argue by contradiction. If there exists $t_0 \in (-A, A)$ such that $\phi_A(t_0) < d^{-1}(\psi(-A))$, then the minimum of $\phi_A(t)$ on $[-A, A]$ is less than $d^{-1}(\psi(-A))$ and is attained at some inner point $t^* \in (-A, A)$ since at the endpoints $\phi_A(\pm A) \ge d^{-1}(\psi(-A))$. At this point t^*, $\phi_A'(t^*) = 0$, $\phi_A''(t^*) \ge 0$, and by (6.2)

$$\psi(t^*) = c\phi_A'(t^*) - D\big(m(|\phi_A(t^*)|^2 + 1/A)^{(m-1)/2}\phi_A'(t^*)\big)' + d(\phi_A(t^*)) < \psi(-A),$$

which contradicts to the fact $\psi(t) \ge \psi(-A)$ for all $t \in [-A, A]$. The proof of $\phi_A(t) \le d^{-1}(\sup \psi)$ is similar.

We assert that $\phi'_A(t) \geq 0$ for $t \in [-A, t_0]$. Otherwise, there exists a $t_* \in (-A, t_0)$ such that $\phi'_A(t_*) < 0$. Let (t_1, t_2) be the maximal interval such that $t_* \in (t_1, t_2)$ and $\phi'_A(t) < 0$ for $t \in (t_1, t_2)$. We note that $\phi_A(t)$ attains its minimum at $-A$, which implies $\phi'_A(-A) \geq 0$. Thus, $t_1 \in [-A, t_*)$, $\phi'_A(t_1) = 0$,

$$(m(|\phi_A(t)|^2 + 1/A)^{(m-1)/2}\phi'_A(t))'\big|_{t=t_1} \leq 0,$$

and

$$\psi(t_1) = c\phi'_A(t_1) - D\big(m(|\phi_A(t_1)|^2 + 1/A)^{(m-1)/2}\phi'_A(t_1)\big)' + d(\phi_A(t_1)) \geq d(\phi_A(t_1)),$$

which shows

$$\phi_A(t_1) \leq d^{-1}(\psi(t_1)) \leq d^{-1}(\psi(t_*)) \leq d^{-1}(\psi(t_0)) \leq d^{-1}(\psi(A)) = \phi_A(A)$$

as $t_1 \leq t_* < t_0 < A$, $\psi(t)$ is increasing on $(-\infty, t_0]$ and $\psi(t) > \psi(t_0) > 0$ for all $t > t_0$. Therefore, $\phi_A(A) = d^{-1}(\psi(A)) \geq \phi_A(t_1)$ and $\phi_A(t)$ cannot always decrease on the whole (t_1, A). Then $t_2 < A$ and $\phi'_A(t_2) = 0$, $\phi_A(t_1) > \phi_A(t_2)$,

$$(m(|\phi_A(t)|^2 + 1/A)^{(m-1)/2}\phi'_A(t))'\big|_{t=t_2} \geq 0,$$

and

$$\begin{aligned}\psi(t_1) &= c\phi'_A(t_1) - D\big(m(|\phi_A(t_1)|^2 + 1/A)^{(m-1)/2}\phi'_A(t_1)\big)' + d(\phi_A(t_1)) \\ &> c\phi'_A(t_2) - D\big(m(|\phi_A(t_2)|^2 + 1/A)^{(m-1)/2}\phi'_A(t_2)\big)' + d(\phi_A(t_2)) \\ &= \psi(t_2), \quad t_1 < t_2,\end{aligned}$$

which contradicts to the monotonically increasing of ψ on $(-\infty, t_0)$ and $\psi(t) \geq \psi(t_0) > 0$ for all $t > t_0$.

Next, we show that $\phi_A(t) \geq \phi_A(t_0) > 0$ for all $t > t_0$. Otherwise, there exists a number $t_1 \in (t_0, A)$ such that $\phi_A(t_1) < \phi_A(t_0)$. Noticing that $\phi_A(t)$ is increasing on $(-A, t_0)$, we see that there exists a maximum point $t^* \in [t_0, t_1)$. Similar to the above analysis at this point t^*, we find that

$$\phi_A(A) \geq \phi_A(t^*) \geq \phi_A(t_0) > \phi_A(t_1)$$

and $\phi_A(t)$ cannot decrease on the whole (t^*, A). Then there exist $t_a \in (t^*, t_1)$ and $t_b \in (t_1, A)$ such that $\phi_A(t_a) = \phi_A(t_b) = \phi_A(t_0)$ and $\phi_A(t)$ satisfies

$$\begin{cases} c\phi'(t) = D\big(m(|\phi(t)|^2 + 1/A)^{(m-1)/2}\phi'(t)\big)' - d(\phi(t)) + \psi(t), & t \in (t_a, t_b), \\ \phi(t_a) = \phi_A(t_0), \quad \phi(t_b) = \phi_A(t_0). \end{cases}$$

Applying the maximum principle of elliptic equations with $\psi(t) \geq \psi(t_0)$ for all $t > t_0$, we find that $\phi_A(t) \geq \phi_A(t_0)$ for $t \in (t_a, t_b)$, which contradicts to $t_1 \in (t_a, t_b)$ and $\phi_A(t_1) < \phi_A(t_0)$.

For any $1 < B < A$, let $\eta(t)$ be the cut-off function such that $0 \leq \eta(t) \leq 1$, $\eta \in C_0^2((-B, B))$, $|\eta'(t)| \leq 2$ for $t \in (-B, B)$, $\eta(t) = 1$ for $t \in (-B+1, B-1)$. Multiply (6.2) by $\eta^2(t)\phi_A(t)$ and integrate over $(-A, A)$, we have

$$\int_{-A}^{A} c\eta^2\phi_A(t)\phi_A'(t)dt + \int_{-A}^{A} Dm\eta^2(|\phi_A(t)|^2+1/A)^{(m-1)/2}|\phi_A'(t)|^2dt$$
$$+\int_{-A}^{A} \eta^2 d(\phi_A(t))\phi_A(t)dt$$
$$\le \int_{-A}^{A} 2Dm\eta(|\phi_A(t)|^2+1/A)^{(m-1)/2}\phi_A(t)\phi_A'(t)|\eta'(t)|dt + \int_{-A}^{A}\eta^2\phi_A(t)\psi(t)dt$$
$$\le \frac{1}{2}\int_{-A}^{A} Dm\eta^2(|\phi_A(t)|^2+1/A)^{(m-1)/2}|\phi_A'(t)|^2dt$$
$$+\int_{-A}^{A} 2Dm(|\phi_A(t)|^2+1/A)^{(m-1)/2}|\phi_A(t)|^2|\eta'(t)|^2dt + 2d^{-1}(\sup\psi)\sup\psi B.$$

Therefore,

$$\frac{1}{2}\int_{-B+1}^{B-1} Dm(|\phi_A(t)|^2+1/A)^{(m-1)/2}|\phi_A'(t)|^2dt + \int_{-B+1}^{B-1} d(\phi_A(t))\phi_A(t)dt$$
$$\le \int_{-B}^{-B+1} + \int_{B-1}^{B} 2Dm(|\phi_A(t)|^2+1/A)^{(m-1)/2}|\phi_A(t)|^2|\eta'(t)|^2dt + 2d^{-1}(\sup\psi)\sup\psi B$$
$$\le 16Dm((\sup\psi)^2+1)^{(m-1)/2}(\sup\psi)^2 + 2d^{-1}(\sup\psi)\sup\psi B.$$

It follows that $\|\phi_A^m\|_{W^{1,2}(-B+1,B-1)}$ is uniformly bounded and independent of A. We note that the embedding $W^{1,2}(-B+1,B-1)$ into $C^\gamma([-B+1,B-1])$ with $\gamma\in(0,\frac{1}{2})$ is compact, and $\phi_A^m\in C^\gamma([-B+1,B-1])$ implies $\phi_A\in C^{\gamma/m}([-B+1,B-1])$. There exist a subsequence of $\{\phi_A(t)\}_{A>1}$ denoted by $\{\phi_{A_n}(t)\}_{n\in\mathbb{N}}$ and a function $\phi(t)\in C^{\gamma/m}(\mathbb{R})$ such that $\phi^m\in W^{1,2}_{\mathrm{loc}}(\mathbb{R})$, $0\le\phi\le K$, and $\phi_{A_n}(t)$ uniformly converges to $\phi(t)$ on any compact interval, $\phi_{A_n}^m(t)$ weakly converges to $\phi^m(t)$ in $W^{1,2}_{\mathrm{loc}}(\mathbb{R})$. Since each $\phi_{A_n}(t)$ is monotonically increasing on $(-\infty,t_0)$, we see that $\phi(t)$ is also increasing on $(-\infty,t_0)$. We can verify that $\phi(t)$ is a solution of (6.1). □

The existence of traveling waves is deduced by Schauder fixed point theorem on an appropriate profile set Γ_ϵ constructed with upper and lower profiles ϕ^* and ϕ_ϵ for two auxiliary problems, where ϕ^*,ϕ_ϵ will be specified in the following. Inspired by Theorem 1.1 in [21], we construct two auxiliary degenerate diffusion equations with quasi-monotonicity. Since $b(\zeta_1)>0$, there is a small $\epsilon_0\in(0,\zeta_1)$ such that $b(\zeta_1-\epsilon)>0$ for every $\epsilon\in[0,\epsilon_0]$. If $b(s)$ satisfies (3.10), for any

$\epsilon \in (0, \epsilon_0)$, define two continuous functions as follows

$$b^*(u) = \begin{cases} \min\left\{b'(0)u, M\right\}, & u \in [0, \zeta_2], \\ \max\left\{M, b(u)\right\}, & u > \zeta_2, \end{cases}$$

and

$$b_\epsilon(u) = \begin{cases} \inf\limits_{\eta \in [u, \zeta_2]} \left\{b(\eta), d\left(\zeta_1 - \epsilon\right)\right\}, & u \in [0, \zeta_2], \\ \min\left\{b(u), d\left(\zeta_1 - \epsilon\right)\right\}, & u > \zeta_2. \end{cases}$$

If $b(s)$ satisfies the unimodality condition (2.3), then the above functions are simplified as

$$b^*(u) = \min\left\{b'(0)u, M\right\},$$

and

$$b_\epsilon(u) = \min\{b(u), d(\zeta_1 - \epsilon)\}.$$

According to the definition, we have

Lemma 6.3. *Both b^* and b_ϵ are continuous on $[0, +\infty)$ and monotonically increasing on $[0, \zeta_2]$; $b^*(s) \geq b(s) \geq b_\epsilon(s)$ for all $s \geq 0$; $b^*(\zeta_2) = d(\zeta_2) = M$ and $b^*(s) > d(s)$ for $s \in (0, \zeta_2)$; $b_\epsilon(\zeta_1 - \epsilon) = d(\zeta_1 - \epsilon) < d(\zeta_1)$ and $b_\epsilon(s) > d(s)$ for $s \in (0, \zeta_1 - \epsilon)$.*

Proof. The above statements are obvious and their proofs are omitted for the sake of simplicity. □

Consider the following two auxiliary delayed degenerate diffusion equations

$$w_t(t, x) = D(w^m)_{xx}(t, x) - d(w(t, x)) + b^*(w(t - r, y)), \tag{6.3}$$

and

$$w_t(t, x) = D(w^m)_{xx}(t, x) - d(w(t, x)) + b_\epsilon(w(t - r, y)). \tag{6.4}$$

The wave equations corresponding to (6.3) and (6.4) are

$$cU'(t) - DU^{m''}(t) + d(U(t)) - b^*(U(t - cr)) = 0, \tag{6.5}$$

and

$$cU'(t) - DU^{m''}(t) + d(U(t)) - b_\epsilon(U(t - cr)) = 0. \tag{6.6}$$

We note that the characteristic functions of (6.5) and (6.6) near 0 are identical to (5.1), i.e., the characteristic function of (2.1) near 0. However, we will show that the critical wave speed is not determined by this characteristic function near 0.

Now we recall the existence of monotone traveling wavefronts for the above two auxiliary degenerate diffusion equations with time delay.

Lemma 6.4. *For any given $m > 1$, $D > 0$ and $r \geq 0$, there exist a constant $\hat{c}(m, r, b^*, d)$ (depending on m, r and the structure of $b^*(\cdot), d(\cdot)$) and a constant $\hat{c}(m, r, b_\epsilon, d)$ (depending on m, r and the structure of $b_\epsilon(\cdot), d(\cdot)$) such that (6.5) and (6.6) admit monotonically increasing wavefronts $\phi^*(t)$ and $\phi_\epsilon(t)$ for $c_1 > \hat{c}(m, r, b^*, d)$ and $c_2 > \hat{c}(m, r, b_\epsilon, d)$, respectively, with $0 < \phi^*(t) < \zeta_2$, $0 < \phi_\epsilon(t) < \zeta_1 - \epsilon$,*

$$|\phi^*(t) - \zeta_2 e^{\lambda_{c_1} t}| \leq C^* e^{\Lambda_{c_1} t}, \quad |\phi_\epsilon(t) - (\zeta_1 - \epsilon)e^{\lambda_{c_2} t}| \leq C_\epsilon e^{\Lambda_{c_2} t}, \quad t < 0, \tag{6.7}$$

where $\lambda_{c_1}, \lambda_{c_2} > 0$ are the unique roots of $\chi_0(\lambda) = 0$ corresponding to c_1 and c_2 respectively, (χ_0 is defined in (5.1)) and $\Lambda_{c_i} > \lambda_{c_i}$ for $i = 1, 2$, $C^*, C_\epsilon > 0$ are constants. Moreover, C_ϵ is uniformly bounded from above and $\hat{c}(m, r, b_\epsilon, d)$ is uniformly bounded from above and below with respect to $\epsilon \in (0, \epsilon_0)$.

Proof. The existence result for degenerate diffusion equations with small time delay and monotone birth function is proved in Theorem 2.4 of [36]. We point out that the restriction of the smallness of time delay lies in the proof of the positive lower bound of the critical wave speed, i.e., Lemma 3.11 of [36]. Here in the proof of Lemma 4.1, we develop the approach further such that it is applicable to all the time delay. Therefore, the result of Theorem 2.4 in [36] is valid for all time delay as we proved here in Lemma 4.1. We apply this conclusion to the two auxiliary degenerate diffusion equations with time delay separately. The dependence of C_ϵ with respect to ϵ is shown by the construction of the upper and lower solutions in [36], which is uniformly bounded since $\zeta_1 - \epsilon$ is uniformly bounded for $\epsilon \in (0, \epsilon_0)$. According to Lemma 4.1, we see that $\hat{c}(m, r, b_\epsilon, d)$ is uniformly bounded from below with respect to ϵ since the structure of the family of functions b_ϵ near zero is the same. The uniform upper bound of $\hat{c}(m, r, b_\epsilon, d)$ with respect to ϵ is trivial as we can construct an upper solution with suitably large wave speed. The proof is completed. □

Proof of Theorem 2.6. For any given $c > \max\{\hat{c}(m, r, b^*, d), \hat{c}(m, r, b_\epsilon, d)\}$, let $\phi^*(t)$ and $\phi_\epsilon(t)$ be the monotonically increasing wavefronts of (6.5) and (6.6), respectively, corresponding to the same wave speed c. According to (6.7),

$$|\phi^*(t) - \zeta_2 e^{\lambda t}| \leq C e^{\Lambda t}, \quad |\phi_\epsilon(t) - (\zeta_1 - \epsilon)e^{\lambda t}| \leq C e^{\Lambda t}, \quad t < 0,$$

for $\Lambda > \lambda$ and $C > 0$ with $\lambda > 0$ being the unique root of $\chi_0(\lambda) = 0$ corresponding to c. We may assume that

$$\phi_*(t) \geq \phi_\epsilon(t), \qquad \text{for all } t \in \mathbb{R}.$$

Otherwise, let $t_0 < 0$ be sufficiently small such that

$$\phi^*(t) \geq \frac{\zeta_2 + \zeta_1}{2} e^{\lambda t} \geq \phi_\epsilon(t), \ \forall t < t_0, \tag{6.8}$$

and choose t_1 such that $\phi^*(t_1) \geq \frac{\zeta_2+\zeta_1}{2}$. Then we shift $\phi_\epsilon(t)$ to $\phi_\epsilon(t - \max\{t_1 - t_0, 0\})$. Therefore,

$$\phi_\epsilon(t - \max\{t_1 - t_0, 0\}) \leq \frac{\zeta_2 + \zeta_1}{2} \leq \phi^*(t), \quad \forall t \geq t_1,$$

and

$$\phi_\epsilon(t-\max\{t_1-t_0,0\}) \le \phi^*(t-\max\{t_1-t_0,0\}) \le \phi^*(t), \quad \forall t<t_1,$$

according to (6.8) and the monotonicity of $\phi^*(t)$. We replace $\phi_\epsilon(t)$ by $\phi_\epsilon(t-\max\{t_1-t_0,0\})$.

Define

$$H^*[\phi](t)=b^*(\phi(t-cr)), \quad t\in\mathbb{R},$$

and

$$H_\epsilon[\phi](t)=b_\epsilon(\phi(t-cr)), \quad t\in\mathbb{R},$$

then for any $\phi,\psi\in C(\mathbb{R},[0,\zeta_2])$ with $\phi(t)\ge\psi(t), t\in\mathbb{R}$, we have

$$H^*[\phi](t)\ge H^*[\psi](t) \quad \text{and} \quad H_\epsilon[\phi](t)\ge H_\epsilon[\psi](t) \quad \text{for all } t\in\mathbb{R},$$

since b^* and b_ϵ are monotonically increasing on $[0,\zeta_2]$. Set

$$\Gamma_\epsilon := \Big\{\phi\in C(\mathbb{R};\mathbb{R});\, \phi_\epsilon(t)\le\phi(t)\le\phi^*(t),\ \phi(t) \text{ is monotonically increasing on } (-\infty,t_\Gamma], \text{and } \phi(t)\ge\phi(t_\Gamma) \text{ for all } t>t_\Gamma\Big\}, \tag{6.9}$$

where $t_\Gamma\in\mathbb{R}$ is a fixed constant such that

$$0<\delta_0(\zeta_1-\epsilon_0)\le\phi_\epsilon(t_\Gamma)\le\phi^*(t_\Gamma)\le\phi^*(t_\Gamma+cr)<\zeta_1$$

with $\delta_0\in(0,1/2)$ being sufficiently small. We note that $\phi_\epsilon(t)$ is depending on ϵ, but the constants in (6.7) can be selected independent of ϵ in Lemma 6.4, and so is δ_0. Then we see that Γ_ϵ is nonempty and convex in $\mathscr{E}$, where $\mathscr{E}$ is the linear space $C^b_{\text{unif}}(\mathbb{R})$ endowed with the norm

$$\|\phi\|_* = \sum_{n=1}^{\infty}\frac{1}{2^n}\|\phi\|_{L^\infty([-n,n])}.$$

For any $\psi(t)\in\Gamma_\epsilon$, we solve the following degenerate equation

$$\begin{cases} c\phi'(t)-D(\phi^m(t))''+d(\phi(t))=b(\psi(t-cr)), \quad t\in\mathbb{R}, \quad \lim_{t\to-\infty}\phi(t)=0,\\ 0<d^{-1}(\liminf_{t\to+\infty} b(\psi(t)))\le\liminf_{t\to+\infty}\phi(t)\\ \qquad\le\limsup_{t\to+\infty}\phi(t)\le d^{-1}(\limsup_{t\to+\infty} b(\psi(t)))<+\infty. \end{cases} \tag{6.10}$$

Denote

$$\hat{\psi}(t):=H[\psi](t):=b(\psi(t-cr)).$$

Since $\psi(t-cr)$ is increasing on $(-\infty,t_\Gamma+cr)$, $\psi(t)\le\phi^*(t)\le\zeta_1$ for all $t\le t_\Gamma$, and $b(s)$ is increasing for $s\in[0,\zeta_1]$, we see that $\hat{\psi}(t)$ is monotonically increasing on $(-\infty,t_\Gamma]$ and $\hat{\psi}(t)\ge\hat{\psi}(t_\Gamma)$ for all $t>t_\Gamma$. According to Lemma 6.2, (6.10) admits a solution $\phi(t)$ such that $\phi(t)$ is monotonically increasing on $(-\infty,t_\Gamma]$ and $\phi(t)\ge\phi(t_\Gamma)$ for all $t>t_\Gamma$. Define $F^*:\Gamma_\epsilon\to C(\mathbb{R},[0,\zeta_2])$ by $F^*(\psi)=\phi$ with $\phi(t)$ being the solution of (6.10) corresponding to $\psi(t)\in\Gamma_\epsilon$.

We need to prove that $F^*(\Gamma_\epsilon) \subset \Gamma_\epsilon$. For any $\psi(t) \in \Gamma_\epsilon$, we have $\phi_\epsilon(t) \le \psi(t) \le \phi^*(t)$, then

$$H[\psi](t-cr) \le H^*[\psi](t-cr) \le H^*[\phi^*](t-cr),$$

and

$$\begin{cases} c\phi^{*\prime}(t) - D(\phi^{*m}(t))'' + d(\phi^*(t)) \ge c\phi'(t) - D(\phi^m(t))'' + d(\phi(t)), \quad t \in \mathbb{R}, \\ \liminf\limits_{t\to-\infty}(\phi^*(t) - \phi(t)) = \liminf\limits_{t\to-\infty}\phi^*(t) - \liminf\limits_{t\to-\infty}\phi(t) = 0, \\ \limsup\limits_{t\to+\infty}(\phi^*(t) - \phi(t)) \ge \lim\limits_{t\to+\infty}\phi^*(t) - \limsup\limits_{t\to-\infty}\phi(t) \\ \qquad \ge \zeta_2 - d^{-1}(\limsup\limits_{t\to+\infty} b(\psi(t))) \ge \zeta_2 - d^{-1}(\limsup\limits_{t\to+\infty} b^*(\phi^*(t))) = 0, \end{cases}$$

since $\phi(t)$ and $\phi^*(t)$ are solutions of (6.10) and (6.5). Applying the comparison principle Lemma 6.1, we find $\phi(t) \le \phi^*(t)$ for all $t \in \mathbb{R}$. In a similar way, the property $\phi(t) \ge \phi_\epsilon(t)$ follows from the comparison principle Lemma 6.1 and the inequality

$$H[\psi](t-cr) \ge H_\epsilon[\psi](t-cr) \ge H_\epsilon[\phi_\epsilon](t-cr).$$

From the proof of Lemma 6.2, we see that the solutions $\phi(t)$ of (6.10) are uniformly bounded in $C^\alpha([-n,n])$ with some $\alpha \in (0, 1/(2m))$, $\phi^m(t)$ are uniformly bounded in $W^{1,2}([-n,n])$ for any compact interval $[-n,n]$, and $\phi(t)$ are uniformly bounded in $L^\infty(\mathbb{R})$. According to the definition of the function space $\mathscr{E}$, $F^*(\Gamma_\epsilon)$ is compact in $\mathscr{E}$. By the Schauder's fixed point theorem, it follows that F^* has a fixed point U in $\Gamma_\epsilon \subset \mathscr{E}$, which satisfies

$$cU'(t) - DU^{m\prime\prime}(t) + d(U(t)) - b(U(t-cr)) = 0,$$

and

$$\phi_\epsilon(t) \le U(t) \le \phi^*(t) \quad \text{for all} \quad t \in \mathbb{R}. \tag{6.11}$$

Moreover, $U(-\infty) = 0$ and

$$\zeta_1 - \epsilon \le \liminf_{t\to+\infty} U(t) \le \limsup_{t\to+\infty} U(t) \le \zeta_2.$$

Since $U(t)$ is independent of ϵ, taking the limit as $\epsilon \to 0^+$, we have

$$\zeta_1 \le \liminf_{t\to+\infty} U(t) \le \limsup_{t\to+\infty} U(t) \le \zeta_2.$$

The proof is completed. □

6.2. *Classification of wave profiles*

To conclude, the time delay r and the degenerate diffusion have a strong influence on the geometry of wave profiles. Here, we depict the shape of the traveling waves characterized by the wave speed c and time delay r. From Theorem 2.6, we know that when the wave speed $c > \hat{c}$,

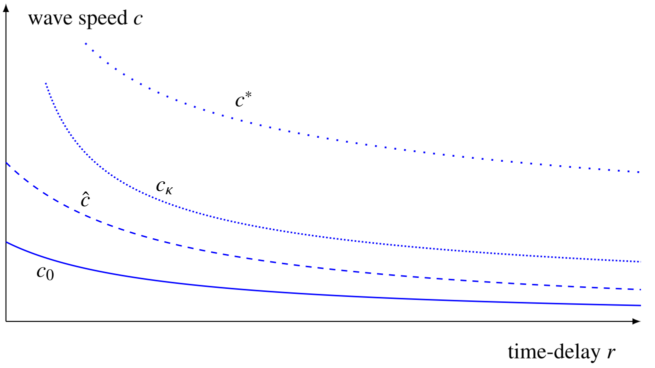

Fig. 5. Different types of traveling waves for the degenerate diffusion equation with time delay (1.1) correspond to time delay r and wave speed c: the case that the curve $c_0(m, r, b, d)$ never intersects with the curves $c_\kappa(m, r, b'(\kappa), d'(\kappa))$ and $c^*(m, r, b'(\kappa), d'(\kappa))$.

there exist smooth traveling wave solutions. Theorem 2.1 implies there exists a sharp traveling wave with the wave speed c_0. After investigate the geometry of leading edge, it is naturally to consider the convergence of the semi-wavefronts. So we have Theorem 5.1 and Theorem 5.2 indicating the oscillating properties for both the sharp type and smooth type traveling waves.

According to the above theorems, Fig. 5 and Fig. 6 illustrate two main possible sketches of the corresponding wave behaviors varying with the traveling wave speed c and time delay r. The critical lines of the wave speeds depend on the time delay and divide the (r, c) plane into several parts relating to different wave behaviors. The slopes and structures of these curves depend on the functions $b(\cdot)$ and $d(\cdot)$. It is worth to mention that there exist sharp-oscillating waves for some proper parameters, which is different from the former literatures (see Fig. 2 and Fig. 3).

In the case that the curve $c_0(m, r, b, d)$ never intersects with the curve $c_\kappa(m, r, b'(\kappa), d'(\kappa))$ and the curve $c^*(m, r, b'(\kappa), d'(\kappa))$ as illustrated in Fig. 5, we have the following different types of waves: the curve c_0 is the wave speed of sharp type traveling waves; the waves with the parameters (r, c) above the curve $\hat{c}$ are positive and smooth and the types (C1), (C2) and (C3) in Fig. 4 are possible; the waves with (r, c) above the curve c_κ are oscillatory; the waves with (r, c) above the curve c^* are non-decaying oscillatory. If $b'(\kappa) \geq 0$, then $c_\kappa(m, r, b'(\kappa), d'(\kappa)) = +\infty$ and $c^*(m, r, b'(\kappa), d'(\kappa)) = +\infty$, and the curve $c_0(m, r, b, d)$ never intersects with the curves $c_\kappa(m, r, b'(\kappa), d'(\kappa))$ and $c^*(m, r, b'(\kappa), d'(\kappa))$. Actually, for the monotonically increasing function $b(\cdot)$, the traveling waves are monotone.

Wave dynamics are rather complicated when the curve $c_0(m, r, b, d)$ or $\hat{c}(m, r, b, d)$ intersects with the curve $c_\kappa(m, r, b'(\kappa), d'(\kappa))$ or $c^*(m, r, b'(\kappa), d'(\kappa))$. It raises the possibility of nine types of traveling waves as shown in Fig. 2, Fig. 3 and Fig. 4. Fig. 6 shows the case that the curve $c_0(m, r, b, d)$ intersects with the curves $c_\kappa(m, r, b'(\kappa), d'(\kappa))$ and $c^*(m, r, b'(\kappa), d'(\kappa))$ at r_κ and r^* respectively. In this situation, many types of waves occur depending on the wave speed c, the time delay r and the degeneracy m as follows:

(i) if the degeneracy is strong with $m \geq 2$, then along the curve $c_0(m, r, b, d)$, the non-C^1 sharp type wave is monotone (A1) for small time delay or non-monotone (A2) if $r > r_\kappa$ or non-decaying oscillatory (A3) if $r > r^*$;

(ii) if the degeneracy is weak with $1 < m < 2$, then along the curve $c_0(m, r, b, d)$, the sharp waves are C^1 type, that is, (A1), (A2), (A3) are replaced by (B1), (B2) and (B3);

(iii) the waves with the parameters (r, c) above the curve $\hat{c}(m, r, b, d)$ are positive and smooth, that is, (C1), (C2), (C3) are possible if the time delay is small;

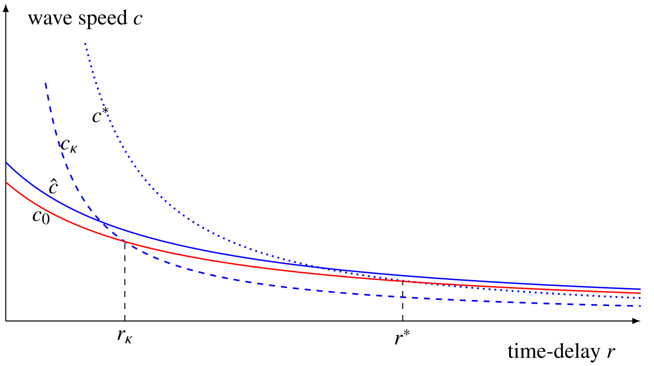

Fig. 6. Different types of traveling waves for the degenerate diffusion equation with time delay (1.1) correspond to time delay r and wave speed c: the case that the curve $c_0(m, r, b, d)$ intersects with the curves $c_\kappa(m, r, b'(\kappa), d'(\kappa))$ and $c^*(m, r, b'(\kappa), d'(\kappa))$ at r_κ and r^* respectively.

(iv) after the curve $\hat{c}(m, r, b, d)$ intersects with $c_\kappa(m, r, b'(\kappa), d'(\kappa))$ (it happens if $\dot{c}(m, r, b, d)$ intersects with $c_\kappa(m, r, b'(\kappa), d'(\kappa))$ since $\hat{c} \geq \dot{c}$), the monotone waves are impossible, that is, only (C2) and (C3) of smooth type exist;

(v) after the curve $\hat{c}(m, r, b, d)$ intersects with $c^*(m, r, b'(\kappa), d'(\kappa))$ (it happens if $\dot{c}(m, r, b, d)$ intersects with $c^*(m, r, b'(\kappa), d'(\kappa))$), the smooth wave has to develop non-decaying oscillations, that is, only (C3) of the smooth type exists.

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References

[1] M. Alfaro, A. Ducrot, T. Giletti, Travelling waves for a non-monotone bistable equation with delay: existence and oscillations, Proc. Lond. Math. Soc. 116 (2018) 729–759.
[2] D.G. Aronson, Density-dependent interaction-diffusion systems, in: W.E. Stewart, W.H. Ray, C.C. Conley (Eds.), Dynamics and Modelling of Reactive Systems, Academic Press, New York, 1980, pp. 161–176.
[3] E.A. Carl, Population control in Arctic ground squirrels, Ecology 52 (1971) 395–413.
[4] I.L. Chern, M. Mei, X. Yang, Q. Zhang, Stability of non-monotone critical traveling waves for reaction-diffusion equations with time-delay, J. Differ. Equ. 259 (2015) 1503–1541.
[5] A. De Pablo, J.L. Vàzquez, Travelling waves and finite propagation in a reaction-diffusion equation, J. Differ. Equ. 93 (1991) 19–61.
[6] T. Faria, S. Trofimchuk, Nonmonotone travelling waves in a single species reaction-diffusion equation with delay, J. Differ. Equ. 228 (2006) 357–376.

[7] B.H. Gilding, R. Kersner, A Fisher/KPP-type equation with density-dependent diffusion and convection: travelling-wave solutions, J. Phys. A 38 (2005) 3367–3379.
[8] A. Gomez, S. Trofimchuk, Global continuation of monotone wavefronts, J. Lond. Math. Soc. 89 (2014) 47–68.
[9] W.S.C. Gurney, R.M. Nisbet, The regulation of inhomogeneous population, J. Theor. Biol. 52 (1975) 441–457.
[10] S.A. Gourley, Travelling front solutions of a nonlocal Fisher equation, J. Math. Biol. 41 (2000) 272–284.
[11] S.A. Gourley, Y. Kuang, A stage structured predator-prey model and its dependence on maturation delay and death rate, J. Math. Biol. 49 (2004) 188–200.
[12] J.K. Hale, S.M. Verduyn Lunel, Introduction to Functional Differential Equations, Applied Mathematical Sciences, Springer-Verlag, 1993.
[13] R. Huang, C. Jin, M. Mei, J. Yin, Existence and stability of traveling waves for degenerate reaction-diffusion equation with time delay, J. Nonlinear Sci. 28 (2018) 1011–1042.
[14] H.J. Hupkes, S.M. Verduyn Lunel, Analysis of Newton's method to compute travelling wave solutions to lattice differential equations, Technical Report 2003-09, Mathematical Institute Leiden, 2003.
[15] Y. Kuang, S.A. Gourley, Wavefronts and global stability in a time-delayed population model with stage structure, Proc. R. Soc. Lond. Ser. A 459 (2003) 1563–1579.
[16] M.K. Kwong, C. Ou, Existence and nonexistence of monotone traveling waves for the delayed Fisher equation, J. Differ. Equ. 249 (2010) 728–745.
[17] M.A. Lewis, B. Li, H.F. Weinberger, Spreading speed and linear determinacy for two-species competition models, J. Math. Biol. 45 (2002) 219–233.
[18] G. Li, M. Mei, Y.S. Wong, Nonlinear stability of travelling wavefronts in an age-structured reaction-diffusion population model, Math. Biosci. Eng. 5 (2008) 85–100.
[19] W.T. Li, S. Ruan, Z.C. Wang, On the diffusive Nicholson's blowflies equation with nonlocal delay, J. Nonlinear Sci. 17 (2007) 505–525.
[20] C.K. Lin, C.T. Lin, Y. Lin, M. Mei, Exponential stability of nonmonotone traveling waves for Nicholson's blowflies equation, SIAM J. Math. Anal. 46 (2014) 1053–1084.
[21] S. Ma, Traveling waves for non-local delayed diffusion equations via auxiliary equations, J. Differ. Equ. 237 (2007) 259–277.
[22] J. Mallet-Paret, The Fredholm alternative for functional differential equations of mixed type, J. Dyn. Differ. Equ. 11 (1999) 1–48.
[23] E. Matthysen, Density-dependent dispersal in birds and mammals, Ecography 28 (2005) 403–416.
[24] M. Mei, C.K. Lin, C.T. Lin, J.W.-H. So, Traveling wavefronts for time-delayed reaction-diffusion equation: (i) local nonlinearity, J. Differ. Equ. 247 (2009) 495–510.
[25] M. Mei, J. So, M. Li, S. Shen, Asymptotic stability of travelling waves for Nicholson's blowflies equation with diffusion, Proc. R. Soc. Edinb., Sect. A, Math. 134 (2004) 579–594.
[26] M. Mei, K. Zhang, Q. Zhang, Global stability of critical traveling waves with oscillations for time-delayed reaction-diffusion equations, Int. J. Numer. Anal. Model. 16 (2019) 375–397.
[27] M. Morisita, Measuring of habitat value by "environmental density" method, in: G.P. Patil, E.C. Pielou, W.E. Waters (Eds.), Statistical Ecology, I: Spatial Patterns and Statistical Distributions, Pennsylvania State University Press, University Park, 1971.
[28] J.D. Murry, Mathematical Biology I: An Introduction, Springer, New York, USA, 2002.
[29] A. Okubo, S.A. Levin, Diffusion and Ecological Problems: Modern Perspectives, Springer Science Business Media, 2013.
[30] J.W.-H. So, J. Wu, X. Zou, A reaction-diffusion model for a single species with age structure travelling wavefronts on unbounded domains, Proc. R. Soc. Lond. Ser. A 2012 (2001) 1841–1853.
[31] L.L. Sullivan, B. Li, T.E.X. Miller, M.G. Neubert, A.K. Shaw, Density dependence in demography and dispersal generates fluctuating invasion speeds, Proc. Natl. Acad. Sci. 114 (19) (2017) 5053–5058.
[32] E. Trofimchuk, V. Tkachenko, S. Trofimchuk, Slowly oscillating wave solutions of a single species reaction-diffusion equation with delay, J. Differ. Equ. 245 (2008) 2307–2332.
[33] E. Trofimchuk, S. Trofimchuk, Admissible wavefront speeds for a single species reaction-diffusion equation with delay, Discrete Contin. Dyn. Syst., Ser. A 20 (2008) 407–423.
[34] J.L. Vàzquez, The Porous Medium Equation: Mathematical Theory, Oxford Univ. Press, 2006.
[35] Z. Wu, J. Zhao, J. Yin, H. Li, Nonlinear Diffusion Equations, World Scientific Publishing Co. Put. Ltd., 2001.
[36] T.Y. Xu, S.M. Ji, M. Mei, J.X. Yin, Traveling waves for time-delayed reaction diffusion equations with degenerate diffusion, J. Differ. Equ. 265 (2018) 4442–4485.
[37] T.Y. Xu, S.M. Ji, M. Mei, J.X. Yin, Variational approach of critical sharp front speeds in density-dependent diffusion model with time delay, Nonlinearity, https://doi.org/10.1088/1361-6544/ab801d, 2020.