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Asymptotic behavior for the fast diffusion equation with absorption and singularity

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Abstract

This paper is concerned with the weak solution for the fast diffusion equation with absorption and singularity in the form of $u_t = \triangle u^m - u^p$. We first prove the existence and decay estimate of weak solution when the fast diffusion index satisfies $0 < m < 1$ and the absorption index is $p > 1$. Then we show the asymptotic convergence of weak solution to the corresponding Barenblatt solution for $\frac{n-1}{n} < m < 1$ and $p > m + \frac{2}{n}$ via the entropy dissipation method combining the generalized Shannon's inequality and Csiszár-Kullback inequality. The singularity of spatial diffusion causes us the technical challenges for the asymptotic behavior of weak solution.

Keywords: Fast diffusion; Decay estimate; Asymptotic behavior; Generalized Shannon's inequality; Entropy dissipation method

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1. Introduction

We consider the following fast diffusion equation with absorption and singularity:

$$u_t = \Delta u^m - u^p, \quad x \in \mathbb{R}^n,\ t > 0, \tag{1.1}$$

with initial data

$$u(x,0) = u_0(x), \quad x \in \mathbb{R}^n, \tag{1.2}$$

where $n \geq 2$, $p > 1$, $0 < m < 1$ and the initial data $u_0(x)$ is a non-negative function in $L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$. The unknown function $u = u(x,t)$ is the density of particles. The absorption term u^p can cause the mass conservation law fail to be true, more precisely, the mass is monotonically decreasing. Δu^m with $0 < m < 1$ is the singular diffusion with more physical sense, that is, the diffusive velocity of particles depends on the current particles density, and the smaller density, the larger diffusion, in particular, when $u = 0$, the diffusion speed is ∞, which is the so-called fast diffusion.

Background of study. The fast diffusion equation

$$u_t = \Delta u^m, \quad x \in \mathbb{R}^n,\ t > 0, \tag{1.3}$$

with $0 < m < 1$, which is an important model for singular nonlinear diffusive phenomena, can be used to model gas-kinetics, thin liquid film dynamics and plasma in nuclear reactors [3,8,33]. Since the 1970s, the equation (1.3) has been thoroughly studied and a series of celebrated results have been obtained, see [5–7,9,18,34] and the references therein. On the other hand, when $m > 1$, the equation (1.3) is the slow diffusion equation, the so-called degenerate diffusion equation, which has been also extensively studied [23,24,33,35,36], see also the references therein.

When the fast diffusion equations are affected by the external sources, the mass of flows is no long conservative. There are two different external sources. One is with external supplement, which is called the expansion phenomenon and presented in the form of

$$u_t = \Delta u^m + u^p, \quad x \in \mathbb{R}^n,\ t > 0. \tag{1.4}$$

The other is with the external omission, the so-called absorption phenomenon, which is described by (1.1), namely,

$$u_t = \Delta u^m - u^p, \quad x \in \mathbb{R}^n,\ t > 0.$$

For the fast diffusion equation with expansion (1.4) subjected to the initial data (1.2), the mass is monotonically increasing. In 1991, Anderson [1,2] first studied the existence and uniqueness of weak solution for (1.4) and (1.2) with expansion phenomenon. Galaktionov and Vázquez [20] further studied the "peak-like" incomplete blow-up solution, and showed the possible continuation of incomplete blow-up solutions to (1.4) after the blow-up time, and discussed the different continuation modes in terms of the exponents m and p. But when the same technique of analysis was applied to (1.1) with absorption phenomenon, they [20] recognized that no such a continuation of incomplete blow-up solution exists if $p+m \leq 0$, namely, the complete extinction solution, and there exists a nontrivial continuation if $p+m > 0$.

For the fast diffusion equation with absorption (1.1) subjected to the initial data (1.2), the mass is monotonically decreasing and the limiting mass is strictly positive (see Lemma 4.1 later). In 1991, Peletier [28] first investigated the existence and uniqueness of solution to problem (1.1)-(1.2), which was then significantly developed by Borelli and Ughi [10] and Leoni [22], respectively. For asymptotic behavior, the critical exponent $p = m + \frac{2}{n}$ is crucial, as the competition between the diffusion $-\Delta u^m$ and the absorption u^p in (1.1). More precisely, Peletier and Zhao [29] proved the asymptotic results uniformly on moving boundary set of $\{x \in \mathbb{R}^n : |x| < at^{\frac{1}{b}}\}$ for $p > 1$, $p > m + \frac{2}{n}$ and $m < p < m + \frac{2}{n}$, where a and b are some numbers specified in [29]. Then, Ferreira and Vázquez [19] showed the finite-time extinction for problem (1.1)-(1.2) in one-dimension, which was the main qualitative feature of the bounded solutions in the range $0 < p < 1$. Furthermore, Benachour et al. [4] gave the asymptotic behavior related to Barenblatt profile with a logarithmic scaling in the critical exponent $p = m + \frac{2}{n}$.

However, as far as we know, for the fast diffusion equation with absorption (1.1)-(1.2), the asymptotic convergence of weak solution to the Barenblatt profile in full space is still open, due to some technical issues. To attach this problem is the main purpose of this paper. The adopted approach is the technical entropy dissipation method.

Regarding the entropy dissipation method, developed first by Toscani and Carrillo [15,32] to investigate the asymptotic behavior of the solution to diffusion equations and Fokker-Planck-type equations, it was applied to prove the asymptotic behavior of the solution to the degenerate diffusion equation ($m > 1$) and fast diffusion equation ($\frac{n-2}{n} < m < 1$) by Carrillo and Toscani [14] and by Carrillo and Vázquez [16], respectively. They [14,16] showed the convergence of the weak solution to Barenblatt profiles with algebraic rates in the corresponding exponent interval. For $\frac{n-2}{n} < m < 1$, the Barenblatt solution to (1.2)-(1.3) is given by the formula [16]

$$U_M(x,t) = \left(\frac{\alpha t}{|x|^2 + \beta t^{\frac{2}{\lambda}}} \right)^{\frac{1}{1-m}}, \tag{1.5}$$

where $\lambda = 2 - n(1-m)$, $\alpha = 2m\lambda/(1-m)$ and $\beta > 0$ is a constant such that $\|U_M\|_1 = M$. The stationary solution to the nonlinear Fokker-Planck equation

$$\rho_s = \mathrm{div}_y(\rho y + \nabla_y \rho^m), \quad y \in \mathbb{R}^n,\ s > 0, \tag{1.6}$$

can be directly obtained by rescaling the time shifted profiles

$$U_M(x, t + \lambda^{-1}) \to \rho_M(y) = \left(\frac{\gamma}{|y|^2 + \theta} \right)^{\frac{1}{1-m}}, \tag{1.7}$$

where $\gamma = \alpha/\lambda = 2m/(1-m)$, $\theta = \beta\lambda^{-2/\lambda}$. More research on the entropy dissipation method can be found in [12,13,25,26] and the literatures therein.

Difficulty and strategy. Regarding the fast diffusion problem with absorption (1.1)-(1.2), in order to investigate the asymptotic behavior of the weak solution, we first need to clarify what will be the targeted asymptotic profile. Note that, in the absorbing fast diffusion case, the total mass of flow is monotonically decreasing and the limiting mass is strictly positive (see Lemma 4.1 later), therefore the expected asymptotic profile for the weak solution is not the standard Barenblatt solution as showed in (1.5). Let us define the total mass of $u(x,t)$ by $\widetilde{M}(t) :=$

$\int_{\mathbb{R}^n} u(x,t)dx$, we are reasonable to construct a special Barenblatt solution to the standard fast diffusion equation (1.3) related to the total mass $\widetilde{M}(t)$ of (1.1) by

$$U_{\widetilde{M}(t)}(x,t) = \left(\frac{\alpha t}{|x|^2 + \beta_{\widetilde{M}(t)} t^{\frac{2}{\lambda}}} \right)^{\frac{1}{1-m}}, \tag{1.8}$$

where $\beta_{\widetilde{M}(t)} > 0$ is a function such that $\|U_{\widetilde{M}(t)}\|_{L^1(\mathbb{R}^n)} = \widetilde{M}(t)$. We expect this modified Barenbalatt solution $U_{\widetilde{M}(t)}(x,t)$ as the asymptotic profile to the original fast diffusion equation with absorption (1.1)-(1.2). The corresponding stationary solution to (1.6) can be derived as

$$U_{\widetilde{M}(t)}(x, t+\lambda^{-1}) \to \rho_{M(s)}(y) = \left(\frac{\gamma}{|y|^2 + \theta_{M(s)}} \right)^{\frac{1}{1-m}}, \tag{1.9}$$

where $\theta_{M(s)} = \beta_{\widetilde{M}(t)} \lambda^{-2/\lambda} > 0$ is a function such that $\|\rho_{M(s)}\|_{L^1(\mathbb{R}^n)} = M(s) = \widetilde{M}(t)$.

In order to prove the convergence of the weak solution to the specially constructed Barenblatt profile (1.8), we adopt the entropy dissipation method as mentioned above. After carrying out the suitable variable scalings (see (4.1) later), we reduce the original IVP (1.1)-(1.2) to the new system presented in (4.3)-(4.4). Then we show that the rescaled solution $\rho(y,s)$ eventually converges to the profile $\rho_{M_\infty}(y)$, by first showing that it converges to the time dependent profile $\rho_{M(s)}(y)$.

Here are some technical issues in the proof. It is worth noting that Carrillo and Fellner [12] studied the long-time asymptotic of the solution to the degenerate (slow) diffusion equation (1.1)-(1.2) with $m > 1$. Since the diffusion coefficient of (1.1) is mu^{m-1}, the larger u, the larger diffusion power for $m > 1$, which can easily derive the L^m boundedness of u. The L^m-bound of the solution plays an important role in showing the asymptotic convergence. However, in the fast diffusion case of $m < 1$, we come across the reverse situation. When $m < 1$, the diffusion coefficient $mu^{m-1} \to \infty$ as $u \to 0$, that is, u^{m-1} is unbounded, and $u \notin L^m$ in general, which requires a number of changes in the arguments. In order to overcome these obstacles, we use the generalized Shannon's inequality to prove the boundedness of the second moment, and then prove L^m boundedness of u by the L^1 decay estimate and the bound of the second moment of u. After a series of meticulous estimates, we can prove the asymptotic convergence of weak solution to the corresponding Barenblatt solution.

Main results. Before stating our main results, we first impose the following assumption:

Assumption. Let $n \geq 2$. The initial data u_0 is a non-negative function satisfying

$$u_0 \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n) \text{ with } u_0^m \in H^1(\mathbb{R}^n).$$

For simplification, we denote the norm of the space $L^r(\mathbb{R}^n)$ for $1 \leq r \leq \infty$ by

$$\|u\|_{L^r(\mathbb{R}^n)} = \|u\|_r.$$

Although the existence of the defined weak solution has been proved in [28], in order to study the asymptotic property of (1.1)-(1.2), we need higher regularities for the weak solution, and then we give another definition of the weak solution.

Definition 1.1 *(Weak solution).* Let the Assumption hold and $0 < m < 1$. For $T > 0$ and $Q_T = \mathbb{R}^n \times [0, T)$, $u(x,t)$ is called a weak solution to (1.1)-(1.2) if
(i) $u(x,t) \geq 0$ for almost all $(x,t) \in Q_T$.
(ii) $u \in L^\infty(0,T; L^2(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n))$ and $\nabla u^m \in L^\infty(0,T; L^2(\mathbb{R}^n))$.
(iii) For any $\phi \in C^\infty(Q_T)$ with supp $\phi(\cdot,t) \subset\subset \mathbb{R}^n$ and $0 < t < T$,

$$\begin{aligned}&\int u(t)\phi(t)dx - \int u_0\phi(0)dx \\ &= \iint\limits_0^t u(\tau)\partial_\tau\phi(\tau)dxd\tau - \iint\limits_0^t \left(\nabla u(\tau)^m \cdot \nabla\phi(\tau) + u(\tau)^p\phi(\tau)\right)dxd\tau.\end{aligned} \tag{1.10}$$

Before we give the asymptotic result of the weak solution to (1.1)-(1.2), we show the time-local existence and decay estimate of weak solutions to (1.1)-(1.2).

Theorem 1.1 *(Local existence). Let the Assumption hold. Suppose $0 < m < 1$ and $p > 1$. Then there exists a weak solution $u(x,t)$ on $[0,T)$ to (1.1)-(1.2) in the sense of Definition 1.1.*

For every $T < \infty$, if $u(x,t)$ is a weak solution to (1.1)-(1.2) in Q_T, we say that $u(x,t)$ is global. Then we present the decay properties for the weak solution to (1.1)-(1.2).

Theorem 1.2 *(Decay estimate). Let the Assumption hold. Suppose $0 < m < 1$ and $p > 1$. Then for $1 \leq r \leq \infty$, problem (1.1)-(1.2) has a global weak solution $u(x,t)$ satisfying*

$$\|u(t)\|_r \leq \|u_0\|_r, \quad t \geq 0, \tag{1.11}$$

and

$$\|u(t)\|_r \leq C(\|u_0\|_1, \|u_0\|_\infty, r, p)(1+t)^{-\frac{r-1}{r(p-1)}}, \quad t \geq 0. \tag{1.12}$$

For the case of $r \geq 3 - m$, we have

$$\|u(t)\|_r \leq C(\|u_0\|_1, \|u_0\|_\infty, n, r, p, m)(1+t)^{-\max\{\frac{r-1}{r(p-1)}, \frac{n(r-1)}{2r(2-m)}\}}, \quad t \geq 0. \tag{1.13}$$

At last, we show the asymptotic behavior of weak solution to (1.1)-(1.2).

Theorem 1.3 *(Convergence to Barenblatt solution). Let the Assumption hold. Suppose $\frac{n-1}{n} < m < 1$, $p > m + \frac{2}{n}$ and $\int u_0|x|^2dx < \infty$. Then the weak solution $u(x,t)$ to (1.1)-(1.2) in the sense of Definition 1.1 converges to the corresponding Barenblatt solution in the form:*

$$\|u - U_{\widetilde{M}(t)}\|_1 \leq C(1+\lambda t)^{-\frac{1}{\lambda}\min\{1,\delta\}}, \quad \text{for } \delta \neq 1, \tag{1.14}$$

and

$$\|u - U_{\widetilde{M}(t)}\|_1 \leq C(1+\lambda t)^{-\frac{1}{\lambda}}\ln(1+\lambda t), \quad \text{for } \delta = 1, \tag{1.15}$$

where $\delta = np - nm - 2$, $U_{\widetilde{M}(t)}$ is defined as (1.8) and the constant $C > 0$ is depending on $\|u_0\|_1$, $\|u_0\|_\infty$, n, p and m. Moreover, for $1 < r < \infty$, we have

$$\|u - U_{\widetilde{M}(t)}\|_r \le C(1+\lambda t)^{-\frac{n(r-1)+\min\{1,\delta\}}{\lambda r}}, \quad \text{for } \delta \ne 1, \tag{1.16}$$

and

$$\|u - U_{\widetilde{M}(t)}\|_r \le C(1+\lambda t)^{-\frac{n(r-1)+1}{\lambda r}} \ln(1+\lambda t), \quad \text{for } \delta = 1, \tag{1.17}$$

where the constant $C > 0$ is depending on $\|u_0\|_1$, $\|u_0\|_\infty$, n, p, m and r.

This paper is organized as follows. In Section 2, we prepare some notations and lemmas which will be used often in the other sections. In Section 3, we prove Theorem 1.1 and Theorem 1.2, namely the existence and decay estimate for the weak solution to (1.1)-(1.2). Then, we prove Theorem 1.3 in Section 4, that is, the asymptotic behavior of the weak solution to (1.1)-(1.2).

2. Preliminaries

In this section, we show some notations and lemmas which will be used often in the other sections.

For simplicity, we denote $\|\cdot\|_{L^r(\mathbb{R}^n)}$ by $\|\cdot\|_r$, $\int_{\mathbb{R}^n} \cdot dx$ by $\int \cdot dx$, and $L^1_2(\mathbb{R}^n) \equiv \{f \in L^1(\mathbb{R}^n); |x|^2 f \in L^1(\mathbb{R}^n)\}$. C represents the constant which may be different from line by line and $C = C(\cdot, \cdots, \cdot)$ denotes the constant that depends only on the variables in parentheses.

Lemma 2.1 *([31]). Let a and b be arbitrary positive numbers. Then we have*

$$|a^r - b^r| \le \begin{cases} 2^{r-1} r(a^{r-1} + b^{r-1})|a-b|, & r > 1, \\ |a-b|^r, & 0 < r \le 1. \end{cases} \tag{2.1}$$

Lemma 2.2 *(Nash inequality [11,13]). There exists a constant $C > 0$ such that for all $f \in L^1(\mathbb{R}^n) \cap H^1(\mathbb{R}^n)$,*

$$\|f\|_2^{1+\frac{2}{n}} \le C\|f\|_1^{\frac{2}{n}} \|\nabla f\|_2, \tag{2.2}$$

where the constant C depend only on n.

Lemma 2.3 *(Cut-off function [30]). Let $R > 0$ and $\eta(x) = \eta(|x|)$ be defined as*

$$\eta(x) = \begin{cases} 1, & 0 \le |x| < R, \\ \exp\left(1 - \frac{R}{2R-|x|}\right), & R \le |x| < 2R, \\ 0, & |x| \ge 2R. \end{cases} \tag{2.3}$$

Then, it holds that

$$|\nabla\eta(x)| \le \frac{C}{a^2R}\eta(x)^{1-a}, \tag{2.4}$$

$$|\Delta\eta(x)| \le \frac{C}{a^4R^2}\eta(x)^{1-a}, \tag{2.5}$$

for all $x \in \mathbb{R}^n$ and all $0 < a < 1$, where C is a constant depending only on n.

Lemma 2.4 *(Generalized Shannon's inequality [21]). Let $n \ge 2$, and $\frac{n}{n+2} < m < 1$. Then there exists a constant $C = C(n) > 0$ such that for any $f \in L^1_2(\mathbb{R}^n)$, we have*

$$\int |f|^m dx \le C\|f\|_1^{m(1-\sigma)} \left(\int |x|^2|f(x)|dx\right)^{m\sigma}, \tag{2.6}$$

where $\sigma = \frac{n(1-m)}{2m}$.

Lemma 2.5 *(Csiszár-Kullback inequality [13,27]). Let $0 < m < 1$ and a nonnegative function $f \in L^1(\mathbb{R}^n)$ with $\|f\|_1 = M$. Suppose that a nonnegative function $g \in L^1(\mathbb{R}^n) \cap L^{2-m}(\mathbb{R}^n)$ satisfies $\|g\|_1 = M$ and*

$$c_{m,g} = M^{\frac{m-2}{2}}\left(\frac{2}{m}\int g(x)^{2-m}dx\right)^{\frac{1}{2}},$$

then it holds that

$$\|f - g\|_1 \le c_{m,g}\sqrt{D(f|g)}, \tag{2.7}$$

where

$$\begin{aligned} D(f|g) = &\frac{1}{m-1}\int f(x)^m dx - \frac{1}{m-1}\int g(x)^m dx \\ &-\frac{m}{m-1}\int g(x)^{m-1}(f(x)-g(x))dx. \end{aligned}$$

3. Existence and decay estimate

As the equation (1.1) is a quasi-linear parabolic equation of singular type, we consider the following approximating problem of (1.1)-(1.2) to justify all the formal arguments:

$$\partial_t u_\varepsilon = \Delta\left(u_\varepsilon^m + \varepsilon u_\varepsilon\right) - u_\varepsilon^p, \qquad x \in \mathbb{R}^n,\ t \in (0,T), \tag{3.1}$$

$$u_\varepsilon(x,0) = (u_0 * \widetilde{\eta}_\varepsilon)(x) = u_{0\varepsilon}(x), \qquad x \in \mathbb{R}^n, \tag{3.2}$$

where $\varepsilon > 0$ and non-negative mollifier $\widetilde{\eta}_\varepsilon \in C_0^\infty(\mathbb{R}^n)$ satisfies $\widetilde{\eta}_\varepsilon \subset \overline{B_\varepsilon(0)}$ and $\|\widetilde{\eta}_\varepsilon\|_1 = 1$. By the similar argument as in [35], we know that (3.1)-(3.2) has a unique classical solution u_ε, and

$$0 \le u_\varepsilon \le C, \tag{3.3}$$

where C is a positive constant which does not depend on ε. Furthermore, there exists a subsequence, which we still define as $\{u_\varepsilon\}$, such that

$$u_\varepsilon \to u, \quad \text{in } C(K) \tag{3.4}$$

for every compact set $K \subset (0, T; \mathbb{R}^n)$. In what follows, we will begin the proof of Theorem 1.1.

Proof of Theorem 1.1. Multiplying the equation (3.1) by $\partial_t \left(u_\varepsilon^m + \varepsilon u_\varepsilon\right)$ and integrating with respect to x and t, we get

$$\begin{aligned}
&\frac{4m}{(m+1)^2} \iint_0^T \left|\partial_t u_\varepsilon^{\frac{m+1}{2}}\right|^2 dxdt + \varepsilon \iint_0^T |\partial_t u_\varepsilon|^2 dxdt \\
&+\frac{1}{2} \sup_{0<t<T} \int |\nabla \left(u_\varepsilon^m + \varepsilon u_\varepsilon\right)|^2 dx \\
&+\frac{m}{p+m} \sup_{0<t<T} \int u_\varepsilon^{p+m} dx + \frac{\varepsilon}{p+1} \sup_{0<t<T} \int u_\varepsilon^{p+1} dx \\
=&\frac{1}{2} \int |\nabla \left(u_{0\varepsilon}^m + \varepsilon u_{0\varepsilon}\right)|^2 dx + \frac{m}{p+m} \int u_{0\varepsilon}^{p+m} dx + \frac{\varepsilon}{p+1} \int u_{0\varepsilon}^{p+1} dx,
\end{aligned} \tag{3.5}$$

which combined with (3.3) can yield

$$\begin{aligned}
&\sup_{0<t<T} \int \left|\nabla u_\varepsilon^{\frac{m+1}{2}}\right|^2 dx \\
&\leq \frac{(m+1)^2}{4m^2} \|u_\varepsilon\|_\infty^{1-m} \sup_{0<t<T} \int |\nabla \left(u_\varepsilon^m + \varepsilon u_\varepsilon\right)|^2 dx \\
&\leq C(\|u_0\|_1, \|u_0\|_\infty, \|\nabla u_0^m\|_2, p, m).
\end{aligned} \tag{3.6}$$

On the other hand, we multiply the equation (3.1) by u_ε^m and integrate with respect to x and t, and then we obtain

$$\begin{aligned}
&\frac{1}{m+1} \sup_{0<t<T} \int u_\varepsilon^{m+1} dx + \frac{4m\varepsilon}{(m+1)^2} \iint_0^T \left|\nabla u_\varepsilon^{\frac{m+1}{2}}\right|^2 dxdt \\
&+ \iint_0^T |\nabla u_\varepsilon^m|^2 dxdt + \iint_0^T u_\varepsilon^{p+m} dxdt \\
&= \frac{1}{m+1} \int u_{0\varepsilon}^{m+1} dx.
\end{aligned} \tag{3.7}$$

From (3.5), (3.6) and (3.7), we have $u_\varepsilon^{\frac{m+1}{2}} \in L^\infty(0, T; H^1(\mathbb{R}^n)) \cap H^1(0, T; L^2(\mathbb{R}^n))$. Thus, we can extract a subsequence, which we still define as $\{u_\varepsilon\}$, such that

$$u_\varepsilon^{\frac{m+1}{2}} \to \xi, \quad \text{strongly in } C(0,T;L^2_{loc}(\mathbb{R}^n)), \tag{3.8}$$

which implies

$$u_\varepsilon^{\frac{m+1}{2}}(x,t) \to \xi(x,t), \quad \text{a.a. } x \in \mathbb{R}^n, t \in (0,T),$$

and then

$$u_\varepsilon(x,t) \to \xi^{\frac{2}{m+1}}(x,t), \quad \text{a.a. } x \in \mathbb{R}^n, t \in (0,T). \tag{3.9}$$

Since it holds by Lebesgue dominated convergence theorem, (3.3), (3.4), and (3.9) that

$$u_\varepsilon \to \xi^{\frac{2}{m+1}} = u, \quad \text{strongly in } L^\infty(0,T;L^1_{loc}(\mathbb{R}^n)), \tag{3.10}$$

we have

$$u_\varepsilon(x,t) \to \xi^{\frac{2}{m+1}}(x,t) = u(x,t), \quad \text{a.a. } x \in \mathbb{R}^n, t \in (0,T). \tag{3.11}$$

Combining (3.8) with (3.11), we obtain

$$u_\varepsilon^{\frac{m+1}{2}} \to u^{\frac{m+1}{2}}, \quad \text{strongly in } C(0,T;L^2_{loc}(\mathbb{R}^n)). \tag{3.12}$$

By virtue of Lemma 2.1, for $0 \le b \le a$, $r \ge 1$ and $0 < m < 1$, we see

$$\begin{aligned}
|a-b|^r &\le |a^r - b^r| \\
&= |(a^{\frac{m+1}{2}})^{\frac{2r}{m+1}} - (b^{\frac{m+1}{2}})^{\frac{2r}{m+1}}| \\
&\le \left(2^{\frac{2r}{m+1}-1} \cdot \frac{2r}{m+1}\right)\left((a^{\frac{m+1}{2}})^{\frac{2r}{m+1}-1} + (b^{\frac{m+1}{2}})^{\frac{2r}{m+1}-1}\right)|a^{\frac{m+1}{2}} - b^{\frac{m+1}{2}}|,
\end{aligned}$$

which combined Hölder's inequality, (3.3) with (3.12) can yield

$$\begin{aligned}
&\sup_{0<t<T} \int_K |u_\varepsilon(x,t) - u(x,t)|^r \\
&\le C \sup_{0<t<T} \int_K |u_\varepsilon^{\frac{m+1}{2}}(x,t) - u^{\frac{m+1}{2}}(x,t)| dx \\
&\le C \sup_{0<t<T} \|u_\varepsilon^{\frac{m+1}{2}}(t) - u^{\frac{m+1}{2}}(t)\|_{L^2(K)} |K|^{\frac{1}{2}} \to 0,
\end{aligned}$$

for all compact sets K as $\varepsilon \to 0$, where $C = C(\|u_0\|_\infty, p, m, r)$. Then for $1 \le r < \infty$, we can see

$$u_\varepsilon \to u, \quad \text{strongly in } C(0,T;L^r_{loc}(\mathbb{R}^n)). \tag{3.13}$$

In addition, from (3.3), (3.5) and (3.7), we obtain

$$u_\varepsilon \to u, \qquad \text{weakly star in } L^\infty(0,T;L^2(\mathbb{R}^n)\cap L^\infty(\mathbb{R}^n)), \tag{3.14}$$

$$\nabla u_\varepsilon^m \to \nabla u^m, \qquad \text{weakly star in } L^\infty(0,T;L^2(\mathbb{R}^n)). \tag{3.15}$$

For any $\phi \in C^\infty(Q_T)$ with supp $\phi(\cdot,t) \subset\subset \mathbb{R}^n$ and $0<t<T$, we see that u_ε satisfies

$$\int u_\varepsilon(t)\phi(t)dx - \int u_{0\varepsilon}\phi(0)dx - \iint_0^t u_\varepsilon(\tau)\partial_\tau\phi(\tau)dxd\tau$$

$$= -\iint_0^t \left(\nabla u_\varepsilon(\tau)^m\cdot\nabla\phi(\tau) + \varepsilon\nabla u_\varepsilon(\tau)\cdot\nabla\phi(\tau) + u_\varepsilon(\tau)^p\phi(\tau)\right)dxd\tau,$$

which yields (1.10) by (3.13)-(3.15). □

Next, we prove the decay property of the weak solution to (1.1)-(1.2).

Proof of Theorem 1.2. Let $\eta_R(x)$ be the cut-off function defined in Lemma 2.3. We first show that the weak solution u on $[0,T)$ given by Theorem 1.1 satisfies (1.11).

For $r=1$, we multiply (3.1) by η_R and integrate it over $\mathbb{R}^n$, which yields that

$$\frac{d}{dt}\int u_\varepsilon\eta_R dx = -\int \nabla u_\varepsilon^m\nabla\eta_R dx - \varepsilon\int\nabla u_\varepsilon\nabla\eta_R dx - \int u_\varepsilon^p\eta_R dx. \tag{3.16}$$

We obtain from (3.13)-(3.16) as the limit $\varepsilon\to 0$ that

$$\frac{d}{dt}\int u\eta_R dx = -\int\nabla u^m\nabla\eta_R dx - \int u^p\eta_R dx. \tag{3.17}$$

Since it holds by Lemma 2.3 that

$$|\nabla\eta_R(x)| \le \frac{C}{R},$$

we obtain by letting $R\to\infty$ in (3.17) that

$$\frac{d}{dt}\int udx = -\int u^p dx \le 0.$$

Then we have (1.11) for $r=1$ on $0\le t<T$.

For $r>1$, multiplying (3.1) by $u_\varepsilon^{r-1}\eta_R$ and integrating it over $\mathbb{R}^n$, we have

$$\frac{1}{r}\frac{d}{dt}\int u_\varepsilon^r\eta_R dx$$

$$= -\frac{4m(r-1)}{(r+m-1)^2}\int\left|\nabla u_\varepsilon^{\frac{r+m-1}{2}}\right|^2\eta_R dx$$

$$
\begin{aligned}
&-\frac{m}{r+m-1}\int \nabla u_\varepsilon^{r+m-1}\nabla \eta_R dx \\
&-\frac{4\varepsilon(r-1)}{r^2}\int \left|\nabla u_\varepsilon^{\frac{r}{2}}\right|^2 \eta_R dx - \frac{\varepsilon}{r}\int \nabla u_\varepsilon^r \nabla \eta_R dx - \int u_\varepsilon^{r+p-1}\eta_R dx \\
\le& -\frac{m}{r+m-1}\int \nabla u_\varepsilon^{r+m-1}\nabla \eta_R dx \\
&-\frac{\varepsilon}{r}\int \nabla u_\varepsilon^r \nabla \eta_R dx - \int u_\varepsilon^{r+p-1}\eta_R dx, \qquad (3.18)
\end{aligned}
$$

which implies by (3.13)-(3.15) as $\varepsilon \to 0$ that

$$
\frac{1}{r}\frac{d}{dt}\int u^r \eta_R dx \le -\frac{m}{r+m-1}\int \nabla u^{r+m-1}\nabla \eta_R dx - \int u^{r+p-1}\eta_R dx. \qquad (3.19)
$$

Let $R \to \infty$ in (3.19), we have

$$
\frac{1}{r}\frac{d}{dt}\int u^r dx = -\int u^{r+p-1} dx \le 0.
$$

Thus, $\|u\|_r \le \|u_0\|_r$, which is also true for $r = \infty$. Then we have (1.11) for $r > 1$ on $0 \le t < T$.

Since $T < \infty$ is arbitrary taken, then we obtain (1.11) for $t \ge 0$.

Moreover, for $r > 1$, we can derive by (3.18) and Lemma 2.3 that

$$
\begin{aligned}
&\frac{1}{r}\frac{d}{dt}\int u_\varepsilon^r \eta_R dx \\
=& -\frac{4m(r-1)}{(r+m-1)^2}\int \left|\nabla u_\varepsilon^{\frac{r+m-1}{2}}\right|^2 \eta_R dx \\
&+\frac{m}{r+m-1}\int u_\varepsilon^{r+m-1}\Delta \eta_R dx \\
&-\frac{4\varepsilon(r-1)}{r^2}\int \left|\nabla u_\varepsilon^{\frac{r}{2}}\right|^2 \eta_R dx + \frac{\varepsilon}{r}\int u_\varepsilon^r \Delta \eta_R dx - \int u_\varepsilon^{r+p-1}\eta_R dx \\
\le& -\frac{4m(r-1)}{(r+m-1)^2}\int \left|\nabla u_\varepsilon^{\frac{r+m-1}{2}}\right|^2 \eta_R dx \\
&+\frac{Cm}{(r+m-1)R^2}\int_{\operatorname{supp}\eta_R} u_\varepsilon^{r+m-1} dx \\
&-\frac{4\varepsilon(r-1)}{r^2}\int \left|\nabla u_\varepsilon^{\frac{r}{2}}\right|^2 \eta_R dx \\
&+\frac{C\varepsilon}{rR^2}\int_{\operatorname{supp}\eta_R} u_\varepsilon^r dx - \int u_\varepsilon^{r+p-1}\eta_R dx. \qquad (3.20)
\end{aligned}
$$

On the one hand, we can get by (3.20) and Hölder's inequality that

$$
\begin{aligned}
&\frac{1}{r}\frac{d}{dt}\int u_\varepsilon^r \eta_R dx \\
&\le \frac{C}{R^2}\int_{\operatorname{supp}\eta_R} u_\varepsilon^{r+m-1}dx + \frac{C\varepsilon}{R^2}\int_{\operatorname{supp}\eta_R} u_\varepsilon^r dx - \int u_\varepsilon^{r+p-1}\eta_R dx \\
&\le \frac{C}{R^2}\int_{\operatorname{supp}\eta_R} u_\varepsilon^{r+m-1}dx + \frac{C\varepsilon}{R^2}\int_{\operatorname{supp}\eta_R} u_\varepsilon^r dx \\
&\quad - \frac{\|u_\varepsilon \eta_R^{\frac{1}{r+p-1}}\|_r^{\frac{r(r+p-2)}{r-1}}}{\|u_\varepsilon \eta_R^{\frac{1}{r+p-1}}\|_1^{\frac{p-1}{r-1}}}.
\end{aligned} \tag{3.21}
$$

Let $\varepsilon \to 0$ and $R \to \infty$ in (3.21), we have

$$
\frac{1}{r}\frac{d}{dt}\|u(t)\|_r^r \le -\frac{\|u\|_r^{\frac{r(r+p-2)}{r-1}}}{\|u\|_1^{\frac{p-1}{r-1}}} \le -\frac{\|u\|_r^{\frac{r(r+p-2)}{r-1}}}{\|u_0\|_1^{\frac{p-1}{r-1}}},
$$

which implies (1.12). It is clear that (1.12) still hold for $r = 1$.

Furthermore, for $r \ge 3 - m$, it follows by Hölder's inequality, Nash inequality (2.2) and Lemma 2.3 that

$$
\begin{aligned}
&\left\| u_\varepsilon \eta_R^{\frac{1}{r+m-1}} \right\|_r^{\frac{r(n+2)}{n}} \\
&\le \left\| u_\varepsilon \eta_R^{\frac{1}{r+m-1}} \right\|_\infty^{\frac{(1-m)(n+2)}{n}} \left\| u_\varepsilon \eta_R^{\frac{1}{r+m-1}} \right\|_{r+m-1}^{\frac{(r+m-1)(n+2)}{n}} \\
&\le C \left\| u_\varepsilon \eta_R^{\frac{1}{r+m-1}} \right\|_\infty^{\frac{(1-m)(n+2)}{n}} \left\| u_\varepsilon \eta_R^{\frac{1}{r+m-1}} \right\|_{\frac{r+m-1}{2}}^{\frac{2(r+m-1)}{n}} \left\| \nabla \left(u_\varepsilon^{\frac{r+m-1}{2}} \eta_R^{\frac{1}{2}} \right) \right\|_2^2 \\
&\le C \left\| u_\varepsilon \eta_R^{\frac{1}{r+m-1}} \right\|_\infty^{\frac{(1-m)(n+2)}{n}} \left\| u_\varepsilon \eta_R^{\frac{1}{r+m-1}} \right\|_1^{\frac{2(r-m+1)}{n(r-1)}} \left\| u_\varepsilon \eta_R^{\frac{1}{r+m-1}} \right\|_r^{\frac{2r(r+m-3)}{n(r-1)}} \\
&\quad \times \left(\int \left| \nabla u_\varepsilon^{\frac{r+m-1}{2}} \right|^2 \eta_R dx + \frac{1}{4}\int u_\varepsilon^{r+m-1} \frac{|\nabla \eta_R|^2}{\eta_R} dx \right) \\
&\le C \left\| u_\varepsilon \eta_R^{\frac{1}{r+m-1}} \right\|_\infty^{\frac{(1-m)(n+2)}{n}} \left\| u_\varepsilon \eta_R^{\frac{1}{r+m-1}} \right\|_1^{\frac{2(r-m+1)}{n(r-1)}} \left\| u_\varepsilon \eta_R^{\frac{1}{r+m-1}} \right\|_r^{\frac{2r(r+m-3)}{n(r-1)}} \\
&\quad \times \left(\int \left| \nabla u_\varepsilon^{\frac{r+m-1}{2}} \right|^2 \eta_R dx + \frac{C}{4R^2}\int u_\varepsilon^{r+m-1} \eta_R^{1-2a} dx \right),
\end{aligned} \tag{3.22}
$$

where $0 < a < \frac{1}{2}$. Then we find by (3.18) and (3.22) that

$$
\begin{aligned}
&\frac{1}{r}\frac{d}{dt}\int u_\varepsilon^r \eta_R dx\\
&\le \frac{C}{R^2}\int\limits_{\operatorname{supp}\eta_R} u_\varepsilon^{r+m-1}dx+\frac{C\varepsilon}{R^2}\int\limits_{\operatorname{supp}\eta_R} u_\varepsilon^{r}dx-\frac{4m(r-1)}{(r+m-1)^2}\int\left|\nabla u_\varepsilon^{\frac{r+m-1}{2}}\right|^2\eta_R dx\\
&\le \frac{C}{R^2}\int\limits_{\operatorname{supp}\eta_R} u_\varepsilon^{r+m-1}dx+\frac{C\varepsilon}{R^2}\int\limits_{\operatorname{supp}\eta_R} u_\varepsilon^{r}dx\\
&\quad-\frac{4m(r-1)}{(r+m-1)^2}\left(C\left\|u_\varepsilon\eta_R^{\frac{1}{r+m-1}}\right\|_r^{\frac{r(nr-n-2m+4)}{n(r-1)}}\times\right.\\
&\quad\left.\times\left\|u_\varepsilon\eta_R^{\frac{1}{r+m-1}}\right\|_\infty^{-\frac{(1-m)(n+2)}{n}}\left\|u_\varepsilon\eta_R^{\frac{1}{r+m-1}}\right\|_1^{-\frac{2(r-m+1)}{n(r-1)}}\right),
\end{aligned}
\tag{3.23}
$$

which yields by letting $\varepsilon\to 0$ and $R\to\infty$ in (3.23), we have

$$
\begin{aligned}
&\frac{1}{r}\frac{d}{dt}\|u(t)\|_r^r\\
&\le -\frac{4Cm(r-1)}{(r+m-1)^2}\|u\|_r^{\frac{r(nr-n-2m+4)}{n(r-1)}}\|u(t)\|_\infty^{-\frac{(1-m)(n+2)}{n}}\|u(t)\|_1^{-\frac{2(r-m+1)}{n(r-1)}}\\
&\le -\frac{4Cm(r-1)}{(r+m-1)^2}\|u\|_r^{\frac{r(nr-n-2m+4)}{n(r-1)}}\|u_0\|_\infty^{-\frac{(1-m)(n+2)}{n}}\|u_0\|_1^{-\frac{2(r-m+1)}{n(r-1)}}.
\end{aligned}
$$

Then we have

$$
\|u(t)\|_r\le C\,(1+t)^{-\frac{n(r-1)}{2r(2-m)}},\quad t\ge 0,
\tag{3.24}
$$

where $C=C(\|u_0\|_1,\|u_0\|_\infty,n,r,m)$.

Combining (1.12) with (3.24), we can have (1.13). □

Remark 3.1. For $r>3-m$, there exists a $n_0>\frac{2(2-m)}{p-1}$ such that for $n>n_0$, we can see that (1.13) degenerates into (3.24).

4. Asymptotic convergence

In order to study the asymptotic convergence of the solution to (1.1)-(1.2), we adopt the following change of similarity variables:

$$
x=yR(t),\ R(t)=(1+\lambda t)^{1/\lambda},\ s=\frac{1}{\lambda}\ln(1+\lambda t),
\tag{4.1}
$$

where $\lambda=2-n(1-m)$. And the new unknown function $\rho=\rho(y,s)$ is given by

$$
\rho(y,s)=R(t)^n u(x,t),
\tag{4.2}
$$

which leads (1.1)-(1.2) to

$$\rho_s = \mathrm{div}_y(\rho y + \nabla_y \rho^m) - e^{-\delta s}\rho^p, \quad y \in \mathbb{R}^n,\ s > 0, \tag{4.3}$$

with initial data

$$\rho(y,0) = \rho_0(y) = u_0(x), \quad y \in \mathbb{R}^n, \tag{4.4}$$

where $\delta = np - nm - 2$. In this section, we consider the diffusion-dominated regime

$$\delta = np - nm - 2 > 0, \text{ i.e. } p > m + \frac{2}{n},$$

where the absorption becomes small perturbation in determining the long-time behavior. In addition, we can directly derive the following corollary of the Theorem 1.2, which we omit the proof here.

Corollary 4.1. *Let the Assumption hold. Suppose* $0 < m < 1$ *and* $p > 1$. *Then for* $1 \le r \le \infty$ *and* $s_0 > 0$, *the global weak solution* $\rho(y,s)$ *to* (4.3)-(4.4) *satisfying*

$$\|\rho(s)\|_r \le e^{\frac{n(r-1)}{r}s}\|u_0\|_r, \quad s \ge 0,$$

$$\|\rho(s)\|_r \le C(\|u_0\|_1, \|u_0\|_\infty, r, p, m), \quad s > s_0,$$

and

$$\|\rho(s)\|_r \le C(\|u_0\|_1, \|u_0\|_\infty, r, p, m)s^{-\frac{r-1}{r(p-1)}}, \quad 0 < s < s_0.$$

Moreover, for the case of $r \ge 3 - m$, *we have*

$$\|\rho(s)\|_r \le C(\|u_0\|_1, \|u_0\|_\infty, n, r, p, m), \quad s > s_0,$$

and

$$\|\rho(s)\|_r \le C(\|u_0\|_1, \|u_0\|_\infty, n, r, p, m)s^{-\max\{\frac{r-1}{r(p-1)}, \frac{n(r-1)}{2r(2-m)}\}}, \quad 0 < s < s_0.$$

Let us define the total mass of $\rho(x,s)$ at time s by

$$M(s) := \int \rho(y,s)dy.$$

Then we have the following lemma.

Lemma 4.1. *Let the Assumption hold. Suppose* $\frac{n-2}{n} < m < 1$. *Then the mass* $M(s)$ *is decreasing and the limiting mass* $M_\infty = M(s = \infty)$ *is strictly positive.*

Proof. Let $\eta_R(y)$ be the cut-off function defined by Lemma 2.3. We consider the approximated problem of (4.3)-(4.4) as follows

$$\partial_s \rho_\varepsilon = \mathrm{div}_y(\rho_\varepsilon y + \nabla_y \rho_\varepsilon^m + \varepsilon e^{(\lambda-2)s}\nabla_y \rho_\varepsilon) - e^{-\delta s}\rho_\varepsilon^p, \quad y \in \mathbb{R}^n,\ s>0, \tag{4.5}$$

$$\rho_\varepsilon(y,0) = (\rho_0 * \widetilde{\eta}_\varepsilon)(y) = \rho_{0\varepsilon}, \quad y \in \mathbb{R}^n, \tag{4.6}$$

for $\varepsilon > 0$. We multiply (4.5) by η_R and integrate it over $\mathbb{R}^n$, which yields that

$$\begin{aligned} &\frac{d}{ds}\int \rho_\varepsilon \eta_R dy \\ &= -\int (\rho_\varepsilon y + \nabla_y \rho_\varepsilon^m + \varepsilon e^{(\lambda-2)s}\nabla_y \rho_\varepsilon)\nabla \eta_R dy - e^{-\delta s}\int \rho_\varepsilon^p \eta_R dy. \end{aligned} \tag{4.7}$$

By (3.13) and (3.15), we have

$$\rho_\varepsilon \to \rho, \qquad \text{strongly in } C(0,T; L^r_{loc}(\mathbb{R}^n)), \tag{4.8}$$

$$\nabla \rho_\varepsilon^m \to \nabla \rho^m, \qquad \text{weakly star in } L^\infty(0,T; L^2(\mathbb{R}^n)), \tag{4.9}$$

for $r \ge 1$. Thus, let $\varepsilon \to 0$ and $R \to \infty$ in (4.7), we have

$$\frac{d}{ds}M(s) = -e^{-\delta s}\int \rho^p dy \ge 0,$$

which yields that the mass $M(s)$ is decreasing. Moreover, combined Corollary 4.1 with $p > m + \frac{2}{n} > 1$, we have

$$\frac{d}{ds}M(s) = -e^{-\delta s}\int \rho^p dy \ge -C(\|u_0\|_1, \|u_0\|_\infty, n, p, m)e^{-\delta s}M(s),$$

which yields

$$M(s) \ge M_\infty \ge M(0)e^{-C(\|u_0\|_1,\|u_0\|_\infty,n,p,m)\int_0^\infty e^{-\delta s}ds} > 0. \tag{4.10}$$

This completes the proof. □

Next, we will begin to prove Theorem 1.3, which is divided to some steps.

4.1. Uniform bound of the second moment

First, we prove the uniformly bound of the second moment of the weak solution $\rho(y,s)$ to (4.3)-(4.4) by the generalized Shanon's inequality (2.4).

Proposition 4.1. *Let the Assumption hold. Suppose $u_0 \in L^1_2(\mathbb{R}^n)$, $\frac{n}{n+2} < m < 1$ and $p > m + \frac{2}{n}$, for the weak solution $\rho(y,s)$ to (4.3)-(4.4), there exists $C > 0$ such that*

$$\int \rho|y|^2 dy \le C, \tag{4.11}$$

where $C = C\left(\int u_0|x|^2dx, \|u_0\|_1, n, m, p\right)$.

Proof. Let $\eta_R(y)$ be the cut-off function defined by Lemma 2.3. We multiply (4.5) by $|y|^2\eta_R$ and integrate it over $\mathbb{R}^n$, which yields that

$$\begin{aligned}
&\frac{d}{ds}\int \rho_\varepsilon|y|^2\eta_R dy\\
&=-\int(\rho_\varepsilon y+\nabla_y\rho_\varepsilon^m+\varepsilon e^{(\lambda-2)s}\nabla_y\rho_\varepsilon)(2y\eta_R+|y|^2\nabla\eta_R)dy\\
&\quad -e^{-\delta s}\int\rho_\varepsilon^p|y|^2\eta_R dy. \qquad (4.12)
\end{aligned}$$

By (4.8)-(4.9), letting $\varepsilon\to 0$ and $R\to\infty$ in (4.12), we have

$$\frac{d}{ds}\int\rho|y|^2dy = 2n\int\rho^m dy - 2\int\rho|y|^2dy - e^{-\delta s}\int\rho^p|y|^2dy. \qquad (4.13)$$

Since it holds by the generalized Shanon's inequality (2.6) and Corollary 4.1 that

$$\int\rho^m dy \le C\left(\int\rho|y|^2dy\right)^{m\sigma}, \qquad (4.14)$$

substituting (4.14) into (4.13), we have

$$\frac{d}{ds}\int\rho|y|^2dy \le C\left(\int\rho|y|^2dy\right)^{m\sigma} - 2\int\rho|y|^2dy, \qquad (4.15)$$

where $\sigma = \frac{n(1-m)}{2m}$ and $C = C(\|u_0\|_1, n, m, p)$. Thus, we obtain

$$\begin{aligned}
\int\rho|y|^2dy &\le e^{-2(1-m\sigma)s}\left(\left(\int\rho_0|y|^2dy\right)^{1-m\sigma}+C(1-m\sigma)s\right)^{\frac{1}{1-m\sigma}}\\
&\le C\left(\int u_0|x|^2dx, \|u_0\|_1, n, m, p\right),
\end{aligned}$$

since $1-m\sigma>0$ for $\frac{n}{n+2}<m<1$. □

4.2. Entropy production

Next, we derive the entropy production of the (4.3)-(4.4).

Proposition 4.2. *Let the Assumption hold. For the weak solution* $\rho(y,s)$ *to* (4.3)-(4.4)*, define the entropy energy*

$$E(\rho)(s) = \frac{1}{m-1}\int\rho^m dy + \frac{1}{2}\int|y|^2\rho dy. \qquad (4.16)$$

Then, for $u_0 \in L^1_2(\mathbb{R}^n)$, $\frac{n}{n+2} < m < 1$ *and* $p > m + \frac{2}{n}$, *it holds that*

$$\begin{aligned} &E(\rho)(s) - E(\rho_0) \\ &\le -\iint_0^s \rho(\tau) \left|\nabla\left(\frac{m}{m-1}\rho(\tau)^{m-1} + \frac{1}{2}|y|^2\right)\right|^2 dyd\tau \\ &\quad -\int_0^s e^{-\delta s}\int \rho(\tau)^p \left(\frac{m}{m-1}\rho(\tau)^{m-1} + \frac{1}{2}|y|^2\right) dyd\tau. \end{aligned} \tag{4.17}$$

Proof. For $R_1 < R_2$, let $\eta_{R_1}(y)$ and $\eta_{R_2}(y)$ be the smooth cut-off function defined by Lemma 2.3 with $R = R_1$ and $R = R_2$, respectively. We set

$$A_\varepsilon \equiv \frac{m}{m-1}\rho_\varepsilon^{m-1} + \frac{1}{2}|y|^2 + \varepsilon e^{(\lambda-2)s}(\log \rho_\varepsilon + 1),$$
$$\tilde{A}_\varepsilon \equiv \frac{m}{m-1}\rho_\varepsilon^{m-1}\eta_{R_1} + \frac{1}{2}|y|^2\eta_{R_1} + \varepsilon e^{(\lambda-2)s}(\log \rho_\varepsilon + 1)\eta_{R_1}.$$

Then (4.5) can be converted to

$$\partial_s \rho_\varepsilon = \mathrm{div}_y(\rho_\varepsilon \nabla_y A_\varepsilon) - e^{-\delta s}\rho_\varepsilon^p,$$

which, by multiplying by $\tilde{A}_\varepsilon \eta_{R_2}$ and integrating with respect to y over $\mathbb{R}^n$, can yield

$$\int \partial_s \rho_\varepsilon \tilde{A}_\varepsilon \eta_{R_2} dy = -\int \rho_\varepsilon \nabla_y A_\varepsilon \nabla_y(\tilde{A}_\varepsilon \eta_{R_2}) dy - e^{-\delta s}\int \rho_\varepsilon^p \tilde{A}_\varepsilon \eta_{R_2} dy. \tag{4.18}$$

Since

$$A_\varepsilon = A_\varepsilon(1-\eta_{R_1}) + \tilde{A}_\varepsilon,$$

the first term of the right hand side in (4.18) is

$$\begin{aligned} &-\int \rho_\varepsilon \nabla_y A_\varepsilon \nabla_y(\tilde{A}_\varepsilon \eta_{R_2}) dy \\ &= -\int \rho_\varepsilon \nabla_y(A_\varepsilon(1-\eta_{R_1}))\nabla_y \tilde{A}_\varepsilon \eta_{R_2} dy - \int \rho_\varepsilon |\nabla_y \tilde{A}_\varepsilon|^2 \eta_{R_2} dy \\ &\quad -\int \rho_\varepsilon \nabla_y A_\varepsilon \tilde{A}_\varepsilon \nabla_y \eta_{R_2} dy. \end{aligned} \tag{4.19}$$

Moreover, the first term of the right hand side in (4.19) is

$$
\begin{aligned}
&-\int \rho_\varepsilon \nabla_y (A_\varepsilon(1-\eta_{R_1}))\nabla_y \tilde{A}_\varepsilon \eta_{R_2} dy \\
&= -\int \rho_\varepsilon |\nabla_y A_\varepsilon|^2 \eta_{R_1}(1-\eta_{R_1})\eta_{R_2} dy \\
&\quad -\frac{1}{2}\int \rho_\varepsilon \nabla_y |A_\varepsilon|^2 \nabla_y \eta_{R_1}(1-\eta_{R_1})\eta_{R_2} dy \\
&\quad +\int \rho_\varepsilon A_\varepsilon \nabla_y \tilde{A}_\varepsilon \nabla_y \eta_{R_1}\eta_{R_2} dy.
\end{aligned} \tag{4.20}
$$

On the other hand, the left hand side in (4.18) is

$$
\begin{aligned}
&\int \partial_s \rho_\varepsilon \tilde{A}_\varepsilon \eta_{R_2} dy \\
&= \frac{d}{ds}\int \left(\frac{1}{m-1}\rho_\varepsilon^m + \frac{1}{2}|y|^2\rho_\varepsilon + \varepsilon e^{(\lambda-2)s}\rho_\varepsilon \log \rho_\varepsilon\right)\eta_{R_1}\eta_{R_2} dy.
\end{aligned} \tag{4.21}
$$

Substituting (4.19), (4.20) and (4.21) into (4.18), and integrating (4.18) in s, we obtain

$$
\begin{aligned}
&\int \left(\frac{1}{m-1}\rho_\varepsilon^m + \frac{1}{2}|y|^2\rho_\varepsilon + \varepsilon e^{(\lambda-2)s}\rho_\varepsilon \log \rho_\varepsilon\right)\eta_{R_1}\eta_{R_2} dy \\
&= \int \left(\frac{1}{m-1}\rho_{0\varepsilon}^m + \frac{1}{2}|y|^2\rho_{0\varepsilon} + \varepsilon e^{(\lambda-2)s}\rho_{0\varepsilon} \log \rho_{0\varepsilon}\right)\eta_{R_1}\eta_{R_2} dy \\
&\quad -\int_0^s e^{-\delta s}\int \rho_\varepsilon^p \tilde{A}_\varepsilon \eta_{R_2} dy d\tau - \iint_0^s \rho_\varepsilon |\nabla_y A_\varepsilon|^2 \eta_{R_1}(1-\eta_{R_1})\eta_{R_2} dy\tau \\
&\quad -\frac{1}{2}\iint_0^s \rho_\varepsilon \nabla_y |A_\varepsilon|^2 \nabla_y \eta_{R_1}(1-\eta_{R_1})\eta_{R_2} dy d\tau \\
&\quad +\iint_0^s \rho_\varepsilon A_\varepsilon \nabla_y \tilde{A}_\varepsilon \nabla_y \eta_{R_1}\eta_{R_2} dy d\tau \\
&\quad -\iint_0^s \rho_\varepsilon |\nabla_y \tilde{A}_\varepsilon|^2 \eta_{R_2} dy d\tau - \iint_0^s \rho_\varepsilon \nabla_y A_\varepsilon \tilde{A}_\varepsilon \nabla_y \eta_{R_2} dy d\tau,
\end{aligned} \tag{4.22}
$$

which yields by letting $R_2 \to \infty$ that

$$
\begin{aligned}
&\int \left(\frac{1}{m-1}\rho_\varepsilon^m + \frac{1}{2}|y|^2\rho_\varepsilon + \varepsilon e^{(\lambda-2)s}\rho_\varepsilon \log \rho_\varepsilon\right)\eta_{R_1} dy \\
&\le \int \left(\frac{1}{m-1}\rho_{0\varepsilon}^m + \frac{1}{2}|y|^2\rho_{0\varepsilon} + \varepsilon e^{(\lambda-2)s}\rho_{0\varepsilon} \log \rho_{0\varepsilon}\right)\eta_{R_1} dy
\end{aligned}
$$

$$
\begin{aligned}
&-\int_0^s e^{-\delta s}\int \rho_\varepsilon^p \tilde{A}_\varepsilon dyd\tau-\iint_0^s \rho_\varepsilon|\nabla_y A_\varepsilon|^2\eta_{R_1}(1-\eta_{R_1})dyd\tau\\
&-\frac{1}{2}\iint_0^s \rho_\varepsilon\nabla_y|A_\varepsilon|^2\nabla_y\eta_{R_1}(1-\eta_{R_1})dyd\tau+\iint_0^s \rho_\varepsilon A_\varepsilon\nabla_y\tilde{A}_\varepsilon\nabla_y\eta_{R_1}dyd\tau\\
&-\iint_0^s \rho_\varepsilon|\nabla_y\tilde{A}_\varepsilon|^2 dyd\tau. \qquad (4.23)
\end{aligned}
$$

By Theorem 1.1, Lemma 2.4 and Proposition 4.1, we have

$$
\begin{aligned}
&\rho_\varepsilon\to\rho, && \text{weakly star in } L^\infty(0,T;L^m(\mathbb{R}^n)),\\
&\nabla\rho_\varepsilon^{m-\frac{1}{2}}\to\nabla\rho^{m-\frac{1}{2}}, && \text{weakly in } L^2(0,T;L^2(\mathbb{R}^n)),
\end{aligned}
$$

which, combined with (4.8) and (4.9), can yield by passing $\varepsilon\to 0$ that

$$
\begin{aligned}
&\int\left(\frac{1}{m-1}\rho^m+\frac{1}{2}|y|^2\rho\right)\eta_{R_1}dy-\int\left(\frac{1}{m-1}\rho_0^m+\frac{1}{2}|y|^2\rho_0\right)\eta_{R_1}dy\\
&\le-\iint_0^s \rho\left|\nabla_y\left(\frac{m}{m-1}\rho_\varepsilon^{m-1}+\frac{1}{2}|y|^2\right)\right|^2\eta(1-\eta_{R_1})dyd\tau\\
&-\int_0^s e^{-\delta\tau}\int\rho^p\left(\frac{m}{m-1}\rho_\varepsilon^{m-1}\eta_{R_1}+\frac{1}{2}|y|^2\eta_{R_1}\right)dyd\tau\\
&-\frac{1}{2}\iint_0^s \rho\nabla_y\left|\frac{m}{m-1}\rho_\varepsilon^{m-1}+\frac{1}{2}|y|^2\right|^2\nabla_y\eta_{R_1}(1-\eta_{R_1})dyd\tau\\
&+\iint_0^s \rho\left(\frac{m}{m-1}\rho_\varepsilon^{m-1}+\frac{1}{2}|y|^2\right)\nabla_y\left(\frac{m}{m-1}\rho_\varepsilon^{m-1}\eta_{R_1}+\frac{1}{2}|y|^2\eta_{R_1}\right)\nabla_y\eta_{R_1}dyd\tau\\
&-\iint_0^s \rho\left|\nabla_y\left(\frac{m}{m-1}\rho_\varepsilon^{m-1}\eta_{R_1}+\frac{1}{2}|y|^2\eta_{R_1}\right)\right|^2 dyd\tau,
\end{aligned}
$$

which yields (4.17) for $R_1\to\infty$. □

4.3. *Properties of the relative entropy*

Then we will show two properties of the relative entropy:

$$
E(f|g)=E(f(y))-E(g(y)). \qquad (4.24)
$$

Proposition 4.3 *([17]). Let $m > \frac{n-1}{n}$ and $m \neq 1$. Then, for any non-negative function $f \in L^1_2(\mathbb{R}^n)$ such that $\nabla f^{m-\frac{1}{2}} \in L^2(\mathbb{R}^n)$ and $\|f\|_1 = M$, it holds that*

$$E(f|\rho_{M(s)}) \leq \frac{1}{2}\int f(y)\left|\frac{m}{m-1}\nabla f(y)^{m-1} + y\right|^2 dy. \tag{4.25}$$

It is clear that Proposition 4.3 implies the upper bound of the relative entropy. Moreover, we consider the lower bound of the relative entropy by Lemma 2.5 with $f = \rho$ and $g = \rho_{M(s)}$. Since it holds by the decreasing of $M(s)$ and the increasing of $\theta_{M(s)}$ that

$$\|\rho_{M(s)}\|_{2-m} \leq \|\rho_{M(s)}\|_\infty^{1-m}\|\rho_{M(s)}\|_1 \leq \frac{\gamma}{\theta_{M(0)}}M(0),$$

for $\frac{n-2}{n} < m < 1$, we have $\rho_{M(s)} \in L^{2-m}(\mathbb{R}^n)$. In addition,

$$\frac{1}{2}|y|^2 = \frac{m}{1-m}\left(\rho_{M(s)}^{m-1} - \frac{\theta_{M(s)}}{\gamma}\right).$$

Thus, we have

$$\begin{aligned}
D(\rho|\rho_{M(s)}) &= \frac{1}{m-1}\int \rho^m dy - \frac{1}{m-1}\int \rho_{M(s)}^m dy \\
&\quad - \frac{m}{m-1}\int \rho_{M(s)}^{m-1}(\rho - \rho_{M(s)})dy \\
&= \frac{1}{m-1}\int \rho^m dy - \frac{1}{m-1}\int \rho_{M(s)}^m dy + \frac{1}{2}\int |y|^2(\rho - \rho_{M(s)})dy \\
&= E(\rho|\rho_{M(s)}),
\end{aligned}$$

which yields by Csiszár-Kullback inequality (2.7) that

$$\|\rho - \rho_{M(s)}\|_1 \leq C\sqrt{E(\rho|\rho_{M(s)})}, \tag{4.26}$$

where

$$\begin{aligned}
C &= M(s)^{\frac{m-2}{2}}\left(\frac{2}{m}\int \rho_{M(s)}^{2-m}dx\right)^{\frac{1}{2}} \\
&\leq \left(M(0)e^{-C(\|u_0\|_1,\|u_0\|_\infty,n,p,m)\int_0^\infty e^{-\delta s}ds}\right)^{\frac{m-2}{2}}\left(\frac{2}{m}\int \rho_{M(s)}^{2-m}dx\right)^{\frac{1}{2}} \\
&< \infty,
\end{aligned}$$

since (4.10) and $\frac{n-2}{n} < \frac{n-1}{n} < m < 1$.

4.4. Asymptotic convergence

Let us consider

$$\begin{aligned}\frac{d}{ds}E(\rho_{M(s)}) &= \frac{d}{ds}\int \rho_{M(s)}\left(\frac{1}{m-1}\rho_{M(s)}^{m-1}+\frac{1}{2}|y|^2\right)dy\\ &= \int \partial_s\rho_{M(s)}\left(\frac{m}{m-1}\rho_{M(s)}^{m-1}+\frac{1}{2}|y|^2\right)dy. \end{aligned}\tag{4.27}$$

Note that

$$\frac{m}{m-1}\rho_{M(s)}^{m-1}+\frac{1}{2}|y|^2=-\frac{1}{2}\theta_{M(s)},\tag{4.28}$$

then we have, from (4.27) and (4.28), that

$$\begin{aligned}\frac{d}{ds}E(\rho_{M(s)}) &= -\frac{1}{2}\theta_{M(s)}\int \partial_s\rho_{M(s)}dy\\ &= -\frac{1}{2}\theta_{M(s)}\frac{d}{ds}\int \rho_{M(s)}dy\\ &= -\frac{1}{2}\theta_{M(s)}\frac{d}{ds}M(s)\\ &= \frac{1}{2}\theta_{M(s)}e^{-\delta s}\int \rho^p dy\\ &= -e^{-\delta s}\int \rho^p\left(\frac{m}{m-1}\rho_{M(s)}^{m-1}+\frac{1}{2}|y|^2\right)dy. \end{aligned}\tag{4.29}$$

On the other hand, by virtue of (4.17), we have

$$\begin{aligned}\frac{d}{ds}E(\rho) \le & -\int \rho\left|\nabla\left(\frac{m}{m-1}\rho^{m-1}+\frac{1}{2}|y|^2\right)\right|^2dy\\ & -e^{-\delta s}\int \rho^p\left(\frac{m}{m-1}\rho^{m-1}+\frac{1}{2}|y|^2\right)dy. \end{aligned}\tag{4.30}$$

Combining (4.29) and (4.30), we obtain by Mean-Value Theorem that

$$\begin{aligned}&\frac{d}{ds}E(\rho|\rho_{M(s)})\\ &\le -\int \rho\left|\nabla\left(\frac{m}{m-1}\rho^{m-1}+\frac{1}{2}|y|^2\right)\right|^2dy\\ &\quad -\frac{m}{m-1}e^{-\delta s}\int \rho^p\left(\rho^{m-1}-\rho_{M(s)}^{m-1}\right)dy\\ &\le -\int \rho\left|\nabla\left(\frac{m}{m-1}\rho^{m-1}+\frac{1}{2}|y|^2\right)\right|^2dy \end{aligned}$$

$$
\begin{aligned}
&-me^{-\delta s}\int \rho^{p+m-2}\left(\rho-\rho_{M(s)}\right)dy \\
&\le -\int \rho\left|\nabla\left(\frac{m}{m-1}\rho^{m-1}+\frac{1}{2}|y|^2\right)\right|^2 dy \\
&+me^{-\delta s}\|\rho\|_\infty^{p+m-2}\|\rho-\rho_{M(s)}\|_1,
\end{aligned}
\tag{4.31}
$$

which implies by (4.25), (4.26) and Corollary 4.1 that

$$
\frac{d}{ds}E(\rho|\rho_{M(s)}) \le -2E(\rho|\rho_{M(s)}) + Ce^{-\delta s}\sqrt{E(\rho|\rho_{M(s)})}, \tag{4.32}
$$

where $C = C(\|u_0\|_1, \|u_0\|_\infty, n, p, m)$. If the relative entropy $E(\rho|\rho_{M(s)})$ is strictly positive on any time interval (s_1, s_2), then

$$
\frac{d}{ds}E(\rho|\rho_{M(s)})^{\frac{1}{2}} \le -E(\rho|\rho_{M(s)})^{\frac{1}{2}} + Ce^{-\delta s}, \tag{4.33}
$$

which yields

$$
\frac{d}{ds}\left(e^{s}E(\rho|\rho_{M(s)})^{\frac{1}{2}}\right) \le Ce^{(1-\delta)s} \tag{4.34}
$$

for $s \in (s_1, s_2)$. By choosing s_1 as small as possible, we have either $s_1 = 0$ or $E(\rho(s_1)|\rho_{M(s_1)}) = 0$, and then we get the following exponential convergence for any time $s > 0$

$$
E(\rho|\rho_{M(s)}) \le Ce^{-2\min\{1,\delta\}s}, \quad \text{for } \delta \neq 1, \tag{4.35}
$$

and

$$
E(\rho|\rho_{M(s)}) \le C(1+s)^2 e^{-2s}, \quad \text{for } \delta = 1. \tag{4.36}
$$

By the Csiszár-Kullback inequality (4.26), we have

$$
\|\rho - \rho_{M(s)}\|_1 \le Ce^{-\min\{1,\delta\}s}, \quad \text{for } \delta \neq 1,
$$

and

$$
\|\rho - \rho_{M(s)}\|_1 \le C(1+s)e^{-s}, \quad \text{for } \delta = 1.
$$

Going back to the original variables, it follows that

$$
\|u - U_{\widetilde{M}(t)}\|_1 \le C(1+\lambda t)^{-\frac{1}{\lambda}\min\{1,\delta\}}, \quad \text{for } \delta \neq 1,
$$

and

$$
\|u - U_{\widetilde{M}(t)}\|_1 \le C(1+\lambda t)^{-\frac{1}{\lambda}}\ln(1+\lambda t), \quad \text{for } \delta = 1,
$$

where $C = C(\|u_0\|_1, \|u_0\|_\infty, n, p, m)$. Moreover, for $1 < r < \infty$, we have

$$\begin{aligned}\|u - U_{\widetilde{M}(t)}\|_r &= e^{-\frac{n(r-1)}{r}s}\left(\int |\rho - \rho_{M(s)}|^r dy\right)^{\frac{1}{r}}\\ &\le e^{-\frac{n(r-1)}{r}s}\|\rho - \rho_{M(s)}\|_\infty^{\frac{r-1}{r}}\|\rho - \rho_{M(s)}\|_1^{\frac{1}{r}}\\ &\le C(1+\lambda t)^{-\frac{n(r-1)+\min\{1,\delta\}}{\lambda r}}, \quad \text{for } \delta \ne 1,\end{aligned}$$

and

$$\|u - U_{\widetilde{M}(t)}\|_r \le C(1+\lambda t)^{-\frac{n(r-1)+1}{\lambda r}} \ln(1+\lambda t), \quad \text{for } \delta = 1,$$

where $C = C(\|u_0\|_1, \|u_0\|_\infty, n, p, m, r)$.

Data availability

No data was used for the research described in the article.

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