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Global Stability of Sharp Traveling Waves for Combustion Model with Degenerate Diffusion

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Abstract
This paper is concerned with the global nonlinear stability of sharp traveling waves for combustion model with degenerate diffusion. We prove that the solutions to the Cauchy problem converge to the unique sharp traveling wave (with a shift depending on the initial data), even the perturbation can be allowed arbitrarily large within the admissible range. Note that the traveling wave and the solution are at most Hölder continuous, and their supports are semi-compact. The degenerate diffusion and the low regularity make the study nontrivial. The proof is based on the monotonicity principle to which we add some new aspects.

Keywords Sharp traveling wave · Nonlinear stability · Combustion nonlinearity · Degenerate diffusion

1 Introduction and Main Results

We consider the Cauchy problem for the following reaction diffusion equation with degenerate diffusion and combustion nonlinearity

$$u_t = (D(u)u_x)_x + f(u), \tag{1.1}$$

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where u is the renormalized temperature, $0 \le u \le 1$, the reaction term $f(u)$ satisfies

$$\begin{aligned} &\text{there exists } \theta \in (0,1), \text{ such that } f(u) = 0, \forall u \in [0,\theta] \cup \{1\}, \\ &f(u) > 0, \forall u \in (\theta, 1),\ f \in C([0,1]) \cap C^1([\theta,1]),\ \text{and } f'(1) < 0. \end{aligned} \tag{1.2}$$

If f is merely defined on $[0, 1]$, we extend $f(u) = f'(1)(u - 1)$ for $u > 1$; otherwise we further assume that $f \in C^1([\theta, +\infty))$, and $f'(u) < 0$ for $u > 1$. We refer to this source term as combustion nonlinearity and θ represents the ignition-temperature [1]. We assume the diffusion coefficient $D(u)$ is temperature-dependent, which is degenerate at $u = 0$. A natural choice in chemical engineering is $D(u) = mu^{m-1}$ with $m > 1$, the so called porous medium type diffusion [2].

In the pioneering work [3], Berestycki, Nicolaenko and Scheurer proposed the combustion model (1.1) describing the premixed Bunsen flame, where $f(u)$ is combustion-type, diffusion coefficients $D(u) \ge D_0 > 0$ is a smooth function of temperature u with strictly positive infimum. They assumed both heat conductivity and diffusion coefficients are temperature-dependent. Since then, several works have considered the combustion models of the form (1.1) with constant diffusion coefficients $D = D_0 > 0$. Bonnet and Hamel [4] established the existence result of two-dimensional V-shaped traveling fronts. Then Hamel, Monneau and Roquejoffre [5] proved the asymptotic stability of V-shaped traveling fronts. After that, Hamel and Monneau [6] studied the stability of conical traveling fronts in higher dimension. Recently, Bu and Wang [7–9] proved existence, uniqueness and asymptotic stability of three-dimensional pyramidal traveling fronts.

Degenerate diffusion equations arise in many natural phenomena, with applications to developmental biology, population dynamics, combustion theory, chemical reaction and fluid in porous media (see [2, 10–13]). The existence of traveling waves for degenerate diffusion equations has been well-studied in [12–22] for kinetics of Fisher-KPP (monostable) type and in [23] of Nagumo (bistable) type. Different from the mentioned works above, Gilding and Kernser [24] employed the integral equation approach instead of the phase plane analysis method to study the existence of traveling waves of the following equation

$$u_t = (D(u)u_x)_x + B(u)u_x + F(u)$$

with general diffusion, convection and reaction.

In the combustion case with degenerate diffusion, the dynamics of traveling waves are much richer, due to the degeneracy of both the diffusion and reaction at equilibrium point $u = 0$. However, the propagation properties of Cauchy problem for the combustion model (1.1) with degenerate diffusion are challenging and still remain open. In this paper, we first show that, due to the effect of degeneracy of diffusion, the traveling waves of (1.1) are sharp type (with semi-compact supports) for unique wave speed $c^* > 0$, and further prove the time asymptotic stability of the sharp traveling wave, which indicates the finite propagation property of solutions to Cauchy problems.

For the reaction diffusion equations with degenerate diffusion, the traveling waves usually loss the regularity due to the degeneracy. Stability results of traveling waves for degenerate diffusion equations are quite limited. In the case of degenerate diffusion equations with Fisher-KPP nonlinearity $f(u) = u(1 - u)$, Biró [25] first proved their nonlinear stability by upper and lower solutions method for the sharp fronts with critical wave speed $c = c^*$. For the non-critical traveling waves with $c > c^*$, Huang–Jin–Mei–Yin [26] and Liu–Mei–Yang [27] obtained the stability of these smooth non-critical traveling waves to the local/nonlocal degenerate diffusion equations with time-delay by the weighted energy method and the viscosity vanishing technique respectively. In [28], Leyva and Plaza established the spectral stability in exponentially weighted spaces of non-critical traveling waves for degenerate diffusion equations of Fisher-KPP type. After that, the spectral stability of traveling waves for degenerate diffusion equation with Nagumo (bistable) type reaction was studied in a companion paper [29]. Recently, Dalibard et al. [30] established the nonlinear stability of degenerate diffusion sharp traveling fronts in the porous medium case. However, the stability and the corresponding convergence rate of sharp traveling waves for the degenerate diffusion equations with combustion type source term remain open due to some technical issues. The effect of the degenerate diffusion Eq. (1.1) causes essential difficulty. In this paper, we are going to prove the nonlinear stability of sharp traveling waves for the combustion equations with degenerate diffusion (1.1), where the nonlinear diffusion function $D(u)$ vanishes at $u = 0$.

We are interested in the propagation properties of the Cauchy problem (1.1) of combustion model with porous medium type degenerate diffusion $D(u) = mu^{m-1}$ $(m > 1)$. The traveling wave solutions of (1.1) connecting the two equilibria 1 and 0 with the form $u(x, t) = \phi(\xi) := \phi(x - ct)$ satisfy the following degenerate elliptic equation

$$\begin{cases} c\phi' + (\phi^m)'' + f(\phi) = 0, \\ \phi(-\infty) = 1, \ \phi(+\infty) = 0. \end{cases} \tag{1.3}$$

It is equivalent to the case with $\hat{\xi} = x + ct$ such that the traveling wave propagates to the left with opposite status.

The existence of traveling wave is already known for the reaction diffusion equation (1.1) of porous medium type, which is unique up to translation for composition type reaction. See the reference book of Gilding and Kernser [24] employing the integral equation approach. The unique traveling wave is sharp type (with semi-compact support) for degenerate diffusion.

We summarize the properties of the sharp traveling waves of the combustion model (1.1) with degenerate diffusion.

Proposition 1.1 (Existence and properties of sharp traveling wave) *For the degenerate diffusion equation (1.1) with composition type reaction $f(u)$ satisfying (1.2) and porous medium type diffusion $D(u) = mu^{m-1}$ $(m > 1)$, there exists a constant $c^* = c^*(m, f) > 0$ such that*

(i) (1.1) admits a unique (up to translation) traveling wave solution $\phi(x - c^*t)$ connecting 1 and 0 with $\phi(-\infty) = 1$ and $\phi(+\infty) = 0$;

(ii) the traveling wave is of sharp type (with semi-compact support), i.e., there exists $\xi_0 \in \mathbb{R}$, such that $\phi(\xi) = 0$ for all $\xi \geq \xi_0$, and $0 < \phi(\xi) < 1$ with $\phi'(\xi) < 0$ for all $\xi < \xi_0$;

(iii) the optimal regularity is $\phi \in C^\alpha(\mathbb{R})$ (Hölder spaces) with $\alpha = \frac{1}{m-1}$, and

$$\phi(\xi) \in (\theta, 1), \xi < \xi_1, \quad \phi(\xi) = \begin{cases} \left(\dfrac{m-1}{m} c^* \cdot |\xi - \xi_0|\right)^{\frac{1}{m-1}}, & \xi \in [\xi_1, \xi_0], \\ 0, & \xi > \xi_0, \end{cases}$$

where $\xi_1 < \xi_0$ is such that $\phi(\xi_1) = \theta$.

Without loss of generality, we always assume that $\xi_0 = 0$. The Hölder spaces C^α are interpreted as Lipschitz spaces $C^{\alpha-1,1}$ for positive integer α, and are interpreted as Hölder spaces $C^{[\alpha],\{\alpha\}}$ for positive non-integer α.

We show that the solution to the Cauchy problem (1.1) of the combustion model with degenerate diffusion propagates in the form of sharp traveling wave $\phi(\xi)$ and also converges to the sharp traveling wave with a shift.

Theorem 1.1 (Stability of sharp traveling waves) *For any solution $u(x,t)$ of the Cauchy problem (1.1) with initial data u_0 such that*

$$u_0 \in L^\infty(\mathbb{R}),\, u_0(x) \geq 0,\, \operatorname{supp} u_0 \subset (-\infty, x_*] \text{ for some } x_* \in \mathbb{R},\, \liminf_{x\to-\infty} u_0(x) > \theta, \quad (1.4)$$

there exists a $x_0 \in \mathbb{R}$ such that $u(x,t)$ converges to $\phi(x - c^*t - x_0)$:

$$\lim_{t\to+\infty} \sup_{x\in\mathbb{R}} |u(x,t) - \phi(x - c^*t - x_0)| = 0, \quad (1.5)$$

where $\phi(x - c^*t)$ is the sharp traveling wave with critical wave speed c^* in Proposition 1.1 (with $\xi_0 = 0$).

Remark The condition $\liminf_{x\to-\infty} u_0(x) > \theta$ of the initial data in (1.4) is almost necessary. Otherwise for some initial data $u_0(x)$ such that $u_0(x) \leq \theta$ for all $x \in \mathbb{R}$, we can see that $u(x,t) \leq \theta$ for all $x \in \mathbb{R}$ and $t > 0$ according to the comparison principle with supper-solution $\bar{u}(x,t) \equiv \theta$. It follows that the solution does not converge to any traveling wave connecting 1 and 0.

Theorem 1.1 shows the development of the free boundary of the solution to Cauchy problem.

Corollary 1.1 *Let $u(x,t)$ be the solution of the Cauchy problem (1.1) with initial data u_0 satisfying (1.4), and let $\zeta(t)$ be the free boundary of the semi-compact support of the solution $u(x,t)$, i.e., $\zeta(t) := \sup\{x \in \mathbb{R}; u(x,t) > 0\}$. Then there exists an $x_0 \in \mathbb{R}$ such that*

$$\lim_{t\to+\infty} (\zeta(t) - c^*t) = x_0.$$

The approach adopted here is based on weak sub- and super-solutions with semi-compact supports, in order to characterize the motion of the free boundary of the solution. Note that both the solution $u(x,t)$ of the Cauchy problem (1.1) in Theorem 1.1 and the sharp traveling wave solutions with shifts $\bar{u}(x,t) = \phi(x - c^*t - x_0)$ have the distinct free boundaries $\zeta(t) := \sup\{x \in \mathbb{R}; u(x,t) > 0\}$ and $\bar{\zeta}(t) := \sup\{x \in \mathbb{R}; \bar{u}(x,t) > 0\} = x_0 + c^*t$, respectively. Our approach differs from the energy method such that it avoids the weak regularity of the solution $u(x,t)$ near the free boundary $\zeta(t)$ and the traveling wave solution $\bar{u}(x,t)$ near $\bar{\zeta}(t)$, where the solutions are at most Hölder continuous.

2 Proof of the Main Results

We first show the existence and properties of sharp traveling wave in Proposition 1.1.

Proof of Proposition 1.1. The existence results have already been proved by Gilding and Kernser [24] employing the integral equation approach for a large class of nonlinear diffusion equations with reaction and convection. Their results include the combustion model (1.1) with degenerate diffusion. The existence results can also be shown by studying the asymptotic behavior with a phase plane analysis, see the references [26, 31] for example. We note that for the combustion reaction $f(u)$ such that $f(u) \equiv 0$ for $u \in [0,\theta]$, an explicit formula of the traveling wave $\phi(\xi)$ is obtained locally for $\phi(\xi) \in [0,\theta]$. According to (1.3), $(\phi^m)' = -c\phi$ for $\phi \in [0,\theta]$, it follows that $(\phi^{m-1})' = -\frac{m-1}{m}c$ and then

$$\phi^{m-1}(\xi) - \phi^{m-1}(\xi_0) = -\frac{m-1}{m} c \cdot (\xi - \xi_0), \quad \xi < \xi_0,$$

where ξ_0 is the edge of the support of sharp traveling wave such that $\phi(\xi_0) = 0$ and $\phi(\xi) > 0$ for $\xi < \xi_0$. Other properties follow from the locally explicit formula and the asymptotic behaviors of $\phi(\xi)$. □

Note that both the unique sharp traveling wave $\phi(x - c^*t)$ and the solution $u(x, t)$ to the Cauchy problem are semi-compactly supported, and they are at most Hölder continuous at the free boundaries. Therefore the key step of the global L^∞ stability of sharp traveling wave is to estimate the evolution of the free boundary.

The characterization of the motion of the free boundary via weak sub- and super-solutions with semi-compact supports is developed by Biró [25] for degenerate diffusion equations of porous medium type $D(u) = mu^{m-1}$ and the Fisher-KPP type reaction $f(u) = u^p - u^q$ with $1 \le p < \min\{m, q\}$. Later, Xu et al. [32] extended the above method to degenerate diffusion equations with advection of Burgers type and a general Fisher-KPP type reaction term $f(u)$ satisfying the following growth condition

$$\sup_{s\in(0,1)} \frac{sf'(s)}{f(s)} < m. \tag{2.1}$$

However, the growth condition (2.1) fails for the combustion type reaction in this paper, which plays an important role in the construction of weak sub- and super-solutions with semi-compact supports in [32] (see Lemma 2.4 therein). Therefore, we construct weak solutions more carefully and take advantage of the combustion structure under the almost necessary condition $\liminf_{x\to-\infty} u_0(x) > \theta$ of the initial data in (1.4), which allows us to formulate convergence results without the growth condition (2.1).

The weak sub- and super-solutions with semi-compact supports are constructed based on the unique sharp traveling wave $\phi(x - c^*t)$ with variable amplitude and propagation speed, of the following form

$$W(x, t) := F(t) \cdot \phi(x - G(t)),$$

where $F(t)$ denotes the amplitude of the weak solution, and $G(t)$ characterize the evolution of the free boundary.

Lemma 2.1 ([25, 32]) *Let $F(t)$ and $G(t)$ be the solutions to the following ordinary differential system*

$$\begin{cases} F'(t) = \varepsilon_1 F(t)(1 - F(t)), \\ G'(t) = c^* F^{m-1}(t) - \varepsilon_2(1 - F(t)), \\ F(0) = F_0 > 0, \quad G(0) = G_0, \end{cases} \tag{2.2}$$

where ε_1 and ε_2 are positive constants. Then

(i) $\lim_{t\to+\infty} F(t) = 1$*, and the convergence rate is exponential;*

(ii) if $F_0 < 1$, then $F(t)$ is strictly increasing, while if $F_0 > 1$, then $F(t)$ is strictly decreasing, such that $F(t) \in [\min\{F_0, 1\}, \max\{F_0, 1\}]$;

(iii) $\lim_{t\to+\infty} G'(t) = c^*$*;*

(iv) there exists $x_0 \in \mathbb{R}$ such that $\lim_{t\to+\infty}(G(t) - c^*t) = x_0$*, and the convergence rate is exponential.*

For the sake of convenience, we fix two constants

$$F_1 \in (\theta, \liminf_{x\to-\infty} u_0(x)), \qquad F_2 > \sup_{x\in\mathbb{R}} u_0(x), \tag{2.3}$$

for any given initial data u_0 satisfying (1.4) in Theorem 1.1 since $\liminf_{x\to-\infty} u_0(x) > \theta$. Without loss of generality, we may assume that $F_1 < 1$ and $F_2 > 1$; otherwise we replace them by $\hat{F}_1 = \min\{F_1, 1-\varepsilon\}$ and $\hat{F}_2 = \max\{F_2, 1+\varepsilon\}$ with a small constant $\varepsilon > 0$.

The validity of $W(x,t) := F(t)\cdot\phi(x-G(t))$ being a weak sub- or super-solution lies in the verification of a differential inequality in the sense of distributions. In the previous works, for $f(u) = u^p - u^q$ with $1 \le p < \min\{m, q\}$ in [25] and for $f(u)$ satisfying the growth condition (2.1) in [32], the mono-stable Fisher-KPP type reaction plays an essential role such that $\frac{\lambda^m f(u) - f(\lambda u)}{(\lambda-1)u}$ admits a positive infimum. However, the above existence of positive infimum is not true for a combustion type reaction satisfying (1.2).

Denote

$$H(F,\varphi) := \frac{F^m f(\varphi) - f(F\varphi)}{F-1}, \qquad F \in [F_1, F_2]\backslash\{1\},\ \varphi \in [0,1]. \tag{2.4}$$

For combustion type reaction $f(u)$ satisfying (1.2), we see that $f'_-(\theta) = 0$ and $f'_+(\theta) \ge 0$ such that $H(F,\varphi)$ may have jump discontinuity for $\varphi = \theta$ and F near 1 since

$$\lim_{F\to 1^+} H(F,\theta) = -\theta f'_+(\theta), \qquad \lim_{F\to 1^-} H(F,\theta) = 0,$$

and then $H(F,\varphi)$ cannot be continuously extended to $F = 1$ and $\varphi \in [0,1]$.

Lemma 2.2 *For combustion type reaction $f(u)$ satisfying (1.2), and for any given $F_1 \in (\theta, 1)$ and $F_2 > 1$, there exists a positive constant $M_0 > 0$ such that*

$$\inf_{F\in[F_1,F_2]\backslash\{1\},\varphi\in[0,1]} H(F,\varphi) \ge -M_0, \qquad \inf_{F\in[F_1,F_2]\backslash\{1\},\varphi\in(0,1]} \frac{H(F,\varphi)}{\varphi} \ge -M_0.$$

Proof We rewrite the definition (2.4) of $H(F,\varphi)$ into the following equality

$$H(F,\varphi) = \frac{F^m f(\varphi) - f(F\varphi)}{F-1} = \frac{F^m - 1}{F-1} f(\varphi) - \frac{f(F\varphi) - f(\varphi)}{F-1}. \tag{2.5}$$

The first term in (2.5) is always nonnegative. It suffices to show that $\frac{f(F\varphi)-f(\varphi)}{F-1}$ is bounded above for the lower bound of $H(F,\varphi)$ and similarly the upper bound of $\frac{f(F\varphi)-f(\varphi)}{(F-1)\varphi}$ yields the lower bound of $\frac{H(F,\varphi)}{\varphi}$. For the combustion type reaction $f(u)$ satisfying (1.2), we see that $f(u) \equiv 0$ for $u \in [0,\theta]$ and $f \in C([0,1]) \cap C^1([\theta, F_2])$ with $f'_+(\theta) \ge 0$. Although it may happen that $f \notin C^1([0,F_2])$, we always have that f is absolutely continuous on $[0, F_2]$ and the almost everywhere derivative

$$\tilde{f}'(u) := \begin{cases} 0, & u \in [0,\theta), \\ f'(u), & u \in (\theta, F_2], \end{cases}$$

is the weak derivative (generalized derivative) in the sense of distributions. Hence the Newton-Leibniz formula is valid. It follows that

$$f(F\varphi) - f(\varphi) = f(\varphi + s(F-1)\varphi)\Big|_{s=0}^{s=1} = (F-1)\varphi \cdot \int_0^1 \tilde{f}'(\varphi + s(F-1)\varphi)\mathrm{d}s. \tag{2.6}$$

Therefore,

$$\begin{aligned} \frac{f(F\varphi) - f(\varphi)}{F-1} &= \varphi \cdot \int_0^1 \tilde{f}'(\varphi + s(F-1)\varphi)\mathrm{d}s \\ &\le \sup_{u\in[0,F_2]} \tilde{f}'(u) \le \sup_{u\in(\theta,F_2]} f'(u) \le \|f\|_{C^1([\theta,F_2])} =: M_0. \end{aligned}$$

Similarly, $\frac{f(F\varphi)-f(\varphi)}{(F-1)\varphi} \leq M_0$. □

Lemma 2.3 *For combustion type reaction $f(u)$ satisfying (1.2), and for any given $F_1 \in (\theta, 1)$ and $F_2 > 1$, there exist positive constants $\mu_1 \in (0, 1-\theta)$, $\lambda_1 \in (0, 1-\theta)$, and $\delta_1 > 0$, such that*

$$\inf_{F\in[1-\lambda_1,1+\lambda_1]\backslash\{1\},\varphi\in[1-\mu_1,1]} H(F,\varphi) \geq \delta_1 > 0.$$

Proof Since f is combustion type reaction satisfying (1.2), we see that $f \in C^1([\theta, F_2])$ with $f'(1) < 0$. There exists a positive constant $\delta_1^* > 0$ (it suffices to take $\delta_1^* = |f'(1)|/2$), and a neighborhood $[1-\mu_1^*, 1+\mu_1^*]$ with $\mu_1^* \in (0, 1-\theta)$ such that $f'(u) \leq -\delta_1^* < 0$ for all $u \in [1-\mu_1^*, 1+\mu_1^*]$.

We choose positive constants $\mu_1 \in (0, 1-\theta)$, $\lambda_1 \in (0, 1-\theta)$ such that both φ and $F\varphi$ lie in the neighborhood $[1-\mu_1^*, 1+\mu_1^*]$ for all $F \in [1-\lambda_1, 1+\lambda_1]$ and $\varphi \in [1-\mu_1, 1]$. For example, we take $\mu_1 = \mu_1^*/2$ and $\lambda_1 = \mu_1^*/2$, then $F\varphi \leq 1+\lambda_1 < 1+\mu_1^*$ and $F\varphi \geq (1-\lambda_1)(1-\mu_1) \geq (1-\mu_1^*/2)^2 > 1-\mu_1^*$.

According to (2.5) and (2.6), we have

$$\begin{aligned} H(F,\varphi) =& \frac{F^m-1}{F-1} f(\varphi) - \frac{f(F\varphi)-f(\varphi)}{F-1} \\ \geq& -\varphi \cdot \int_0^1 \tilde{f}'(\varphi + s(F-1)\varphi)\mathrm{d}s \\ \geq& -\varphi \cdot \sup_{u\in[\min\{\varphi,F\varphi\},\max\{\varphi,F\varphi\}]} \tilde{f}'(u) \\ \geq& (1-\mu_1)\delta_1^* =: \delta_1 > 0, \end{aligned} \tag{2.7}$$

for all $F \in [1-\lambda_1, 1+\lambda_1]\backslash\{1\}$ and $\varphi \in [1-\mu_1, 1]$. □

Lemma 2.4 *For combustion type reaction $f(u)$ satisfying (1.2), and for any given $F_1 \in (\theta, 1)$ and $F_2 > 1$, let $\mu_1 \in (0, 1-\theta)$ and $\lambda_1 \in (0, 1-\theta)$ be the constants in Lemma 2.3, then there exists a positive constant $\delta_2 > 0$, such that*

$$\inf_{F\in[1+\lambda_1,F_2],\varphi\in[1-\mu_1,1]} H(F,\varphi) \geq \delta_2 > 0.$$

Proof According to the proof of Lemma 2.3, $f'(u) \leq -\delta_1^* < 0$ for all $u \in [1-\mu_1^*, 1+\mu_1^*]$. By the assumptions on f we have $f \in C^1([\theta, F_2])$ and $f'(u) < 0$ for all $u \in [1, F_2]$. There exists a positive constant $\delta_2^* > 0$ (we may assume that $\delta_2^* \in (0, \delta_1^*]$) such that $f'(u) \leq -\delta_2^* < 0$ for all $u \in [1, F_2]$. Then for all $F \in [1+\lambda_1, F_2]$ and $\varphi \in [1-\mu_1, 1]$, both φ and $F\varphi$ lie in the interval $[1-\mu_1^*, F_2]$.

Similar to (2.7) in the proof of Lemma 2.3, we have

$$\begin{aligned} H(F,\varphi) \geq& -\varphi \cdot \int_0^1 \tilde{f}'(\varphi + s(F-1)\varphi)\mathrm{d}s \\ \geq& -\varphi \cdot \sup_{u\in[\min\{\varphi,F\varphi\},\max\{\varphi,F\varphi\}]} \tilde{f}'(u) \\ \geq& (1-\mu_1)\delta_2^* =: \delta_2 > 0. \end{aligned}$$

The proof is complete. □

Lemma 2.5 *For combustion type reaction $f(u)$ satisfying (1.2), and for any given $F_1 \in (\theta, 1)$ and $F_2 > 1$, let $\lambda_1 \in (0, 1-\theta)$ be the constant in Lemma 2.3, then there exist positive constants $\mu_2 \in (0, 1-\theta)$ and $\delta_3 > 0$, such that*

$$\inf_{F \in [F_1, 1-\lambda_1], \varphi \in [1-\mu_2, 1]} H(F, \varphi) \geq \delta_3 > 0.$$

Proof Note that $F_1 \in (\theta, 1)$ and $\lambda_1 > 0$, then $[F_1, 1-\lambda_1] \subset (\theta, 1)$. Take $F_1^* \in (\theta, F_1)$, such as $F_1 = (\theta + F_1)/2$, then $[F_1, 1-\lambda_1] \subset [F_1^*, 1-\lambda_1] \subset (\theta, 1)$. According to the combustion structure, there exists a positive constant $\delta_3^* > 0$ such that

$$\inf_{u \in [F_1^*, 1-\lambda_1]} f(u) \geq \delta_3^* > 0,$$

since f is positive and continuous on $(\theta, 1)$. Noticing that $f(1) = 0$ and $f'(1) < 0$, we see that there exists a positive constant $\mu_2^* > 0$ and $\mu_2^* < \lambda_1$, such that

$$\sup_{u \in [1-\mu_2^*, 1]} f(u) \leq \frac{\delta_3^*}{2}.$$

The last step is to take $\mu_2 \in (0, \mu_2^*)$ such that $F\varphi$ lies in the interval $[F_1^*, 1-\lambda_1]$ for all $F \in [F_1, 1-\lambda_1]$ and $\varphi \in [1-\mu_2, 1]$. It suffices to take $\mu_2 > 0$ sufficiently small such that $(1-\mu_2)F_1 > F_1^*$, which is possible since $F_1 > F_1^*$.

According to (2.5) and the above selection, we have

$$\begin{aligned} H(F, \varphi) &= \frac{F^m - 1}{F - 1} f(\varphi) - \frac{f(F\varphi) - f(\varphi)}{F - 1} \\ &\geq \frac{f(F\varphi) - f(\varphi)}{1 - F} \geq f(F\varphi) - f(\varphi) \\ &\geq \inf_{u \in [F_1^*, 1-\lambda_1]} f(u) - \sup_{u \in [1-\mu_2^*, 1]} f(u) \geq \frac{\delta_3^*}{2} =: \delta_3 > 0, \end{aligned}$$

for any $F \in [F_1, 1-\lambda_1]$ and $\varphi \in [1-\mu_2, 1]$. □

We conclude from the above Lemmas 2.3–2.5 the positive infimum of $H(F, \varphi)$ for φ sufficiently close to 1.

Lemma 2.6 *For combustion type reaction $f(u)$ satisfying (1.2), and for any given $F_1 \in (\theta, 1)$ and $F_2 > 1$, there exist positive constants $\mu_0 \in (0, 1-\theta)$ and $\delta_0 > 0$ such that*

$$\inf_{F \in [F_1, F_2] \setminus \{1\}, \varphi \in [1-\mu_0, 1]} H(F, \varphi) \geq \delta_0 > 0.$$

Proof Taking $\mu_0 = \min\{\mu_1, \mu_2\} > 0$ and $\delta_0 = \min\{\delta_1, \delta_2, \delta_3\} > 0$, we arrive at the positive infimum of $H(F, \varphi)$ over $([F_1, F_2] \setminus \{1\}) \times [1-\mu_0, 1]$ according to Lemma 2.3, Lemma 2.4, and Lemma 2.5. □

Lemma 2.7 *Let $\phi(\xi) = \phi(x - c^* t)$ be the unique sharp traveling wave in Proposition 1.1. For combustion type reaction $f(u)$ satisfying (1.2), and for any given $F_1 \in (\theta, 1)$ and $F_2 > 1$, there exist a sufficiently small $\varepsilon_1 > 0$ and a sufficiently large $\varepsilon_2 > 0$, such that*

$$\varepsilon_1 F \phi(\xi) + \varepsilon_2 F \phi'(\xi) - H(F, \phi(\xi)) \leq 0, \tag{2.8}$$

for any $F \in [F_1, F_2]$ and any $\xi \in \mathbb{R}$. The above inequality (2.8) is valid for any smaller ε_1 and any larger ε_2.

Proof Note that the sharp traveling wave $\phi(\xi) \equiv 0$, for all $\xi \geq 0$, we only need to consider the case $\xi < 0$. According to properties in Proposition 1.1, $\phi(-\infty) = 1$, $\phi'(\xi) < 0$ for all $\xi \in (-\infty, 0)$. There exists a unique $\xi_* \in (-\infty, 0)$ such that $\phi(\xi_*) = 1 - \mu_0$, where μ_0 is the positive constant in Lemma 2.6. We next divide the proof into two cases.

For $\xi \in [\xi^*, 0)$, we see that $\phi(\xi) \in (0, \phi(\xi_*)] = (0, 1-\mu_0]$. According to the asymptotic behavior in Proposition 1.1

$$\phi(\xi) = \left(\frac{m-1}{m}c^* \cdot |\xi - \xi_0|\right)^{\frac{1}{m-1}}, \quad \xi \in [\xi_1, \xi_0], \text{ with } \xi_0 = 0,$$

there exists a positive constant $\delta_4^1 > 0$ such that

$$\inf_{\xi \in [\xi_1, 0)} \frac{|\phi'(\xi)|}{\phi(\xi)} \geq \delta_4^1 > 0.$$

Since $\phi(\xi)$ is monotone decreasing and $\phi(\xi_*) = 1-\mu_0 > \theta = \phi(\xi_1)$, it is clear that $\xi_* < \xi_1$. The following continuous function admits its positive minimum

$$\min_{\xi \in [\xi_*, \xi_1]} \frac{|\phi'(\xi)|}{\phi(\xi)} = \delta_4^2 > 0.$$

For simplicity, we denote $\delta_4 := \min\{\delta_4^1, \delta_4^2\} > 0$. Consider the following function

$$J(F, \xi) := -\varepsilon_1 F - \varepsilon_2 F \frac{\phi'(\xi)}{\phi(\xi)} + \frac{H(F, \phi(\xi))}{\phi(\xi)}, \qquad F \in [F_1, F_2], \xi \in [\xi^*, 0),$$

for positive constants ε_1 and ε_2 to be determined. We calculate according to Lemma 2.2 that

$$\begin{aligned} J(F, \xi) &= -\varepsilon_1 F + \varepsilon_2 F \frac{|\phi'(\xi)|}{\phi(\xi)} + \frac{H(F, \phi(\xi))}{\phi(\xi)} \\ &\geq -\varepsilon_1 F_2 + \varepsilon_2 F_1 \delta_4 - M_0 \geq 0, \quad \xi \in [\xi^*, 0), \end{aligned}$$

provided that $\varepsilon_2 \geq 2M_0/(F_1\delta_4)$ and $\varepsilon_1 < M_0/F_2$. Therefore,

$$\varepsilon_1 F \phi(\xi) + \varepsilon_2 F \phi'(\xi) - H(F, \phi(\xi)) = -J(F, \xi) \cdot \phi(\xi) \leq 0.$$

For $\xi \in (-\infty, \xi^*)$, we see that $\phi(\xi) \in (\phi(\xi_*), 1) = (1-\mu_0, 1)$. In this case, $H(F, \phi(\xi))$ admits positive infimum $\delta_0 > 0$ according to Lemma 2.6. Hence,

$$\varepsilon_1 F \phi(\xi) + \varepsilon_2 F \phi'(\xi) - H(F, \phi(\xi)) \leq \varepsilon_1 F_2 - \delta_0 \leq 0,$$

provided that $\varepsilon_1 < \delta_0/F_2$. The proof is complete. □

Now we are ready to construct weak sub- and super-solutions with semi-compact supports. The comparison principle for degenerate parabolic equations, see [2, 25] for example, implies that the sub- and super-solutions (with the comparison of the initial data) enclose the solution to the Cauchy problem.

Lemma 2.8 *Define*

$$W(x, t) := F(t)\phi(x - G(t)), \tag{2.9}$$

where $F(t)$ and $G(t)$ are the solutions in Lemma 2.1 with initial data (F_0, G_0), ε_1 and ε_2 are the constants in Lemma 2.7, $\phi(\xi) = \phi(x - c^*t)$ is the unique sharp traveling wave *in Proposition 1.1. Then $W(x,t)$ is a sub-solution to (1.1) if $F_0 \in [F_1, 1)$; $W(x,t)$ is a super-solution to (1.1) if $F_0 \in (1, F_2]$. Moreover there exists $x_0 \in \mathbb{R}$ such that*

$$\lim_{t\to+\infty} \sup_{\xi\in\mathbb{R}} |W(\xi + c^*t, t) - \phi(\xi - x_0)| = 0.$$

Proof Lemma 2.1 shows that $F(t) \in [F_0, 1) \subset [F_1, 1)$ if $F_0 \in [F_1, 1)$ and $F(t) \in (1, F_2]$ if $F_0 \in (1, F_2]$. Note that $\phi(\xi) \equiv 0$ for all $\xi \geq 0$, and $\phi(\xi) > 0$ for all $\xi < 0$. It suffices for sub-solutions to prove that for any $t > 0$ and $x < G(t)$,

$$F'(t)\phi(\eta) - F(t)\phi'(\eta)G'(t) - F^m(t)(\phi^m)''(\eta) - f(F(t)\phi(\eta)) \leq 0, \tag{2.10}$$

where $\eta := x - G(t) < 0$. Since $\phi(\xi)$ is a sharp traveling wave, satisfying (1.3), substituting (1.3) into (2.10), we see (2.10) is equivalent to

$$F'(t)\phi(\eta) - (F(t)G'(t) - c^*F^m(t))\phi'(\eta) + F^m(t)f(\phi(\eta)) - f(F(t)\phi(\eta)) \leq 0.$$

According to the differential system (2.2), (2.10) is further transformed into

$$\varepsilon_1 F(t)(1 - F(t))\phi(\eta) + \varepsilon_2 F(t)(1 - F(t))\phi'(\eta) + F^m(t)f(\phi) - f(F(t)\phi(\eta)) \leq 0,$$

which is valid according to Lemma 2.7 since

$$\begin{aligned}
&\varepsilon_1 F(t)(1 - F(t))\phi(\eta) + \varepsilon_2 F(t)(1 - F(t))\phi'(\eta) + F^m(t)f(\phi) - f(F(t)\phi(\eta)) \\
=&(1 - F(t)) \cdot (\varepsilon_1 F(t)\phi(\eta) + \varepsilon_2 F(t)\phi'(\eta) - H(F(t), \phi(\eta))) \leq 0,
\end{aligned}$$

for $F(t) \in [F_1, 1)$ with $F_0 \in [F_1, 1)$. It follows that $W(x,t)$ is a sub-solution if $F_0 \in [F_1, 1)$. The above inequality is reversed for $F(t) \in (1, F_2]$ with $F_0 \in (1, F_2]$ such that $W(x,t)$ corresponds to super-solutions.

The convergence follows from Lemma 2.1 such that

$$\begin{aligned}
&\lim_{t\to+\infty} \sup_{\xi\in\mathbb{R}} |W(\xi + c^*t, t) - \phi(\xi - x_0)| \\
=&\lim_{t\to+\infty} \sup_{\xi\in\mathbb{R}} |F(t)\phi(\xi + c^*t - G(t)) - \phi(\xi - x_0)| \\
\leq&\lim_{t\to+\infty} \sup_{\xi\in\mathbb{R}} \big(|(F(t) - 1)\phi(\xi + c^*t - G(t))| + |\phi(\xi + c^*t - G(t)) - \phi(\xi - x_0)|\big) \\
\leq&\lim_{t\to+\infty} |F(t) - 1| + \lim_{t\to+\infty} \sup_{\xi\in\mathbb{R}} |\phi(\xi + c^*t - G(t)) - \phi(\xi - x_0)| \\
=&0,
\end{aligned}$$

since $\lim_{t\to+\infty} F(t) = 1$ and $\lim_{t\to+\infty}(G(t) - c^*t) = x_0$. The proof is complete. □

Lemma 2.9 *Let $F(t)$ and $G(t)$ be the solutions in Lemma 2.1 with initial data (F_0, G_0). Then there exists a function $\omega(\cdot) \geq 0$ such that*

$$|G(t) - c^*t - G_0| \leq \omega(|F_0 - 1|), \qquad \lim_{F_0\to 1} \omega(|F_0 - 1|) = 0.$$

Proof This is proved in the proof of Lemma 2.5 in [32]. Note that the above properties only depend on the ordinary differential system (2.2), which is valid here. □

The last step is to confirm the global L^∞ stability result.

Lemma 2.10 *Let $u(x,t)$ be the solution of the Cauchy problem (1.1) with initial data u_0 satisfying (1.4). Then there exists a $x_0 \in \mathbb{R}$ such that $u(x,t)$ converges to $\phi(x - c^*t - x_0)$ in the sense that for any $0 < \varepsilon < 1$, there exists $T > 0$ such that*

$$(1-\varepsilon)\phi(\xi - x_0 + \varepsilon) \le u(\xi + c^*t, t) \le (1+\varepsilon)\phi(\xi - x_0 - \varepsilon), \tag{2.11}$$

for all $t \ge T$ and $\xi \in \mathbb{R}$.

Proof The global L^∞ stability result for mono-stable Fisher-KPP reaction such that $f(u) = u^p - u^q$ with $1 \le p < \min\{m, q\}$ was first proved by Z. Biró in [25]. For general mono-stable Fisher-KPP reaction $f(u)$ satisfying growth condition (2.1), the L^∞ stability result was proved by Xu et al. [32]. Here we show that the L^∞ stability result is valid for combustion reaction f satisfying (1.2) once the sub- and super-solutions with semi-compact supports are constructed as in Lemma 2.8.

For any u_0 satisfying (1.4), let F_1 and F_2 be the constants as defined in (2.3). There exist sub- and super- solutions in the form

$$\overline{W}(x,t) := \overline{F}(t)\phi(x - \overline{G}(t)), \quad \underline{W}(x,t) := \underline{F}(t)\phi(x - \underline{G}(t)),$$

where $\overline{F}(t)$ and $\overline{G}(t)$ are solutions to problem (2.2) in Lemma 2.1 with initial data $\overline{F}(0) = F_2$ and $\overline{G}(0) = x_2$, similarly, $\underline{F}(t)$ and $\underline{G}(t)$ are solutions to problem (2.2) with initial data $\underline{F}(0) = F_1$ and $\underline{G}(0) = x_1$, such that

$$\underline{W}(x,0) < u_0(x) < \overline{W}(x,0). \tag{2.12}$$

It is achieved for x_1, x_2 chosen appropriately. Therefore, Lemma 2.7 shows that $\underline{W}(x,t)$ and $\overline{W}(x,t)$ are sub- and super-solutions to the equation (1.1), which together with the comparison of initial data (2.12) imply that $\underline{W}(x,t)$ and $\overline{W}(x,t)$ are sub- and super-solutions to the Cauchy problem with initial data $u_0(x)$.

According to Lemma 2.1, both $\underline{G}(t) - c^*t$ and $\overline{G}(t) - c^*t$ converge to some points ξ_1^* and ξ_2^* and lie in a bounded interval $[\xi_1, \xi_2]$. It follows from the comparison with the sub- and super-solutions $\underline{W}(x,t)$ and $\overline{W}(x,t)$ that

$$\underline{F}(t)\phi(\xi - \xi_1) \le u(\xi + c^*t, t) \le \overline{F}(t)\phi(\xi - \xi_2), \quad t > 0, \xi \in \mathbb{R}, \tag{2.13}$$

which provides bounds of the free boundary in the moving coordinates.

Applying the compactness arguments, we can show the convergence of $u(x,t)$ to the sharp wave $\phi(x - c^*t - x_0)$ with a shift. The proof is quite similar to the proof in [25] and also the proof of Lemma 2.6 in [32]. So we only sketch the outline and omit the details. Denote $z(\xi,t) := u(\xi + c^*t, t)$ in the moving coordinates. For any sequence $\{t_n\}$ with $\lim_{n\to\infty} t_n = +\infty$, denote $z_n(\xi) := u(\xi + c^*t_n, t_n)$. The uniform estimate (2.13) implies that

$$\underline{F}(t_n)\phi(\xi - \xi_1) \le z_n(\xi) \le \overline{F}(t_n)\phi(\xi - \xi_2),$$

which means

$$z_n(\xi) - 1 \le (z_n(\xi) - \phi(\xi - \xi_2)) + (\phi(\xi - \xi_2) - 1) \le C_1 \mathrm{e}^{-\beta_1 t_n} + C_2 \mathrm{e}^{-\beta_2|\xi|}, \tag{2.14}$$

and similarly

$$z_n(\xi) - 1 \ge (z_n(\xi) - \phi(\xi - \xi_1)) + (\phi(\xi - \xi_1) - 1) \ge -C_3 \mathrm{e}^{-\beta_3 t_n} - C_4 \mathrm{e}^{-\beta_4|\xi|}, \tag{2.15}$$

for some positive constants C_i and β_i, $i = 1, 2, 3, 4$, independent of n. The compactness arguments in $C^\alpha(\mathbb{R})$, together with the above uniform asymptotic behavior (2.14) and (2.15), show the existence of a function $z(\xi) \in C^\alpha(\mathbb{R})$ and a convergent subsequence of $\{z_n\}$, denoted by $\{z_n\}$ itself for simplicity, such that

$$\lim_{n\to\infty} \sup_{\xi\in\mathbb{R}} |z_n(\xi) - z(\xi)| = 0. \tag{2.16}$$

There exists $\theta_n \in (t_n, t_{n+1})$ such that $\{z(\xi, \theta_n)\}$ converges to the unique sharp traveling wave $\phi(\xi - x_0)$ with some shift x_0, and the free boundary of $z(\xi, \theta_n)$, denoted by ζ_n, converges to x_0,

$$\lim_{n\to\infty} \zeta_n = x_0, \qquad \lim_{n\to\infty} \sup_{\xi\in\mathbb{R}} |z(\xi, \theta_n) - \phi(\xi - x_0)| = 0. \tag{2.17}$$

Lemma 4 of [25] shows that the closeness of boundaries ($|\zeta_n - x_0|$ being sufficiently small) and the L^∞ convergence of function sequences ($\sup_{\xi\in\mathbb{R}} |z(\xi, \theta_n) - \phi(\xi - x_0)|$ being sufficiently small) imply the uniform similarity in shape, i.e., for any $\varepsilon_* > 0$, there exists $\delta > 0$ such that

$$(1 - \varepsilon_*)\phi(\xi - x_0 + \varepsilon_*) \le z(\xi, \theta_n) \le (1 + \varepsilon_*)\phi(\xi - x_0 - \varepsilon_*),$$

provided that $|\zeta_n - x_0| \le \delta$ and $\sup_{\xi\in\mathbb{R}} |z(\xi, \theta_n) - \phi(\xi - x_0)| \le \delta$. Actually, this is true for any bounded monotone function with sharp edge. Starting from time $T = \theta_n$, we construct weak sub- and super-solutions in the same form as in Lemma 2.8 but with more accurate amplitude $F(t)$ and free boundary $G(t)$. We may take $\varepsilon_* > 0$ smaller, such that $\varepsilon_* + \omega(\varepsilon_*) < \varepsilon$, where $\omega(\cdot)$ is the function in Lemma 2.9. Lemma 2.9 shows that if the amplitude $F(t)$ is sufficiently close to 1, then the bound of the evolution of the free boundary $|G(t) - c^*t - G_0|$ can be sufficiently small. Therefore,

$$(1 - \varepsilon)\phi(\xi - x_0 + \varepsilon) < u(\xi + c^*t, t) < (1 + \varepsilon)\phi(\xi - x_0 - \varepsilon),$$

for all $t \ge T$. The proof is completed. □

Proof of Theorem 1.1. This is proved in Lemma 2.10. □

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Declarations

Conflict of interest The authors declare no conflict of interest.

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