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Characteristic boundary layers in the vanishing viscosity limit for the Hunter-Saxton equation

Lei Peng [a], Jingyu Li [a,*], Ming Mei [b,c], Kaijun Zhang [a]

[a] *School of Mathematics and Statistics, Northeast Normal University, Changchun 130024, PR China*
[b] *Department of Mathematics, Champlain College Saint-Lambert, Saint-Lambert, Quebec, J4P 3P2, Canada*
[c] *Department of Mathematics and Statistics, McGill University, Montreal, Quebec, H3A 2K6, Canada*

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Abstract

The Hunter-Saxton equation models the propagation of weakly nonlinear orientation waves in a massive director field of the nematic liquid crystal. In this paper, we study the vanishing viscosity limit for an initial boundary value problem of the Hunter-Saxton equation with the characteristic boundary condition. By the formal multiscale analysis, we first derive the characteristic boundary layer profile, which satisfies a nonlinear parabolic equation. On the base of the Galerkin method along with a compactness argument, we then establish the global well-posedness of the boundary layer equation. Finally, we prove the global stability of the boundary layer profiles together with the optimal convergence rate of the vanishing viscosity limit by the energy method.

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* Corresponding author.
E-mail addresses: pengl923@nenu.edu.cn (L. Peng), lijy645@nenu.edu.cn (J. Li), ming.mei@mcgill.ca (M. Mei), zhangkj201@nenu.edu.cn (K. Zhang).

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1. Introduction

To model the dynamics of the director field of a nematic liquid crystal, Hunter and Saxton [15] proposed the following Hunter-Saxton equation:

$$(u_t + uu_x)_x = \frac{1}{2}u_x^2, \tag{1.1}$$

where $u(x,t)$ describes the director field of the nematic liquid crystal, x is the space variable in a reference frame moving with the linearized wave speed, and t is a slow time variable. Mathematically, the Hunter-Saxton equation, as the high-frequency limit of the Camassa-Holm equation [4,5], is a completely integrable system with a bi-Hamiltonian structure [16].

In the past decade, the Hunter-Saxton equation has been extensively studied. By the characteristics method, Hunter and Saxton [15] considered the 1D initial value problem and showed that the smooth solutions break down in finite time. Then, Hunter and Zheng [17] constructed global dissipative/conservative weak solutions to the Hunter-Saxton equation in the following version:

$$\begin{cases} v_t + uv_x = -\frac{1}{2}v^2, & x>0, t>0,\\ u_x = v, & x>0, t>0,\\ u(0,t)=0, & \\ v(x,0) = v_0(x), & \end{cases} \tag{1.2}$$

where the initial value $v_0(x)$ is compactly supported and $v_0(x) \in BV(\mathbb{R}^+)$. By employing the Young measure theory, Zhang and Zheng [41] established the global existence of dissipative weak solutions to (1.2) with nonnegative $v_0 \in L^p(\mathbb{R}^+)$ for any $p > 2$. After that, Zhang and Zheng [42] proved the global existence and uniqueness of dissipative weak solutions to (1.2) with nonnegative $v_0 \in L^2(\mathbb{R}^+)$. By combining the Young measure theory with the mollification technique, they [43] further established the global well-posedness of admissible weak solutions to (1.2) with general data $v_0 \in L^2(\mathbb{R}^+)$. Moreover, Bressan and Constantin [2] technically constructed a continuous semigroup of global dissipative weak solutions.

An alternative strategy to find global weak solutions to the Hunter-Saxton is based on the vanishing viscosity/dispersion approximation. Indeed, Hunter and Zheng [18] was the first to construct global dissipative/conservative weak solutions to (1.2) with a specific initial datum via the vanishing viscosity/dispersion approximation on the half space. Later on, by the L^p Young measure theory, Li and Zhang [20] obtained the global existence of dissipative weak solutions to (1.2) with general L^2 initial data. Both the works of [18] and [20] are interested in the global well-posedness of weak solutions where the boundary conditions are of little influence.

In this paper, we are interested in the behaviors of global strong solutions to the Hunter-Saxton equation after the vanishing viscosity procedure. The Hunter-Saxton equation with an artificial viscosity reads

$$\begin{cases} v_t^\varepsilon + u^\varepsilon v_x^\varepsilon = -\frac{1}{2}(v^\varepsilon)^2 + \varepsilon v_{xx}^\varepsilon, & x>0, t>0,\\ u_x^\varepsilon = v^\varepsilon, & x>0, t>0,\\ u^\varepsilon(0,t)=0, & \\ v^\varepsilon(x,0) = v_0(x), & \\ v^\varepsilon(0,t) = h(t), & \end{cases} \tag{1.3}$$

where $h(t)$ is a given time-dependent function. Formally, as $\varepsilon \to 0$, the associated inviscid hyperbolic equation is the Hunter-Saxton equation (1.2). It is natural to ask whether there exists a boundary layer near the boundary $x = 0$ in this singular limit. Recently, we studied in [30] this singular limit where the homogeneous Neumann boundary condition at $x = 0$ was described on v^ε, and showed that no boundary layer exists. In other words, for the Neumann problem the viscous approximate solutions converge to the dissipative solutions of (1.2) uniformly in ε. In [31] we proceeded to investigate the singular limit of the solutions in a bounded interval $[0, l]$, where a Dirichlet boundary condition was imposed on v^ε at the right boundary $x = l$. We showed that, in contrast to the findings in [30], there exists a *noncharacteristic* boundary layer profile near the boundary $x = l$. Moreover, we [31] obtained the global-in-time convergence rate as $\varepsilon \to 0$. In the present paper, we are interested in the existence and stability of *characteristic* boundary layers in the vanishing viscosity limit. Comparing (1.2) and (1.3), one may observe that there is a mismatch between the viscous solution v^ε and inviscid solution v at the boundary $x = 0$. According to the singular perturbation theory [13], there exists a boundary layer near $x = 0$, in which the solution v^ε changes dramatically. The size of the boundary layer can be $\sqrt{\varepsilon}$ or ε depending on the type of the boundary condition [10]. Here the boundary condition of (1.3) is characteristic since v^ε satisfies the Dirichlet boundary condition and $u^\varepsilon(0, t) = 0$. Thus, the size of the boundary layer is $\sqrt{\varepsilon}$ and the solution of (1.3) is anticipated to have the following expansion

$$\begin{cases} v^\varepsilon(x,t) = v^0(x,t) + v^{B,0}(\frac{x}{\sqrt{\varepsilon}}, t) + O(\sqrt{\varepsilon}), \\ u^\varepsilon(x,t) = u^0(x,t) + O(\sqrt{\varepsilon}), \end{cases} \tag{1.4}$$

where (v^0, u^0) is exactly the solution of (1.2), and $v^{B,0}$ is the leading order boundary layer profile. After a formal calculation, one can see that the boundary layer equation is exactly a nonlinear Prandtl type parabolic equation which might make the problem challenging.

The purpose of this paper is to establish the global well-posedness of the nonlinear boundary layer equation, and rigorously justify the stability of the characteristic boundary layer profile by providing the optimal convergence rate in L^∞ sense as $\varepsilon \to 0$. To achieve this, some new strategies are developed: (i) To overcome the difficulties caused by the nonlinearity and nonlocality of the boundary layer equation, we construct global solutions in the algebraic weighted Sobolev space, and adopt the Hardy inequality rather than the Poincaré inequality to establish the *a priori* estimates. (ii) To obtain the optimal convergence rate in L^∞ sense, we construct *higher-order* approximation solutions in the form

$$\begin{cases} v^\varepsilon = v^{I,0}(x,t) + v^{B,0}(\frac{x}{\sqrt{\varepsilon}}, t) + \varepsilon^{\frac{1}{2}} v^{I,1}(x,t) + \varepsilon^{\frac{1}{2}} v^{B,1}(\frac{x}{\sqrt{\varepsilon}}, t) + \varepsilon R_1^\varepsilon, \\ u^\varepsilon = u^{I,0}(x,t) + \varepsilon^{\frac{1}{2}} u^{I,1}(x,t) + \varepsilon^{\frac{1}{2}} u^{B,1}(\frac{x}{\sqrt{\varepsilon}}, t) + \varepsilon u^{B,2}(\frac{x}{\sqrt{\varepsilon}}, t) - \varepsilon u^{B,2}(0,t) + \varepsilon R_2^\varepsilon, \end{cases}$$

rather than the first order approximation since the latter leads to a singular quantity of order $\varepsilon^{-\frac{1}{2}}$. Here $\varepsilon u^{B,2}(\frac{x}{\sqrt{\varepsilon}}, t)$ and $\varepsilon u^{B,2}(0,t)$ are introduced to homogenize the boundary condition of R_2^ε.

Before proceeding, let us recall some related results on the classical Prandtl boundary layer theory. The effect of boundary layer has been one of the fundamental problems in fluid dynamics [37] since the pioneering work of Prandtl [32]. See also the significant developments [1,11,19,22,23,27,29,33,35,36,39,40]. Prandtl explained that the flow past a solid body can be divided into two parts: away from the boundary the flow is mainly driven by convection so that the viscosity can be neglected, and the flow corresponds to the inviscid limiting solution, while near the boundary the viscosity plays an important role in the fluid motion. More precisely, there exists a thin transition layer near the boundary, in which the behavior of flow changes dramatically, this transition layer is called the boundary layer. By using a formal asymptotic expansion, Prandtl studied the zero-viscosity limit for the viscous incompressible flow with non-slip boundary condition, and also formally derived the Prandtl boundary layer equation, which described by a nonlinear degenerate parabolic-elliptic coupled system. Mathematically, the formal derivation of the Prandtl equation results in two mathematical problems: the well-posedness of the Prandtl equations and the rigorous justification of the boundary layer expansion. Up to now, both problems remain open for general case. There have been many outstanding works concerning the well-posedness or ill-posedness of the Prandtl boundary layer equation, see [1,3,6,8,9,11,12,14,21,26–29,40]. We also refer the interested readers to [7,24,25,35,36,38] for the validity of Prandtl expansion.

The rest of this paper is organized as follows. In Section 2, we derive the equations for inner profiles and boundary layer profiles by the method of matched asymptotic expansions and state the main convergence results. In Section 3, we obtain the global well-posedness of the nonlinear boundary layer equations. Section 4 is devoted to rigorously justifying the asymptotic expansion.

2. Preliminaries and main results

In this section, we first derive the equations for the inner and boundary layer profiles by applying the matched asymptotic expansion method, and then state the main results of this paper.

2.1. Asymptotic expansions

Based on the WKB method [13], the solution $(v^\varepsilon, u^\varepsilon)$ to (1.3) has the following ansatz:

$$\begin{cases} v^\varepsilon(x,t) = \sum_{j\geq 0} \varepsilon^{\frac{j}{2}}(v^{I,j}(x,t) + v^{B,j}(z,t)), \\ u^\varepsilon(x,t) = \sum_{j\geq 0} \varepsilon^{\frac{j}{2}}(u^{I,j}(x,t) + u^{B,j}(z,t)), \end{cases} \tag{2.1}$$

where the boundary layer coordinate is defined by $z = \frac{x}{\sqrt{\varepsilon}}$, $x \in [0, \infty)$. We assume that each term in the expansion (2.1) is smooth, and the boundary layer profiles $v^{B,j}$ and $u^{B,j}$ decay to zero fast enough as $z \to +\infty$.

Substituting (2.1) into the initial condition of (1.3), and noting that the initial value $v_0(x)$ is independent of ε, we obtain

$$\begin{cases} v^{I,0}(x,0) = v_0(x), \ \ v^{B,0}(z,0) = 0, \\ v^{I,j}(x,0) = v^{B,j}(z,0) = 0, \quad j \geq 1. \end{cases} \tag{2.2}$$

Plugging (2.1) into the boundary conditions of (1.3) and matching terms of the same order in ε, we have

$$\begin{cases} u^{I,j}(0,t) + u^{B,j}(0,t) = 0, & j \geq 0, \\ v^{I,0}(0,t) + v^{B,0}(0,t) = h(t), & j = 0, \\ v^{I,j}(0,t) + v^{B,j}(0,t) = 0, & j \geq 1. \end{cases} \tag{2.3}$$

Similarly, plugging (2.1) into the second equation of (1.3) yields

$$u_x^{I,j} - v^{I,j} = 0, \quad j \ge 0, \tag{2.4}$$

and

$$u_z^{B,0} = 0, \ u_z^{B,j+1} - v^{B,j} = 0, \quad j \ge 0, \tag{2.5}$$

which implies, by noting $u^{B,0}$ decays to zero fast enough as $z \to +\infty$, that

$$u^{B,0}(z,t) = 0. \tag{2.6}$$

Substituting (2.1) into the first equation of (1.3), it follows that

$$\begin{aligned}
&\sum_{j\ge 0} \varepsilon^{\frac{j}{2}} (v^{I,j} + v^{B,j})_t + \sum_{j\ge 0} \varepsilon^{\frac{j}{2}} \sum_{k=0}^{j} (u^{I,k} + u^{B,k}) v_x^{I,j-k} \\
&\quad + \varepsilon^{-\frac{1}{2}} (u^{I,0} + u^{B,0}) v_z^{B,0} + \sum_{j\ge 0} \varepsilon^{\frac{j}{2}} \sum_{k=0}^{j+1} (u^{I,k} + u^{B,k}) v_z^{B,j+1-k} \\
&= -\frac{1}{2} (\sum_{j\ge 0} \varepsilon^{\frac{j}{2}} (v^{I,j} + v^{B,j}))^2 + v_{zz}^{B,0} + \varepsilon^{\frac{1}{2}} v_{zz}^{B,1} + \sum_{j\ge 0} \varepsilon^{1+\frac{j}{2}} (v_{xx}^{I,j} + v_{zz}^{B,j+2}).
\end{aligned} \tag{2.7}$$

Letting $z \to +\infty$ in (2.7), we get

$$\begin{cases}
v_t^{I,0} + u^{I,0} v_x^{I,0} + \dfrac{1}{2} (v^{I,0})^2 = 0, \\
v_t^{I,1} + u^{I,1} v_x^{I,0} + u^{I,0} v_x^{I,1} + v^{I,0} v^{I,1} = 0, \\
v_t^{I,j} + \sum\limits_{k=0}^{j} u^{I,k} v_x^{I,j-k} + \dfrac{1}{2} \sum\limits_{k=0}^{j} v^{I,k} v^{I,j-k} - v_{xx}^{I,j-2} = 0, \quad j \ge 2.
\end{cases} \tag{2.8}$$

It then follows that

$$\sum_{j\ge -1} \varepsilon^{\frac{j}{2}} G_j(x,z,t) = 0, \tag{2.9}$$

where

$$\begin{aligned}
G_{-1} &= (u^{I,0} + u^{B,0}) v_z^{B,0}, \\
G_0 &= v_t^{B,0} + (u^{I,0} + u^{B,0}) v_z^{B,1} + (u^{I,1} + u^{B,1}) v_z^{B,0} + u^{B,0} v_x^{I,0} \\
&\quad + \frac{1}{2} (v^{B,0})^2 + v^{I,0} v^{B,0} - v_{zz}^{B,0}, \\
G_1 &= v_t^{B,1} + (u^{I,0} + u^{B,0}) v_z^{B,2} + (u^{I,1} + u^{B,1}) v_z^{B,1} + (u^{I,2} + u^{B,2}) v_z^{B,0} \\
&\quad + u^{B,0} v_x^{I,1} + u^{B,1} v_x^{I,0} + (v^{I,0} v^{B,1} + v^{I,1} v^{B,0} + v^{B,0} v^{B,1}) - v_{zz}^{B,1}, \\
&\cdots \ \cdots\cdot
\end{aligned}$$

Recalling $x = \varepsilon^{\frac{1}{2}} z$ and expanding $G_j(x, z, t)$ formally in x by the Taylor expansion, one has

$$G_j(x,z,t) = G_j(\varepsilon^{\frac{1}{2}} z, z, t) = G_j(0,z,t) + \sum_{k=1}^{\infty} \frac{1}{k!} (\varepsilon^{\frac{1}{2}} z)^k \partial_x^k G_j(0,z,t). \tag{2.10}$$

Then applying (2.10) into (2.9), we get

$$\sum_{j \geq -1} \varepsilon^{\frac{j}{2}} \tilde{G}_j(z,t) = 0, \tag{2.11}$$

where

$$\begin{aligned}
&\tilde{G}_{-1} = (u^{I,0}(0,t) + u^{B,0}) v_z^{B,0}, \\
&\tilde{G}_0 = v_t^{B,0} + (u^{I,0}(0,t) + u^{B,0}) v_z^{B,1} + (u^{I,1}(0,t) + u^{B,1}) v_z^{B,0} + u^{B,0} v_x^{I,0}(0,t) \\
&\qquad + \frac{1}{2} (v^{B,0})^2 + v^{I,0}(0,t) v^{B,0} - v_{zz}^{B,0} + z u_x^{I,0}(0,t) v_z^{B,0}, \\
&\tilde{G}_1 = v_t^{B,1} + (u^{I,0}(0,t) + u^{B,0}) v_z^{B,2} + (u^{I,1}(0,t) + u^{B,1}) v_z^{B,1} + (u^{I,2}(0,t) + u^{B,2}) v_z^{B,0} \\
&\qquad + u^{B,0} v_x^{I,1}(0,t) + u^{B,1} v_x^{I,0}(0,t) + v^{I,0}(0,t) v^{B,1} + v^{I,1}(0,t) v^{B,0} + v^{B,0} v^{B,1} \\
&\qquad - v_{zz}^{B,1} + z u_x^{I,0}(0,t) v_z^{B,1} + z u_x^{I,1}(0,t) v_z^{B,0} + z u^{B,0} v_{xx}^{I,0}(0,t) \\
&\qquad + z v_x^{I,0}(0,t) v^{B,0} + \frac{1}{2} z^2 u_{xx}^{I,0}(0,t) v_z^{B,0}, \\
&\ldots \quad \ldots.
\end{aligned}$$

From (2.4) with $j = 0$, (2.8), (2.2), (2.3) and (2.6), we obtain

$$\begin{cases}
v_t^{I,0} + u^{I,0} v_x^{I,0} + \frac{1}{2} (v^{I,0})^2 = 0, \\
u_x^{I,0} - v^{I,0} = 0, \\
v^{I,0}(x,0) = v_0(x), \\
u^{I,0}(0,t) = 0.
\end{cases} \tag{2.12}$$

Combining (2.5)-(2.6), (2.11)-(2.2) and (2.3), one has the following initial-boundary value problem for a nonlinear parabolic system:

$$\begin{cases}
v_t^{B,0} + \left(u^{I,1}(0,t) + u^{B,1} + z u_x^{I,0}(0,t) \right) v_z^{B,0} + \frac{1}{2} (v^{B,0})^2 + v^{I,0}(0,t) v^{B,0} = v_{zz}^{B,0}, \\
u_z^{B,1} = v^{B,0}, \\
v^{B,0}(z,0) = 0, \\
v^{B,0}(0,t) = h(t) - v^{I,0}(0,t), \qquad \lim_{z \to +\infty} (v^{B,0}, u^{B,1})(z,t) = 0.
\end{cases} \tag{2.13}$$

Here $u^{B,1}(z,t)$ is given by

$$u^{B,1}(z,t) = -\int_z^\infty v^{B,0}(y,t)dy. \tag{2.14}$$

Similarly, from (2.4) with $j = 1$, (2.2)-(2.3) and (2.8), we get

$$\begin{cases} v_t^{I,1} + u^{I,1}v_x^{I,0} + u^{I,0}v_x^{I,1} + v^{I,0}v^{I,1} = 0, \\ u_x^{I,1} - v^{I,1} = 0, \\ v^{I,1}(x,0) = 0, \\ u^{I,1}(0,t) = -u^{B,1}(0,t). \end{cases} \tag{2.15}$$

Moreover, the boundary layer profiles $v^{B,1}$ and $u^{B,2}$ satisfy an initial-boundary value problem for a linear parabolic system:

$$\begin{cases} v_t^{B,1} + \left(u^{I,1}(0,t) + u^{B,1} + zu_x^{I,0}(0,t)\right) v_z^{B,1} \\ \qquad + \left(u^{I,2}(0,t) + u^{B,2} + zu_x^{I,1}(0,t) + \frac{1}{2}z^2 u_{xx}^{I,0}(0,t)\right) v_z^{B,0} \\ \qquad + u^{B,1}v_x^{I,0}(0,t) + v^{I,0}(0,t)v^{B,1} + v^{I,1}(0,t)v^{B,0} + zv_x^{I,0}(0,t)v^{B,0} \\ \qquad + v^{B,0}v^{B,1} = v_{zz}^{B,1}, \\ u_z^{B,2} = v^{B,1}, \\ v^{B,1}(z,0) = 0, \\ v^{B,1}(0,t) = -v^{I,1}(0,t), \quad \lim_{z\to+\infty} (v^{B,1}, u^{B,2})(z,t) = 0. \end{cases} \tag{2.16}$$

Here $u^{B,2}$ is determined by

$$u^{B,2}(z,t) = -\int_z^\infty v^{B,1}(y,t)dy. \tag{2.17}$$

2.2. *Main results*

For convenience, we first introduce some notation. C is a generic positive constant independent of ε that may change from one line to another. We use the notation $A \lesssim B$ to stand for $A \le CB$. For $1 \le p \le \infty$, we use L^p and L_z^p to denote the Lebesgue space $L^p(0,\infty)$ with respect to x and z whose norms are denoted by $\|\cdot\|_{L^p}$ and $\|\cdot\|_{L_z^p}$, respectively. Similarly, H^k, H_z^k denote the Sobolev space $W^{k,2}(0,\infty)$ with respect to x and z with norms $\|\cdot\|_{H^k}$ and $\|\cdot\|_{H_z^k}$, respectively. We use $(u,v) = \int uv$ to denote the inner product in the Hilbert space L^2.

We now state our main results.

Theorem 2.1. *Assume that the initial and boundary data satisfy*

$$v_0 \in H^3,\ v_0(x) \ge 0,\ \int_0^\infty xv_{0x}^2 dx < +\infty,\ \int_0^\infty xv_{0xx}^2 dx < +\infty,\ \|h(t)\|_{H^2(0,T)} < \infty,$$

and the compatibility condition $h(0) = v_0(0)$. *Let* $(v^\varepsilon, u^\varepsilon)$ *be the solution of* (1.3) *with initial data* v_0. *Then for any fixed* $T > 0$ *there exists a unique solution* $(v^{I,0}, u^{I,0}) \in L^\infty(0, T; H^3 \times H^4)$ *to the problem* (2.12) *with initial data* v_0, *and a unique boundary layer profile* $v^{B,0}$ *to the problem* (2.13) *satisfying for any* $k \geq 0$

$$\langle z\rangle^k v^{B,0} \in L^2(0, T; H_z^3) \cap L^\infty(0, T; H_z^2),$$

$$\langle z\rangle^k v_t^{B,0} \in L^2(0, T; H_z^1) \cap L^\infty(0, T; L_z^2),$$

with $\langle z\rangle \triangleq \sqrt{1+z^2}$. *Moreover, the following convergence rate holds:*

$$\sup_{0\leqslant t\leqslant T} \|v^\varepsilon - v^{I,0} - v^{B,0}\|_{L^\infty} \leq C\varepsilon^{\frac{1}{2}}. \tag{2.18}$$

Remark 2.1.

1. In comparison with the works of [30,31], the system (1.3) of this paper exhibits the characteristic boundary condition. And hence the size of the boundary layer is of $\sqrt{\varepsilon}$, and the boundary layer equation is of a nonlinear parabolic equation, both of which require new strategies to handle.
2. If the initial data $v_0(x) < 0$, then the conclusion of Theorem 2.1 is still valid before the blow-up time of the inviscid equation (1.2).
3. As in [18], one can obtain the global existence and uniqueness of the viscous system (1.3). Namely, under the conditions of Theorem 2.1, the system (1.3) has a unique solution satisfying

$$(v^\varepsilon, u^\varepsilon) \in L^\infty(0, T; H^3) \cap L^2(0, T; H^4) \text{ for any } T > 0.$$

4. It is easy to see that if $h(t) \equiv 0$ in the system (1.3), then $v^{B,0}(x,t) \equiv 0$. Indeed, applying the characteristic method to the first equation of (2.12), observing $u^{I,0}(0,t) = 0$, it follows from the compatibility condition $h(0) = v_0(0)$ that

$$v^{I,0}(0,t) = v_0(0) = h(0) = 0.$$

Then $v^{B,0}$, the solution of (2.13), satisfies the boundary condition

$$v^{B,0}(0,t) = h(t) - v^{I,0}(0,t) \equiv 0.$$

Thus, we have $v^{B,0}(x,t) \equiv 0$ owing to the uniqueness of strong solutions to (2.13).

3. Well-posedness of the inner and boundary layer equations

In this section we study the global well-posedness of the inner and boundary layer equations.

3.1. Well-posedness of inner equations

Proposition 3.1. *Under the same assumptions of Theorem 2.1, there exists a unique solution* $(v^{I,0}, u^{I,0})$ *to* (2.12) *satisfying*

$$v^{I,0} \in C([0,\infty); H^3), u^{I,0} \in C([0,\infty); H^4). \tag{3.1}$$

Similarly, there exists a unique solution $(v^{I,1}, u^{I,1})$ *to the problem* (2.15) *such that*

$$v^{I,1} \in C([0,\infty); H^2), \quad u^{I,1} \in C([0,\infty); H^3). \tag{3.2}$$

Proof. The global existence and uniqueness of solutions to (2.12) follows from [20,41,42] and we omit the details for brevity. It only remains to prove (3.1) and (3.2). By the method of characteristics, (2.12) is equivalent to the following system

$$\begin{cases} \dfrac{d\Phi_t(x)}{dt} = u^{I,0}(t, \Phi_t(x)) = \int_0^{\Phi_t(x)} v^{I,0}(t,y)dy, \quad \Phi_0(x) = x, \\ \dfrac{dv^{I,0}(t,\Phi_t(x))}{dt} = -\dfrac{1}{2}(v^{I,0})^2(t,\Phi_t(x)), \\ v^{I,0}(t,\Phi_t(x))|_{t=0} = v_0(x). \end{cases} \tag{3.3}$$

Then a direct computation gives

$$0 \le v^{I,0}(\Phi_t(x), t) = \frac{2v_0(x)}{2 + v_0(x)t}. \tag{3.4}$$

Furthermore, by the first equation of (3.3), one has

$$1 \le \frac{\partial \Phi_t(x)}{\partial x} = \exp\left(\int_0^t v^{I,0}(\Phi_s(x), s)ds\right) = \left(\frac{2 + v_0(x)t}{2}\right)^2. \tag{3.5}$$

Thus, by (3.4) and (3.5),

$$v^{I,0}(x,t) = \frac{2v_0(\Phi_t^{-1}(x))}{2 + v_0(\Phi_t^{-1}(x))t}, \tag{3.6}$$

where $\Phi_t^{-1}(x)$ is the inverse mapping of $\Phi_t(x)$. Thus, (3.1) follows directly from (3.6) and the regularity of $v_0(x)$. Similarly, one can see that $(v^{I,1}, u^{I,1})$, the solution of (2.15), satisfies (3.2). □

3.2. Well-posedness of boundary layer equations

We will adopt the Galerkin method to study the well-posedness of the problem (2.13). In order to homogenize the boundary conditions of (2.13), we modify $v^{B,0}$ and $u^{B,1}$ by

$$\begin{cases} \widetilde{v^{B,0}} \triangleq v^{B,0} - \phi(z)b(t), \\ \widetilde{u^{B,1}} \triangleq u^{B,1} - \psi(z)b(t), \end{cases} \tag{3.7}$$

where $b(t) \triangleq h(t) - v^{I,0}(0,t)$ and (ϕ, ψ) satisfy

$$\psi_z = \phi,$$
$$\phi(0) = 1, \ \psi(0) = 0,$$
$$\phi(z) = \psi(z) = 0, \ \text{for } z > 1.$$

Substituting (3.7) into (2.13), and omitting the tildes for simplicity of notation, we obtain

$$\begin{cases} v_t^{B,0} + \Big(u^{B,1} - u^{B,1}(0,t) + (z-\psi)v^{I,0}(0,t) + \psi(z)h(t)\Big) v_z^{B,0} \\ \qquad + \Big(u^{B,1} - u^{B,1}(0,t)\Big)\phi_z b(t) + \frac{1}{2}(v^{B,0})^2 + \phi(z)b(t)v^{B,0} \\ \qquad + v^{I,0}(0,t)v^{B,0} = v_{zz}^{B,0} + f(z,t), \\ u_z^{B,1} = v^{B,0}, \\ v^{B,0}(z,0) = 0, \\ v^{B,0}(0,t) = 0, \qquad \lim_{z\to+\infty} (v^{B,0}, u^{B,1})(z,t) = 0, \end{cases} \tag{3.8}$$

where

$$f \triangleq \phi_{zz} b(t) - \phi(z)b_t(t) - \Big((z-\psi)v^{I,0}(0,t) + \psi(z)h(t)\Big)\phi_z b(t)$$
$$- \frac{1}{2}\phi^2(z)b^2(t) - v^{I,0}(0,t)\phi(z)b(t).$$

It is straightforward to check that

$$\langle z\rangle^k f \in L^\infty(0,T; C_0^\infty(\mathbb{R}^+)), \quad \langle z\rangle^k f_t \in L^\infty(0,T; C_0^\infty(\mathbb{R}^+)), \quad \forall k \geq 0. \tag{3.9}$$

Lemma 3.1. *Let $(v^{B,0}, u^{B,1})$ be the solutions to the problem (3.8). Under the assumptions of Theorem 2.1, for any $T > 0$ and $k \geq 0$, it holds that*

$$\|\langle z\rangle^k v^{B,0}\|^2_{L^\infty(0,T;L_z^2)} + \|\langle z\rangle^k v_z^{B,0}\|^2_{L^2(0,T;L_z^2)} \leq C, \tag{3.10}$$

$$\|\langle z\rangle^k v_z^{B,0}\|^2_{L^\infty(0,T;L_z^2)} + \|\langle z\rangle^k v_{zz}^{B,0}\|^2_{L^2(0,T;L_z^2)} \leq C, \tag{3.11}$$

$$\|\langle z\rangle^k v_t^{B,0}\|^2_{L^\infty(0,T;L_z^2)} + \|\langle z\rangle^k v_{tz}^{B,0}\|^2_{L^2(0,T;L_z^2)} \leq C, \tag{3.12}$$

where $C > 0$ is a constant depending on T, k, ϕ, ψ, $|v^{I,0}(0,t)|$ and $|h(t)|$, but not on ε.

Remark 3.1. Using the equation of $v^{B,0}$, we further have

$$\|\langle z\rangle^k v_{zz}^{B,0}\|^2_{L^\infty(0,T;L_z^2)} + \|\langle z\rangle^k v_{zzz}^{B,0}\|^2_{L^2(0,T;L_z^2)} \le C.$$

Proof. *Step 1*. Multiplying $(3.8)_1$ by $\langle z\rangle^{2k} v^{B,0}$ and integrating the equation over $(0, \infty)$, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\langle z\rangle^k v^{B,0}\|^2_{L_z^2} + \|\langle z\rangle^k v_z^{B,0}\|^2_{L_z^2}\\
&= -k\int_0^\infty z\langle z\rangle^{2k-2}\left(u^{B,1} - u^{B,1}(0,t) + (z-\psi)v^{I,0}(0,t) + \psi(z)h(t)\right)|v^{B,0}|^2 dz\\
&\quad - \int_0^\infty \langle z\rangle^{2k} v^{B,0}\left(u^{B,1} - u^{B,1}(0,t)\right)\phi_z b(t)dz - \frac{1}{2}\int_0^\infty \langle z\rangle^{2k} v^{I,0}(0,t)|v^{B,0}|^2 dz\\
&\quad - \frac{1}{2}\int_0^\infty \langle z\rangle^{2k}\phi(z)b(t)|v^{B,0}|^2 dz + k\int_0^\infty \partial_z(z\langle z\rangle^{2k-2})|v^{B,0}|^2 dz + \int_0^\infty \langle z\rangle^{2k} v^{B,0} f dz\\
&\triangleq \sum_{k=1}^{6} I_k.
\end{aligned} \tag{3.13}$$

Let us next estimate the terms on the right hand side of (3.13). For I_1, we rewrite it as

$$\begin{aligned}
I_1 =& -k\int_0^\infty z^2\langle z\rangle^{2k-2}\left(\frac{u^{B,1} - u^{B,1}(0,t)}{z}\right)|v^{B,0}|^2 dz\\
&- k\int_0^\infty z^2\langle z\rangle^{2k-2}\left(\frac{z-\psi}{z}\right)v^{I,0}(0,t)|v^{B,0}|^2 dz\\
&- k\int_0^\infty z^2\langle z\rangle^{2k-2}\frac{\psi(z)}{z}h(t)|v^{B,0}|^2 dz\\
\triangleq& I_{11} + I_{12} + I_{13}.
\end{aligned} \tag{3.14}$$

For the first term I_{11}, it holds

$$\begin{aligned}
I_{11} &\le k\left\|\frac{u^{B,1} - u^{B,1}(0,t)}{z}\right\|_{L_z^2}\|\langle z\rangle^k v^{B,0}\|_{L_z^\infty}\|\langle z\rangle^k v^{B,0}\|_{L_z^2}\\
&\le k\|v^{B,0}\|_{L_z^2}\|\langle z\rangle^k v^{B,0}\|^{\frac{1}{2}}_{L_z^2}\|\langle z\rangle^k v_z^{B,0}\|^{\frac{1}{2}}_{L_z^2}\|\langle z\rangle^k v^{B,0}\|_{L_z^2}\\
&\le \frac{1}{2}\|\langle z\rangle^k v_z^{B,0}\|^2_{L_z^2} + Ck\|v^{B,0}\|^{\frac{4}{3}}_{L_z^2}\|\langle z\rangle^k v^{B,0}\|^2_{L_z^2},
\end{aligned} \tag{3.15}$$

where we have used Hölder's inequality in the first inequality, Hardy's inequality and Sobolev's inequality in the second inequality and the Young's inequality in the third inequality. For I_{12} and I_{13}, by Hölder's inequality, we have

$$\begin{aligned} I_{12}+I_{13} &\le k|v^{I,0}(0,t)|\left\|(1-\frac{\psi}{z})\right\|_{L_z^\infty}\|\langle z\rangle^k v^{B,0}\|_{L_z^2}^2+k|h(t)|\left\|\frac{\psi}{z}\right\|_{L_z^\infty}\|\langle z\rangle^k v^{B,0}\|_{L_z^2}^2 \\ &\le Ck\|\langle z\rangle^k v^{B,0}\|_{L_z^2}^2. \end{aligned} \tag{3.16}$$

For I_2, one gets

$$\begin{aligned} I_2 &\le \|\frac{u^{B,1}-u^{B,1}(0,t)}{z}\|_{L_z^2}\|\langle z\rangle^{k+1}\phi_z b(t)\|_{L_z^\infty}\|\langle z\rangle^k v^{B,0}\|_{L_z^2} \\ &\le \|v^{B,0}\|_{L_z^2}\|\langle z\rangle^{k+1}\phi_z b(t)\|_{L_z^\infty}\|\langle z\rangle^k v^{B,0}\|_{L_z^2} \\ &\le C\|\langle z\rangle^k v^{B,0}\|_{L_z^2}^2, \end{aligned} \tag{3.17}$$

where we have used Hölder's inequality in the first inequality, Hardy's inequality in the second inequality. Moreover, one can obtain

$$\begin{aligned} I_3+I_4 &\le \frac{1}{2}(|v^{I,0}(0,t)|\|\langle z\rangle^k v^{B,0}\|_{L_z^2}^2+|b(t)|\|\phi\|_{L_z^\infty}\|\langle z\rangle^k v^{B,0}\|_{L_z^2}^2) \\ &\le C\|\langle z\rangle^k v^{B,0}\|_{L_z^2}^2. \end{aligned} \tag{3.18}$$

By Young's inequality, the last two terms I_5 and I_6 can be estimated as

$$I_5+I_6 \le (k+\frac{1}{2})\|\langle z\rangle^k v^{B,0}\|_{L_z^2}^2+\frac{1}{2}\|\langle z\rangle^k f\|_{L_z^2}^2. \tag{3.19}$$

Hence, substituting (3.14)-(3.19) into (3.13), we have

$$\frac{d}{dt}\|\langle z\rangle^k v^{B,0}\|_{L_z^2}^2+\|\langle z\rangle^k v_z^{B,0}\|_{L_z^2}^2 \le C\|\langle z\rangle^k v^{B,0}\|_{L_z^2}^2(1+k\|v^{B,0}\|_{L_z^2}^{\frac{4}{3}})+\|\langle z\rangle^k f\|_{L_z^2}^2. \tag{3.20}$$

In order to apply the Gronwall's inequality for (3.20), the term $\int_0^t\|v^{B,0}(s)\|_{L_z^2}^{\frac{4}{3}}ds$ must be controlled. For that, let $k=0$ in (3.20), we obtain

$$\frac{d}{dt}\|v^{B,0}\|_{L_z^2}^2+\|v_z^{B,0}\|_{L_z^2}^2 \le C\|v^{B,0}\|_{L_z^2}^2+\|f\|_{L_z^2}^2. \tag{3.21}$$

Applying the Gronwall's inequality to the above inequality, it follows

$$\|v^{B,0}\|_{L^\infty(0,T;L_z^2)}^2+\|v_z^{B,0}\|_{L^2(0,T;L_z^2)}^2 \le C. \tag{3.22}$$

Finally, applying the Gronwall's inequality to (3.20), by (3.22), we have

$$\sup_{0\leqslant t\leqslant T}\|\langle z\rangle^k v^{B,0}\|^2_{L^2_z}+\|\langle z\rangle^k v^{B,0}_z\|^2_{L^2(0,T;L^2_z)}\leq C. \tag{3.23}$$

Step 2. Multiplying the first equation of (3.8) by $-\langle z\rangle^{2k}v^{B,0}_{zz}$, noting

$$-\int_0^\infty v^{B,0}_t\langle z\rangle^{2k}v^{B,0}_{zz}=\frac{1}{2}\frac{d}{dt}\|\langle z\rangle^k v^{B,0}_z\|^2_{L^2_z}+2k\int_0^\infty z\langle z\rangle^{2k-2}v^{B,0}_t v^{B,0}_z,$$

we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\langle z\rangle^k v^{B,0}_z\|^2_{L^2_z}+\|\langle z\rangle^k v^{B,0}_{zz}\|^2_{L^2_z}\\
=&k\int_0^\infty z\langle z\rangle^{2k-2}\Big(u^{B,1}-u^{B,1}(0,t)+(z-\psi)v^{I,0}(0,t)+\psi(z)h(t)\Big)|v^{B,0}_z|^2dz\\
&-\int_0^\infty\langle z\rangle^{2k}\Big(u^{B,1}-u^{B,1}(0,t)\Big)\phi_{zz}b(t)v^{B,0}_zdz-2\int_0^\infty\langle z\rangle^{2k}\phi_z b(t)v^{B,0}v^{B,0}_zdz\\
&-\frac{3}{2}\int_0^\infty\langle z\rangle^{2k}v^{B,0}|v^{B,0}_z|^2dz-\frac{3}{2}\int_0^\infty\langle z\rangle^{2k}v^{I,0}(0,t)|v^{B,0}_z|^2dz\\
&-\frac{3}{2}\int_0^\infty\langle z\rangle^{2k}\phi(z)b(t)|v^{B,0}_z|^2dz-2k\int_0^\infty z\langle z\rangle^{2k-2}(v^{B,0}_{zz}+f)v^{B,0}_zdz-\int_0^\infty\langle z\rangle^{2k}v^{B,0}_{zz}fdz\\
\triangleq&\sum_{k=1}^{8}J_k.
\end{aligned}\tag{3.24}$$

We now estimate the terms on the right side of (3.24). Firstly, we have

$$\begin{aligned}
J_1=&k\int_0^\infty z^2\langle z\rangle^{2k-2}\left(\frac{u^{B,1}-u^{B,1}(0,t)}{z}\right)|v^{B,0}_z|^2dz\\
&+k\int_0^\infty z^2\langle z\rangle^{2k-2}\left(\frac{z-\psi}{z}\right)v^{I,0}(0,t)|v^{B,0}_z|^2dz\\
&+k\int_0^\infty z^2\langle z\rangle^{2k-2}\frac{\psi(z)}{z}h(t)|v^{B,0}_z|^2dz\\
\triangleq&J_{11}+J_{12}+J_{13}.
\end{aligned}\tag{3.25}$$

For J_{11}, we have

$$
\begin{aligned}
J_{11} &\le k \left\| \frac{u^{B,1}-u^{B,1}(0,t)}{z} \right\|_{L_z^2} \|\langle z\rangle^k v_z^{B,0}\|_{L_z^4}^2 \\
&\le k\|v^{B,0}\|_{L_z^2}\|\langle z\rangle^k v_z^{B,0}\|_{L_z^4}^2 \\
&\le k\|v^{B,0}\|_{L_z^2}\|\langle z\rangle^k v^{B,0}\|_{L_z^2}^{\frac{3}{4}}\|\langle z\rangle^k v_{zz}^{B,0}\|_{L_z^2}^{\frac{5}{4}} \\
&\le \frac{1}{8}\|\langle z\rangle^k v_{zz}^{B,0}\|_{L_z^2}^2 + Ck\|v^{B,0}\|_{L_z^2}^{\frac{8}{3}}\|\langle z\rangle^k v^{B,0}\|_{L_z^2}^2,
\end{aligned}
\tag{3.26}
$$

where we have used Hölder's inequality in the first inequality, Hardy's inequality in the second inequality, Gagliardo-Nirenberg's inequality in the third inequality and Young's inequality in the last inequality. By Hölder's inequality, we get

$$
\begin{aligned}
J_{12}+J_{13} &\le k|v^{I,0}(0,t)| \left\| (1-\frac{\psi}{z}) \right\|_{L_z^\infty} \|\langle z\rangle^k v_z^{B,0}\|_{L_z^2}^2 + k|h(t)| \left\| \frac{\psi}{z} \right\|_{L_z^\infty} \|\langle z\rangle^k v_z^{B,0}\|_{L_z^2}^2 \\
&\le Ck\|\langle z\rangle^k v_z^{B,0}\|_{L_z^2}^2.
\end{aligned}
\tag{3.27}
$$

Similarly, it holds

$$
\begin{aligned}
J_2 &\le \left\| \frac{u^{B,1}-u^{B,1}(0,t)}{z} \right\|_{L_z^2} \|\langle z\rangle^{k+1}\phi_{zz}b(t)\|_{L_z^\infty}\|\langle z\rangle^k v_z^{B,0}\|_{L_z^2} \\
&\le \|v^{B,0}\|_{L_z^2}\|\langle z\rangle^{k+1}\phi_{zz}b(t)\|_{L_z^\infty}\|\langle z\rangle^k v_z^{B,0}\|_{L_z^2} \\
&\le C\|v^{B,0}\|_{L_z^2}^2 + C\|\langle z\rangle^k v_z^{B,0}\|_{L_z^2}^2.
\end{aligned}
\tag{3.28}
$$

The term J_3 can be estimated as follows

$$
\begin{aligned}
J_3 &\le \|v^{B,0}\|_{L_z^2}\|\langle z\rangle^k \phi_z b(t)\|_{L_z^\infty}\|\langle z\rangle^k v_z^{B,0}\|_{L_z^2} \\
&\le C\|v^{B,0}\|_{L_z^2}^2 + C\|\langle z\rangle^k v_z^{B,0}\|_{L_z^2}^2.
\end{aligned}
\tag{3.29}
$$

By Hölder's inequality, Gagliardo-Nirenberg's inequality and Young's inequality, J_4 satisfies

$$
\begin{aligned}
J_4 &\le \frac{3}{2}\|v^{B,0}\|_{L_z^2}\|\langle z\rangle^k v_z^{B,0}\|_{L_z^4}^2 \\
&\lesssim \|v^{B,0}\|_{L_z^2}\|\langle z\rangle^k v^{B,0}\|_{L_z^2}^{\frac{3}{4}}\|\langle z\rangle^k v_{zz}^{B,0}\|_{L_z^2}^{\frac{5}{4}} \\
&\le \frac{1}{8}\|\langle z\rangle^k v_{zz}^{B,0}\|_{L_z^2}^2 + C\|v^{B,0}\|_{L_z^2}^{\frac{8}{3}}\|\langle z\rangle^k v^{B,0}\|_{L_z^2}^2.
\end{aligned}
\tag{3.30}
$$

Moreover, one can get

$$
\begin{aligned}
J_5+J_6 &\le \frac{3}{2}(|v^{I,0}(0,t)|\|\langle z\rangle^k v_z^{B,0}\|_{L_z^2}^2 + |b(t)|\|\phi\|_{L_z^\infty}\|\langle z\rangle^k v_z^{B,0}\|_{L_z^2}^2) \\
&\le C\|\langle z\rangle^k v_z^{B,0}\|_{L_z^2}^2.
\end{aligned}
\tag{3.31}
$$

Finally, by Cauchy-Schwarz inequality, we obtain

$$J_7 + J_8 \le \frac{1}{4}\|\langle z\rangle^k v_{zz}^{B,0}\|_{L_z^2}^2 + (k+2)\|\langle z\rangle^k f\|_{L_z^2}^2 + Ck\|\langle z\rangle^k v_z^{B,0}\|_{L_z^2}^2. \tag{3.32}$$

Now plugging (3.25)-(3.32) into (3.24), we have

$$\begin{aligned} &\frac{d}{dt}\|\langle z\rangle^k v_z^{B,0}\|_{L_z^2}^2 + \|\langle z\rangle^k v_{zz}^{B,0}\|_{L_z^2}^2 \\ \le& C(1+k)\|\langle z\rangle^k v_z^{B,0}\|_{L_z^2}^2 + C(1+k)\|\langle z\rangle^k v^{B,0}\|_{L_z^2}^2 \|v^{B,0}\|_{L_z^2}^{\frac{8}{3}} \\ &+ C\|v^{B,0}\|_{L_z^2}^2 + (4+2k)\|\langle z\rangle^k f\|_{L_z^2}^2. \end{aligned} \tag{3.33}$$

Hence it follows from the Gronwall's inequality and (3.23) that

$$\sup_{0\leqslant t\leqslant T}\|\langle z\rangle^k v_z^{B,0}\|_{L_z^2}^2 + \|\langle z\rangle^k v_{zz}^{B,0}\|_{L^2(0,T;L_z^2)}^2 \le C. \tag{3.34}$$

Step 3. Differentiating the first equation of (3.8) in t gives

$$\begin{aligned} v_{tt}^{B,0} =& -\left(u_t^{B,1} - u_t^{B,1}(0,t) + (z-\psi)v_t^{I,0}(0,t) + \psi(z)h_t(t)\right)v_z^{B,0} \\ &- \left(u^{B,1} - u^{B,1}(0,t) + (z-\psi)v^{I,0}(0,t) + \psi(z)h(t)\right)v_{zt}^{B,0} \\ &- \left(u_t^{B,1} - u_t^{B,1}(0,t)\right)\phi_z b(t) - \left(u^{B,1} - u^{B,1}(0,t)\right)\phi_z b_t(t) - v^{B,0}v_t^{B,0} \\ &- \phi(z)b_t(t)v^{B,0} - \phi(z)b(t)v_t^{B,0} - v_t^{I,0}(0,t)v^{B,0} - v^{I,0}(0,t)v_t^{B,0} \\ &+ v_{tzz}^{B,0} + f_t. \end{aligned} \tag{3.35}$$

Multiplying (3.35) by $\langle z\rangle^{2k} v_t^{B,0}$, we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|\langle z\rangle^k v_t^{B,0}\|_{L_z^2}^2 + \|\langle z\rangle^k v_{tz}^{B,0}\|_{L_z^2}^2 \\ =& -\int_0^\infty \langle z\rangle^{2k}\left(u_t^{B,1} - u_t^{B,1}(0,t) + (z-\psi)v_t^{I,0}(0,t) + \psi(z)h_t(t)\right)v_z^{B,0}v_t^{B,0} \\ &+ k\int_0^\infty z\langle z\rangle^{2k-2}\left(u^{B,1} - u^{B,1}(0,t) + (z-\psi)v^{I,0}(0,t) + \psi(z)h(t)\right)|v_t^{B,0}|^2 dz \\ &- \int_0^\infty \langle z\rangle^{2k}\left(u_t^{B,1} - u_t^{B,1}(0,t)\right)\phi_z b(t)v_t^{B,0}dz - \int_0^\infty \langle z\rangle^{2k}\left(u^{B,1} - u^{B,1}(0,t)\right)\phi_z b_t(t)v_t^{B,0}dz \\ &- \frac{1}{2}\int_0^\infty \langle z\rangle^{2k}v^{B,0}|v_t^{B,0}|^2 dz - \int_0^\infty \langle z\rangle^{2k}\phi(z)b_t(t)v^{B,0}v_t^{B,0}dz - \int_0^\infty \langle z\rangle^{2k}v_t^{I,0}(0,t)v^{B,0}v_t^{B,0}dz \end{aligned}$$

$$
\begin{aligned}
&-\frac{1}{2}\int_0^\infty \langle z\rangle^{2k}\phi(z)b(t)|v_t^{B,0}|^2dz-\frac{1}{2}\int_0^\infty \langle z\rangle^{2k}v^{I,0}(0,t)|v_t^{B,0}|^2dz\\
&-2k\int_0^\infty z\langle z\rangle^{2k-2}v_{tz}^{B,0}v_t^{B,0}dz+\int_0^\infty \langle z\rangle^{2k}v_t^{B,0}f_tdz\\
\triangleq&\sum_{i=1}^{11}K_i. \qquad (3.36)
\end{aligned}
$$

The term K_1 can be rewritten as

$$
\begin{aligned}
K_1=&\int_0^\infty z\langle z\rangle^{2k}\left(\frac{u_t^{B,1}-u_t^{B,1}(0,t)+(z-\psi)v_t^{I,0}(0,t)+\psi(z)h_t(t)}{z}\right)v_z^{B,0}v_t^{B,0}dz\\
=&\int_0^\infty z\langle z\rangle^{2k}\left(\frac{u_t^{B,1}-u_t^{B,1}(0,t)}{z}\right)v_z^{B,0}v_t^{B,0}dz+\int_0^\infty z\langle z\rangle^{2k}\left(\frac{z-\psi}{z}\right)v_t^{I,0}(0,t)v_z^{B,0}v_t^{B,0}dz\\
&+\int_0^\infty z\langle z\rangle^{2k}\frac{\psi(z)}{z}h_t(t)v_z^{B,0}v_t^{B,0}dz\\
\triangleq&K_{1a}+K_{1b}+K_{1c}.
\end{aligned}
\qquad (3.37)
$$

For K_{1a}, we have

$$
\begin{aligned}
K_{1a}\le&\left\|\frac{u_t^{B,1}-u_t^{B,1}(0,t)}{z}\right\|_{L_z^2}\|\langle z\rangle^{k+1}v_z^{B,0}\|_{L_z^2}\|\langle z\rangle^{k}v_t^{B,0}\|_{L_z^\infty}\\
\le&\|v_t^{B,0}\|_{L_z^2}\|\langle z\rangle^{k+1}v_z^{B,0}\|_{L_z^2}\|\langle z\rangle^{k}v_t^{B,0}\|_{L_z^2}^{\frac{1}{2}}\|\langle z\rangle^{k}v_{zt}^{B,0}\|_{L_z^2}^{\frac{1}{2}}\\
\le&\|\langle z\rangle^{k+1}v_z^{B,0}\|_{L_z^2}\|\langle z\rangle^{k}v_t^{B,0}\|_{L_z^2}^{\frac{3}{2}}\|\langle z\rangle^{k}v_{zt}^{B,0}\|_{L_z^2}^{\frac{1}{2}}\\
\le&\frac{1}{8}\|\langle z\rangle^{k}v_z^{B,0}\|_{L_z^2}^{2}+C\|\langle z\rangle^{k+1}v_z^{B,0}\|_{L_z^2}^{\frac{4}{3}}\|\langle z\rangle^{k}v_t^{B,0}\|_{L_z^2}^{2},
\end{aligned}
\qquad (3.38)
$$

where we have used Hölder's inequality in the first inequality, Hardy's inequality and Gagliardo-Nirenberg's inequality in the second inequality, Young's inequality in the last inequality. By Hölder's inequality and Young's inequality, it holds that

$$
\begin{aligned}
K_{1b}+K_{1c}\le&|v_t^{I,0}(0,t)|\left\|(1-\frac{\psi}{z})\right\|_{L_z^\infty}\|\langle z\rangle^{k+1}v_z^{B,0}\|_{L_z^2}\|\langle z\rangle^{k}v_t^{B,0}\|_{L_z^2}\\
&+|h_t(t)|\|\frac{\psi}{z}\|_{L_z^\infty}\|\langle z\rangle^{k+1}v_z^{B,0}\|_{L_z^2}\|\langle z\rangle^{k}v_t^{B,0}\|_{L_z^2}
\end{aligned}
$$

$$\leq C\|\langle z\rangle^{k+1}v_z^{B,0}\|_{L_z^2}^2+C\|\langle z\rangle^k v_t^{B,0}\|_{L_z^2}^2. \tag{3.39}$$

Similarly, the second term on the right hand side of (3.36) can be rewritten as

$$\begin{aligned}K_2=&k\int_0^\infty z^2\langle z\rangle^{2k-2}\left(\frac{u^{B,1}-u^{B,1}(0,t)}{z}\right)|v_t^{B,0}|^2dz\\&+k\int_0^\infty z^2\langle z\rangle^{2k-2}\left(\frac{z-\psi}{z}\right)v^{I,0}(0,t)|v_t^{B,0}|^2dz\\&+k\int_0^\infty z^2\langle z\rangle^{2k-2}\frac{\psi(z)}{z}h(t)|v_t^{B,0}|^2dz\\\triangleq&K_{21}+K_{22}+K_{23}.\end{aligned} \tag{3.40}$$

As in (3.38), we have

$$\begin{aligned}K_{21}&\leq k\left\|\frac{u^{B,1}-u^{B,1}(0,t)}{z}\right\|_{L_z^2}\|\langle z\rangle^k v_t^{B,0}\|_{L_z^4}^2\\&\leq k\|v^{B,0}\|_{L_z^2}\|\langle z\rangle^k v_t^{B,0}\|_{L_z^2}^{\frac32}\|\langle z\rangle^k v_{tz}^{B,0}\|_{L_z^2}^{\frac12}\\&\leq\frac18\|\langle z\rangle^k v_{tz}^{B,0}\|_{L_z^2}^2+Ck\|v^{B,0}\|_{L_z^2}^{\frac43}\|\langle z\rangle^k v_t^{B,0}\|_{L_z^2}^2.\end{aligned} \tag{3.41}$$

By Hölder's inequality, one can get

$$\begin{aligned}K_{22}+K_{23}&\leq k|v^{I,0}(0,t)|\left\|(1-\frac{\psi}{z})\right\|_{L_z^\infty}\|\langle z\rangle^k v_t^{B,0}\|_{L_z^2}^2+k|h(t)|\left\|\frac{\psi}{z}\right\|_{L_z^\infty}\|\langle z\rangle^k v_t^{B,0}\|_{L_z^2}^2\\&\leq Ck\|\langle z\rangle^k v_t^{B,0}\|_{L_z^2}^2.\end{aligned} \tag{3.42}$$

As in (3.17), we have

$$\begin{aligned}K_3&\leq\left\|\frac{u_t^{B,1}-u_t^{B,1}(0,t)}{z}\right\|_{L_z^2}\|\langle z\rangle^{k+1}\phi_z b(t)\|_{L_z^\infty}\|\langle z\rangle^k v_t^{B,0}\|_{L_z^2}\\&\leq\|v_t^{B,0}\|_{L_z^2}\|\langle z\rangle^{k+1}\phi_z b(t)\|_{L_z^\infty}\|\langle z\rangle^k v_t^{B,0}\|_{L_z^2}\\&\leq C\|\langle z\rangle^k v_t^{B,0}\|_{L_z^2}^2.\end{aligned} \tag{3.43}$$

It follows from Hölder's inequality, Hardy's inequality and Young's inequality that

$$
\begin{aligned}
K_4 &\le \left\| \frac{u^{B,1}-u^{B,1}(0,t)}{z} \right\|_{L_z^2} \|\langle z\rangle^{k+1}\phi_z b_t(t)\|_{L_z^\infty} \|\langle z\rangle^k v_t^{B,0}\|_{L_z^2} \\
&\le \|v^{B,0}\|_{L_z^2} \|\langle z\rangle^{k+1}\phi_z b(t)\|_{L_z^\infty} \|\langle z\rangle^k v_t^{B,0}\|_{L_z^2} \\
&\le C\|v^{B,0}\|_{L_z^2}^2 + C\|\langle z\rangle^k v_t^{B,0}\|_{L_z^2}^2 .
\end{aligned}
\tag{3.44}
$$

By Hölder's inequality, Gagliardo-Nirenberg's inequality and Young's inequality, one has

$$
\begin{aligned}
K_5 &\le \frac{1}{2}\|v^{B,0}\|_{L_z^2} \|\langle z\rangle^k v_t^{B,0}\|_{L_z^4}^2 \\
&\le \frac{1}{2}\|v^{B,0}\|_{L_z^2} \|\langle z\rangle^k v_t^{B,0}\|_{L_z^2}^{\frac{3}{2}} \|\langle z\rangle^k v_{tz}^{B,0}\|_{L_z^2}^{\frac{1}{2}} \\
&\le \frac{1}{8}\|\langle z\rangle^k v_{tz}^{B,0}\|_{L_z^2}^2 + C\|v^{B,0}\|_{L_z^2}^{\frac{4}{3}} \|\langle z\rangle^k v_t^{B,0}\|_{L_z^2}^2 .
\end{aligned}
\tag{3.45}
$$

Similar to (3.39), the terms K_6 and K_7 can be estimated as

$$
\begin{aligned}
K_6 + K_7 &\le |b_t(t)|\|\phi\|_{L_z^\infty} \|\langle z\rangle^k v^{B,0}\|_{L_z^2} \|\langle z\rangle^k v_t^{B,0}\|_{L_z^2} + |v_t^{I,0}(0,t)|\|\langle z\rangle^k v^{B,0}\|_{L_z^2} \|\langle z\rangle^k v_t^{B,0}\|_{L_z^2}^2 \\
&\le C\|\langle z\rangle^k v^{B,0}\|_{L_z^2}^2 + C\|\langle z\rangle^k v_t^{B,0}\|_{L_z^2}^2 .
\end{aligned}
\tag{3.46}
$$

By Hölder's inequality, similar to (3.42), it holds

$$
\begin{aligned}
K_8 + K_9 &\le \frac{1}{2}(|b(t)|\|\phi\|_{L_z^\infty} \|\langle z\rangle^k v_t^{B,0}\|_{L_z^2}^2 + |v^{I,0}(0,t)|\|\langle z\rangle^k v_t^{B,0}\|_{L_z^2}^2) \\
&\le C\|\langle z\rangle^k v_t^{B,0}\|_{L_z^2}^2 .
\end{aligned}
\tag{3.47}
$$

By Hölder's inequality and Young's inequality, we have

$$
K_{10} \le \frac{1}{8}\|\langle z\rangle^k v_{tz}^{B,0}\|_{L_z^2}^2 + Ck\|\langle z\rangle^k v_t^{B,0}\|_{L_z^2}^2 ,
\tag{3.48}
$$

and

$$
K_{11} \le \frac{1}{2}\|\langle z\rangle^k v_t^{B,0}\|_{L_z^2}^2 + \frac{1}{2}\|\langle z\rangle^k f_t\|_{L_z^2}^2 .
\tag{3.49}
$$

Plugging (3.37)-(3.49) into (3.36), we get

$$
\begin{aligned}
&\frac{d}{dt}\|\langle z\rangle^k v_t^{B,0}\|_{L_z^2}^2 + \|\langle z\rangle^k v_{tz}^{B,0}\|_{L_z^2}^2 \\
&\le C\|\langle z\rangle^k v_t^{B,0}\|_{L_z^2}^2 [1 + k + (1+k)\|v^{B,0}\|_{L_z^2}^{\frac{4}{3}} + \|\langle z\rangle^{k+1} v_z^{B,0}\|_{L_z^2}^{\frac{4}{3}}] \\
&\quad + C\|\langle z\rangle^k v^{B,0}\|_{L_z^2}^2 + C\|\langle z\rangle^{k+1} v_z^{B,0}\|_{L_z^2}^2 + \|\langle z\rangle^k f_t\|_{L_z^2}^2 .
\end{aligned}
\tag{3.50}
$$

Observing that k in (3.34) is arbitrary, (3.34) still holds when k is replaced by $k+1$. Therefore, by Gronwall's inequality, (3.23) and (3.34), we obtain

$$\sup_{0\leqslant t\leqslant T}\|\langle z\rangle^k v_t^{B,0}\|_{L_z^2}^2+\|\langle z\rangle^k v_{tz}^{B,0}\|_{L^2(0,T;L_z^2)}^2\leq C. \tag{3.51}$$

The proof is completed. □

We now establish the global well-posedness of the problem (3.8). By virtue of (3.7), we immediately get the same properties of the boundary layer profiles $(v^{B,0},u^{B,1})$ to the problem (2.13). By Lemma 3.1, we have for any given $T>0$

$$\sup_{0\leqslant t\leqslant T}(\|\langle z\rangle^k v^{B,0}\|_{H_z^1}^2+\|\langle z\rangle^k v_t^{B,0}\|_{L_z^2}^2)+\|\langle z\rangle^k v^{B,0}\|_{L^2(0,T;H_z^2)}^2+\|\langle z\rangle^k v_t^{B,0}\|_{L^2(0,T;H_z^1)}^2\leq C. \tag{3.52}$$

By (2.14) and the Hardy's inequality, we have

$$\|u^{B,1}\|_{L_z^2}\lesssim\|zv^{B,0}\|_{L_z^2}\lesssim\|\langle z\rangle v^{B,0}\|_{L_z^2}. \tag{3.53}$$

Thus, we get from (3.52) and (3.53) that

$$\sup_{0\leqslant t\leqslant T}(\|\langle z\rangle^k u^{B,1}\|_{H_z^1}^2+\|\langle z\rangle^k u_t^{B,1}\|_{L_z^2}^2)+\|\langle z\rangle^k u^{B,1}\|_{L^2(0,T;H_z^2)}^2+\|\langle z\rangle^k u_t^{B,1}\|_{L^2(0,T;H_z^1)}^2\leq C. \tag{3.54}$$

With the above *a priori* estimates in hand, one can obtain the existence of solution $(v^{B,0},u^{B,1})$ to the problem (3.8) by using the Galerkin method. Indeed let $\{w_j\}_{j=1}^{\infty}$ be a complete orthonormal basis of $H_0^1(0,r)\cap H^2(0,r)$. For each $m\in\mathbb{N}$ we define an approximate solution sequence $(v_m^{B,0},u_m^{B,1})$ as follows: for any $t\in[0,T]$

$$v_m^{B,0}(z,t)\triangleq\sum_{j=1}^m g_{jm}(t)w_j(z), \tag{3.55}$$

$$u_m^{B,1}(z,t)\triangleq u_m^{B,1}(0,t)+\int_0^z v_m^{B,0}(y,t)dy=u_m^{B,1}(0,t)+\sum_{j=1}^m g_{jm}(t)\int_0^z w_j(y)dy. \tag{3.56}$$

Then a straightforward calculation yields

$$\begin{aligned}&(\frac{d}{dt}v_m^{B,0}(t),w_k)+\Big((u_m^{B,1}-u_m^{B,1}(0,t)+(z-\psi)v^{I,0}(0,t)+\psi(z)h(t))v_{mz}^{B,0},w_k\Big)\\&\quad+\Big((u_m^{B,1}-u_m^{B,1}(0,t))\phi_z b(t),w_k\Big)+\frac{1}{2}\Big((v_m^{B,0})^2,w_k\Big)\\&\quad+\Big(\phi(z)b(t)v_m^{B,0},w_k\Big)+\Big(v^{I,0}(0,t)v_m^{B,0},w_k\Big)\\&=\Big(v_{mzz}^{B,0},w_k\Big)+(f,w_k),\quad t\in[0,T],\ k=1,2,\dots,m,\end{aligned} \tag{3.57}$$

and

$$v_m^{B,0}(0) = 0. \tag{3.58}$$

This forms a nonlinear system of ODE for the functions $g_{jm}(t)$, $j = 1, \cdots, m$:

$$\begin{aligned}
&\frac{d}{dt} g_{km}(t) + \sum_{j,i=1}^{m} g_{jm}(t) g_{im}(t) \left(\int_0^z w_j(y) dy w_j', w_k \right) + \sum_{j=1}^{m} g_{jm}(t) \left((z-\psi) v^{I,0}(0,t) w_j', w_k \right) \\
&\quad + \sum_{j=1}^{m} g_{jm}(t) \left(\psi(z) h(t) w_j', w_k \right) + \sum_{j=1}^{m} g_{jm}(t) \left(\int_0^z w_j(y) dy \phi_z b(t), w_k \right) \\
&\quad + \frac{1}{2} \sum_{j,i=1}^{m} g_{jm}(t) g_{im}(t) (w_j w_i, w_k) + \sum_{j=1}^{m} g_{jm}(t) \left(\phi(z) b(t) w_j, w_k \right) \\
&\quad + \sum_{j=1}^{m} g_{jm}(t) \left(v^{I,0}(0,t) w_j, w_k \right) = - \sum_{j=1}^{m} g_{jm}(t) (w_j', w_k') + (f, w_k),
\end{aligned} \tag{3.59}$$

and

$$g_{km}(0) = 0. \tag{3.60}$$

By means of the standard existence theory of ODE, we conclude that the nonlinear system (3.59)-(3.60) admits a unique solution $g_{jm}(t) \in C^1([0, t_m))$ for some $t_m > 0$. Moreover if $t_m < T$, then $\|v_m^{B,0}\|_{L^\infty(0,t;L^2(0,r))}$ must tend to ∞ as $t \to t_m$. But by the *a priori* estimates obtained previously, we shall see that $\|v_m^{B,0}\|_{L^\infty(0,t_m;L^2(0,r))}$ is bounded above and therefore $t_m = T$.

In the same way as we get the *a priori* estimates (3.52) and (3.54), one can deduce that $v_m^{B,0}$ and $u_m^{B,1}$ are uniformly bounded in $L^\infty(0, T; H^1(0, r)) \cap L^2(0, T; H^2(0, r))$, $v_{mt}^{B,0}$ and $u_{mt}^{B,1}$ are uniformly bounded in $L^\infty(0, T; L^2(0, r))$ for any $r > 0$. Therefore, using the Banach-Alaoglu theorem and the diagonal argument, there exist a subsequence, still denoted by $(v_m^{B,0}, u_m^{B,1})$, and a pair of function $(v^{B,0}, u^{B,1})$, such that

$$(v_m^{B,0}, u_m^{B,1}) \rightharpoonup (v^{B,0}, u^{B,1}) \text{ in } L^\infty(0, T; H^1(0, r)) \text{ weak-star}, \tag{3.61}$$

$$(v_m^{B,0}, u_m^{B,1}) \rightharpoonup (v^{B,0}, u^{B,1}) \text{ in } L^2(0, T; H^2(0, r)) \text{ weakly}, \tag{3.62}$$

$$(v_{mt}^{B,0}, u_{mt}^{B,1}) \rightharpoonup (v_t^{B,0}, u_t^{B,1}) \text{ in } L^\infty(0, T; L^2(0, r)) \text{ weak-star}. \tag{3.63}$$

We also need the following compactness lemma (see. [34]) to handle the nonlinear terms.

Lemma 3.2 *(Aubin-Lions-Simon lemma). Let X_0, X, X_1 be three Banach spaces such that $X_0 \subset X \subset X_1$. Suppose that X_0 is compactly embedded in X and that X is continuously embedded in X_1. For $1 \le p, q \le \infty$, let*

$$\mathcal{Y} \triangleq \{f \in L^p([0,T];X_0) | \partial_t f \in L^q([0,T];X_1)\}.$$

(i) If $p=\infty$ and $q>1$, then the embedding of $\mathcal{Y}$ into $C([0,T];X)$ is compact;
(ii) If $p<\infty$, then the embedding of $\mathcal{Y}$ into $L^p([0,T];X)$ is compact.

In view of (3.61)-(3.63), and the compactness of the embeddings $H^2(0,r) \hookrightarrow H^1(0,r)$ and $H^1(0,r) \hookrightarrow L^\infty(0,r)$, we obtain from Lemma 3.2 that

$$(v_m^{B,0}, u_m^{B,1}) \to (v^{B,0}, u^{B,1}) \text{ in } L^2(0,T;H^1(0,r)) \cap C(0,T;L^\infty(0,r)) \text{ strongly.} \tag{3.64}$$

Let $\alpha(t)$ be a differentiable function on $[0,T]$ with $\alpha(T)=0$. Multiplying both sides of (3.57) by $\alpha(t)$, and integrating the resulting equation over $[0,T]$, we get

$$\begin{aligned}
&-\int_0^T \left(v_m^{B,0}(t), \alpha_t(t) w_k\right) dt \\
&+\int_0^T \left((u_m^{B,1} - u_m^{B,1}(0,t) + (z-\psi)v^{I,0}(0,t) + \psi(z)h(t))v_{mz}^{B,0}, \alpha(t) w_k\right) dt \\
&+\int_0^T \left((u_m^{B,1} - u_m^{B,1}(0,t))\phi_z b(t), \alpha(t) w_k\right) dt + \frac{1}{2}\int_0^T \left((v_m^{B,0})^2, \alpha(t) w_k\right) dt \\
&+\int_0^T \left(\phi(z) b(t) v_m^{B,0}, \alpha(t) w_k\right) dt + \int_0^T \left(v^{I,0}(0,t) v_m^{B,0}, \alpha(t) w_k\right) dt \\
&= \int_0^T \left(v_{mzz}^{B,0}, \alpha(t) w_k\right) dt + \int_0^T (f, \alpha(t) w_k)\, dt.
\end{aligned} \tag{3.65}$$

Utilizing (3.61)-(3.64) and passing to the limit in (3.65), noting $\{w_k\}_{k=1}^\infty$ forms a complete orthonormal basis of $H_0^1(0,r) \cap H^2(0,r)$, we obtain

$$\begin{aligned}
&-\int_0^T \left(v^{B,0}(t), \alpha_t(t) w\right) dt \\
&+\int_0^T \left((u^{B,1} - u^{B,1}(0,t) + (z-\psi)v^{I,0}(0,t) + \psi(z)h(t))v_z^{B,0}, \alpha(t) w\right) dt \\
&+\int_0^T \left((u^{B,1} - u^{B,1}(0,t))\phi_z b(t), \alpha(t) w\right) dt + \frac{1}{2}\int_0^T \left((v^{B,0})^2, \alpha(t) w\right) dt
\end{aligned}$$

$$
\begin{aligned}
&+\int_0^T \left(\phi(z)b(t)v^{B,0},\alpha(t)w\right)dt+\int_0^T \left(v^{I,0}(0,t)v^{B,0},\alpha(t)w\right)dt\\
&=\int_0^T \left(v^{B,0}_{zz},\alpha(t)w\right)dt+\int_0^T (f,\alpha(t)w)\,dt,
\end{aligned}
$$

for all $w\in H_0^1(0,r)\cap H^2(0,r)$. Letting $r\to\infty$, by the diagonal argument, one can see that $(v^{B,0},u^{B,1})$ satisfies (3.8) in the sense of distribution.

Proposition 3.2. *Under the same assumptions of Theorem 2.1, the system (2.13) has a unique solution $(v^{B,0},u^{B,1})$ satisfying*

$$
\begin{aligned}
&\langle z\rangle^k v^{B,0}\in L^2(0,T;H_z^3)\cap L^\infty(0,T;H_z^2),\\
&\langle z\rangle^k v_t^{B,0}\in L^2(0,T;H_z^1)\cap L^\infty(0,T;L_z^2),\\
&\langle z\rangle^k u^{B,1}\in L^2(0,T;H_z^3)\cap L^\infty(0,T;H_z^2),
\end{aligned}
$$

for any $0<T<\infty$ *and* $k\geq 0$.

Proof. The existence and regularity of the solution $(v^{B,0},u^{B,1})$ of (2.13) follow from the above analysis. Next we prove the uniqueness of the solution. Let $(v_1^{B,0},u_1^{B,1})$ and $(v_2^{B,0},u_2^{B,1})$ be two solutions of (3.8) and set $W^{B,0}=v_1^{B,0}-v_2^{B,0}$, $U^{B,1}=u_1^{B,1}-u_2^{B,1}$. Then $(W^{B,0},U^{B,1})$ satisfies

$$
\begin{cases}
W_t^{B,0}+\left(u_1^{B,1}-u_1^{B,1}(0,t)\right)W_z^{B,0}+\left(U^{B,1}-U^{B,1}(0,t)\right)v_{2z}^{B,0}\\
\qquad+\left((z-\psi)v^{I,0}(0,t)+\psi(z)h(t)\right)W_z^{B,0}+\left(U^{B,1}-U^{B,1}(0,t)\right)\phi_z b(t)\\
\qquad+\frac{1}{2}(v_1^{B,0}+v_2^{B,0})W^{B,0}+\phi(z)b(t)W^{B,0}+v^{I,0}(0,t)W^{B,0}=W_{zz}^{B,0},\\
U_z^{B,1}=W^{B,0},\\
W^{B,0}(z,0)=0,\\
W^{B,0}(0,t)=0,\quad \lim\limits_{z\to+\infty}(W^{B,0},U^{B,1})(z,t)=0.
\end{cases}
\tag{3.66}
$$

Multiplying the first equation of (3.66) by $W^{B,0}$, we get

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|W^{B,0}\|_{L_z^2}^2+\|W_z^{B,0}\|_{L_z^2}^2\\
&=-\int_0^\infty \left(U^{B,1}-U^{B,1}(0,t)\right)v_{2z}^{B,0}W^{B,0}dz-\int_0^\infty \left(U^{B,1}-U^{B,1}(0,t)\right)\phi_z b(t)W^{B,0}dz\\
&\quad-\frac{1}{2}\int_0^\infty v^{I,0}(0,t)|W^{B,0}|^2dz-\frac{1}{2}\int_0^\infty \phi(z)b(t)|W^{B,0}|^2dz-\frac{1}{2}\int_0^\infty v_2^{B,0}|W^{B,0}|^2dz
\end{aligned}
\tag{3.67}
$$

Next, we estimate the terms on the right side of (3.67). First, it holds that

$$
\begin{aligned}
-\int_0^\infty \left(U^{B,1}-U^{B,1}(0,t)\right) v_{2z}^{B,0} W^{B,0} dz &\le \left\| \frac{U^{B,1}-U^{B,1}(0,t)}{z} \right\|_{L_z^2} \|z v_{2z}^{B,0}\|_{L_z^\infty} \|W^{B,0}\|_{L_z^2} \\
&\le \|U_z^{B,1}\|_{L_z^2} \|z v_{2z}^{B,0}\|_{L_z^\infty} \|W^{B,0}\|_{L_z^2}, \\
&\le \|z v_{2z}^{B,0}\|_{L_z^\infty} \|W^{B,0}\|_{L_z^2}^2,
\end{aligned}
\tag{3.68}
$$

where we have used Hölder's inequality in the first inequality, Hardy's inequality in the second inequality and $U_z^{B,1} = W^{B,0}$ in the last inequality. Similarly, we have

$$
\begin{aligned}
-\int_0^\infty \left(U^{B,1}-U^{B,1}(0,t)\right) \phi_z b(t) W^{B,0} dz &\le \left\| \frac{U^{B,1}-U^{B,1}(0,t)}{z} \right\|_{L_z^2} \|z \phi_z b(t)\|_{L_z^\infty} \|W^{B,0}\|_{L_z^2} \\
&\le \|W^{B,0}\|_{L_z^2} \|z \phi_z b(t)\|_{L_z^\infty} \|W^{B,0}\|_{L_z^2} \\
&\le C\|W^{B,0}\|_{L_z^2}^2.
\end{aligned}
\tag{3.69}
$$

Using Hölder's inequality, we have

$$
\begin{aligned}
&-\frac{1}{2}\int_0^\infty v^{I,0}(0,t)|W^{B,0}|^2 dz - \frac{1}{2}\int_0^\infty \phi(z) b(t) |W^{B,0}|^2 dz \\
&\le \frac{1}{2}\left(|v^{I,0}(0,t)| \|W^{B,0}\|_{L_z^2}^2 + |b(t)| \|\phi\|_{L_z^\infty} \|W^{B,0}\|_{L_z^2}^2 \right) \\
&\le C\|W^{B,0}\|_{L_z^2}^2,
\end{aligned}
\tag{3.70}
$$

and

$$
-\frac{1}{2}\int_0^\infty v_2^{B,0} |W^{B,0}|^2 dz \le \frac{1}{2} \|v_2^{B,0}\|_{L_z^\infty} \|W^{B,0}\|_{L_z^2}^2. \tag{3.71}
$$

Thus, plugging (3.68)-(3.71) into (3.67) and using the estimates in Lemma 3.1, we can easily deduce that

$$
\frac{d}{dt}\|W^{B,0}\|_{L_z^2}^2 \le C\|W^{B,0}\|_{L_z^2}^2. \tag{3.72}
$$

It then follows from the Gronwall's inequality that $\|W^{B,0}\|_{L_z^2}^2 = 0$. Thus $v_1^{B,0} = v_2^{B,0}$ and $u_1^{B,1} = u_2^{B,1}$. This completes the proof of Proposition 3.2. □

Applying the similar arguments as that for the system (2.13), one can obtain the well-posedness of the linear parabolic system (2.16).

Proposition 3.3. *Under the same assumptions of Theorem 2.1, there exists a unique solution $(v^{B,1}, u^{B,2})$ of (2.16), which satisfies for any $0 < T < \infty$ and $k \geq 0$*

$$\langle z\rangle^k v^{B,1} \in L^2(0,T;H_z^3) \cap L^\infty(0,T;H_z^2), \tag{3.73}$$

$$\langle z\rangle^k v_t^{B,1} \in L^2(0,T;H_z^1) \cap L^\infty(0,T;L_z^2), \tag{3.74}$$

$$\langle z\rangle^k u^{B,2} \in L^2(0,T;H_z^3) \cap L^\infty(0,T;H_z^2).$$

4. Convergence rates

In this section, we will justify the asymptotic expansion by providing an optimal error estimate. Our strategy is based on the basic energy method. We construct the approximate solutions as follows:

$$\begin{cases} v^{\varepsilon,a} \triangleq v^{I,0}(x,t) + v^{B,0}(\frac{x}{\sqrt{\varepsilon}},t) + \varepsilon^{\frac12} v^{I,1}(x,t) + \varepsilon^{\frac12} v^{B,1}(\frac{x}{\sqrt{\varepsilon}},t), \\ u^{\varepsilon,a} \triangleq u^{I,0}(x,t) + \varepsilon^{\frac12} u^{I,1}(x,t) + \varepsilon^{\frac12} u^{B,1}(\frac{x}{\sqrt{\varepsilon}},t) + \varepsilon u^{B,2}(\frac{x}{\sqrt{\varepsilon}},t) - \varepsilon u^{B,2}(0,t), \end{cases} \tag{4.1}$$

and denote

$$\begin{cases} R_1^\varepsilon \triangleq \varepsilon^{-1}(v^\varepsilon - v^{\varepsilon,a}), \\ R_2^\varepsilon \triangleq \varepsilon^{-1}(u^\varepsilon - u^{\varepsilon,a}), \end{cases} \tag{4.2}$$

where $\varepsilon u^{B,2}(0,t)$ is presented to homogenize the boundary condition of R_2^ε.

Substituting (4.1)-(4.2) into the equation (1.3), and using the initial and boundary conditions of (2.12)-(2.16), one can see that the error $(R_1^\varepsilon, R_2^\varepsilon)$ satisfies

$$\begin{cases} R_{1t}^\varepsilon + R_2^\varepsilon v_x^{\varepsilon,a} + u^{\varepsilon,a} R_{1x}^\varepsilon + \varepsilon R_2^\varepsilon R_{1x}^\varepsilon = -\frac{1}{2}\varepsilon (R_1^\varepsilon)^2 - v^{\varepsilon,a} R_1^\varepsilon + \varepsilon R_{1xx}^\varepsilon + \varepsilon^{-1} f^\varepsilon, \\ R_{2x}^\varepsilon = R_1^\varepsilon, \\ R_1^\varepsilon(x,0) = R_2^\varepsilon(0,t) = R_1^\varepsilon(0,t) = 0, \end{cases} \tag{4.3}$$

where

$$f^\varepsilon = -v_t^{\varepsilon,a} - u^{\varepsilon,a} v_x^{\varepsilon,a} - \frac{1}{2}(v^{\varepsilon,a})^2 + \varepsilon v_{xx}^{\varepsilon,a}. \tag{4.4}$$

Before proceeding, we introduce some basic facts which will be used later. For any $F(z,t) \in H_z^m$ with $m \in \mathbb{N}$, we have

$$\left\| \partial_x^m F(\frac{x}{\sqrt{\varepsilon}},t) \right\|_{L^2} = \varepsilon^{\frac14 - \frac{m}{2}} \left\| \partial_z^m F(z,t) \right\|_{L_z^2}. \tag{4.5}$$

For $\phi(\cdot,t) \in H^1$ with $\phi|_{x=0} = 0$, one gets

$$\|\phi(\cdot,t)\|_{L^\infty} \le \sqrt{2}\|\phi(\cdot,t)\|_{L^2}^{\frac{1}{2}}\|\phi_x(\cdot,t)\|_{L^2}^{\frac{1}{2}}. \tag{4.6}$$

Lemma 4.1. *Let* $0 < T < \infty, 0 < \varepsilon < 1$. *Then there exists a constant* C *independent of* ε, *such that*

$$\|f^\varepsilon\|_{L^\infty(0,T;L^2)} \le C\varepsilon^{3/4}. \tag{4.7}$$

Proof. Recalling the transformation $z = \frac{x}{\sqrt{\varepsilon}}$, one gets from (2.13) that

$$\begin{aligned}\varepsilon v_{xx}^{B,0} =& v_t^{B,0} + \varepsilon^{\frac{1}{2}}u^{I,1}(0,t)v_x^{B,0} + \varepsilon^{\frac{1}{2}}u^{B,1}v_x^{B,0} \\ &+ \frac{1}{2}(v^{B,0})^2 + v^{I,0}(0,t)v^{B,0} + xu_x^{I,0}(0,t)v_x^{B,0}.\end{aligned} \tag{4.8}$$

This identity along with (4.4), (2.12) and (2.15), gives

$$\begin{aligned}f^\varepsilon =& [\varepsilon v_{xx}^{I,0} + \varepsilon^{\frac{3}{2}}v_{xx}^{I,1}] + [\varepsilon^{\frac{3}{2}}v_{xx}^{B,1} - \varepsilon^{\frac{1}{2}}v_t^{B,1}] - [(u^{I,0}(x,t) - u^{I,0}(0,t) - xu_x^{I,0}(0,t))v_x^{B,0}] \\ &- [\varepsilon^{\frac{1}{2}}u^{I,0}v_x^{B,1}] - [\varepsilon^{\frac{1}{2}}(u^{I,1}(x,t) - u^{I,1}(0,t))v_x^{B,0}] - [\varepsilon^{\frac{1}{2}}u^{B,1}v_x^{I,0}] \\ &- [\varepsilon(u^{I,1} + u^{B,1})(v_x^{I,1} + v_x^{B,1})] - [\varepsilon u^{B,2}(v_x^{I,0} + v_x^{B,0} + \varepsilon^{\frac{1}{2}}v_x^{I,1} + \varepsilon^{\frac{1}{2}}v_x^{B,1})] \\ &- [\varepsilon u^{B,2}(0,t)(v_x^{I,0} + v_x^{B,0} + \varepsilon^{\frac{1}{2}}v_x^{I,1} + \varepsilon^{\frac{1}{2}}v_x^{B,1})] - [(v^{I,0} - v^{I,0}(0,t))v^{B,0}] \\ &- [\varepsilon^{\frac{1}{2}}(v^{I,0}v^{B,1} + v^{I,1}v^{B,0} + v^{B,0}v^{B,1})] - [\frac{1}{2}\varepsilon(v^{I,1})^2 - \frac{1}{2}\varepsilon(v^{B,1})^2 - \varepsilon v^{I,1}v^{B,1}] \\ \triangleq& \sum_{k=1}^{12} M_k.\end{aligned} \tag{4.9}$$

We proceed to estimate M_k $(1 \le k \le 12)$. By (3.1) and (3.2), we have

$$\begin{aligned}\|M_1\|_{L^\infty(0,T;L^2)} &= \varepsilon\|v_{xx}^{I,0}\|_{L^\infty(0,T;L^2)} + \varepsilon^{\frac{3}{2}}\|v_{xx}^{I,1}\|_{L^\infty(0,T;L^2)} \\ &\le \varepsilon\|v^{I,0}\|_{L^\infty(0,T;H^2)} + \varepsilon^{\frac{3}{2}}\|v^{I,1}\|_{L^\infty(0,T;H^2)} \\ &\le C\varepsilon.\end{aligned}$$

It follows from (4.5), (3.73) and (3.74) that

$$\begin{aligned}\|M_2\|_{L^\infty(0,T;L^2)} &\le \varepsilon^{\frac{3}{4}}(\|v_{zz}^{B,1}\|_{L^\infty(0,T;L_z^2)} + \|v_t^{B,1}\|_{L^\infty(0,T;L_z^2)}) \\ &\le C\varepsilon^{\frac{3}{4}}.\end{aligned}$$

By Taylor expansion, Sobolev inequality, (4.5), Propositions 3.1-3.2, we obtain

$$\begin{aligned}\|M_3\|_{L^\infty(0,T;L^2)} &= \varepsilon \left\| \frac{u^{I,0}(x,t) - u^{I,0}(0,t) - x u_x^{I,0}(0,t)}{x^2} z^2 v_x^{B,0} \right\|_{L^\infty(0,T;L^2)} \\ &\le C\varepsilon \|u_{xx}^{I,0}\|_{L^\infty(0,T;L^\infty)} \|z^2 v_x^{B,0}\|_{L^\infty(0,T;L^2)} \\ &\le C\varepsilon^{\frac{3}{4}} \|u^{I,0}\|_{L^\infty(0,T;H^3)} \|z^2 v_z^{B,0}\|_{L^\infty(0,T;L_z^2)} \\ &\le C\varepsilon^{\frac{3}{4}}.\end{aligned}$$

Similar arguments further give rise to the estimates for M_4, M_5 and M_{10} as follows:

$$\begin{aligned}\|M_4\|_{L^\infty(0,T;L^2)} &= \varepsilon^{\frac{1}{2}} \left\| \frac{u^{I,0}(x,t)}{x} \cdot \varepsilon^{\frac{1}{2}} z v_x^{B,1} \right\|_{L^\infty(0,T;L^2)} \\ &\le \varepsilon \|u_x^{I,0}\|_{L^\infty(0,T;L^\infty)} \|z v_x^{B,1}\|_{L^\infty(0,T;L^2)} \\ &\le C\varepsilon^{\frac{3}{4}} \|u^{I,0}\|_{L^\infty(0,T;H^2)} \|z v_z^{B,1}\|_{L^\infty(0,T;L_z^2)} \\ &\le C\varepsilon^{\frac{3}{4}}, \\ \|M_5\|_{L^\infty(0,T;L^2)} &= \varepsilon \left\| \frac{u^{I,1}(x,t) - u^{I,1}(0,t)}{x} z v_x^{B,0} \right\|_{L^\infty(0,T;L^2)} \\ &\le C\varepsilon^{\frac{3}{4}} \|u^{I,1}\|_{L^\infty(0,T;H^2)} \|z v_z^{B,0}\|_{L^\infty(0,T;L_z^2)} \\ &\le C\varepsilon^{\frac{3}{4}},\end{aligned}$$

and

$$\begin{aligned}\|M_{10}\|_{L^\infty(0,T;L^2)} &= \varepsilon^{\frac{1}{2}} \left\| \frac{v^{I,0}(x,t) - v^{I,0}(0,t)}{x} z v_x^{B,0} \right\|_{L^\infty(0,T;L^2)} \\ &\le C\varepsilon^{\frac{3}{4}} \|v^{I,0}\|_{L^\infty(0,T;H^2)} \|z v_z^{B,0}\|_{L^\infty(0,T;L_z^2)} \\ &\le C\varepsilon^{\frac{3}{4}}.\end{aligned}$$

Then using (4.5) and Propositions 3.1-3.2, we estimate M_6 as follows:

$$\begin{aligned}\|M_6\|_{L^\infty(0,T;L^2)} &\le C\varepsilon^{\frac{1}{2}} \|u^{B,1}\|_{L^\infty(0,T;L^2)} \|v_x^{I,0}\|_{L^\infty(0,T;L^\infty)} \\ &\le C\varepsilon^{\frac{3}{4}} \|u^{B,1}\|_{L^\infty(0,T;L_z^2)} \|v^{I,0}\|_{L^\infty(0,T;H^2)} \\ &\le C\varepsilon^{\frac{3}{4}}\end{aligned}$$

Moreover, using a similar argument as estimating M_6, we have

$$\|M_7\|_{L^\infty(0,T;L^2)} + \|M_8\|_{L^\infty(0,T;L^2)} + \|M_{11}\|_{L^\infty(0,T;L^2)} \le C\varepsilon^{\frac{3}{4}},$$

and

$$\begin{aligned}\|M_{12}\|_{L^\infty(0,T;L^2)} \le& \frac{1}{2}\varepsilon\|v^{I,1}\|_{L^\infty(0,T;H^1)}\|v^{I,1}\|_{L^\infty(0,T;L^2)}\\&+\frac{1}{2}\varepsilon^{\frac{5}{4}}\|v^{B,1}\|_{L^\infty(0,T;H_z^1)}\|v^{B,1}\|_{L^\infty(0,T;L_z^2)}\\&+\varepsilon^{\frac{5}{4}}\|v^{I,1}\|_{L^\infty(0,T;H^1)}\|v^{B,1}\|_{L^\infty(0,T;L_z^2)}\\ \le& C\varepsilon.\end{aligned}$$

Finally, noticing that

$$\|u^{B,2}(0,t)\|_{L^\infty(0,T)} \le C\|u^{B,2}\|_{L^\infty(0,T;H_z^1)} \le C, \tag{4.10}$$

one obtains from (4.5) and Propositions 3.1-3.3 that

$$\begin{aligned}\|M_9\|_{L^\infty(0,T;L^2)} \le& C\varepsilon\big(\|v^{I,0}\|_{L^\infty(0,T;H^1)}+\varepsilon^{-\frac{1}{4}}\|v_z^{B,0}\|_{L^\infty(0,T;L_z^2)}\\&+\varepsilon^{\frac{1}{2}}\|v^{I,1}\|_{L^\infty(0,T;H^1)}+\varepsilon^{\frac{1}{4}}\|v_z^{B,1}\|_{L^\infty(0,T;L_z^2)}\big)\\ \le& C\varepsilon^{\frac{3}{4}}.\end{aligned}$$

Combining the above estimates for M_1 to M_{12} with (4.9), we deduce (4.7). The proof is completed. □

Next, we derive the L^∞ estimates for the reminder $(R_1^\varepsilon, R_2^\varepsilon)$. By the Sobolev inequality, Propositions 3.1-3.3, we first have

$$\begin{aligned}\|v^{\varepsilon,a}\|_{L^\infty([0,T]\times[0,\infty))} \le& \|v^{I,0}\|_{L^\infty([0,T]\times[0,\infty))}+\|v^{B,0}\|_{L^\infty(0,T;L_z^\infty)}\\&+\varepsilon^{\frac{1}{2}}\|v^{I,1}\|_{L^\infty([0,T]\times[0,\infty))}+\varepsilon^{\frac{1}{2}}\|v^{B,1}\|_{L^\infty(0,T;L_z^\infty)}\\ \le& C.\end{aligned}\tag{4.11}$$

Lemma 4.2. *Let $0 < T < \infty$ and $0 < \varepsilon < 1$. Then there exists a positive constant C independent of ε, such that the following estimate holds:*

$$\sup_{0\leqslant t\leqslant T}\|R_1^\varepsilon\|_{L^2}^2+\varepsilon\|R_{1x}^\varepsilon\|_{L^2(0,T;L^2)}^2 \le C\varepsilon^{-\frac{1}{2}}. \tag{4.12}$$

Proof. Multiplying the first equation of (4.3) by R_1^ε, one gets from integration by parts that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|R_1^\varepsilon\|_{L^2}^2+\varepsilon\|R_{1x}^\varepsilon\|_{L^2}^2 =& -\int_0^\infty R_2^\varepsilon v_x^{\varepsilon,a}R_1^\varepsilon dx-\frac{1}{2}\int_0^\infty v^{\varepsilon,a}(R_1^\varepsilon)^2dx+\varepsilon^{-1}\int_0^\infty f^\varepsilon R_1^\varepsilon dx\\ \triangleq& II_1+II_2+II_3.\end{aligned}\tag{4.13}$$

Invoking the boundary condition $R_2^\varepsilon(0,t)=0$ and using the second equation of (4.3), one has

$$R_2^\varepsilon = \int_0^x R_1^\varepsilon dx \leq \left(\int_0^x |R_1^\varepsilon|^2 dx\right)^{\frac{1}{2}} x^{\frac{1}{2}}. \tag{4.14}$$

Recalling that $x = \varepsilon^{\frac{1}{2}} z$, by the definition of $v^{\varepsilon,a}$ and Propositions 3.2-3.3, we have

$$\begin{aligned}\int_0^\infty x|v_x^{\varepsilon,a}|^2 dx &\lesssim \int_0^\infty x|v_x^{I,0}|^2 dx + \int_0^\infty x|v_x^{B,0}|^2 dx + \varepsilon \int_0^\infty x|v_x^{I,1}|^2 dx + \varepsilon \int_0^\infty x|v_x^{B,1}|^2 dx \\ &= \int_0^\infty x|v_x^{I,0}|^2 dx + \int_0^\infty z|v_z^{B,0}|^2 dz + \varepsilon \int_0^\infty x|v_x^{I,1}|^2 dx + \varepsilon \int_0^\infty z|v_z^{B,1}|^2 dz \\ &\leq C,\end{aligned} \tag{4.15}$$

where we have used $\int_0^\infty x v_{0x}^2 dx < +\infty$ and $\int_0^\infty x v_{0xx}^2 dx < +\infty$. It then follows from Hölder's inequality, (4.14)-(4.15) that

$$\begin{aligned} II_1 &\leq \left(\int_0^\infty |R_1^\varepsilon|^2 dx\right)^{\frac{1}{2}} \int_0^\infty |x^{\frac{1}{2}} v_x^{\varepsilon,a} R_1^\varepsilon| dx \\ &\leq \int_0^\infty |R_1^\varepsilon|^2 dx \left(\int_0^\infty x|v_x^{\varepsilon,a}|^2 dx\right)^{\frac{1}{2}} \\ &\leq C\|R_1^\varepsilon\|_{L^2}^2. \end{aligned} \tag{4.16}$$

By (4.11) and Hölder's inequality, we deduce

$$II_2 \leq \frac{1}{2}\|v^{\varepsilon,a}\|_{L^\infty}\|R_1^\varepsilon\|_{L^2}^2 \leq C\|R_1^\varepsilon\|_{L^2}^2. \tag{4.17}$$

By Hölder's inequality and Young's inequality, it yields

$$II_3 \leq \frac{1}{2}\|R_1^\varepsilon\|_{L^2}^2 + \frac{1}{2}\varepsilon^{-2}\|f^\varepsilon\|_{L^2}^2. \tag{4.18}$$

Substituting (4.16), (4.17) and (4.18) into (4.13), we have

$$\frac{d}{dt}\|R_1^\varepsilon\|_{L^2}^2 + \varepsilon\|R_{1x}^\varepsilon\|_{L^2}^2 \leq C\|R_1^\varepsilon\|_{L^2}^2 + \varepsilon^{-2}\|f^\varepsilon\|_{L^2}^2. \tag{4.19}$$

The desired estimate (4.12) then follows from the Gronwall's inequality and Lemma 4.1. □

Lemma 4.3. *Let $0 < T < \infty$ and $0 < \varepsilon < 1$. Then there exists a positive constant C independent of ε, such that the following estimate holds:*

$$\sup_{0\leqslant t\leqslant T}\|R^{\varepsilon}_{1x}\|^2_{L^2}+\varepsilon\|R^{\varepsilon}_{1xx}\|^2_{L^2(0,T;L^2)}\leq C\varepsilon^{-\frac{3}{2}}. \tag{4.20}$$

Proof. Multiplying the first equation of (4.3) by $\varepsilon R^{\varepsilon}_{1xx}$, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\varepsilon\|R^{\varepsilon}_{1x}\|^2_{L^2})+\varepsilon^2\|R^{\varepsilon}_{1xx}\|^2_{L^2}\\
&=\varepsilon\int_0^\infty R^{\varepsilon}_2 v^{\varepsilon,a}_x R^{\varepsilon}_{1xx}dx-\frac{1}{2}\varepsilon\int_0^\infty v^{\varepsilon,a}(R^{\varepsilon}_{1x})^2dx-\frac{3}{2}\varepsilon^2\int_0^\infty R^{\varepsilon}_1(R^{\varepsilon}_{1x})^2dx\\
&\quad+\varepsilon\int_0^\infty v^{\varepsilon,a}R^{\varepsilon}_1R^{\varepsilon}_{1xx}dx-\int_0^\infty f^{\varepsilon}R^{\varepsilon}_{1xx}dx\\
&\triangleq II_4+II_5+II_6+II_7+II_8.
\end{aligned} \tag{4.21}$$

Now we estimate the terms on the right hand side of the (4.21). By using a similar argument as that for II_1, we have

$$II_4\leq C\varepsilon\|R^{\varepsilon}_1\|_{L^2}\|R^{\varepsilon}_{1xx}\|_{L^2}\leq\frac{1}{8}\varepsilon^2\|R^{\varepsilon}_{1xx}\|^2_{L^2}+C\|R^{\varepsilon}_1\|^2_{L^2}. \tag{4.22}$$

Thanks to (4.11), one obtains by the Hölder's inequality that

$$II_5\leq\frac{1}{2}\varepsilon\|v^a\|_{L^\infty}\|R^{\varepsilon}_{1x}\|^2_{L^2}\leq C\varepsilon\|R^{\varepsilon}_{1x}\|^2_{L^2}. \tag{4.23}$$

It follows from Gagliardo-Nirenberg's inequality that

$$\begin{aligned}
II_6&\leq C\varepsilon^2\|R^{\varepsilon}_1\|_{L^2}\|R^{\varepsilon}_{1x}\|^2_{L^4}\\
&\leq C\varepsilon^2\|R^{\varepsilon}_1\|_{L^2}\|R^{\varepsilon}_1\|^{\frac{3}{4}}_{L^2}\|R^{\varepsilon}_{1xx}\|^{\frac{5}{4}}_{L^2}\\
&\leq\frac{1}{8}\varepsilon^2\|R^{\varepsilon}_{1xx}\|^2_{L^2}+C\varepsilon^2\|R^{\varepsilon}_1\|^{\frac{14}{3}}_{L^2}.
\end{aligned} \tag{4.24}$$

Using Hölder's inequality and (4.11), one gets

$$\begin{aligned}
II_7&\leq\varepsilon\|R^{\varepsilon}_1\|_{L^2}\|v^a\|_{L^\infty}\|R^{\varepsilon}_{1xx}\|_{L^2}\\
&\leq C\varepsilon\|R^{\varepsilon}_1\|_{L^2}\|R^{\varepsilon}_{1xx}\|_{L^2}\\
&\leq\frac{1}{8}\varepsilon^2\|R^{\varepsilon}_{1xx}\|^2_{L^2}+C\|R^{\varepsilon}_1\|^2_{L^2}.
\end{aligned} \tag{4.25}$$

Similarly, the term II_8 satisfies

$$II_8\leq\frac{1}{8}\varepsilon^2\|R^{\varepsilon}_{1xx}\|^2_{L^2}+2\varepsilon^{-2}\|f^{\varepsilon}\|^2_{L^2}. \tag{4.26}$$

Then, substituting all the estimates above into (4.21), one gets

$$\frac{d}{dt}(\varepsilon\|R_{1x}^{\varepsilon}\|_{L^2}^2)+\varepsilon^2\|R_{1xx}^{\varepsilon}\|_{L^2}^2 \le C\varepsilon\|R_{1x}^{\varepsilon}\|_{L^2}^2 + C(\|R_1^{\varepsilon}\|_{L^2}^2+\varepsilon^2\|R_1^{\varepsilon}\|_{L^2}^{\frac{14}{3}}) + 4\varepsilon^{-2}\|f^{\varepsilon}\|_{L^2}^2, \quad (4.27)$$

which along with Gronwall's inequality and Lemmas 4.1-4.2 leads to (4.20). The proof of Lemma 4.3 is completed. □

Proof of Theorem 2.1. It remains to prove the solution convergence rate (2.18). By Lemmas 4.2-4.3, we have

$$\|R_1^{\varepsilon}\|_{L^\infty(0,T;L^2)} \le C\varepsilon^{-\frac{1}{4}}, \ \|R_{1x}^{\varepsilon}\|_{L^\infty(0,T;L^2)} \le C\varepsilon^{-\frac{3}{4}}. \quad (4.28)$$

It then follows from (4.6) and (4.28) that

$$\|R_1^{\varepsilon}\|_{L^\infty(0,T;L^\infty)} \le \sqrt{2}\|R_1^{\varepsilon}\|_{L^\infty(0,T;L^2)}^{\frac{1}{2}}\|R_{1x}^{\varepsilon}\|_{L^\infty(0,T;L^2)}^{\frac{1}{2}} \le C\varepsilon^{-\frac{1}{2}}. \quad (4.29)$$

Thus, by (4.1), (4.2), (4.29), we have

$$\|v^{\varepsilon}-v^{I,0}-v^{B,0}\|_{L^\infty(0,T;L^\infty)} \le C\varepsilon^{\frac{1}{2}}, \quad (4.30)$$

which completes the proof of Theorem 2.1. □

Data availability

No data was used for the research described in the article.

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