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SMOOTH TRANSONIC STEADY STATES OF HYDRODYNAMIC MODEL FOR SEMICONDUCTORS*

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Abstract. In this paper, we investigate the existence and regularity of the smooth transonic steady solutions of Euler–Poisson equations representing for the hydrodynamic model of semiconductors. Different from the previous studies with the various setting boundaries, we observe that the crucial mechanism to affect the structure of the stationary Euler–Poisson equations is the doping profile. When the doping profile is supersonic, regardless of the boundary settings, we prove that the Euler–Poisson system possesses two C^∞-smooth transonic solutions. One is from the supersonic region to the subsonic region, and the other is of the inverse direction. However, when the doping profile is subsonic, the case is more complicated. We prove that there is no continuous transonic solution if the semiconductor effect is small enough, but there will arise two kinds of smooth transonic solutions when the semiconductor effect is large enough. Both of them are from the supersonic region the to subsonic region, where one is a unique C^∞-smooth transonic solution with a relatively large number as its derivative at the sonic point, and the other consists of a class of smooth transonic solutions with another relatively small number as the derivative at the sonic point. This class of solutions are proved mostly to be C^∞ smooth, except for a special case in which we only prove the C^m smoothness. The method adopted is mainly the manifold analysis and the singularity analysis near the sonic line and the singular point.

Key words. Euler–Poisson equations, transonic solutions, hydrodynamic model of semiconductors, doping profile

AMS subject classifications. 35R35, 35Q35, 76N10, 35J70

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1. Introduction.

This paper is concerned with the smooth transonic solutions for the one-dimensional steady hydrodynamic model for semiconductors, which is written as the Euler–Poisson equations with the relaxation effect

$$
\begin{cases}
n_t + (nu)_x = 0, \\
(nu)_t + (nu^2 + P(n))_x = nE - \frac{nu}{\tau}, \\
E_x = n - b(x).
\end{cases} \tag{1.1}
$$

This model is used to describe the motion of electrons in semiconductor devices or plasmas [4, 6, 9, 20]. Here the particle density n, the particle velocity u, and the electric field E are the unknowns, which are coupled through the Euler equation and

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the Poisson equation. $P = P(n)$ is the pressure function of the electron density. For the ideal gas law of isentropic case, the pressure function is physically represented by

$$P(n) = Tn^{\gamma},$$

where $T > 0$ is a constant, and $\gamma > 1$ represents the adiabatic exponent. In this paper, we mainly consider the isothermal case, i.e., $\gamma = 1$ for simplicity of analysis. The function $b(x) > 0$ is the doping profile standing for a background density of charged ions, and we take it as a constant in the present paper. $\tau > 0$ means the momentum relaxation time. We set

$$\alpha := \frac{1}{\tau}, \text{the reciprocal of the delay time of the current,} \tag{1.2}$$

and denote

$$J = nu, \text{the current density.} \tag{1.3}$$

We consider the current driven flow; thus the current density J is a prescribed constant. Then the stationary flow of (1.1) is

$$\begin{cases} J = constant, \\ \left(\frac{J^2}{n} + P(n)\right)_x = nE - \alpha J, \\ E_x = n - b. \end{cases} \tag{1.4}$$

Using the terminology from gas dynamics, we call $c := \sqrt{P'(n)} = \sqrt{T} > 0$ the sound speed for $P(n) = Tn$. Thus, the stationary flow is supersonic/sonic/subsonic if the fluid velocity satisfies

$$\text{fluid velocity} : u = \frac{J}{n} > (\text{ or } \quad =, \quad \text{or } \ <) c = \sqrt{P'(n)} = \sqrt{T} : \text{sound speed.}$$

Note that if $(n(x), E(x))$ is a solution to (1.4) for a given constant J, then $(n(1-x), -E(1-x))$ is a solution to (1.4) with respect to $-J$ and $b(1-x)$. Therefore, we just think about the case of $J > 0$.

Without loss of generality, in the subsequent analysis, assume that $T = 1$, i.e., $P = n$; then the above equations (1.4) can be reduced to

$$\begin{cases} \left(1 - \frac{J^2}{n^2}\right) n_x = nE - \alpha J, \\ E_x = n - b. \end{cases} \tag{1.5}$$

Thus, it is identified that $n > J$ is for the subsonic flow, $n = J$ is for the sonic flow, and $0 < n < J$ represents the supersonic flow.

As we know, with different boundary settings, the stationary Euler–Poisson system (1.4) may or may not possess the subsonic/supersonic/transonic solutions. For subsonic flows, Degond and Markowich first proved that when the system is with a strong subsonic background (subsonic Ohmic contact boundary and subsonic doping profile), the system exists as a unique subsonic smooth solution in the one-dimensional case [8] and for potential flow in the multidimensional case [9]. Under different subsonic boundary conditions and in the multidimensional cases the subsonic flows were then studied in [2, 3, 4, 13, 14, 15, 21, 22].

On the other hand, when the boundary setting is still subsonic but the doping profile is supersonic, Ascher et al. [1] first observed a one-dimensional transonic solution via a phase-plane analysis. Furthermore, Gamba and Morawetz [11, 12] constructed the transonic solutions with a shock by artificial viscosity approximation. The well posedness and the structure of the transonic flow were studied in [10, 19, 24] for Euler–Poisson equations. The structure stability and the time-asymptotic behavior of solutions were investigated in [13, 14, 15, 18, 21, 22].

For the supersonic flow with both the supersonic boundary and doping profile, Peng and Violet [23] and Bae et al. [5] obtained the existence and uniqueness of the supersonic solution.

Recently, when the boundary is subjected to be sonic (the critical case), Li et al. [16, 17] first classified the structure of all-type solutions to (1.4). When the doping profile is subsonic, the steady Euler–Poisson system possesses a unique subsonic solution, at least one supersonic solution, and infinitely many shock-transonic solutions when the semiconductor effect is weak ($\tau \gg 1$) and infinitely many C^1-smooth transonic solutions when the semiconductor effect is strong ($\tau \ll 1$); while when the doping profile is supersonic and far from the sonic line, there is not any physical (subsonic/supersolic/transonic) solution. The supersonic solution and many shock-transonic solutions exist only when the doping profile is sufficiently close to the sonic line. When the doping profile is transonic, according to two cases of the subsonic-dominated and supersonic-dominated doping profile, Chen et al. [7] further classified the structure of all subsonic/supersonic/shock-transonic solutions.

However, from the existing studies [1, 16, 17, 24], there are no more detailed discussions on the structure of the directions of the transonic solutions, what kind regularity of the smooth transonic solutions, and in particular, what will be the crucial mechanism of the system to possess the transonic solutions. To answer these technical questions will be the main goal of the present paper.

In this paper, we observe that the crucial mechanism to affect the structure of the stationary Euler–Poisson equations is the doping profile, regardless of the boundary settings. When the doping profile is supersonic, we prove that the Euler–Poisson system possesses two C^∞-smooth transonic solutions. One is from the supersonic region to the subsonic region, and the other is of the inverse direction. However, when the doping profile is subsonic, the case is more complicated. We prove that there is no continuous transonic solution if the semiconductor effect is small enough, but there will arise two kinds of smooth transonic solutions when the semiconductor effect is large enough. Both of them are from the supersonic region to the subsonic region, where one is a unique C^∞-smooth transonic solution with a relatively large number as its derivative at the sonic point, and the other is a smooth transonic solution with another relatively small number as its derivative at the sonic point. This class of solutions is proved mostly to be C^∞ smooth, except for a special case in which we only prove the C^m smoothness. Remarkably, this is the first result to show the C^∞ smoothness of the transonic solutions. The method adopted is mainly the manifold analysis and singularity analysis near the sonic line and the singular point.

The main results of this paper are as follows.

THEOREM 1.1. *Let the doping profile be supersonic, i.e., $b < J$; then the steady-state Euler–Poisson equations of* (1.5) *possess two C^∞-smooth transonic solutions, one of which is from the subsonic region to the supersonic region, and another is from the supersonic region to the subsonic region.*

THEOREM 1.2. *Let the doping profile be subsonic, i.e., $b > J$; then the steady-state Euler–Poisson equations of* (1.5) *admit the following:*

1. *When $\alpha = \frac{1}{\tau} \geq 0$, but α is sufficiently small, the system* (1.5) *has no continuous transonic solution.*
2. *When $\alpha = \frac{1}{\tau} > 0$, and α is large enough, the system* (1.5) *possesses two kinds of smooth transonic solutions, and they are all from the supersonic region to the subsonic region. One is corresponding to a unique and C^∞-smooth solution, which has a relatively large derivative at the sonic point. The other kind is corresponding to a class of smooth transonic solutions with a relatively small derivative at the sonic point. For the last class of solutions we prove that when $h(\alpha, b, J) \bar{\in} Z$ they are C^∞ smooth, and when $h(\alpha, b, J) \in Z$ they are C^m smooth, where $h(\alpha, b, J) = \frac{8(b-J)}{[\alpha - \sqrt{\alpha^2 - 8(b-J)}]^2}$, $m = \frac{8(b-J)}{[\alpha - \sqrt{\alpha^2 - 8(b-J)}]^2} - 1$, and Z is the set of all integers.*

Remark 1.

1. The proof of the above theorems is mainly by phase-plane analysis and by singularity analysis near the sonic line and around the sonic point.
2. The doping profile b is taken as a constant in the paper, and of course, it will affect the regularity when it is a general function, and it will be more complicated for this case.
3. We take $P(n) = Tn$ just for simplicity, and the method in this paper can be applied for the case of $P(n) = Tn^\gamma$.
4. Due to the technique reason, we can only derive the C^m-smooth result for some special case in Theorem 1.2, and maybe it is not optimal.

The main structure of this paper is as follows. In section 2, the existence and smoothness of the transonic solutions of the system (1.5) are discussed when the doping profile is supersonic. It is divided into two parts. The first part is for $\alpha = 0$; that is, the case of infinite relaxation time $\tau = \infty$. The second part is for $\alpha > 0$, where the trajectory equation can not be solved directly. We make a local analysis on the phase plane near the sonic line and then prove the existence and smoothness of the trajectory that is passing through the singular point. Section 3 is devoted to the case of the subsonic doping profile. It is also divided into two parts. The first part is that when α is small enough, we prove that there is no continuous transonic solution. The second part is that for a α large enough, we prove that the system has infinitely many smooth transonic solutions by combining the local manifold analysis and the singularity analysis around the sonic point. Moreover, we describe these trajectories, which converge to the singular point in detail; that is, only one trajectory converges to the singular point with a large slope, and it is proved to be C^∞ smooth. The other trajectories that converge to the singular point have a relatively small slope, and whether these trajectories can be C^∞ smooth depends on the value of $h(\alpha, b, J)$, which is defined in section 3.

2. The case of $b < J$ (supersonic doping profile).

2.1. The case: $\alpha = 0$. This is corresponding to zero semiconductor effect. By integrating equations and solving the ODE, the trajectory can be solved; thus the proof is simplified. For this case, (1.5) is transformed into the simple case

$$\begin{cases} \left(1 - \frac{J^2}{n^2}\right) n_x = nE, \\ E_x = n - b. \end{cases} \tag{2.1}$$

From the first equation of the system (2.1), it can be seen that a continuous transonic solution must pass through the point $(J, 0)$ for $(n(x), E(x))$. We refer to this point as the singular point. For $n \neq J$, write

$$\begin{cases} n_x = \frac{n^3 E}{(n+J)(n-J)}, \\ E_x = n - b, \end{cases} \tag{2.2}$$

and the trajectory equation of the system (2.1) can be obtained by integrating the equations of (2.2) as follows:

$$\frac{dE}{dn} = \frac{(n+J)(n-J)(n-b)}{n^3 E}. \tag{2.3}$$

Lemma 2.1. *Let the profile $b < J$; then for $\alpha = 0$,* (2.3) *has two continuous transonic trajectories, and they are C^∞, smooth.*

Proof. First, we prove that for the system (2.3) there exist two continuous trajectories that pass through the singular point $(J, 0)$. Because the equation is variable separable in this case, it can be solved directly. From (2.3),

$$EdE = \frac{(n+J)(n-J)(n-b)}{n^3} dn.$$

Now, integrating both sides of the above equation, we have

$$\frac{E^2}{2} = n + \frac{J^2}{n} - b \ln n - \frac{bJ^2}{2n^2} + C.$$

Substituting $(n, E) = (J, 0)$ into the above equation, we then obtain

$$C = \frac{b}{2} + b \ln J - 2J,$$

and thus

$$E^2 = 2n + \frac{2J^2}{n} - 2b \ln n - \frac{bJ^2}{n^2} + 2C. \tag{2.4}$$

Denote

$$g(n) := 2n + \frac{2J^2}{n} - 2b \ln n - \frac{bJ^2}{n^2} + 2C.$$

Then

$$E(n) = \begin{cases} \sqrt{g(n)} & \text{for } E > 0, \\ -\sqrt{g(n)} & \text{for } E < 0. \end{cases}$$

Since the singularity is produced on the sonic line, i.e., $n = J$, therefore, we should do the Taylor expansion of the trajectory equation (2.4) near the singular point $(n, E) = (J, 0)$. That means we just need to do the Taylor expansion of $g(n)$ at the singular point $(n, E) = (J, 0)$; that is,

$$g(n) = \sum_{k=0}^{\infty} \frac{g^{(k)}(J)}{k!} (n - J)^k,$$

where $g(J) = g'(J) = 0$ and $g''(J) > 0$ can be obtained by simple calculation, and then

$$E(n) = \begin{cases} |n-J|\sqrt{\frac{g^{(2)}(J)}{2!} + \sum_{k=3}^{\infty}\frac{g^{(k)}(J)}{k!}(n-J)^{k-2}} & \text{for } E > 0, \\ -|n-J|\sqrt{\frac{g^{(2)}(J)}{2!} + \sum_{k=3}^{\infty}\frac{g^{(k)}(J)}{k!}(n-J)^{k-2}} & \text{for } E < 0. \end{cases}$$

In order to simplify the formula, we denote

$$w(n) := \sqrt{\frac{g^{(2)}(J)}{2!} + \sum_{k=3}^{\infty}\frac{g^{(k)}(J)}{k!}(n-J)^{k-2}}. \tag{2.5}$$

Therefore, for $E > 0$ and $n > J$, it holds

$$E(n) = (n-J)\cdot w(n).$$

Meanwhile, when $E < 0$ and $n < J$, the same result is obtained, namely, $E(n) = (n-J)\cdot w(n)$. Thus we can obtain two branches and denote them as

$$E_1: \quad E(n) = (n-J)\cdot w(n),$$

and

$$E_2: \quad E(n) = (J-n)\cdot w(n).$$

As shown in Figure 1, it is easy to see that the trajectory solutions of this system are symmetric about $E = 0$. According to the direction of the trajectory, we know one of the trajectories is from the supersonic to the subsonic region, and another one is from the subsonic to the supersonic region. Next, we analyze the regularity between the solutions of the original system (2.1) and these trajectories. Because this system is autonomous, let us assume $\lim_{x\to x_0} n(x) = J$; then from the first equation of the system (2.2),

$$\lim_{x\to x_0} n_x = \lim_{n\to J}\frac{n^3}{n+J}\cdot\frac{E}{n-J}. \tag{2.6}$$

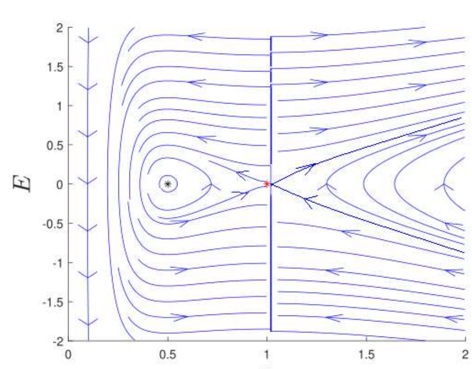

FIG. 1. *Phase plane of (n, E) with $\alpha = 0$, $b = 0.5$, and $J = 1$ in the case of a supersonic doping profile. The red $*$ is the singular point $(1, 0)$; the black $*$ is the stable point $(0.5, 0)$.*

Because $w(n)$ defined in (2.5) is a part of the analytic expansion of $g(n)$, $w(n)$ is infinitely smooth with respect to n; i.e., $w(n) \in C^\infty$. Thus, $E(n) \in C^\infty$ with $E(J) = 0$. In fact, by Hospital's rule, we conclude n_x is continuous. Furthermore, the value of n_x can be calculated in this case. Since $E_x = E_n \cdot n_x$, we calculate by (2.6)

$$\lim_{x\to x_0} n_x = \lim_{n\to J} \frac{n^3}{n+J} \cdot \frac{E_x}{n_x},$$

which gives

$$\left(\lim_{x\to x_0} n_x\right)^2 = \frac{J^2(J-b)}{2}.$$

Since $b < J$, then

$$\lim_{x\to x_0} n_x = \pm\sqrt{\frac{J^2(J-b)}{2}}.$$

Thus system (2.1) has two smooth transonic solutions. In addition, for trajectory E_1,

$$\lim_{x\to x_0} n_x = \sqrt{\frac{J^2(J-b)}{2}},$$

and for trajectory E_2,

$$\lim_{x\to x_0} n_x = -\sqrt{\frac{J^2(J-b)}{2}}.$$

Next, we prove that the transonic solutions are C^∞ smooth. Without loss of generality, we only need to discuss the smoothness of one of the trajectories. From the second equation of the system (2.2), we know that if $n(x) \in C^k$, then $E(x) \in C^{k+1}$; therefore, we only need to prove $n(x) \in C^\infty$.

According to the first equation of (2.1), we have

$$n_x = \frac{n^3}{n+J} \cdot w(n),$$

where $w(n) \in C^\infty$. Now, assume that $n(x) \in C^k$, taking the $k+1$-order derivative for $n(x)$; then

$$n^{(k+1)} = (n_x)^{(k)} = \left[\left(\frac{n^3}{n+J}\right) \cdot w(n)\right]^{(k)} = \sum_{i=0}^{k} C_k^i \left(\frac{n^3}{n+J}\right)^{(i)} (w(n))^{(k-i)},$$

and then $n(x) \in C^{k+1}$. By induction, we reach a conclusion of $n(x) \in C^\infty$.
Now, the proof is completed. ∎

2.2. The case: $\alpha > 0$. Now we consider the system (1.5) with $\alpha > 0$ i.e., the effect of semiconductors is taken into account. Note that for $\alpha > 0$, the equation is not variable separated. Therefore, a local analysis near the singular point on the phase plane is necessary.

In this case, the corresponding singular point is (J, α). Write for $n \neq J$,

$$\begin{cases} n_x = \frac{n^2(nE-\alpha J)}{(n+J)(n-J)}, \\ E_x = n - b. \end{cases} \tag{2.7}$$

Furthermore, the equilibrium point of the system (2.7) is $(b, \frac{\alpha J}{b})$, and the type is a focus or node by calculation. Since the singular point and equilibrium point are on the hyperbolic branch,

$$E = \frac{\alpha J}{n}, \quad n > 0,$$

we call it the singularity line on which $n_x = 0$ and $E_n = \infty$. In order to simplify the proof process, transform the singularity line into a straight line via the transformation

$$\widetilde{E} = E - \frac{\alpha J}{n}, \tag{2.8}$$

and system (2.7) is written as

$$\begin{cases} n_x = \frac{n^3 \widetilde{E}}{(n+J)(n-J)}, \\ \widetilde{E}_x = n - b + \frac{\alpha J n_x}{n^2}. \end{cases} \tag{2.9}$$

The phase diagram of system (2.9) is shown in Figure 2. It can be seen that the singularity line is $\widetilde{E} = 0$, where $\widetilde{E}_n = \infty$. Next, we only need to analyze the trend of the trajectory $\widetilde{E}(n)$ in the $(n, \widetilde{E})$-phase plane. Through integrating the system (2.9), for $\widetilde{E} \neq 0$ we write

$$\widetilde{E}'(n) = \frac{(n+J)(n-J)(n-b)}{n^3 \widetilde{E}} + \frac{\alpha J}{n^2}. \tag{2.10}$$

For express convenience, define

$$f(n) := \frac{(n+J)(n-b)}{n^3},$$

and then

$$\widetilde{E}'(n) = f(n) \cdot \frac{1}{\frac{\widetilde{E}}{n-J}} + \frac{\alpha J}{n^2}. \tag{2.11}$$

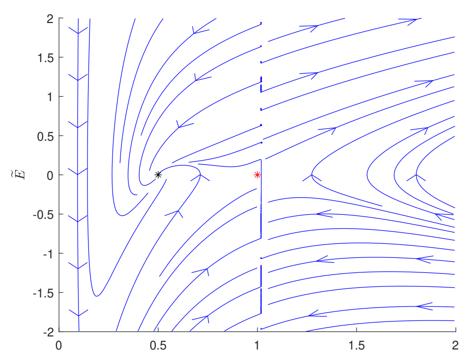

FIG. 2. *Phase plane of* $(n, \widetilde{E})$ *with* $\alpha = 1$, $b = 0.5$, *and* $J = 1$ *in the case of a supersonic doping profile. The red* $*$ *is the singular point* $(1, 0)$*; the black* $*$ *is the stable point* $(0.5, 0)$.

LEMMA 2.2. *Let the doping profile $b < J$, $\alpha > 0$; then* (2.10) *has two smooth transonic trajectories that are C^∞ smooth.*

Proof. The proof is divided into five steps.

Step 1. We prove that there exist continuous trajectories through the singular point $(J, 0)$. Without loss of generality, it only needs to prove that there is a trajectory tracing back to the singular point in the area of $\widetilde{E} > 0$ and $n > J$.

First, let us analyze the trend of these trajectories that start from the part of $\widetilde{E} > 0$ on the sonic line. From (2.10), $\widetilde{E}'(n) = \frac{\alpha}{J} > 0$ for $n = J$ and $\widetilde{E} > 0$. Furthermore, since $b < J$, we know that these trajectories will always keep $\widetilde{E}'(n) > 0$. As shown in Figure 3, these trajectories will intersect with the straight line l^{ε_0} : $n = J + \varepsilon_0$ ($\varepsilon_0 \ll 0$). Choose a starting point sequence, and set the intersection of these trajectories and l^{ε_0} as $A_n(J + \varepsilon_0, a_n), n = 1, 2, \ldots$. Then $\{a_n\}$ will form a monotonically decreasing sequence with a lower bound. According to the monotone bounded theorem, the sequence $\{a_n\}$ is convergent. So, assume that

$$\lim_{n\to\infty} a_n = a_0. \tag{2.12}$$

Then we can easily prove $a_0 > 0$.

Mark this critical trajectory as T_a, which intersects with the line l^{ε_0} at the point $(J + \varepsilon_0, a_0)$. Next, we prove that T_a will trace back to the singular point. In fact, the trajectory T_a cannot trace back to the sonic line of $\widetilde{E} > 0$. If the trajectory T_a starts from the point (J, h), where $h > 0$, then the trajectory that starts from the point $(J, \frac{h}{2})$ will intersect with the line l^{ε_0}. Since the system (2.9) is autonomous, this trajectory will be under the trajectory T_a, which will contradict with (2.12). Furthermore, we can illustrate that it cannot be traced back to the part of $n > J$ on the singularity line in the same way.

Thus, there exists a trajectory that traces back to the singular point $(J, 0)$ in the subsonic region of $\widetilde{E} > 0$. In the same way, there will be a trajectory that traces back to the singular point in the subsonic area of $\widetilde{E} < 0$. Similarly, the same result can be obtained in the corresponding supersonic part.

Step 2. We prove that the trajectory $\widetilde{E}(n)$ obtained in Step 1 is unique in the area of $\widetilde{E} > 0$ and $n > J$. In fact, since the system (2.7) is autonomous, if there are two different continuous transonic trajectories $\widetilde{E}_1(n)$ and $\widetilde{E}_2(n)$, they must satisfy

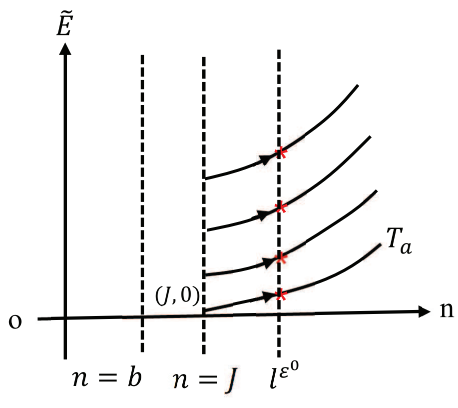

FIG. 3. *Phase plane of $(n, \widetilde{E})$ in the case of a supersonic doping profile.*

$$\lim_{n\to J}\widetilde{E}_1(n)=\lim_{n\to J}\widetilde{E}_2(n)=0. \tag{2.13}$$

Locally, assume that they satisfy the following conditions, i.e.,

$$\tilde{E}_1(n)>\tilde{E}_2(n), n\in(J,J+\varepsilon_0),$$

and denote $\bar{E}(n):=\tilde{E}_1(n)-\tilde{E}_2(n)$; then

$$\bar{E}(n)>0, n\in(J,J+\varepsilon_0). \tag{2.14}$$

According to (2.10), it is obvious that

$$\bar{E}'(n)=\widetilde{E}_1'(n)-\widetilde{E}_2'(n)<0 \text{ for } n\in(J,J+\varepsilon_0). \tag{2.15}$$

So, from (2.14) and (2.15), it follows

$$\lim_{n\to J}\bar{E}(n)>0, \text{ for } n\in(J,J+\varepsilon_0),$$

namely,

$$\lim_{n\to J}\widetilde{E}_1(n)>\lim_{n\to J}\widetilde{E}_2(n),$$

which contradicts with (2.13). Thus, $\widetilde{E}(n)$ is unique in the subsonic area of $\widetilde{E}>0$, and it can also be proved that $\widetilde{E}(n)$ is unique in the other three regions.

Step 3. In this part, we will prove $\widetilde{E}(n)\in C^1$. In order to find the possible convergence slope of the trajectory in advance, first assume

$$\lim_{n\to J}\frac{\widetilde{E}(n)}{n-J}=\lim_{n\to J}\widetilde{E}'(n)=m_0.$$

Take limits on both sides of (2.11); it gives

$$m_0=f(J)\cdot\frac{1}{m_0}+\frac{\alpha}{J}. \tag{2.16}$$

A simple calculation shows that

$$m_0=\frac{\alpha\pm\sqrt{\alpha^2+8(J-b)}}{2J}.$$

Here we only take the possible slope of $m_0=\frac{\alpha+\sqrt{\alpha^2+8(J-b)}}{2J}$ because now we only consider the part of $\widetilde{E}>0$ and $n>J$.

Next, we prove that $\widetilde{E}(n)$ will converge to the singular point with the slope of m_0. It only needs to illustrate that for all $\varepsilon>0$ there exists $\delta_0>0$ such that when $n\in(J,J+\delta_0)$, this trajectory will converge to the region bounded by the line l_0^+ and l_0^-, where l_0^+ and l_0^- are the lines through the singular point $(J,0)$ with the slope of $m_0+\varepsilon$ and $m_0-\varepsilon$, respectively. Next, we argue this by contradiction.

Suppose that the trajectory $\widetilde{E}(n)$ does not converge to this region; that is, there exists $\varepsilon_0>0$ for all $\delta>0$ when $n\in(J,J+\delta)$, and there will always exist a point on this trajectory above the line l_0^+ or below the line l_0^-. Assume this kind of points are above the straight line l_0^+. Since $f(n)>0$ when $n\to J_+$, then from (2.11),

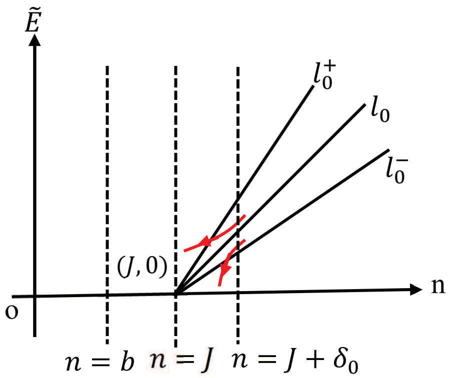

FIG. 4. *Phase plane of* $(n,\widetilde{E})$ *in the case of the supersonic doping profile.*

$\widetilde{E}'(n)$ will decrease with the increasing of the value of $\frac{\widetilde{E}}{n-J}$. That is, there exists $\delta=\delta_0>0$ when $n\in(J,J+\delta_0)$, and we have $\widetilde{E}'(n)<m_0$ for $\frac{\widetilde{E}}{n-J}>m_0+\varepsilon_0$. Therefore, $\widetilde{E}'(n)<m_0<\frac{\widetilde{E}}{n-J}$ at this kind of point, and it is easy to see that the value of $\frac{\widetilde{E}}{n-J}$ will increase after this point as $n\to J_+$. The tracing back directions of the trajectories, which above the line l_0^+ or below the line l_0^-, are shown in Figure 4.

(a) If $\{\frac{\widetilde{E}}{n-J}\}$ is unbounded, i.e., $\lim_{n\to J}\frac{\widetilde{E}}{n-J}=+\infty$, then by taking the limit on both sides of (2.11) we get

$$\lim_{n\to J}\widetilde{E}'(n)=\frac{\alpha}{J}\neq\lim_{n\to J}\frac{\widetilde{E}}{n-J},$$

which is a contradiction.

(b) If $\{\frac{\widetilde{E}}{n-J}\}$ has an upper bound, then according to the monotone bounded theorem, we know that $\{\frac{\widetilde{E}}{n-J}\}$ has a limit. Set $\lim_{n\to J}\frac{\widetilde{E}}{n-J}=\bar{m}_0$. It is obvious $\bar{m}_0>m_0$. So, if the limits on both sides of (2.11) are taken, then $\lim_{n\to J}\widetilde{E}'(n)$ exists with $\lim_{n\to J}\widetilde{E}'(n)<m_0$; i.e.,

$$\lim_{n\to J}\widetilde{E}'(n)\neq\lim_{n\to J}\frac{\widetilde{E}}{n-J},$$

which is also a contradiction. Furthermore, if there always exist points of the trajectory $\widetilde{E}(n)$ that are below the straight line l_0^- , the method of proof is similar. In conclusion, the trajectory $\widetilde{E}(n)$ will trace back to the singular point with the slope of m_0; i.e., $\lim_{n\to J}\frac{\widetilde{E}(n)}{n-J}=m_0$. So, taking the limit on both sides of (2.10), we have $\lim_{n\to J}\widetilde{E}'(n)=m_0$, i.e., $\widetilde{E}(n)\in C^1$.

Step 4. We will further prove that $\widetilde{E}(n)\in C^2$. According to Step 3,

$$\lim_{n\to J}\widetilde{E}'(n)=\lim_{n\to J}\frac{\widetilde{E}(n)}{n-J}=m_0\neq 0.$$

In view of the form of (2.10) and (2.11), we may consider $\frac{\widetilde{E}(n)}{n-J}$ as a whole. Denote

$$\widetilde{B}(n):=\frac{\widetilde{E}(n)}{n-J}. \tag{2.17}$$

It is obvious that $\lim_{n\to J}\widetilde{B}(n)=m_0$, i.e., $\widetilde{B}(n)\in C^0$. From (2.11) and (2.17), we can see that if $\widetilde{B}(n)\in C^1$, then $\widetilde{E}(n)\in C^2$. Thus, let us prove that $\widetilde{B}(n)\in C^1$. Firstly, through (2.17),

$$\widetilde{E}'(n)=(n-J)\widetilde{B}'(n)+\widetilde{B}(n),$$

and (2.11) is transformed into

$$\widetilde{E}'(n)=f(n)\cdot\frac{1}{\widetilde{B}(n)}+\frac{\alpha J}{n^2}, \tag{2.18}$$

and

$$\widetilde{B}'(n)=[\widetilde{E}'(n)-\widetilde{B}(n)]\cdot\frac{1}{n-J}. \tag{2.19}$$

Since $\widetilde{E}(n)\in C^1$, then by taking the limits on both sides of (2.17), we have

$$[\widetilde{E}'(n)-\widetilde{B}(n)]\mid_{n=J}=0. \tag{2.20}$$

According to (2.19) and (2.20), it follows

$$\widetilde{B}'(n)=\{\widetilde{E}'(n)-\widetilde{B}(n)-[\widetilde{E}'(n)-\widetilde{B}(n)]\mid_{n=J}\}\cdot\frac{1}{n-J}.$$

So, from (2.18) we derive

$$\widetilde{B}'(n)=\left\{f(n)\cdot\frac{1}{\widetilde{B}(n)}+\frac{\alpha J}{n^2}-\widetilde{B}(n)-\left[f(n)\cdot\frac{1}{\widetilde{B}(n)}+\frac{\alpha J}{n^2}-\widetilde{B}(n)\right]\mid_{n=J}\right\}\cdot\frac{1}{n-J}.$$

A simple calculation shows that

$$\widetilde{B}'(n)=-\left[\frac{f(n)}{\widetilde{B}(n)\widetilde{B}(J)}+1\right]\cdot\frac{\widetilde{B}(n)-\widetilde{B}(J)}{n-J}+\frac{1}{\widetilde{B}(J)}\cdot\frac{f(n)-f(J)}{n-J}+\frac{\frac{\alpha J}{n^2}-\frac{\alpha J}{n^2}\mid_{n=J}}{n-J}.$$

Obviously, some parts of the equation will not generate singularities, so we set

$$R(n,m_0)=\frac{1}{\widetilde{B}(J)}\cdot\frac{f(n)-f(J)}{n-J}+\frac{\frac{\alpha J}{n^2}-\frac{\alpha J}{n^2}\mid_{n=J}}{n-J}.$$

Therefore,

$$\widetilde{B}'(n)=-\left[\frac{f(n)}{\widetilde{B}(n)\widetilde{B}(J)}+1\right]\cdot\frac{\widetilde{B}(n)-\widetilde{B}(J)}{n-J}+R(n,m_0). \tag{2.21}$$

Since

$$\lim_{n\to J}\widetilde{B}(n)=\widetilde{B}(J)=m_0,$$

we need to consider

$$\bar{B}_1(n):=\widetilde{B}(n)-\widetilde{B}(J), \tag{2.22}$$

and it only needs to prove that $\bar{B}_1(n) \in C^1$. From (2.21),

$$\bar{B}_1'(n) = -\left[\frac{f(n)}{\widetilde{B}(n)\widetilde{B}(J)} + 1\right] \cdot \frac{\bar{B}_1(n)}{n-J} + R(n, m_0). \tag{2.23}$$

The next proof follows Step 3. Suppose in advance that

$$\lim_{n\to J} \frac{\bar{B}_1(n)}{n-J} = \lim_{n\to J} \bar{B}_1'(n) = m_1.$$

Then taking limits on both sides of (2.23), we have

$$m_1 = -\left[\frac{f(J)}{\widetilde{B}(J)^2} + 1\right] m_1 + R(J, m_0),$$

where

$$R(J, m_0) = \frac{1}{\widetilde{B}(J)} \cdot f'(n) \mid_{n=J} + \left(\frac{\alpha J}{n^2}\right)' \mid_{n=J} .$$

Denote

$$g(k) := -\left[\frac{f(J)}{\widetilde{B}^2(J)} + k\right].$$

Since $g(1) < 0$, we have $m_1 = \frac{R(J,m_0)}{1-g(1)}$.

Next, let us prove that $\bar{B}_1(n)$ will converge to the singular point with the slope of m_1. The proof process here is almost exactly the same as Step 3; the major difference is that $\widetilde{E}'(n)$ and $\bar{B}_1'(n)$ satisfy different relations, but we can easily find that the key to Step 3 is that $\widetilde{E}'(n)$ decreases with the increase of the value of $\frac{\widetilde{E}}{n-J}$ when $n \to J_+$. For this case, according to $g(1) < 0$, we can still get the same conclusion from (2.23). Therefore, through the local analysis on the $(n, \bar{B}_1)$-phase plane, we can prove $\bar{B}_1(n) \in C^1$, i.e., $\widetilde{E}(n) \in C^2$.

Step 5. Next, we prove that $\widetilde{E}(n) \in C^\infty$. From Step 4 it is easy to see that we just need to prove $\widetilde{B}(n) \in C^\infty$. According to induction, let us assume that $\widetilde{B}(n) \in C^k$ and go to prove $\widetilde{B}(n) \in C^{k+1}$. Set

$$\widetilde{B}^{(j)}(J) = m_j, (j = 0 \dots k).$$

From (2.17),

$$\begin{aligned}\widetilde{E}^{(k)}(n) &= (n-J)\widetilde{B}^{(k)}(n) + k\widetilde{B}^{(k-1)}(n),\\ \widetilde{E}^{(k+1)}(n) &= (n-J)\widetilde{B}^{(k+1)}(n) + (k+1)\widetilde{B}^{(k)}(n),\end{aligned}$$

and thus

$$\widetilde{B}^{(k)}(n) = \left[\widetilde{E}^{(k)}(n) - k\widetilde{B}^{(k-1)}(n)\right] \cdot \frac{1}{n-J}, \tag{2.24}$$

$$\widetilde{B}^{(k+1)}(n) = \left[\widetilde{E}^{(k+1)}(n) - (k+1)\widetilde{B}^{(k)}(n)\right] \cdot \frac{1}{n-J}. \tag{2.25}$$

Since $\widetilde{B}(n) \in C^k$, by taking the limits on both sides of (2.24), through Hospital's rule,

$$\left[\widetilde{E}^{(k+1)}(n) - (k+1)\widetilde{B}^{(k)}(n)\right] |_{n=J} = 0. \tag{2.26}$$

Thus, from (2.25) and (2.26), it follows

$$\widetilde{B}^{(k+1)}(n) = \left\{\widetilde{E}^{(k+1)}(n) - (k+1)\widetilde{B}^{(k)}(n) - \left[\widetilde{E}^{(k+1)}(n) - (k+1)\widetilde{B}^{(k)}(n)\right] |_{n=J}\right\} \cdot \frac{1}{n-J}.$$

According to (2.18), we have

$$\begin{aligned}\widetilde{B}^{(k+1)}(n) = \Bigg\{ & \left(f(n)\cdot\frac{1}{\widetilde{B}(n)} + \frac{\alpha J}{n^2}\right)^{(k)} - (k+1)\widetilde{B}^{(k)}(n) \\ & - \left[\left(f(n)\cdot\frac{1}{\widetilde{B}(n)} + \frac{\alpha J}{n^2}\right)^{(k)} - (k+1)\widetilde{B}^{(k)}(n)\right] |_{n=J} \Bigg\} \cdot \frac{1}{n-J}.\end{aligned} \tag{2.27}$$

Now, our goal is to extract $\frac{\widetilde{B}^{(k)}(n) - \widetilde{B}^{(k)}(J)}{n-J}$ from the right side of (2.27). A simple calculation shows that

$$\begin{aligned}\widetilde{B}^{(k+1)}(n) = & \left[-f(n)\cdot\frac{\widetilde{B}^{(k)}(n)}{\widetilde{B}^2(n)} + f(J)\cdot\frac{\widetilde{B}^{(k)}(J)}{\widetilde{B}^2(J)}\right]\cdot\frac{1}{n-J} - (k+1)\cdot\frac{\widetilde{B}^{(k)}(n) - \widetilde{B}^{(k)}(J)}{n-J} \\ & + \left[\left(\frac{\alpha J}{n^2}\right)^{(k)} - \left(\frac{\alpha J}{n^2}\right)^{(k)} |_{n=J}\right]\cdot\frac{1}{n-J} + L\left(n, \widetilde{B}(n), \widetilde{B}^{(1)}(n), \ldots, \widetilde{B}^{(k-1)}(n)\right) \\ & - L(J, m_0, m_1, \ldots, m_{k-1}),\end{aligned} \tag{2.28}$$

where

$$L\left(n, \widetilde{B}(n), \widetilde{B}^{(1)}(n), \ldots, \widetilde{B}^{(k-1)}(n)\right) = \left[\frac{f(n)}{\widetilde{B}(n)}\right]^{(k)} + f(n)\cdot\frac{\widetilde{B}^{(k)}(n)}{\widetilde{B}^2(n)}$$

and

$$L\left(J, m_0, m_1, \ldots, m_{k-1}\right) = \left[\frac{f(n)}{\widetilde{B}(n)}\right]^{(k)} |_{n=J} + f(J)\cdot\frac{\widetilde{B}^{(k)}(J)}{\widetilde{B}^2(J)}.$$

Simplify further the first term in (2.28) as follows:

$$\begin{aligned}& \left[-f(n)\cdot\frac{\widetilde{B}^{(k)}(n)}{\widetilde{B}^2(n)} + f(J)\cdot\frac{\widetilde{B}^{(k)}(J)}{\widetilde{B}^2(J)}\right]\cdot\frac{1}{n-J} \\ & = \left[-f(n)\cdot\frac{\widetilde{B}^{(k)}(n)}{\widetilde{B}^2(n)} + f(n)\cdot\frac{\widetilde{B}^{(k)}(J)}{\widetilde{B}^2(n)} - f(n)\cdot\frac{\widetilde{B}^{(k)}(J)}{\widetilde{B}^2(n)} + f(J)\cdot\frac{\widetilde{B}^{(k)}(J)}{\widetilde{B}^2(J)}\right]\cdot\frac{1}{n-J} \\ & = -\frac{f(n)}{\widetilde{B}^2(n)}\cdot\frac{\widetilde{B}^{(k)}(n) - \widetilde{B}^{(k)}(J)}{n-J} - \widetilde{B}^{(k)}(J)\cdot\left[\frac{f(n)}{\widetilde{B}^2(n)} - \frac{f(J)}{\widetilde{B}^{(2)}(J)}\right]\cdot\frac{1}{n-J}.\end{aligned} \tag{2.29}$$

By (2.28) and (2.29), it holds that

$$\widetilde{B}^{(k+1)}(n) = -\left[\frac{f(n)}{\widetilde{B}^2(n)} + (k+1)\right]\frac{\widetilde{B}^{(k)}(n) - \widetilde{B}^{(k)}(J)}{n-J} + R\left(n, \widetilde{B}(n), \ldots, \widetilde{B}^{(k)}(n), m_0, \ldots, m_k\right).$$

Since

$$\lim_{n\to J}\widetilde{B}^{(k)}(n)=\widetilde{B}^{(k)}(J)=m_k,$$

denote

$$\bar{B}_{k+1}(n):=\widetilde{B}^{(k)}(n)-\widetilde{B}^{(k)}(J).$$

Then

(2.30)

$$\bar{B}'_{k+1}(n)=-\left[\frac{f(n)}{\widetilde{B}^2(n)}+(k+1)\right]\frac{\bar{B}_{k+1}(n)}{n-J}+R\left(n,\widetilde{B}(n),\dots,\widetilde{B}^{(k)}(n),m_0,\dots,m_k\right).$$

In order to get the possible convergence slope of $\bar{B}_{k+1}(n)$ in advance, assume that

$$\lim_{n\to J}\frac{\bar{B}_{k+1}(n)}{n-J}=\lim_{n\to J}\bar{B}'_{k+1}(n)=m_{k+1}.$$

Taking limits on both sides of (2.30), we have

$$m_{k+1}=-\left[\frac{f(J)}{\widetilde{B}^2(J)}+(k+1)\right]m_{k+1}+R(J,m_0,\dots,m_k).$$

Since $g(k+1)<0$ for all $k\in N_+$, then use the same method in Step 4; we can get the conclusion that $\bar{B}_{k+1}(n)\in C^{k+1}$, i.e., $\widetilde{B}(n)\in C^{k+1}$.

The above proof is done in the region of $\widetilde{E}>0,n>J$. In the other three parts, the method of the proof is the same, and the C^∞-smooth transonic trajectory solution will be proved. Moreover, we can obtain that the derivatives of the trajectory in parts $\widetilde{E}>0,n>J$ and $\widetilde{E}<0,n<J$ are the same, and the derivatives of the trajectory in parts $\widetilde{E}>0,n<J$ and $\widetilde{E}<0,n>J$ are also the same. Thus, two smooth trajectories passing through the singular point $(J,0)$ are obtained, one of which is from the subsonic region to the supersonic region, and another one is from the supersonic region to the subsonic region. ∎

Now, we are ready to prove Theorem 1.1.

Proof of Theorem 1.1. By Lemma 2.1 and Lemma 2.2, we have got two C^∞-smooth transonic trajectories $\widetilde{E}(n)$. Then the proof method for the smoothness of $n(x)$ is the same as $\alpha=0$, which is given in subsection 2.1, so we obtain two smooth transonic solutions of the original system (1.5). The proof is complete. ∎

3. The case of $b>J$ (subsonic doping profile). When the doping profile is subsonic, the corresponding analysis of the system (1.5) becomes much more complicated. As in subsection 2.2, starting from (2.8) and (2.9), we still analyze the trend of the trajectory in the $(n,\widetilde{E})$-phase plane. It is easy to obtain that the equilibrium point $(b,0)$ is the saddle point through calculation, but when α takes different values, the phase diagram will be completely different. We mainly study the solution of the system when α is sufficiently small, and α is sufficiently large. For α is sufficiently small, we may take $\alpha=0$ to see the phase diagram, which is shown in Figure 5. For α is large, we show the phase diagram of system (1.5) after transformation in Figure 6.

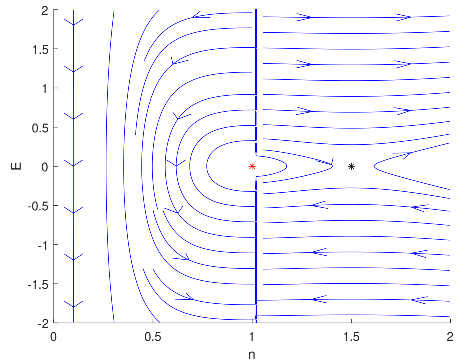

FIG. 5. *Phase plane of* (n, E) *with* $\alpha = 0$*,* $b = 1.5$*, and* $J = 1$ *in the case of a subsonic doping profile. The red* $*$ *is the singular point* $(1, 0)$*; the black* $*$ *is the equilibrium point* $(1.5, 0)$*.*

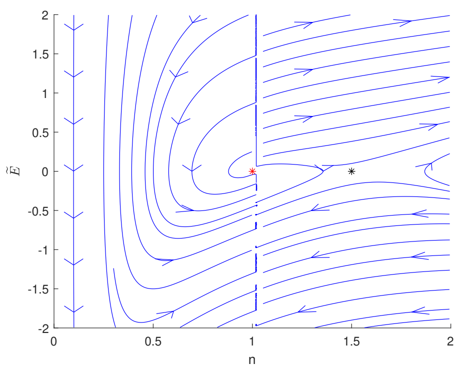

FIG. 6. *Phase plane of* $(n, \widetilde{E})$ *with* $\alpha = 2$*,* $b = 1.5$*, and* $J = 1$ *in the case of a subsonic doping profile. The red* $*$ *is the singular point* $(1, 0)$*; the black* $*$ *is the stable point* $(1.5, 0)$*.*

3.1. $\alpha \geq 0$, and α is sufficiently small.

LEMMA 3.1. *Let the doping profile be subsonic, i.e.,* $b > J$*; then for* $\alpha \geq 0$ *and* α *is sufficiently small,* (2.10) *has no continuous transonic trajectory.*

Proof. First, we analyze the general trend of the trajectory, which starts from the part of $n > J$ on the singularity line. From (2.10), when $\widetilde{E} = 0$ and $n > J$, we can get $\widetilde{E}'(n) = -\infty$. Since $b > J$, then, there exists $\delta_0 > 0$ such that $f(n) < -C$ for $n \in (J, J + \delta_0)$; i.e., $\widetilde{E}'(n) < 0$ as long as $\frac{\widetilde{E}}{n-J} < \frac{CJ}{\alpha}$. Then, there exists a line $\bar{l}$ through the singular point with the slope of $\frac{CJ}{\alpha}$, which will be large when α is small such that $\widetilde{E}'(n) < 0$ for the trajectory in the region bounded by the three lines $\bar{l}$, l^{δ_0}, and $\widetilde{E} = 0$. These trajectories must intersect with the line $\bar{l}$. As shown in Figure 7, they can not trace back to the singular point because $\widetilde{E}'(n) < \frac{\alpha}{J}$, which is small enough for this case. ∎

3.2. The Case: $\alpha > 0$, and α is sufficiently large.

LEMMA 3.2. *Let* $b > J$*,* $\alpha > 0$*, and* α *is large enough;* (2.10) *has infinitely many continuous transonic trajectories. Moreover, one of the transonic trajectory is* C^∞ *smooth, and the rest of the transonic trajectories are also* C^∞ *smooth when*

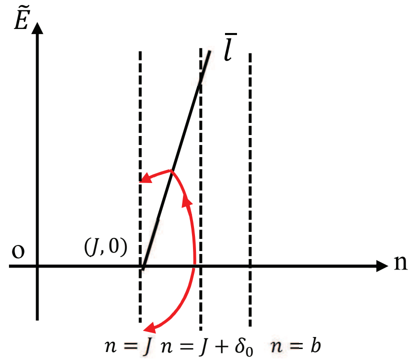

FIG. 7. *Phase plane of $(n,\widetilde{E})$ in the case of a subsonic doping profile.*

$h(\alpha,b,J)\bar{\in}Z$, and they are C^m smooth when $h(\alpha,b,J)\in Z$, where $h(\alpha,b,J)=\frac{8(b-J)}{[\alpha-\sqrt{\alpha^2-8(b-J)}]^2}$ and $m=\frac{8(b-J)}{[\alpha-\sqrt{\alpha^2-8(b-J)}]^2}-1$.

Proof. When the value of α becomes large, the phase diagram of the system changes as shown in Figure 6. For this case, in addition to phase-plane analysis, the local manifold analysis is used to illustrate the smoothness of these transonic solutions. Next, we divide our proof into three steps.

Step 1. We prove the existence of continuous transonic trajectories. Without a loss of generality, we only need to prove that there will exist trajectories passing through the singular point $(J,0)$ in the area of $\widetilde{E}>0$ and $n>J$. From (2.11), $\widetilde{E}'(n)=\frac{\alpha}{J}$, which is large enough when the trajectory starts from the sonic line of $\widetilde{E}>0$. As shown in Figure 8, we may as well assume that for α large enough there exists a straight line $\widetilde{l}$ passing through the singular point with a large slope such that there exists $\delta_0>0$ for $n\in(J,J+\delta_0)$; all the trajectories starting from the $\widetilde{E}>0$ part of the sonic line will all be above this line. Then, if this fact holds, we consider the point $A(n,\widetilde{E}(n))(n\in(J,J+\delta_0))$ below this line. First, the trajectory through the point of A cannot trace back to the sonic line of $\widetilde{E}>0$. Also, this trajectory cannot trace back to the part of $\widetilde{E}=0$ for $n>J$ either because $\widetilde{E}'(n)=-\infty$ when the trajectory starts from the singularity line of $n>J$.

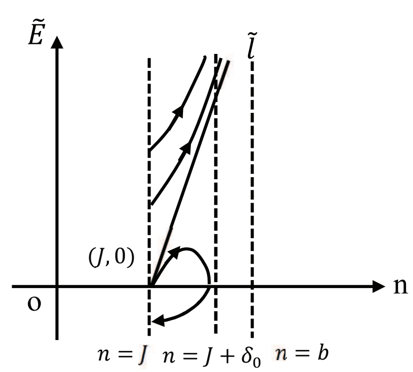

FIG. 8. *Phase plane of $(n,\widetilde{E})$ in the case of a subsonic doping profile.*

Now, we only need to prove the existence of this kind of straight line $\widetilde{l}$. From (2.11), we know there exists $\delta_0 > 0$ such that $\frac{\alpha J}{n^2} > \frac{\alpha}{J} - \frac{1}{2}$ for $n \in (J, J+\delta_0)$. When $n \in (J, J+\delta_0)$, $f(n)$ has no singularity, and we might as well assume that $|f(n)| < C$. Then we can get $|f(n)/\frac{\widetilde{E}}{n-J}| < \frac{1}{2}$ for $\frac{\widetilde{E}}{n-J} > \frac{\alpha}{J} - 1$ with a α large enough. Therefore, there exists $\delta_0 > 0$ and α large enough that, if the date $\frac{\widetilde{E}}{n-J} > \frac{\alpha}{J} - 1$, we have $\widetilde{E}'(n) > \frac{\alpha}{J} - 1$ for $n \in (J, J+\delta_0)$. That is, the slope of the line $\widetilde{l}$ we should choose can be $\frac{\alpha}{J} - 1$ for α large enough.

Step 2. Next, we prove that this kind of $\widetilde{E}(n)$ of the trajectories obtained in the *step* 1 is C^1 smooth. Compared with the previous proof method, we need to find the slope of these trajectories, convergence in advance. Suppose that

$$\lim_{n\to J} \widetilde{E}'(n) = \lim_{n\to J} \frac{\widetilde{E}}{n-J} = s.$$

Then, take limits on both sides of (2.11); we have

$$s = f(J) \cdot \frac{1}{s} + \frac{\alpha}{J}$$

from which two values are obtained:

$$S_0 = \frac{\alpha + \sqrt{\alpha^2 - 8(b-J)}}{2J},$$

and

$$s_0 = \frac{\alpha - \sqrt{\alpha^2 - 8(b-J)}}{2J}.$$

It is obvious that S_0 tends to infinity, and s_0 tends to zero as α tends to infinity. Next, we prove that $\widetilde{E}(n)$ will trace back to the singular point with S_0 or s_0 as the slope.

First of all, let us prove that there are infinitely many trajectories that trace back to the singular point with s_0 as the slope. We still do local analysis near the singular point. As shown in Figure 9, we may as well make two straight lines L_0^- and T_0^-, which are through the singular point with $S_0 - \varepsilon_1$ and $s_0 - \eta_1$ as the slopes, respectively. First, we prove that there exists $\delta > 0$ such that for $n \in (J, J+\delta)$ these

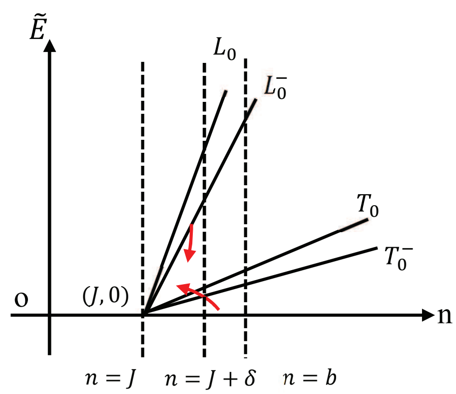

FIG. 9. *Phase-Plane of* $(n, \widetilde{E})$ *in the case of subsonic doping profile.*

trajectories, which pass through the line of L_0^-, will converge and stay in the region between the line L_0^- and T_0^- when $n \to J_+$. Suppose that one of the trajectories intersects with the line L_0^- at point $A_0(n_0, \widetilde{E}(n_0))$. Then we analyze the trend of this trajectory and compare the magnitude of $\widetilde{E}'(n_0)$ and $\frac{\widetilde{E}}{n_0-J}$ for this aim. From (2.11),

$$\widetilde{E}'(n_0) = f(n_0) \cdot \frac{1}{\frac{\widetilde{E}}{n_0-J}} + \frac{\alpha J}{n_0^2}.$$

Then, we just need to judge the sign of

$$\frac{\widetilde{E}}{n_0 - J} - \left[f(n_0) \cdot \frac{1}{\frac{\widetilde{E}}{n_0-J}} + \frac{\alpha J}{n_0^2} \right].$$

Since $\frac{\widetilde{E}}{n_0-J} > 0$, it is equivalent to judge the sign of

$$\left(\frac{\widetilde{E}}{n_0 - J} \right)^2 - \frac{\alpha J}{n_0^2} \cdot \frac{\widetilde{E}}{n_0 - J} - f(n_0).$$

Consider the following quadratic manifold:

$$F(z; n) := z^2 - \frac{\alpha J}{n^2} z - f(n). \tag{3.1}$$

For $F(z, n) = 0$, there are two roots:

$$z_1(n) = \frac{1}{2} \left(\frac{\alpha J}{n^2} - \sqrt{\frac{\alpha^2 J^2}{n^4} + 4f(n)} \right),$$

and

$$z_2(n) = \frac{1}{2} \left(\frac{\alpha J}{n^2} + \sqrt{\frac{\alpha^2 J^2}{n^4} + 4f(n)} \right).$$

It is obvious that $z_1(J) = s_0$ and $z_2(J) = S_0$; i.e., as $n \to J_+$, it holds $z_1(n) \to s_0$ and $z_2(n) \to S_0$. Then, for $z = S_0 - \varepsilon_1$, there exists $\delta_1 > 0$ such that $F(z; n) < 0$ for $n \in (J, J + \delta_1)$. In the same way, for $z = s_0 - \eta_1$, there exists $\delta_2 > 0$ such that $F(z; n) > 0$ for $n \in (J, J + \delta_2)$. Thus $\delta = min\{\delta_1, \delta_2\}$ is satisfied; that is, these trajectories will converge and stay in this region, which will trace back to the singular point.

Next, make two straight lines, M_0^+ and M_0^-, which are through the point $(J, 0)$ with the slope of $s_0 + \eta_2$ and $s_0 - \eta_2$. We prove that there exists $\delta' < \delta$ such that for $n \in (J, J + \delta')$, the above trajectories will converge and stay in the region bounded by M_0^+ and M_0^- for any fixed η_2 with $\eta_2 \leq \eta_1$. Suppose that when $n \to J_+$, there always exist some points on the trajectory between L_0^- and M_0^+. Then, according to the discussion of the manifold, the trajectory cannot intersect with the straight line M_0^+ because if they intersect with M_0^+ they will enter and stay under M_0^+ for a small δ_1. That is, there exists $\delta_1 > 0$ when $n \in (J, J + \delta_1)$; the trajectory will always stay between the straight line L_0^- and M_0^+. Therefore, we can get $F(Z; n) < 0$ for $n \in (J, J + \delta_1)$; i.e., in the process of trajectory tracing, the value of $\frac{\widetilde{E}}{n-J}$ will be monotonically decreasing, which is the slope of the line connecting the point on the trajectory and the singular point. It is easy to see $\frac{\widetilde{E}}{n-J}$ has a lower bound, and then it has a limit. Set

$$\lim_{n \to J} \frac{\widetilde{E}}{n - J} = \bar{s}_0,$$

and it is obvious that $\bar{s}_0 > s_0$; then according to (2.11), $\widetilde{E}'(n)$ has a limit when $n \to J_+$ and

$$\lim_{n\to J} \widetilde{E}'(n) < \bar{s}_0,$$

namely,

$$\lim_{n\to J} \widetilde{E}'(n) \neq \lim_{n\to J} \frac{\widetilde{E}(n)}{n-J},$$

which is a contradiction. Similarly, the trajectory must also converge above the line M_0^-. Then, according to the arbitrariness of η_2, these trajectories will eventually trace back to the singular point with the slope of s_0.

Now we will continue to prove that there will be a unique trajectory going back to the singular point with the slope of S_0. From the above analysis, we have known that there will be infinitely many trajectories that trace back to the singular point with the slope of s_0. These trajectories will intersect with the straight line $l^\delta : n = J + \delta$. We may as well mark these intersections as $B_n(J+\delta, b_n)$ after choosing a sequence, where $\{b_n\}$ is monotonically increasing and has an upper bound. According to the monotone bounded theorem, the sequence $\{b_n\}$ has a limit. Set

$$\lim_{n\to\infty} b_n = b_0. \tag{3.2}$$

Let T_b denote the trajectory through the point $B_0(J+\delta, b_0)$. Now we just need to prove that T_b will go back to the singular point with the slope of S_0.

It is clear that T_b must trace back to the singular point. If not, T_b will go back to the sonic line of $\widetilde{E} > 0$, which contradicts with (3.2). Also, the trajectory T_b can't go back to the singular point with the slope of s_0. Otherwise, if the trajectory T_b goes back to the singular point with the slope of s_0, then there is still a trajectory with the slope of s_0 going back to the singular point above the trajectory T_b due to the former discussion. This would also contradict with (3.2).

Thus, when $n \to J_+$, the trajectory T_b will always be above the straight line L_0^-. Otherwise, if there is always a point on the trajectory below the straight line L_0^-, then when n is close enough to J, the trajectory will enter and stay below the straight line L_0^-, and finally it will trace back to the singular point with s_0 as the slope. Similarly, when $n \to J_+$, the trajectory T_b can't be above the straight line L_0^+. Otherwise, according to the manifold analysis, the trajectory will always be above the straight line L_0^+ when $n \to J_+$. Thus, the value of $\frac{\widetilde{E}}{n-J}$ will increases monotonically, where $\frac{\widetilde{E}}{n-J}$ denotes the slope of the line connecting the point on the trajectory and the singular point.

(a) If $\{\frac{\widetilde{E}}{n-J}\}$ is unbounded, i.e., $\lim_{n\to J} \frac{\widetilde{E}}{n-J} = +\infty$, then taking limits on both sides of (2.11), we have

$$\lim_{n\to J} \widetilde{E}'(n) = \frac{\alpha}{J} \neq +\infty,$$

which is a contradiction.

(b) If $\{\frac{\widetilde{E}}{n-J}\}$ has an upper bound on the basis of the monotone bounded theorem, then $\{\frac{\widetilde{E}}{n-J}\}$ has a limit. Set

$$\lim_{n\to J} \frac{\widetilde{E}}{n-J} = \bar{S}_0.$$

It is obvious that $\bar{S}_0 > S_0$; then $\widetilde{E}'(n)$ has a limit and

$$\lim_{n\to J} \widetilde{E}'(n) < \bar{S}_0,$$

namely,

$$\lim_{n\to J} \widetilde{E}'(n) \neq \lim_{n\to J} \frac{\widetilde{E}(n)}{n-J},$$

which is also a contradiction.

Therefore, according to the arbitrariness of ε, we prove that T_b will eventually trace back to the singular point with the slope of S_0.

Finally, we prove this kind of trajectory is unique. In fact, if there are two trajectories T_b and T_b' satisfying the conditions, that is, (2.9) has two trajectory solutions, $\widetilde{E}_1(n)$, $\widetilde{E}_2(n)$, and they satisfy

$$\lim_{n\to J} \widetilde{E}_1'(n) = \lim_{n\to J} \widetilde{E}_2'(n) = S_0, \tag{3.3}$$

then assume that $\widetilde{E}_1(n) > \widetilde{E}_2(n), n \in (J, J+\varepsilon_0)$; from (2.17), we have $\widetilde{B}_1(n) = \frac{\widetilde{E}_1(n)}{n-J}$ and $\widetilde{B}_2(n) = \frac{\widetilde{E}_2(n)}{n-J}$. Set

$$\bar{B}(n) = \widetilde{B}_1(n) - \widetilde{B}_2(n),$$

and it is obvious that

$$\bar{B}(n) > 0, n \in (J, J+\varepsilon_0). \tag{3.4}$$

From (2.19),

$$\bar{B}'(n) = \widetilde{B}_1'(n) - \widetilde{B}_2'(n) = \left(-f(n) \cdot \frac{1}{\widetilde{B}_1(n)\widetilde{B}_2(n)} - 1\right) \cdot \frac{\widetilde{B}_1(n) - \widetilde{B}_2(n)}{n-J}.$$

Since

$$\lim_{n\to J} \widetilde{B}_1(n) = \lim_{n\to J} \widetilde{B}_2(n) = S_0,$$

thus for all $\varepsilon > 0$, there exists $\delta > 0$ such that for $n \in (J, J+\delta)$ it holds $\widetilde{B}_1(n) \in (S_0-\varepsilon, S_0+\varepsilon)$ and $\widetilde{B}_2(n) \in (S_0-\varepsilon, S_0+\varepsilon)$. Therefore, as long as $\varepsilon_0 < \delta$, we can get

$$\widetilde{B}_1(n)\widetilde{B}_2(n) > (S_0-\varepsilon)^2, n \in (J, J+\varepsilon_0).$$

Since S_0 is large enough, then it must hold

$$\bar{B}'(n) < 0, n \in (J, J+\varepsilon_0). \tag{3.5}$$

Thus, from (3.4) and (3.5) we deduce

$$\lim_{n\to J} \bar{B}(n) > 0,$$

that is,

$$\lim_{n\to J} \widetilde{E}_1(n) > \lim_{n\to J} \widetilde{E}_2(n),$$

which contradicts with (3.3).

Step 3. Next, we need to discuss further the regularity of these trajectories. First of all, according to the proof process of Lemma 2.2, we can always get $g(k) < 0$ when $\widetilde{B}(J)$ is sufficiently large. Thus, one can easily know that the trajectory tracing back to the singular point with the slope of S_0 is C^∞ smooth, and we omit the proof here. Then we mainly discuss the regularity of the family of trajectories that trace back to the singular point with s_0 as the slope. Without a loss of generality, we just need to choose one of the trajectories to prove. Let us still use $\widetilde{E}(n)$ to represent this trajectory.

For clarification, we continue to prove $\widetilde{E}(n) \in C^2$. It is the same as the proof of Lemma 2.2 except for a slight difference here. By the transforming of (2.17)–(2.22), we still have

$$\bar{B}_1'(n) = -\left[\frac{f(n)}{\widetilde{B}(n)\widetilde{B}(J)} + 1\right] \cdot \frac{\bar{B}_1(n)}{n-J} + R(n, s_0), \tag{3.6}$$

where

$$R(n, s_0) = \frac{1}{\widetilde{B}(J)} \cdot \frac{f(n) - f(J)}{n-J} + \frac{\frac{\alpha J}{n^2} - \frac{\alpha J}{n^2}\,|_{n=J}}{n-J}.$$

We have known that $\bar{B}_1(n) \in C^0$ and $\lim_{n\to J} \bar{B}_1(n) = s_0$. Next, we will prove $\bar{B}_1(n) \in C^1$. In order to obtain the slope of trajectory in advance, assume that

$$\lim_{n\to J} \bar{B}_1'(n) = \lim_{n\to J} \frac{\bar{B}_1(n)}{n-J} = s_1.$$

Then take the limits on both sides of (3.6); it follows

$$s_1 = g(1)s_1 + R(J, s_0).$$

It is easy to see that $g(1) > 1$ when $\widetilde{B}(J)$ is sufficiently small. Next, we just need to prove that this trajectory will converge to the singular point with the slope of s_1. Still do the local manifold analysis in the $(n, \bar{B}_1)$-phase plane. As shown in Figure 10, the trajectory is making two straight lines T_1^+ and T_1^- pass through the singular point, where the slope of T_1^+ is $s_1 + \varepsilon_1$ and the slope of T_1^- is $s_1 - \varepsilon_1$.

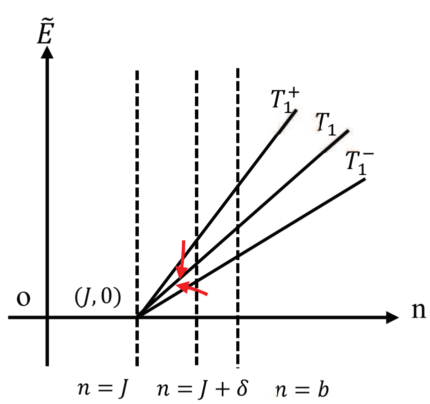

FIG. 10. *Phase plane of $(n, \widetilde{E})$ in the case of a subsonic doping profile.*

Then, we analyze the trend of the trajectory above T_1^+ and below T_1^- for which it needs to compare $\bar{B}_1'(n)$ and $\frac{\bar{B}_1}{n-J}$, that is, to judge the sign of $\frac{\bar{B}_1}{n-J} - \bar{B}_1'(n)$; i.e., $\frac{\bar{B}_1}{n-J} - \{-[\frac{f(n)}{\widetilde{B}(n)\widetilde{B}(J)} + 1] \cdot \frac{\bar{B}_1(n)}{n-J} + R(n, s_0)\}$ for $\frac{\bar{B}_1}{n-J} > s_1 + \varepsilon_1$ or $\frac{\bar{B}_1}{n-J} < s_1 - \varepsilon_1$ as $n \to J_+$. To do this, introduce the following manifold:

$$G(z; n) =: z - \left[-\left(\frac{f(n)}{\widetilde{B}(n)\widetilde{B}(J)} + 1 \right) \right] \cdot z - R(n, s_0).$$

For $G(z; n) = 0$, we have

$$z(n) = \frac{R(n, s_0)}{1 - \left[-\left(\frac{f(n)}{\widetilde{B}(n)\widetilde{B}(J)} + 1 \right) \right]},$$

and it is obvious that $z(J) = s_1$; i.e., $z(n) \to s_1$ for $n \to J_+$. Thus, for $z \geq s_1 + \varepsilon_1$ there exists $\delta_1 > 0$ such that $G(z; n) < 0$ with $n \in (J, J + \delta_1)$. In the same way, for $z \leq s_1 - \varepsilon_1$ there exists $\delta_2 > 0$ such that $G(z; n) > 0$ for $n \in (J, J + \delta_2)$. Take $\delta = min\{\delta_1, \delta_2\}$, and then for $n \in (J, J + \delta)$ we have $\bar{B}_1'(n) > \frac{\bar{B}_1}{n-J}$ for $\frac{\bar{B}_1}{n-J} \geq s_1 + \varepsilon_1$ and $\bar{B}_1'(n) < \frac{\bar{B}_1}{n-J}$ for $\frac{\bar{B}_1}{n-J} \leq s_1 - \varepsilon_1$. Thus, for this reason, as $n \to J_+$, this trajectory will converge and stay in the region bounded by T_1^+ and T_1^-. Otherwise, if there are always points on this trajectory above the straight line T_1^+ as $n \to J_+$, then there must exist $\delta_0 > 0$ such that the trajectory will always above the straight line T_1^+ for $n \in (J, J + \delta_0)$. It is easy to see that the value of $\frac{\bar{B}_1}{n-J}$ will also decrease monotonically. Set

$$\lim_{n \to J} \frac{\bar{B}_1}{n - J} = \bar{s}_1.$$

It is obvious that $\bar{s}_1 > s_1$; then

$$\lim_{n \to J} \bar{B}_1'(n) < \bar{s}_1,$$

namely,

$$\lim_{n \to J} \bar{B}_1'(n) \neq \lim_{n \to J} \frac{\bar{B}_1(n)}{n - J},$$

which is a contradiction. In the same way, we can also prove that there will not always exist points on this trajectory below the line T_1^- as $n \to J_+$. Therefore, according to the arbitrariness of ε_1, we prove that this trajectory will eventually trace back to the singular point with the slope of s_1, i.e., $\bar{B}_1(n) \in C^1$, and then $\widetilde{E}(n) \in C^2$.

Next, we discuss the higher regularity of $\widetilde{E}(n)$. According to Step 5 in the proof of Lemma 2.2, we also have

$$\bar{B}_k'(n) = -\left(\frac{f(n)}{\widetilde{B}^2(n)} + k \right) \frac{\bar{B}_k(n)}{n - J} + R\left(n, \widetilde{B}(n), \ldots, \widetilde{B}^{(k-1)}(n), s_0, \ldots, s_{k-1}\right).$$

We find that if

$$g(k) = -\left(\frac{f(J)}{\widetilde{B}^2(J)} + k \right) \bar{\in} Z, k = 1, 2, \ldots,$$

namely,

$$h(\alpha, b, J) := -\frac{f(J)}{\widetilde{B}^2(J)} = \frac{8(b-J)}{[\alpha - \sqrt{\alpha^2 - 8(b-J)}]^2} \bar{\in} Z, k = 1, 2, \ldots,$$

then, according to the above method and the proof method of Lemma 2.2, we can continue the above discussion and finally get that $\widetilde{B}(n) \in C^\infty$, i.e., $\widetilde{E}(n) \in C^\infty$. It should be noticed, however, that when $h(\alpha, b, J) \in Z$, there will be $k \in N$ such that $g(k) = 1$. Then we cannot calculate s_k in advance. But for this case, we can compute that $\bar{B}_1(n)$ can reach to C^{m-1}, where

$$m = \frac{8(b-J)}{[\alpha - \sqrt{\alpha^2 - 8(b-J)}]^2} - 1,$$

that is, $\widetilde{E}(n) \in C^m$. Thus the proof of Lemma 3.2 is obtained. ∎

Now, by Lemma 3.1 and Lemma 3.2, we can prove Theorem 1.2.

Proof of Theorem 1.2. By the same method of Lemma 3.1 and Lemma 3.2, we can prove that there are also two classes of trajectories in the supersonic side, which pass through the singular point, and their slopes are also s_0 and S_0, respectively. Thus, we get two classes of smooth trajectories, which cross the sonic point from the supersonic region to the subsonic region. By the discussion of regularity between the solution of the original system (1.5) and the trajectory, we can obtain these conclusions. ∎

REFERENCES

[1] U. Ascher, P. Markowich, P. Pietra, and C. Schmeiser, *A phase plane analysis of transonic solutions for the hydrodynamic semiconductor model*, Math. Mod. Meth. Appl. Sci., 1 (1991), pp. 347–376, https://doi.org/10.1142/S0218202591000174.

[2] P. Amster, M. P. B. Varela, A. Jüngel, and M. C. Mariani, *Subsonic solutions to a one-dimensional non-isentropic hydrodynamic model for semiconductors*, J. Math. Anal. Appl., 258 (2001), pp. 52–62, https://doi.org/10.1006/jmaa.2000.7359.

[3] M. Bae, B. Duan, and C. Xie, *Subsonic solutions for steady Euler–Poisson system in two-dimensional nozzles*, SIAM J. Math. Anal., 46 (2014), pp. 3455–3480, https://doi.org/10.1137/13094222X.

[4] M. Bae, B. Duan, and C. Xie, *Subsonic flow for the multidimensional Euler-Poisson system*, Arch. Ration. Mech. Anal., 220 (2016), pp. 155–191, https://doi.org/10.1007/s00205-015-0930-6.

[5] M. Bae, B. Duan, J. J Xiao, and C. Xie, *Two dimensional supersonic solutions for the steady Euler-Poisson system in a bounded domain*, preprint, arXiv:1602.01892v1 2016, http://arxiv.org/abs/1602.01892vl.

[6] K. Bløtekjær, *Transport equations for electrons in two-valley semiconductors*, IEEE Trans. Electron Devices, 17 (1970), pp. 38–47, https://doi.org/10.1109/T-ED.1970.16921.

[7] L. Chen, M. Mei, G. Zhang, and K. Zhang, *Steady hydrodynamic model of semiconductors with sonic boundary and transonic doping profile*, J. Differ. Equ., 269 (2020), pp. 8173–8211, https://doi.org/10.1016/j.jde.2020.06.022.

[8] P. Degond and P. Markowich, *On a one-dimensional steady-state hydrodynamic model for semiconductors*, Appl. Math. Lett., 3 (1990), pp. 25–29, https://doi.org/10.1016/0893-9659(90)90130-4.

[9] P. Degond and P. Markowich, *A steady state potential flow model for semiconductors*, Ann. Mat. Pura Appl., 4 (1993), pp. 87–98, https://doi.org/10.1007/BF01765842.

[10] B. Duan and Y. Zhou *Non-isentropic multi-transonic solutions of Euler-Poisson system*, J. Differ. Equ., 268 (2020), pp. 7029–7046, https://doi.org/10.1016/j.jde.2019.11.073.

[11] I. M. Gamba, *Stationary transonic solutions of a one-dimensional hydrodynamic model for semiconductors*, Commun. Partial Diferf. Eq., 17 (1992), pp. 553–577, https://doi.org/10.1080/03605309208820853.

[12] I. M. Gamba and C. S. Morawetz, *A viscous approximation for a 2-D steady semiconductor or transonic gas dynamic flow: Existence for potntial flow*, Commun. Pure Appl. Math., 49 (1996), pp. 999–1049, https://doi.org/10.1002/(SICI)1097-0312(199610)49:10⟨999::AID-CPA1⟩3.0.CO;2-2.

[13] F. Huang, M. Mei, Y. Wang, and H. Yu, *Asymptotic convergence to stationary waves for unipolar hydrodynamic model of semiconductors*, SIAM J. Math. Anal., 43 (2011), pp. 411–429, https://doi.org/10.1137/100793025.

[14] F. Huang, M. Mei, Y. Wang, and H. Yu, *Asymptotic convergence to planar stationary waves for multi-dimensional unipolar hydrodynamic model of semiconductors*, J. Differ. Equ., 251 (2011), pp. 1305–1331, https://doi.org/10.1016/j.jde.2011.04.007.

[15] H. L. Li, P. Markowich, and M. Mei, *Asymptotic behavior of solutions of the hydrodynamic model of semiconductors*, Proc. Roy. Soc. Edinburgh, Sect. A, 132 (2002), pp. 359–378, https://doi.org/10.1017/S0308210500001670.

[16] J. Y. Li, M. Mei, G. J. Zhang, and K. J. Zhang, *Steady hydrodynamic model of semiconductors with sonic boundary:* (I) *Subsonic doping profile*, SIAM J. Math. Anal., 49 (2017), pp. 4767–4811, https://doi.org/10.1137/17M1127235.

[17] J. Y. Li, M. Mei, G. J. Zhang, and K. J. Zhang, *Steady hydrodynamic model of semiconductors with sonic boundary:* (II) *Supersonic doping profile*, SIAM J. Math. Anal., 50 (2018), pp. 718–734, https://doi.org/10.1137/17M1129477.

[18] T. Luo, J. Rauch, C. Xie, and Z. Xin, *Stability of transonic shock solutions for one-dimensional Euler-Poisson equations*, Arch. Ration. Mech. Anal., 202 (2011), pp. 787–827, https://doi.org/10.1007/s00205-011-0433-z.

[19] T. Luo and Z. Xin, *Transonic shock solutions for a system of Euler-Poisson equations*, Commun. Math. Sci., 10 (2012), pp. 419–462, https://doi.org/10.4310/CMS.2012.v10.n2.a1.

[20] P. Markowich, C. Ringhofer, and C. Schmeiser, *Semiconductor Equations*, Springer Verlag, Vienna, Austria, 1990.

[21] S. Nishibata and M. Suzuki, *Asymptotic stability of a stationary solution to a hydrodynamic model of semiconductors*, Osaka J. Math., 44 (2007), pp. 639–665.

[22] S. Nishibata and M. Suzuki, *Asymptotic stability of a stationary Solution to a thermal hydrodynamic model for semiconductors*, Arch. Ration. Mech. Anal., 192 (2009), pp. 187–215, https://doi.org/10.1007/s00205-008-0129-1.

[23] Y. Peng and I. Violet, *Example of supersonic solutions to a steady state Euler-Poisson system*, Appl. Math. Lett., 19 (2006), pp. 1335–1340, https://doi.org/10.1016/j.aml.2006.01.015.

[24] M. D. Rosini, *A phase analysis of transonic solutions for the hydrodynamic semiconductor model*, Quart. Appl. Math., 63 (2005), pp. 251–268, https://doi.org/10.1090/S0033-569X-05-00942-1.