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Convergence rate of the vanishing viscosity limit for the Hunter-Saxton equation in the half space

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Abstract

In this paper, we study the asymptotic behavior of the solutions to an initial boundary value problem of the Hunter-Saxton equation in the half space when the viscosity tends to zero. By means of the asymptotic analysis with multiple scales, we first formally derive the equations for boundary layer profiles. Next, we study the well-posedness of the equations for the boundary layer profiles by using the compactness argument. Moreover, we construct an accurate approximate solution and use the energy method to obtain the convergence results of the vanishing viscosity limit.

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Keywords: Hunter-Saxton equation; Asymptotic analysis; Boundary layer; Well-posedness; Energy method; Vanishing viscosity limit

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1. Introduction

The Hunter-Saxton equation

$$(u_t + uu_x)_x = \frac{1}{2}u_x^2 \tag{1.1}$$

was introduced in [11] as an asymptotic equation of a nonlinear variational wave equation, which models waves in a massive director field of a nematic liquid crystal. Here, $u(t,x)$ describes the director field of a nematic liquid crystal, x being the space variable in a reference frame moving with the linearized wave speed, and t being a slow time variable. It is known that the Hunter-Saxton equation is a completely integrable system with a bi-Hamiltonian structure [12]. The Hunter-Saxton equation also arises from a completely different situation as the high-frequency limit of the Camassa-Holm equation, which is an integrable model equation for shallow water waves [3,4,35,36].

Hunter and Saxton [11] studied the initial value problem for the Hunter-Saxton equation on the line by using the method of characteristics and showed that smooth solutions break down in finite time. Yin [37] proved the local existence of strong solutions of the periodic Hunter-Saxton equation and showed that all strong solutions except space-independent solutions blow up in finite time. Hunter and Zheng [13] established the global existence of dissipative and conservative weak solutions to the following problem

$$\begin{cases} \partial_t v + u\partial_x v = -\dfrac{1}{2}v^2, & x>0, t>0, \\ \partial_x u = v, & x>0, t>0, \\ u(0,t) = 0, & \\ v(x,0) = v_0(x), & \end{cases} \tag{1.2}$$

where initial data $v_0(x)$ is compactly supported and $v_0(x) \in BV(\mathbb{R}^+)$. Hunter and Zheng also considered the viscosity and dispersion regularities of the Hunter-Saxton equation (1.1) in [14]. By employing the theory of Young measures, Zhang and Zheng [38] established the global existence of dissipative weak solutions to (1.2) with nonnegative $v_0 \in L^p(\mathbb{R}^+)$ for any $p > 2$. In [39], Zhang and Zheng proved the global existence and uniqueness of dissipative weak solutions to (1.2) with nonnegative $v_0 \in L^2(\mathbb{R}^+)$. Using the method of Young measures and mollification techniques, they [40] also established the global existence and uniqueness of admissible weak solutions to (1.2) with general data $v_0 \in L^2(\mathbb{R}^+)$. In recent years, the Hunter-Saxton equation has attracted more attention. Bressan and Constantin [2] constructed a continuous semigroup of global dissipative weak solutions. Using the vanishing viscosity method, Li and Zhang [17] proved the global existence of dissipative weak solutions to the Hunter-Saxton equation. Li and Yin [18] established the local well-posedness, blow-up phenomena and traveling wave solutions to the periodic integrable dispersive Hunter-Saxton equation. In this paper, we consider the following viscous problem

$$\begin{cases} \partial_t v^\epsilon + u^\epsilon \partial_x v^\epsilon = -\dfrac{1}{2}(v^\epsilon)^2 + \epsilon\partial_x^2 v^\epsilon, & x>0, t>0, \\ \partial_x u^\epsilon = v^\epsilon, & x>0, t>0, \\ u^\epsilon(0,t) = 0, & \\ \partial_x v^\epsilon(0,t) = 0, & \\ v^\epsilon(x,0) = v_0(x). & \end{cases} \tag{1.3}$$

The effect of boundary layer has been one of the fundamental problems in hydrodynamics [27] since the classical work by Prandtl [22] in 1904. Due to the inconsistency of boundary conditions between the Navier-Stokes equations and Euler equations, the boundary layer phenomenon usually occurs near a boundary when the viscosity ϵ vanishes (cf. [1,5,7,15,16,19–21, 23,25,26,29–34]). It is also observed when the change from a viscous parabolic equation to an associated inviscid hyperbolic one is preformed (cf. [6,8,24,28]). Motivated by these works, in this paper, we consider the asymptotic behavior of the solutions to the problem (1.3) in the half space as ϵ tend to zero. To our best knowledge, the boundary layer problem of the Hunter-Saxton equation has not been related yet due to some technical reason. Thus, we hope that our results presented in this paper can shed some light on the further theoretical study of boundary layers for the Hunter-Saxton equation.

The rest of this paper is organized as follows. In Section 2, we derive the equations for outer and boundary layer profiles by the method of matched asymptotic expansions and state the main convergence results. In Section 3, we study the well-posedness of outer and boundary layer problem. In Section 4, we construct an approximate solution and prove the main convergence results. Finally, we prove the existence of approximate solutions to (3.10) by the fixed point argument in Appendix A.

2. Asymptotic expansions and main results

In this section, we are going to derive the equations for outer and boundary layer profiles by the method of matched asymptotic expansions and state the main results in this paper.

2.1. Asymptotic expansions

In this subsection, by means of the formal asymptotic analysis with multiple scales, we derive the equations of boundary layer profiles of (1.3) with respect to ϵ.

Hence, based on the WKB method [8–10,24], the solution (v^ϵ, u^ϵ) to the problem (1.3) with respect to ϵ has the following ansatz:

$$\begin{cases} v^\epsilon(x,t) = \sum\limits_{j\geq 0} \epsilon^{\frac{j}{2}}(v^{I,j}(x,t) + v^{B,j}(z,t)), \\ u^\epsilon(x,t) = \sum\limits_{j\geq 0} \epsilon^{\frac{j}{2}}(u^{I,j}(x,t) + u^{B,j}(z,t)), \end{cases} \tag{2.1}$$

where the boundary layer coordinate is defined by

$$z = \frac{x}{\sqrt{\epsilon}}, \quad x \in [0,\infty). \tag{2.2}$$

We assume that each term in the expansion (2.1) is smooth, and the boundary layer solutions $v^{B,j}$ and $u^{B,j}$ fast decay as $z \to +\infty$.

Plugging (2.1) into initial condition in (1.3), and noticing that $v_0(x)$ is independent of ϵ, we obtain

$$\begin{cases} v^{I,0}(x,0) = v_0(x), \ \ v^{B,0}(z,0) = 0, \\ v^{I,j}(x,0) = v^{B,j}(z,0) = 0, \quad j \geq 1. \end{cases} \tag{2.3}$$

Plugging (2.1) into boundary conditions in (1.3) and matching terms of the same magnitude, it is easy to get that

$$\begin{cases} u^{I,j}(0,t)+u^{B,j}(0,t)=0, & j\geq 0, \\ v_z^{B,0}(0,t)=0, \quad v_x^{I,j}(0,t)+v_z^{B,j+1}(0,t)=0, & j\geq 0. \end{cases} \tag{2.4}$$

Plugging (2.1) into the second equation of (1.3), it follows that

$$u_x^{I,j}-v^{I,j}=0, \quad j\geq 0, \tag{2.5}$$

and

$$\begin{cases} u_z^{B,0}=0, \\ u_z^{B,j+1}-v^{B,j}=0, \quad j\geq 0, \end{cases} \tag{2.6}$$

which implies

$$u^{B,0}(z,t)=0, \tag{2.7}$$

by noting that $u^{B,0}$ decays to zero fast enough when $z\to+\infty$.

Substituting the expansions (2.1) into the first equation of (1.3), it follows

$$\begin{aligned} &\sum_{j\geq 0}\epsilon^{\frac{j}{2}}(v^{I,j}+v^{B,j})_t+\sum_{j\geq 0}\epsilon^{\frac{j}{2}}\sum_{k=0}^{j}(u^{I,k}+u^{B,k})v_x^{I,j-k} \\ &+\epsilon^{-\frac{1}{2}}(u^{I,0}+u^{B,0})v_z^{B,0}+\sum_{j\geq 0}\epsilon^{\frac{j}{2}}\sum_{k=0}^{j+1}(u^{I,k}+u^{B,k})v_z^{B,j+1-k} \\ &=-\frac{1}{2}(\sum_{j\geq 0}\epsilon^{\frac{j}{2}}(v^{I,j}+v^{B,j}))^2+v_{zz}^{B,0}+\epsilon^{\frac{1}{2}}v_{zz}^{B,1}+\sum_{j\geq 0}\epsilon^{1+\frac{j}{2}}(v_{xx}^{I,j}+v_{zz}^{B,j+2}). \end{aligned} \tag{2.8}$$

Letting z go to $+\infty$ in (2.8), it yields

$$\begin{cases} v_t^{I,0}+u^{I,0}v_x^{I,0}+\frac{1}{2}(v^{I,0})^2=0, \\ v_t^{I,1}+u^{I,1}v_x^{I,0}+u^{I,0}v_x^{I,1}+v^{I,0}v^{I,1}=0, \\ v_t^{I,j}+\sum\limits_{k=0}^{j}u^{I,k}v_x^{I,j-k}+\frac{1}{2}\sum\limits_{k=0}^{j}v^{I,k}v^{I,j-k}-v_{xx}^{I,j-2}=0, \quad j\geq 2. \end{cases} \tag{2.9}$$

Using (2.9) in (2.8), we obtain

$$\sum_{j\geq -1}\epsilon^{\frac{j}{2}}G_j(x,z,t)=0, \tag{2.10}$$

where

$$\begin{cases} G_{-1} = (u^{I,0} + u^{B,0})v_z^{B,0}, \\ G_0 = v_t^{B,0} + (u^{I,0} + u^{B,0})v_z^{B,1} + (u^{I,1} + u^{B,1})v_z^{B,0} + u^{B,0}v_x^{I,0} \\ \qquad + \frac{1}{2}(v^{B,0})^2 + v^{I,0}v^{B,0} - v_{zz}^{B,0}, \\ G_1 = v_t^{B,1} + (u^{I,0} + u^{B,0})v_z^{B,2} + (u^{I,1} + u^{B,1})v_z^{B,1} + (u^{I,2} + u^{B,2})v_z^{B,0} \\ \qquad + u^{B,0}v_x^{I,1} + u^{B,1}v_x^{I,0} + (v^{I,0}v^{B,1} + v^{I,1}v^{B,0} + v^{B,0}v^{B,1}) - v_{zz}^{B,1}, \\ \cdots\ \cdots \end{cases}$$

Using $x = \epsilon^{\frac{1}{2}}z$ and then expanding $G_j(x, z, t)$ formally in ϵ by the Taylor expansion,

$$G_j(x, z, t) = G_j(\epsilon^{\frac{1}{2}}z, z, t) = G_j(0, z, t) + \sum_{k=1}^{\infty} \frac{1}{k!}(\epsilon^{\frac{1}{2}}z)^k \partial_x^k G_j(0, z, t). \tag{2.11}$$

Then applying (2.11) into (2.10), we get

$$\sum_{j \geq -1} \epsilon^{\frac{j}{2}} \tilde{G}_j(z, t) = 0, \tag{2.12}$$

where

$$\begin{cases} \tilde{G}_{-1} = (u^{I,0}(0,t) + u^{B,0})v_z^{B,0}, \\ \tilde{G}_0 = v_t^{B,0} + (u^{I,0}(0,t) + u^{B,0})v_z^{B,1} + (u^{I,1}(0,t) + u^{B,1})v_z^{B,0} + u^{B,0}v_x^{I,0}(0,t) \\ \qquad + \frac{1}{2}(v^{B,0})^2 + v^{I,0}(0,t)v^{B,0} - v_{zz}^{B,0} + zu_x^{I,0}(0,t)v_z^{B,0}, \\ \tilde{G}_1 = v_t^{B,1} + (u^{I,0}(0,t) + u^{B,0})v_z^{B,2} + (u^{I,1}(0,t) + u^{B,1})v_z^{B,1} + (u^{I,2}(0,t) + u^{B,2})v_z^{B,0} \\ \qquad + u^{B,0}v_x^{I,1}(0,t) + u^{B,1}v_x^{I,0}(0,t) + v^{I,0}(0,t)v^{B,1} + v^{I,1}(0,t)v^{B,0} + v^{B,0}v^{B,1} \\ \qquad - v_{zz}^{B,1} + zu_x^{I,0}(0,t)v_z^{B,1} + zu_x^{I,1}(0,t)v_z^{B,0} + zu^{B,0}v_{xx}^{I,0}(0,t) \\ \qquad + zv_x^{I,0}(0,t)v^{B,0} + \frac{1}{2}z^2u_{xx}^{I,0}(0,t)v_z^{B,0}, \\ \cdots\ \cdots \end{cases}$$

From (2.5) with $j = 0$, (2.9), (2.3), (2.4) and (2.7), we obtain

$$\begin{cases} v_t^{I,0} + u^{I,0}v_x^{I,0} + \frac{1}{2}(v^{I,0})^2 = 0, \\ u_x^{I,0} - v^{I,0} = 0, \\ v^{I,0}(x,0) = v_0(x), \\ u^{I,0}(0,t) = 0. \end{cases} \tag{2.13}$$

Combining (2.6), (2.7), (2.12), (2.3) and (2.4), one has

$$\begin{cases} v_t^{B,0} = -(u^{I,1}(0,t) + u^{B,1})v_z^{B,0} - \frac{1}{2}(v^{B,0})^2 - v^{I,0}(0,t)v^{B,0} + v_{zz}^{B,0} - zu_x^{I,0}(0,t)v_z^{B,0}, \\ v^{B,0}(z,0) = 0, \\ v_z^{B,0}(0,t) = 0, \end{cases} \tag{2.14}$$

and the first-order boundary layer profile $u^{B,1}(z,t)$ is given by

$$u^{B,1}(z,t) = -\int_z^\infty v^{B,0}(y,t)dy. \tag{2.15}$$

It is easy to see that the boundary layer profile $(v^{B,0}, u^{B,1}) = (0,0)$.

Similarly, from (2.5) with $j = 1$, (2.9), (2.3) and (2.4), we get

$$\begin{cases} v_t^{I,1} + u^{I,1}v_x^{I,0} + u^{I,0}v_x^{I,1} + v^{I,0}v^{I,1} = 0, \\ u_x^{I,1} - v^{I,1} = 0, \\ v^{I,1}(x,0) = 0, \\ u^{I,1}(0,t) = \int_0^\infty v^{B,0}(z,t)dz. \end{cases} \tag{2.16}$$

Since $v^{B,0} = 0$, we also have $(v^{I,1}, u^{I,1}) = (0,0)$.

Moreover the boundary layer solution $(v^{B,1}, u^{B,2})$ satisfies

$$\begin{cases} v_t^{B,1} = -v^{I,0}(0,t)v^{B,1} + v_{zz}^{B,1} - zv^{I,0}(0,t)v_z^{B,1}, \\ v^{B,1}(z,0) = 0, \\ v_z^{B,1}(0,t) = -v_x^{I,0}(0,t), \end{cases} \tag{2.17}$$

and

$$u^{B,2}(z,t) = -\int_z^\infty v^{B,1}(y,t)dy. \tag{2.18}$$

Continuing these procedures, one can derive the equations for higher-order layer profiles $(v^{I,j}, u^{I,j})$, $(v^{B,j}, u^{B,j+1})$ for $j \geq 2$.

2.2. Main results

For convenience, we first introduce some notations to be used throughout this paper. Without loss of generality, we assume $0 \leq \epsilon < 1$. We denote by C a generic positive constant that may change from one line to another with C independent of ϵ but dependent on T, while C_0 denotes a generic positive constant independent of both ϵ and T. With $1 \leq p \leq \infty$, we use L^p and L_z^p to denote the Lebesgue space $L^p(0,\infty)$ with respect to x and z respectively, with corresponding norms $\|\cdot\|_{L^p}$ and $\|\cdot\|_{L_z^p}$. Similarly, H^k, H_z^k denote the Sobolev space $W^{k,2}(0,\infty)$ with respect to x and z respectively, with corresponding norms $\|\cdot\|_{H^k}$ and $\|\cdot\|_{H_z^k}$.

Next, we state our main results.

Theorem 2.1. *Assume that $v_0 \in H^3$, $v_0(x) \geq 0$ and $\int_0^\infty x v_{0x}^2 dx < +\infty$. Let (v^ϵ, u^ϵ) be the weak solution of (1.3) with $\epsilon \geq 0$. Then there exists a unique solution $(v^{I,0}, u^{I,0}) \in L^\infty(0,T;H^3) \cap L^\infty(0,T;H^4)$ to the problem (2.13) with initial data v_0. There exists a unique boundary layer solution $(v^{B,1}, u^{B,2})$ to the problem (2.17)-(2.18) satisfying*

$$
\begin{aligned}
v^{B,1} &\in L^2(0,T;H_z^3) \cap L^\infty(0,T;H_z^2),\\
zv^{B,1} &\in L^2(0,T;H_z^2) \cap L^\infty(0,T;H_z^1),\\
v_t^{B,1} &\in L^2(0,T;H_z^1) \cap L^\infty(0,T;L_z^2),\\
u^{B,2} &\in L^2(0,T;H_z^4) \cap L^\infty(0,T;H_z^3),
\end{aligned}
$$

such that as $\epsilon \to 0$, for any fixed $0 < T < \infty$, it holds that

$$\sup_{0\leqslant t\leqslant T} \|v^\epsilon - v^{I,0} - \epsilon^{\frac{1}{2}} v^{B,1}\|_{L^2} \leq C\epsilon^{\frac{3}{4}}, \tag{2.19}$$

$$\sup_{0\leqslant t\leqslant T} \|v^\epsilon - v^{I,0} - \epsilon^{\frac{1}{2}} v^{B,1}\|_{H^1} \leq C\epsilon^{\frac{1}{4}}, \tag{2.20}$$

$$\sup_{0\leqslant t\leqslant T} \|v^\epsilon - v^{I,0} - \epsilon^{\frac{1}{2}} v^{B,1}\|_{L^\infty} \leq C\epsilon^{\frac{1}{2}}, \tag{2.21}$$

and

$$\sup_{0\leqslant t\leqslant T} \|u^\epsilon - u^{I,0}\|_{L^\infty(0,M)} \leq C\epsilon^{\frac{1}{2}}, \tag{2.22}$$

for any $M > 0$.

Remark 2.1. If the initial data $v_0(x) < 0$, the above results in Theorem 2.1 valid before the blow-up time $t = T^*$ of the inviscid equation (1.2).

3. Well-posedness of outer and boundary layer problems

In this section, we consider the well-posedness of the problems for the outer and boundary layer profiles.

3.1. Well-posedness of outer layer problem

Lemma 3.1. *Under the same assumptions as in Theorem 2.1, there exists a unique solution $(v^{I,0}, u^{I,0})$ to (2.13) satisfying*

$$v^{I,0} \in L^\infty(0,T;H^3),\, u^{I,0} \in L^\infty(0,T;H^4). \tag{3.1}$$

Proof. The global existence and uniqueness of solutions to (2.13) can be referred to [17,38,39] and we omit the details here. It only remains to prove (3.1). By the method of characteristics, (2.13) is equivalent to the following system

$$\begin{cases} \dfrac{d\Phi_t(x)}{dt} = u^{I,0}(t,\Phi_t(x)) = \int_0^{\Phi_t(x)} v^{I,0}(t,y)dy, \quad \Phi_0(x) = x, \\ \dfrac{dv^{I,0}(t,\Phi_t(x))}{dt} = -\dfrac{1}{2}(v^{I,0})^2(t,\Phi_t(x)), \\ v^{I,0}(t,\Phi_t(x))|_{t=0} = v_0(x). \end{cases}$$

Then, by a direct computation, we get

$$v^{I,0}(t,x) = \frac{2v_0(x)}{2+v_0(x)t}. \tag{3.2}$$

Thus, (3.1) follows directly from (3.2), (2.13) and the assumptions on $v_0(x)$ in Theorem 2.1. The proof is complete. □

3.2. Well-posedness of boundary layer problem

In this subsection, we study the existence and uniqueness of the solution $(v^{B,1}, u^{B,2})$ to the problem (2.17)-(2.18) and derive the regularity of boundary layer profiles.

Lemma 3.2. *Under the assumptions of Theorem 2.1, there exists a unique solution $v^{B,1}$ of (2.17). Moreover, for any $0 < T < \infty$, it holds that*

$$v^{B,1} \in L^2(0,T;H_z^3) \cap L^\infty(0,T;H_z^2), \tag{3.3}$$

$$zv^{B,1} \in L^2(0,T;H_z^2) \cap L^\infty(0,T;H_z^1), \tag{3.4}$$

and

$$v_t^{B,1} \in L^2(0,T;H_z^1) \cap L^\infty(0,T;L_z^2). \tag{3.5}$$

Consequently it follows from (2.18) that

$$u^{B,2} \in L^2(0,T;H_z^4) \cap L^\infty(0,T;H_z^3). \tag{3.6}$$

Proof. For clarity, we divide the proof into two steps.

Step 1. Existence and uniqueness of $v^{B,1}$ and $u^{B,2}$.

For brevity, we define $A(t) := v^{I,0}(0,t)$ and $B(t) := -v_x^{I,0}(0,t)$. In order to homogenize the boundary condition of (2.17), we set

$$w(z,t) := v^{B,1}(z,t) + B(t)e^{-z}. \tag{3.7}$$

Thus, from (2.17), it follows that $w(z,t)$ satisfies

$$\begin{cases} w_t - w_{zz} + zA(t)w_z + A(t)w = \rho(z,t), \quad (z,t) \in (0,\infty)\times(0,\infty), \\ w(z,0) = B(0)e^{-z}, \\ w_z(0,t) = 0, \end{cases} \tag{3.8}$$

where

$$\rho(z,t) = (B'(t) - B(t) - zA(t)B(t) + A(t)B(t))e^{-z}.$$

By the reflection method with even extension, we can extend the problem (3.8) to the whole space and we obtain

$$\begin{cases} w_t - w_{zz} + zA(t)w_z + A(t)w = \rho(|z|,t), & (z,t) \in (-\infty,\infty)\times(0,\infty), \\ w(z,0) = B(0)e^{-|z|}. \end{cases} \tag{3.9}$$

To overcome the singularity in z of (3.9), we consider the following approximate equation

$$\begin{cases} w_{kt} - w_{kzz} + B(z,k)A(t)w_{kz} + A(t)w_k = \rho(|z|,t), & (z,t) \in (-\infty,\infty)\times(0,\infty), \\ w_k(z,0) = B(0)e^{-|z|}, \end{cases} \tag{3.10}$$

where

$$B(z,k) = \begin{cases} k, & z > k, \\ z, & -k < z < k, \\ -k, & z < -k. \end{cases}$$

It is straightforward to check that

$$z^l\rho \in L^\infty(0,T;L^2_z(\mathbb{R})), \quad z^l\rho_z \in L^\infty(0,T;L^2_z(\mathbb{R})), \quad \forall l \geq 0. \tag{3.11}$$

Indeed, we can use the fundamental solution Φ of the heat equation to represent w_k and show that there exists a unique solution w_k of (3.10) for each $k > 0$ (details are given in the appendix).

Next, we derive some basic a priori estimates for the solution to the equation (3.10). Using these estimates, we can obtain the existence of w to equation (3.9) by applying the compactness argument.

Multiplying the first equation of (3.10) by $(1+z^2)w_k$ and integrating the resulting equation over $\mathbb{R}$, we obtain through integration by parts that

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}(\|w_k\|^2_{L^2_z(\mathbb{R})} + \|zw_k\|^2_{L^2_z(\mathbb{R})}) + (\|w_{kz}\|^2_{L^2_z(\mathbb{R})} + \|zw_{kz}\|^2_{L^2_z(\mathbb{R})}) \\ &= -\int_{\mathbb{R}} B(z,k)A(t)w_{kz}(1+z^2)w_k dz \\ &\quad - \int_{\mathbb{R}} A(t)(1+z^2)w_k^2 dz + \int_{\mathbb{R}} \rho(1+z^2)w_k dz + \int_{\mathbb{R}} w_k^2 dz. \end{aligned} \tag{3.12}$$

First, noticing that for any $F(x,t) \in L^p(0,T;H^1)$ with $1 \leq p \leq \infty$, it follows from Sobolev embedding inequality that

$$\|F(0,t)\|_{L^p(0,T)} \leq \|F\|_{L^p(0,T;L^\infty)} \leq C_0\|F\|_{L^p(0,T;H^1)}. \tag{3.13}$$

By the definition of $A(t)$, (3.13) and Lemma 3.1, we have

$$\|A(t)\|_{L^\infty(0,T)} \le C. \tag{3.14}$$

We next estimate terms on the right side of (3.12). First, we have

$$\begin{aligned}
-\int_{\mathbb{R}} B(z,k)A(t)w_{kz}(1+z^2)w_k dz &= \int_{\mathbb{R}} B(z,k)A(t)zw_k^2 dz + \int_{\mathbb{R}} B_z(z,k)A(t)\frac{1+z^2}{2}w_k^2 dz \\
&\le |A(t)|\int_{\mathbb{R}} z^2 w_k^2 dz + |A(t)|\int_{\mathbb{R}} \frac{1+z^2}{2} w_k^2 dz \\
&\le C(\|w_k\|^2_{L^2_z(\mathbb{R})} + \|zw_k\|^2_{L^2_z(\mathbb{R})})
\end{aligned} \tag{3.15}$$

by using $|B(z,k)| \le |z|$, $|B_z(z,k)| \le 1$ and $\|A(t)\|_{L^\infty(0,T)} \le C$.

Similarly to (3.15), we have

$$\begin{aligned}
-\int_{\mathbb{R}} A(t)(1+z^2)w_k^2 dz &\le |A(t)|\int_{\mathbb{R}} (1+z^2)w_k^2 dz \\
&\le C(\|w_k\|^2_{L^2_z(\mathbb{R})} + \|zw_k\|^2_{L^2_z(\mathbb{R})}).
\end{aligned} \tag{3.16}$$

It follows from Cauchy-Schwarz inequality and (3.11) that

$$\begin{aligned}
\int_{\mathbb{R}} \rho(1+z^2)w_k dz &\le \frac{1}{2}\int_{\mathbb{R}} (1+z^2)w_k^2 dz + \frac{1}{2}\int_{\mathbb{R}} (1+z^2)\rho^2 dz \\
&\le \frac{1}{2}(\|w_k\|^2_{L^2_z(\mathbb{R})} + \|zw_k\|^2_{L^2_z(\mathbb{R})}) + C.
\end{aligned} \tag{3.17}$$

Plugging (3.15), (3.16) and (3.17) into (3.12), it follows that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|w_k\|^2_{L^2_z(\mathbb{R})} + \|zw_k\|^2_{L^2_z(\mathbb{R})}) + (\|w_{kz}\|^2_{L^2_z(\mathbb{R})} + \|zw_{kz}\|^2_{L^2_z(\mathbb{R})}) \\
&\le C(\|w_k\|^2_{L^2_z(\mathbb{R})} + \|zw_k\|^2_{L^2_z(\mathbb{R})}) + C.
\end{aligned} \tag{3.18}$$

Applying the Gronwall's inequality to the above inequality, it yields

$$\sup_{0\leqslant t\leqslant T}(\|w_k\|^2_{L^2_z(\mathbb{R})} + \|zw_k\|^2_{L^2_z(\mathbb{R})}) + \|w_{kz}\|^2_{L^2(0,T;L^2_z(\mathbb{R}))} + \|zw_{kz}\|^2_{L^2(0,T;L^2_z(\mathbb{R}))} \le C, \tag{3.19}$$

where $C = C(T, \|B(0)e^{-|z|}\|_{L^2_z(\mathbb{R})})$ is a constant, independent of ϵ.

Similarly, multiplying the first equation of (3.10) by w_{kt} and integrating over $\mathbb{R}$, one gets from integration by parts that

$$\frac{1}{2}\frac{d}{dt}\|w_{kz}\|^2_{L^2_z(\mathbb{R})} + \|w_{kt}\|^2_{L^2_z(\mathbb{R})} = -\int_{\mathbb{R}} A(t)B(z,k)w_{kz}w_{kt}dz - \int_{\mathbb{R}} A(t)w_k w_{kt}dz + \int_{\mathbb{R}} \rho w_{kt}dz. \tag{3.20}$$

By Cauchy-Schwarz inequality, we get

$$\begin{aligned} -\int_{\mathbb{R}} A(t)B(z,k)w_{kz}w_{kt}dz &\le \int_{\mathbb{R}} |A(t)||B(z,k)||w_{kz}w_{kt}|dz \\ &\le \int_{\mathbb{R}} |z||A(t)||w_{kz}w_{kt}|dz \\ &\le \frac{1}{8}\|w_{kt}\|^2_{L^2_z(\mathbb{R})} + C\|zw_{kz}\|^2_{L^2_z(\mathbb{R})}, \end{aligned} \tag{3.21}$$

and

$$-\int_{\mathbb{R}} A(t)w_k w_{kt}dz \le \frac{1}{8}\|w_{kt}\|^2_{L^2_z(\mathbb{R})} + C\|w_k\|^2_{L^2_z(\mathbb{R})}. \tag{3.22}$$

Similar procedure further gives

$$\begin{aligned} \int_{\mathbb{R}} \rho w_{kt}dz &\le \frac{1}{4}\|w_{kt}\|^2_{L^2_z(\mathbb{R})} + \|\rho\|^2_{L^2_z(\mathbb{R})} \\ &\le \frac{1}{4}\|w_{kt}\|^2_{L^2_z(\mathbb{R})} + C. \end{aligned} \tag{3.23}$$

Hence, by plugging the estimates (3.21), (3.22) and (3.23) into (3.20), we get

$$\frac{d}{dt}\|w_{kz}\|^2_{L^2_z(\mathbb{R})} + \|w_{kt}\|^2_{L^2_z(\mathbb{R})} \le C\|w_k\|^2_{L^2_z(\mathbb{R})} + C\|zw_{kz}\|^2_{L^2_z(\mathbb{R})} + C. \tag{3.24}$$

By using the estimate (3.19), we obtain

$$\sup_{0\leqslant t\leqslant T}\|w_{kz}\|^2_{L^2_z(\mathbb{R})} + \|w_{kt}\|^2_{L^2(0,T;L^2_z(\mathbb{R}))} \le C, \tag{3.25}$$

where $C = C(T, \|B(0)e^{-|z|}\|_{L^2_z(\mathbb{R})})$ is a constant, independent of ϵ.

According to the energy estimates (3.19) and (3.25), one can see that w_k is uniformly bounded in $L^2(0,T;H^1_z(\mathbb{R}))$ and w_{kt} is uniformly bounded in $L^2(0,T;L^2_z(\mathbb{R}))$. Consequently there exists a subsequence of w_k, here we still denote by w_k for simplicity, such that

$$w_k \rightharpoonup w, \qquad \text{weakly in } L^2(0,T;L_z^2(\mathbb{R})), \tag{3.26}$$

$$w_{kz} \rightharpoonup w_z, \qquad \text{weakly in } L^2(0,T;L_z^2(\mathbb{R})), \tag{3.27}$$

$$zw_k \rightharpoonup zw, \qquad \text{weakly in } L^2(0,T;L_z^2(\mathbb{R})), \tag{3.28}$$

$$zw_{kz} \rightharpoonup zw_z, \qquad \text{weakly in } L^2(0,T;L_z^2(\mathbb{R})), \tag{3.29}$$

$$w_{kt} \rightharpoonup w_t, \qquad \text{weakly in } L^2(0,T;L_z^2(\mathbb{R})). \tag{3.30}$$

Since w_k is a weak solution of the parabolic initial value problem (3.10), then

$$\begin{aligned}\int_0^T\int_{\mathbb{R}} w_{kt}\varphi dz + \int_0^T\int_{\mathbb{R}} w_{kz}\varphi_z dz = &-\int_0^T\int_{\mathbb{R}} A(t)B(z,k)w_{kz}\varphi dz \\ &-\int_0^T\int_{\mathbb{R}} A(t)w_k\varphi dz + \int_0^T\int_{\mathbb{R}} \rho\varphi dz,\end{aligned} \tag{3.31}$$

for any $\varphi \in C_0^\infty((0,T)\times\mathbb{R})$.

Passing to limits as $k \to \infty$, we have

$$\int_0^T\int_{\mathbb{R}} w_t\varphi dz + \int_0^T\int_{\mathbb{R}} w_z\varphi_z dz = -\int_0^T\int_{\mathbb{R}} A(t)zw_z\varphi dz - \int_0^T\int_{\mathbb{R}} A(t)w\varphi dz + \int_0^T\int_{\mathbb{R}} \rho\varphi dz. \tag{3.32}$$

Hence $w \in L^2(0,T;H_z^1(\mathbb{R}))$ is a weak solution of (3.9).

Next we prove the uniqueness of the weak solution. Let w_1 and w_2 be two solutions of problem (3.9). Then $\phi := w_1 - w_2$ satisfies the following problem

$$\begin{cases}\phi_t - \phi_{zz} = -A(t)z\phi_z - A(t)\phi, \\ \phi(z,0) = 0.\end{cases} \tag{3.33}$$

Multiplying the first equation in (3.33) by ϕ and integrating on $\mathbb{R}$, then using integration by parts we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\phi\|^2_{L_z^2(\mathbb{R})} + \|\phi_z\|^2_{L_z^2(\mathbb{R})} &= -\int_{\mathbb{R}} A(t)z\phi_z\phi dz - \int_{\mathbb{R}} A(t)\phi^2 dz \\ &= -\int_{\mathbb{R}} A(t)z(\frac{\phi^2}{2})_z dz - \int_{\mathbb{R}} A(t)\phi^2 dz \\ &= -\int_{\mathbb{R}} A(t)(\frac{\phi^2}{2}) dz - \int_{\mathbb{R}} A(t)\phi^2 dz \\ &= -\frac{1}{2}\int_{\mathbb{R}} A(t)\phi^2 dz.\end{aligned} \tag{3.34}$$

Therefore

$$\frac{1}{2}\frac{d}{dt}\|\phi\|^2_{L^2_z(\mathbb{R})}+\|\phi_z\|^2_{L^2_z(\mathbb{R})}\le C\|\phi\|^2_{L^2_z(\mathbb{R})}.$$

Using Gronwall's inequality and noticing that $\phi(z,0)=0$, we deduce that $\|\phi\|_{L^2_z(\mathbb{R})}=0$. Hence $\phi\equiv 0$.

Thus, from the definition of $w(z,t):=v^{B,1}(z,t)+B(t)e^{-z}$, we can easily obtain the same properties for the boundary layer solution $v^{B,1}$ to the problem (2.17). Moreover, according to (2.18), the existence of $u^{B,2}$ is obvious.

Step 2. Regularity of boundary layer profiles.

Taking the L^2 inner product of $(3.8)_1$ with z^2w_{zz} and integrating by parts, we get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|zw_z\|^2_{L^2_z}+\|zw_{zz}\|^2_{L^2_z}&=\int_0^\infty z^3A(t)w_zw_{zz}dz+\int_0^\infty z^2A(t)ww_{zz}dz\\&\quad-\int_0^\infty z^2\rho w_{zz}dz-2\int_0^\infty zw_tw_zdz\\&=-\frac{5}{2}\int_0^\infty z^2A(t)w_z^2dz+\int_0^\infty A(t)w^2dz\\&\quad-\int_0^\infty z^2\rho w_{zz}dz-2\int_0^\infty zw_tw_zdz.\end{aligned}\tag{3.35}$$

It follows from the Cauchy-Schwarz inequality that

$$-\frac{5}{2}\int_0^\infty z^2A(t)w_z^2dz+\int_0^\infty A(t)w^2dz\le C\|zw_z\|^2_{L^2_z}+C\|w\|^2_{L^2_z},\tag{3.36}$$

and

$$-\int_0^\infty z^2\rho w_{zz}dz\le\frac{1}{2}\|zw_{zz}\|^2_{L^2_z}+\frac{1}{2}\|z\rho\|^2_{L^2_z}.\tag{3.37}$$

Similarly, it follows that

$$-2\int_0^\infty zw_tw_zdz\le\|zw_z\|^2_{L^2_z}+\|w_t\|^2_{L^2_z}.\tag{3.38}$$

Plugging (3.36), (3.37) and (3.38) into (3.35) yields

$$\frac{d}{dt}\|zw_z\|^2_{L^2_z} + \|zw_{zz}\|^2_{L^2_z} \le C\|zw_z\|^2_{L^2_z} + C\|w\|^2_{L^2_z} + C\|z\rho\|^2_{L^2_z} + C\|w_t\|^2_{L^2_z}. \tag{3.39}$$

Applying Gronwall's inequality to the above inequality, one gets

$$\sup_{0\leqslant t\leqslant T} \|zw_z\|^2_{L^2_z} + \|zw_{zz}\|^2_{L^2(0,T;L^2_z)} \le C, \tag{3.40}$$

where $C = C(T, \|B(0)e^{-z}\|_{L^2_z})$ is a constant, independent of ϵ.

Next, differentiating equation $(3.8)_1$ in z, we obtain

$$w_{zt} - w_{zzz} = -zA(t)w_{zz} - 2A(t)w + \rho_z. \tag{3.41}$$

Taking the L^2 inner product of (3.41) with w_{zt} and integrating by parts, we deduce

$$\frac{1}{2}\frac{d}{dt}\|w_{zz}\|^2_{L^2_z} + \|w_{zt}\|^2_{L^2_z} = -\int_0^\infty A(t)zw_{zz}w_{zt}dz - 2\int_0^\infty A(t)w_z w_{zt}dz + \int_0^\infty \rho_z w_{zt}dz. \tag{3.42}$$

By employing the Cauchy-Schwarz inequality, we get

$$-\int_0^\infty A(t)zw_{zz}w_{zt}dz \le \frac{1}{8}\|w_{zt}\|^2_{L^2_z} + C\|zw_{zz}\|^2_{L^2_z}, \tag{3.43}$$

and

$$-2\int_0^\infty A(t)w_z w_{zt}dz \le \frac{1}{8}\|w_{zt}\|^2_{L^2_z} + C\|w_z\|^2_{L^2_z}. \tag{3.44}$$

Also, we have

$$\int_0^\infty \rho_z w_{zt}dz \le \frac{1}{4}\|w_{zt}\|^2_{L^2_z} + \|\rho_z\|^2_{L^2_z}. \tag{3.45}$$

Plugging (3.43), (3.44) and (3.45) into (3.42), it follows that

$$\frac{d}{dt}\|w_{zz}\|^2_{L^2_z} + \|w_{zt}\|^2_{L^2_z} \le C\|zw_{zz}\|^2_{L^2_z} + C\|w_z\|^2_{L^2_z} + 2\|\rho_z\|^2_{L^2_z}, \tag{3.46}$$

which along with (3.11), (3.25), (3.27) and (3.40) leads to

$$\sup_{0\leqslant t\leqslant T} \|w_{zz}\|^2_{L^2_z} + \|w_{zt}\|^2_{L^2(0,T;L^2_z)} \le C, \tag{3.47}$$

where $C = C(T, \|B(0)e^{-z}\|_{L^2_z})$ is a constant, independent of ϵ.

By using the equation of (3.41), we can then easily derive that

$$\sup_{0\leqslant t\leqslant T}\|w_{zz}\|^2_{L^2_z}+\|w_{zzz}\|^2_{L^2(0,T;L^2_z)}\leq C, \tag{3.48}$$

which implies

$$w\in L^2(0,T;H^3_z)\cap L^\infty(0,T;H^2_z). \tag{3.49}$$

Thus (3.3) follows from the definition of w and (3.49). Then it follows from (2.17), (3.3), (3.7), (3.25), (3.30), (3.40) and (3.47) that

$$v_t^{B,1}\in L^2(0,T;H^1_z)\cap L^\infty(0,T;L^2_z).$$

Furthermore, the regularity of $u^{B,2}$ follows directly from (2.18) and (3.3). By (3.7), (3.19) and (3.40), we finally obtain that

$$zv^{B,1}\in L^2(0,T;H^2_z)\cap L^\infty(0,T;H^1_z).$$

This verifies (3.4) and the proof is completed. □

4. Error estimates and convergence rates

In this section, we will prove our main convergence results of this paper.

We construct an approximate solution as follows:

$$\begin{cases} v^{\epsilon,a}=v^{I,0}(x,t)+\epsilon^{\frac12}v^{B,1}(z,t),\\ u^{\epsilon,a}=u^{I,0}(x,t)+\epsilon u^{B,2}(z,t)-\epsilon u^{B,2}(0,t),\end{cases} \tag{4.1}$$

and

$$\begin{cases} R_1^\epsilon=\epsilon^{-1}(v^\epsilon-v^{\epsilon,a}),\\ R_2^\epsilon=\epsilon^{-1}(u^\epsilon-u^{\epsilon,a}),\end{cases} \tag{4.2}$$

where $\epsilon u^{B,2}(0,t)$ is constructed to homogenize the boundary condition of R_2^ϵ.

Plugging (4.1) and (4.2) into the equations (1.3) and using the initial and boundary conditions in (2.13) – (2.17), one can see that the reminder $(R_1^\epsilon, R_2^\epsilon)$ satisfies the following equations:

$$\begin{cases} R^\epsilon_{1t}+R^\epsilon_2 v_x^{\epsilon,a}+u^{\epsilon,a}R^\epsilon_{1x}+\epsilon R^\epsilon_2R^\epsilon_{1x}=-\dfrac12\epsilon(R^\epsilon_1)^2-v^{\epsilon,a}R^\epsilon_1+\epsilon R^\epsilon_{1xx}+\epsilon^{-1}f^\epsilon,\\ R^\epsilon_{2x}=R^\epsilon_1,\\ R^\epsilon_2(0,t)=0,\\ R^\epsilon_{1x}(0,t)=0,\\ R^\epsilon_1(x,0)=0,\end{cases} \tag{4.3}$$

where

$$f^{\epsilon}=-v_t^{\epsilon,a}-u^{\epsilon,a}v_x^{\epsilon,a}-\frac{1}{2}(v^{\epsilon,a})^2+\epsilon v_{xx}^{\epsilon,a}. \tag{4.4}$$

Before proceeding, we introduce some basic facts first which will be used later. For any $F(z,t)\in H_z^m$ with $m\in\mathbb{N}$, we have

$$\left\|\partial_x^m F(\frac{x}{\sqrt{\epsilon}},t)\right\|_{L^2}=\epsilon^{\frac{1}{4}-\frac{m}{2}}\left\|\partial_z^m F(z,t)\right\|_{L_z^2}. \tag{4.5}$$

For $f(\cdot,t)\in H^1$ with fixed $t>0$, one gets

$$\|f(\cdot,t)\|_{L^\infty}\le C_0(\|f(\cdot,t)\|_{L^2}^{\frac{1}{2}}\|f_x(\cdot,t)\|_{L^2}^{\frac{1}{2}}+\|f(\cdot,t)\|_{L^2}). \tag{4.6}$$

We next recall the Hardy inequality that

$$\left\|\frac{1}{x}\int_0^x f(t)dt\right\|_{L^2(\mathbb{R}^+)}\le C\|f\|_{L^2(\mathbb{R}^+)}. \tag{4.7}$$

Lemma 4.1. *Let $0<T<\infty, 0<\epsilon<1$. Then there exists a constant C independent of ϵ, such that*

$$\|f^{\epsilon}\|_{L^2(0,T;L^2)}\le C\epsilon^{3/4}. \tag{4.8}$$

Proof. From the definition of $v^{\epsilon,a}$, $u^{\epsilon,a}$, (4.4) and (2.13), we can deduce that

$$\begin{aligned}f^{\epsilon}=&[\epsilon v_{xx}^{I,0}]+[\epsilon^{\frac{3}{2}}v_{xx}^{B,1}-\epsilon^{\frac{1}{2}}v_t^{B,1}]-[\epsilon^{\frac{1}{2}}u^{I,0}v_x^{B,1}]\\&-[\epsilon u^{B,2}(v_x^{I,0}+\epsilon^{\frac{1}{2}}v_x^{B,1})]-[\frac{1}{2}\epsilon(v^{B,1})^2+\epsilon^{\frac{1}{2}}v^{I,0}v^{B,1}]\\&+[\epsilon u^{B,2}(0,t)(v_x^{I,0}+\epsilon^{\frac{1}{2}}v_x^{B,1})]\\:=&\sum_{k=1}^{6}J_k.\end{aligned} \tag{4.9}$$

We proceed to estimate $J_k(1\le k\le 6)$. By (3.1) we have

$$\begin{aligned}\|J_1\|_{L^2(0,T;L^2)}&=\|\epsilon v_{xx}^{I,0}\|_{L^2(0,T;L^2)}\\&\le\epsilon\|v^{I,0}\|_{L^\infty(0,T;H^3)}\\&\le C\epsilon.\end{aligned} \tag{4.10}$$

It follows from (4.5), (3.3) and (3.5) that

$$\begin{aligned}\|J_2\|_{L^2(0,T;L^2)} &\leq \epsilon^{\frac{3}{4}}(\|v_{zz}^{B,1}\|_{L^2(0,T;L_z^2)} + \|v_t^{B,1}\|_{L^2(0,T;L_z^2)}) \\ &\leq C\epsilon^{\frac{3}{4}}.\end{aligned} \tag{4.11}$$

By using (2.2), (4.7), (4.5), (3.1), (3.7), (3.40) and Sobolev embedding inequality, one gets that

$$\begin{aligned}\|J_3\|_{L^2(0,T;L^2)} &= \epsilon^{\frac{1}{2}}\|\frac{u^{I,0}(x,t)}{x}\cdot\epsilon^{\frac{1}{2}}zv_x^{B,1}\|_{L^2(0,T;L^2)} \\ &\leq \epsilon\|u_x^{I,0}\|_{L^2(0,T;L^\infty)}\|zv_x^{B,1}\|_{L^\infty(0,T;L^2)} \\ &\leq C_0\epsilon^{\frac{3}{4}}\|u^{I,0}\|_{L^2(0,T;H^2)}\|zv_z^{B,1}\|_{L^\infty(0,T;L_z^2)} \\ &\leq C\epsilon^{\frac{3}{4}}.\end{aligned} \tag{4.12}$$

Similarly, we have

$$\begin{aligned}\|J_4\|_{L^2(0,T;L^2)} &= \|\epsilon u^{B,2}(v_x^{I,0}+\epsilon^{\frac{1}{2}}v_x^{B,1})\|_{L^2(0,T;L^2)} \\ &\leq \epsilon\|u^{B,2}\|_{L^2(0,T;L^2)}\|v_x^{I,0}\|_{L^\infty(0,T;L^\infty)} \\ &\quad + \epsilon^{\frac{3}{2}}\|u^{B,2}\|_{L^2(0,T;L^\infty)}\|v_x^{B,1}\|_{L^\infty(0,T;L^2)} \\ &\leq C_0\epsilon^{\frac{5}{4}}\|u^{B,2}\|_{L^2(0,T;L_z^2)}\|v^{I,0}\|_{L^\infty(0,T;H^2)} \\ &\quad + C_0\epsilon^{\frac{5}{4}}\|u^{B,2}\|_{L^2(0,T;L_z^\infty)}\|v_z^{B,1}\|_{L^\infty(0,T;L_z^2)} \\ &\leq C\epsilon^{\frac{3}{4}},\end{aligned} \tag{4.13}$$

and

$$\begin{aligned}\|J_5\|_{L^2(0,T;L^2)} &\leq \|\frac{1}{2}\epsilon(v^{B,1})^2\|_{L^2(0,T;L^2)} + \|\epsilon^{\frac{1}{2}}v^{I,0}v^{B,1}\|_{L^2(0,T;L^2)} \\ &\leq C_0\epsilon^{\frac{5}{4}}\|v^{B,1}\|_{L^2(0,T;L^\infty)}\|v^{B,1}\|_{L^\infty(0,T;L_z^2)} \\ &\quad + \epsilon^{\frac{3}{4}}\|v^{I,0}\|_{L^2(0,T;L^\infty)}\|v^{B,1}\|_{L^\infty(0,T;L_z^2)} \\ &\leq C_0\epsilon^{\frac{5}{4}}\|v^{B,1}\|_{L^2(0,T;L_z^\infty)}\|v^{B,1}\|_{L^\infty(0,T;L_z^2)} \\ &\quad + C_0\epsilon^{\frac{3}{4}}\|v^{I,0}\|_{L^2(0,T;H^1)}\|v^{B,1}\|_{L^\infty(0,T;L_z^2)} \\ &\leq C\epsilon^{\frac{3}{4}}.\end{aligned} \tag{4.14}$$

Thanks to

$$\|u^{B,2}(0,t)\|_{L^\infty(0,T)} \leq C_0\|u^{B,2}\|_{L^\infty(0,T;H_z^1)} \leq C, \tag{4.15}$$

we can estimate the J_6 to obtain

$$\begin{aligned}\|J_6\|_{L^2(0,T;L^2)} &\le C\epsilon(\|v^{I,0}\|_{L^2(0,T;H^1)} + \epsilon^{\frac{1}{4}}\|v^{B,1}\|_{L^2(0,T;H_z^1)}) \\ &\le C\epsilon^{\frac{3}{4}}.\end{aligned} \tag{4.16}$$

Combining the above estimates for J_1 to J_6 with (4.9) we deduce that (4.8). The proof is completed. □

Next, we give the estimates of the reminder R_1^ϵ in the spaces $L^\infty(0,T;L^2)$ and $L^\infty(0,T;H^1)$. Before stating the results, we first estimate $v^{\epsilon,a}$ by the Sobolev embedding inequality, and Lemmas 3.1-3.2 as follows

$$\begin{aligned}\|v^{\epsilon,a}\|_{L^\infty([0,T]\times[0,\infty))} &\le \|v^{I,0}\|_{L^\infty([0,T]\times[0,\infty))} + \epsilon^{\frac{1}{2}}\|v^{B,1}\|_{L^\infty(0,T;L_z^\infty)} \\ &\le C(1+\epsilon^{\frac{1}{2}}) \le C.\end{aligned} \tag{4.17}$$

Lemma 4.2. *Let* $0 < T < \infty$ *and* $0 < \epsilon < 1$. *Then there exists a positive constant C independent of* ϵ, *such that the following estimate holds:*

$$\sup_{0\leqslant t\leqslant T} \|R_1^\epsilon\|_{L^2}^2 + \epsilon\|R_{1x}^\epsilon\|_{L^2(0,T;L^2)}^2 \le C\epsilon^{-\frac{1}{2}}. \tag{4.18}$$

Proof. Multiplying the first equation of (4.3) by R_1^ϵ and performing the integration by parts, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|R_1^\epsilon\|_{L^2}^2 + \epsilon\|R_{1x}^\epsilon\|_{L^2}^2 &= -\int_0^\infty R_1^\epsilon R_2^\epsilon v_x^{\epsilon,a}dx - \int_0^\infty R_1^\epsilon u^{\epsilon,a}R_{1x}^\epsilon dx - \epsilon\int_0^\infty R_1^\epsilon R_2^\epsilon R_{1x}^\epsilon dx \\ &\quad - \frac{\epsilon}{2}\int_0^\infty (R_1^\epsilon)^3 dx - \int_0^\infty R_1^\epsilon v^{\epsilon,a}R_1^\epsilon dx + \epsilon^{-1}\int_0^\infty R_1^\epsilon f^\epsilon dx \\ &= -\int_0^\infty R_1^\epsilon v_x^{\epsilon,a}R_2^\epsilon dx - \frac{1}{2}\int_0^\infty v^{\epsilon,a}(R_1^\epsilon)^2 dx + \epsilon^{-1}\int_0^\infty R_1^\epsilon f^\epsilon dx \\ &:= I_1 + I_2 + I_3.\end{aligned} \tag{4.19}$$

In view of the second equation of (4.3),

$$R_2^\epsilon = \int_0^x R_1^\epsilon dx \le \left(\int_0^x |R_1^\epsilon|^2 dx\right)^{\frac{1}{2}} x^{\frac{1}{2}}. \tag{4.20}$$

By the definition of $v^{\epsilon,a}$, (3.3), (3.7) and (3.40), we have

$$\begin{aligned}\int_0^\infty x|v_x^{\epsilon,a}|^2dx &\le \int_0^\infty x|v_x^{I,0}|^2dx + \epsilon\int_0^\infty x|v_x^{B,1}|^2dx \\ &= \int_0^\infty x|v_x^{I,0}|^2dx + \epsilon\int_0^\infty z|v_z^{B,1}|^2dz \\ &\le C,\end{aligned} \tag{4.21}$$

where we have used $\int_0^\infty xv_{0x}^2dx < +\infty$.

It follows from (4.20), (4.21) and Hölder inequality that

$$\begin{aligned}I_1 &\le \left(\int_0^\infty |R_1^\epsilon|^2dx\right)^{\frac{1}{2}}\int_0^\infty |x^{\frac{1}{2}}v_x^{\epsilon,a}R_1^\epsilon|dx \\ &\le \int_0^\infty |R_1^\epsilon|^2dx\left(\int_0^\infty x|v_x^{\epsilon,a}|^2dx\right)^{\frac{1}{2}} \\ &\le C\|R_1^\epsilon\|_{L^2}^2.\end{aligned} \tag{4.22}$$

By (4.17) and Hölder inequality, we deduce

$$I_2 \le C_0\|v^{\epsilon,a}\|_{L^\infty}\|R_1^\epsilon\|_{L^2}^2 \le C\|R_1^\epsilon\|_{L^2}^2. \tag{4.23}$$

It follows from Cauchy-Schwarz inequality that

$$I_3 \le \frac{1}{2}\|R_1^\epsilon\|_{L^2}^2 + \frac{1}{2}\epsilon^{-2}\|f^\epsilon\|_{L^2}^2. \tag{4.24}$$

Plugging (4.22), (4.23), (4.24) into (4.19), it follows that

$$\frac{d}{dt}\|R_1^\epsilon\|_{L^2}^2 + \epsilon\|R_{1x}^\epsilon\|_{L^2}^2 \le C\|R_1^\epsilon\|_{L^2}^2 + \epsilon^{-2}\|f^\epsilon\|_{L^2}^2, \tag{4.25}$$

which implies the estimate (4.18) by using the Gronwall's inequality along with Lemma 4.1. □

Lemma 4.3. *Let $0 < T < \infty$ and $0 < \epsilon < 1$. Then there exists a positive constant C independent of ϵ, such that the following estimate holds:*

$$\sup_{0\leqslant t\leqslant T}\|R_{1x}^\epsilon\|_{L^2}^2 + \epsilon\|R_{1xx}^\epsilon\|_{L^2(0,T;L^2)}^2 \le C\epsilon^{-\frac{3}{2}}. \tag{4.26}$$

Proof. Multiplying the first equation of (4.3) by $\epsilon R_{1xx}^\epsilon$ and integrating by parts over $(0, \infty)$, we obtain that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}(\epsilon\|R^\epsilon_{1x}\|^2_{L^2})+\epsilon^2\|R^\epsilon_{1xx}\|^2_{L^2} &= \epsilon\int_0^\infty R^\epsilon_2 v^{\epsilon,a}_x R^\epsilon_{1xx}dx+\epsilon\int_0^\infty u^{\epsilon,a}R^\epsilon_{1x}R^\epsilon_{1xx}dx\\ &\quad+\epsilon^2\int_0^\infty R^\epsilon_2R^\epsilon_{1x}R^\epsilon_{1xx}dx+\frac{1}{2}\epsilon^2\int_0^\infty (R^\epsilon_1)^2R^\epsilon_{1xx}dx\\ &\quad+\epsilon\int_0^\infty v^{\epsilon,a}R^\epsilon_1R^\epsilon_{1xx}dx-\int_0^\infty f^\epsilon R^\epsilon_{1xx}dx\\ &=\epsilon\int_0^\infty R^\epsilon_2 v^{\epsilon,a}_x R^\epsilon_{1xx}dx-\frac{1}{2}\epsilon\int_0^\infty v^{\epsilon,a}(R^\epsilon_{1x})^2dx\\ &\quad-\frac{3}{2}\epsilon^2\int_0^\infty R^\epsilon_1(R^\epsilon_{1x})^2dx-\int_0^\infty f^\epsilon R^\epsilon_{1xx}dx\\ &:=I_4+I_5+I_6+I_7.\end{aligned}\tag{4.27}$$

By using a similar argument as that for I_1 and using Cauchy-Schwarz inequality, one has

$$I_4\le C\epsilon\|R^\epsilon_1\|_{L^2}\|R^\epsilon_{1xx}\|_{L^2}\le\frac{1}{8}\epsilon^2\|R^\epsilon_{1xx}\|^2_{L^2}+C\|R^\epsilon_1\|^2_{L^2}.\tag{4.28}$$

It is easy to show that

$$I_5\le C\epsilon\|R^\epsilon_{1x}\|^2_{L^2}.\tag{4.29}$$

It follows from Hölder and Gagliardo-Nirenberg and Cauchy-Schwarz inequalities that

$$\begin{aligned}I_6&\le C\epsilon^2\|R^\epsilon_1\|_{L^2}\|R^\epsilon_{1x}\|^2_{L^4}\\ &\le C\epsilon^2\|R^\epsilon_1\|_{L^2}(\|R^\epsilon_1\|^{\frac{3}{4}}_{L^2}\|R^\epsilon_{1xx}\|^{\frac{5}{4}}_{L^2}+\|R^\epsilon_1\|^2_{L^2})\\ &\le C\epsilon^2\|R^\epsilon_1\|^{\frac{7}{4}}_{L^2}\|R^\epsilon_{1xx}\|^{\frac{5}{4}}_{L^2}+C\epsilon^2\|R^\epsilon_1\|^3_{L^2}\\ &\le\frac{1}{8}\epsilon^2\|R^\epsilon_{1xx}\|^2_{L^2}+C\epsilon^2\|R^\epsilon_1\|^{\frac{14}{3}}_{L^2}+C\epsilon^2\|R^\epsilon_1\|^3_{L^2}.\end{aligned}\tag{4.30}$$

Similarly, the estimate for I_7 follows from Cauchy-Schwarz inequality that

$$I_7\le\frac{1}{4}\epsilon^2\|R^\epsilon_{1xx}\|^2_{L^2}+\epsilon^{-2}\|f^\epsilon\|^2_{L^2}.\tag{4.31}$$

Thus, plugging all the estimates above into (4.27), one gets

$$\begin{aligned}\frac{d}{dt}(\epsilon\|R^\epsilon_{1x}\|^2_{L^2})+\epsilon^2\|R^\epsilon_{1xx}\|^2_{L^2}&\le C\epsilon\|R^\epsilon_{1x}\|^2_{L^2}+C(\|R^\epsilon_1\|^2_{L^2}+\epsilon^2\|R^\epsilon_1\|^{\frac{14}{3}}_{L^2}+\epsilon^2\|R^\epsilon_1\|^3_{L^2})\\ &\quad+2\epsilon^{-2}\|f^\epsilon\|^2_{L^2},\end{aligned}\tag{4.32}$$

which implies the estimate (4.26) immediately. □

Now, let's complete the proof of Theorem 2.1.

Proof of Theorem 2.1. Based on the study given in Section 3, it remains only to prove the solution convergence (2.19)-(2.22). By Lemmas 4.2-4.3 we have

$$\|R_1^\epsilon\|_{L^\infty(0,T;L^2)} \le C\epsilon^{-\frac{1}{4}}, \tag{4.33}$$

$$\|R_{1x}^\epsilon\|_{L^\infty(0,T;L^2)} \le C\epsilon^{-\frac{3}{4}}. \tag{4.34}$$

Then the estimate for R_1^ϵ in the space $L^\infty(0,T;L^\infty)$ follows from (4.6), (4.33) and (4.34) that

$$\|R_1^\epsilon\|_{L^\infty(0,T;L^\infty)} \le C_0(\|R_1^\epsilon\|^{\frac{1}{2}}_{L^\infty(0,T;L^2)}\|R_{1x}^\epsilon\|^{\frac{1}{2}}_{L^\infty(0,T;L^2)} + \|R_1^\epsilon\|_{L^\infty(0,T;L^2)}) \le C\epsilon^{-\frac{1}{2}}. \tag{4.35}$$

Thanks to $R_2^\epsilon = \int_0^x R_1^\epsilon dx$, we get

$$\|R_2^\epsilon\|_{L^\infty(0,T;L^\infty(0,M))} \le M\|R_1^\epsilon\|_{L^\infty(0,T;L^\infty)} \le C\epsilon^{-\frac{1}{2}} \tag{4.36}$$

for any $M > 0$.

It follows from (4.1), (4.2), (4.33), (4.34), (4.35) and (4.36) that

$$\|v^\epsilon - v^{I,0} - \epsilon^{\frac{1}{2}} v^{B,1}\|_{L^\infty(0,T;L^2)} \le C\epsilon^{\frac{3}{4}}, \tag{4.37}$$

$$\|v^\epsilon - v^{I,0} - \epsilon^{\frac{1}{2}} v^{B,1}\|_{L^\infty(0,T;H^1)} \le C\epsilon^{\frac{1}{4}}, \tag{4.38}$$

$$\|v^\epsilon - v^{I,0} - \epsilon^{\frac{1}{2}} v^{B,1}\|_{L^\infty(0,T;L^\infty)} \le C\epsilon^{\frac{1}{2}}, \tag{4.39}$$

$$\|u^\epsilon - u^{I,0}\|_{L^\infty(0,T;L^\infty(0,M))} \le C\epsilon^{\frac{1}{2}}, \tag{4.40}$$

for any $M > 0$. Thus, the proof of Theorem 2.1 is completed. □

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Appendix A

In this section, we will use the fundamental solution Φ of the heat equation to represent w_k, and then show that there exists a unique solution w_k of (3.10) for each $k > 0$.

Set $X = L^\infty(0, T; H_z^1(\mathbb{R}))$. For each $v \in X$, we have

$$\begin{cases} w_{kt} - w_{kzz} = -B(z,k)A(t)v_z - A(t)v + \rho(|z|,t) := f(z,t), & (z,t) \in (-\infty,\infty) \times (0,\infty), \\ w_k(z,0) = B(0)e^{-|z|}. \end{cases}$$

We use the fundamental solution Φ of the heat equation to represent w_k in terms of $B(0)e^{-|z|}$ and $f(z,t)$ as follows

$$w_k(z,t) = \int_{-\infty}^{+\infty} \Phi(z-y, t-s)B(0)e^{-|y|}dy + \int_0^t \int_{-\infty}^{+\infty} \Phi(z-y, t-s) f(y,s)dyds, \tag{A.1}$$

where

$$\Phi(z,t) = \frac{1}{2\sqrt{\pi t}} e^{-\frac{z^2}{4t}}.$$

Then take $v^{(1)} \in X$ and let $w_k^{(1)}$ solve

$$\begin{cases} w_{kt}^{(1)} - w_{kzz}^{(1)} = -B(z,k)A(t)v_z^{(1)} - A(t)v^{(1)} + \rho(|z|,t), & (z,t) \in (-\infty,\infty) \times (0,\infty), \\ w_k^{(1)}(z,0) = B(0)e^{-|z|}. \end{cases} \tag{A.2}$$

Similarly, $v^{(2)} \in X$ and $w_k^{(2)}$ satisfy

$$\begin{cases} w_{kt}^{(2)} - w_{kzz}^{(2)} = -B(z,k)A(t)v_z^{(2)} - A(t)v^{(2)} + \rho(|z|,t), & (z,t) \in (-\infty,\infty) \times (0,\infty), \\ w_k^{(2)}(z,0) = B(0)e^{-|z|}. \end{cases} \tag{A.3}$$

Let $\tilde{w} := w_k^{(1)} - w_k^{(2)}$ and $\tilde{v} := v^{(1)} - v^{(2)}$. Subtracting (A.2) from (A.3), we deduce that $\tilde{w}$ satisfies

$$\begin{cases} \tilde{w}_t - \tilde{w}_{zz} = -B(z,k)A(t)\tilde{v}_z - A(t)\tilde{v} := \tilde{f}(z,t), & (z,t) \in (-\infty,\infty) \times (0,\infty), \\ \tilde{w}(z,0) = 0. \end{cases} \tag{A.4}$$

Similarly, we use the fundamental solution Φ of the heat equation to represent $\tilde{w}$ as follows

$$\begin{aligned}\tilde{w}(z,t) &= \int_0^t \int_{-\infty}^{+\infty} \Phi(z-y,t-s)\tilde{f}(y,s)dyds, \\ &= \int_0^t [\Phi(\cdot,t-s) * \tilde{f}(\cdot,s)](z)ds.\end{aligned} \tag{A.5}$$

It follows from (A.5) that

$$\begin{aligned}\|\tilde{w}(t)\|_{L^2_z(\mathbb{R})} &= \left(\int_{-\infty}^{+\infty} \left| \int_0^t [\Phi(\cdot,t-s) * \tilde{f}(\cdot,s)](z)ds \right|^2 dz \right)^{\frac{1}{2}} \\ &\le \int_0^t \left(\int_{-\infty}^{+\infty} \left| [\Phi(\cdot,t-s) * \tilde{f}(\cdot,s)](z) \right|^2 dz \right)^{\frac{1}{2}} ds \\ &= \int_0^t \|\Phi(\cdot,t-s) * \tilde{f}(\cdot,s)\|_{L^2_z(\mathbb{R})}\, ds \\ &\le \int_0^t \|\Phi(\cdot,t-s)\|_{L^1_z(\mathbb{R})} \|\tilde{f}(\cdot,s)\|_{L^2_z(\mathbb{R})}\, ds \\ &= \int_0^t \|\tilde{f}(\cdot,s)\|_{L^2_z(\mathbb{R})}\, ds \\ &\le T\|\tilde{f}\|_{L^\infty(0,T;L^2_z(\mathbb{R}))},\end{aligned} \tag{A.6}$$

where we have used

$$\begin{aligned}\|\Phi(\cdot,t-s)\|_{L^1_z(\mathbb{R})} &= \int_{-\infty}^{+\infty} \frac{1}{2\sqrt{\pi(t-s)}} e^{-\frac{(z-y)^2}{4(t-s)}} dz \\ &= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-y^2} dy \\ &= 1.\end{aligned}$$

Similarly, we have

$$\begin{aligned}\tilde{w}_z(z,t) &= \int_0^t \int_{-\infty}^{+\infty} \Phi_z(z-y,t-s)\tilde{f}(y,s)dyds \\ &= \int_0^t [\Phi_z(\cdot,t-s) * \tilde{f}(\cdot,s)](z)ds.\end{aligned}$$

Note that

$$\begin{aligned}\int_{-\infty}^{+\infty} |\Phi_z(z-y,t-s)|dz &= 2\int_0^{+\infty} \frac{1}{2\sqrt{\pi(t-s)}} \frac{z-y}{2(t-s)} e^{-\frac{(z-y)^2}{4(t-s)}} dz \\ &= \frac{2}{\sqrt{\pi}}(t-s)^{-\frac{1}{2}} \int_0^{+\infty} xe^{-x^2}dx \\ &= \frac{1}{\sqrt{\pi}}(t-s)^{-\frac{1}{2}}.\end{aligned}$$

Thus

$$\begin{aligned}\|\tilde{w}_z(t)\|_{L^2_z(\mathbb{R})} &= \left(\int_{-\infty}^{+\infty}\left|\int_0^t [\Phi_z(\cdot,t-s) * \tilde{f}(\cdot,s)](z)ds\right|^2 dz\right)^{\frac{1}{2}} \\ &\leq \int_0^t \|\Phi_z(\cdot,t-s)\|_{L^1_z(\mathbb{R})} \|\tilde{f}(\cdot,s)\|_{L^2_z(\mathbb{R})}\, ds \\ &\leq C\int_0^t (t-s)^{-\frac{1}{2}} \|\tilde{f}(\cdot,s)\|_{L^2_z(\mathbb{R})}\, ds \\ &\leq C\|\tilde{f}\|_{L^\infty(0,T;L^2_z(\mathbb{R}))} \int_0^t (t-s)^{-\frac{1}{2}}ds \\ &\leq CT^{\frac{1}{2}}\|\tilde{f}\|_{L^\infty(0,T;L^2_z(\mathbb{R}))}.\end{aligned} \tag{A.7}$$

Combining (A.6) and (A.7), one gets

$$\begin{aligned}\|\tilde{w}\|_{L^\infty(0,T;H^1_z(\mathbb{R}))} &\leq C(T+T^{\frac{1}{2}})\|\tilde{f}\|_{L^\infty(0,T;L^2_z(\mathbb{R}))} \\ &\leq C(T+T^{\frac{1}{2}})\|\tilde{v}\|_{L^\infty(0,T;H^1_z(\mathbb{R}))}.\end{aligned}$$

If T is so small that $C(T+T^{\frac{1}{2}}) \leq \frac{1}{2}$, then

$$\|\tilde{w}\|_{L^\infty(0,T;H_z^1(\mathbb{R}))} \leq \frac{1}{2}\|\tilde{v}\|_{L^\infty(0,T;H_z^1(\mathbb{R}))}.$$

Thus

$$\|w_k^{(1)} - w_k^{(2)}\|_{L^\infty(0,T;H_z^1(\mathbb{R}))} \leq \frac{1}{2}\|v^{(1)} - v^{(2)}\|_{L^\infty(0,T;H_z^1(\mathbb{R}))}.$$

According to the contraction mapping principle, the mapping $v \mapsto w_k$ has a unique fixed point.

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