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Monotone reducing mechanism in delayed population model with degenerate diffusion

Ming Mei [a,b], Tianyuan Xu [c,*], Jingxue Yin [d]

[a] Department of Mathematics, Champlain College Saint-Lambert, Quebec, J4P 3P2, Canada
[b] Department of Mathematics and Statistics, McGill University, Montreal, Quebec, H3A 2K6, Canada
[c] School of Mathematics and Statistics, Guangdong University of Technology, Guangzhou, Guangdong, 510520, PR China
[d] School of Mathematical Sciences, South China Normal University, Guangzhou, Guangdong, 510631, PR China

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Abstract

In this paper, we study the monotone reducing mechanism of degenerate diffusion equations with time delay. We show the monotone dependence of critical wave speed on time delay. Due to the complex dynamics arising from degeneracy and time delay, we use a new phase transform approach to analyze the delicate local and global behaviors of critical sharp traveling waves and then derive the comparison of critical speeds.

Keywords: Time delay; Degenerate diffusion; Critical wave speed; Reducing mechanism

1. Introduction

We consider the model of degenerate diffusion equation with time delay that describe the population dynamics of single species with age-structure and density-dependent diffusion. Precisely, the equation is given by

$$\frac{\partial u}{\partial t} = D(u^m)_{xx} - d(u) + b(u(t-r,x)), \quad x \in \mathbb{R}, \ t > 0. \tag{1.1}$$

* Corresponding author.
E-mail address: tyxu93@163.com (T. Xu).

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Here, u denotes the total mature population of the species at space x and time $t > 0$, time delay $r \geq 0$ represents the maturation time, $b(u(t-r,x))$ and $d(u)$ denote the birth rate and the death rate respectively. The degenerate diffusion $D(u^m)_{xx}$ with $m > 1$ models the density dependent dispersal processes due to the population pressure [16,26] possessing the finite speed of propagation property. The reaction term is of Fisher-KPP type such that: there exists two equilibria $u_- = 0$ and $u_+ := K > 0$, $d(0) = b(0) = 0$, $d(u_+) = b(u_+)$, $b(u) > d(u)$ for $u \in (0, u_+)$, $b'(0) > d'(0) \geq 0$, $d'(u_+) > b'(u_+) \geq 0$, and $d'(u) > 0$, $b'(u) > 0$ for $u \in (0, u_+)$. This reaction term includes the Nicholson's blowflies equation, the Mackey-Glass equation, and the delayed Fisher-KPP equations derived by [18].

Porous-medium type degenerate diffusion has been observed in ecology [16,26]. Dispersal velocities increase with population density arising from an anti-crowding mechanism and species migrate from high density areas to sparse areas [8]. Random diffusion equations possess the infinite spreading speed property. However, the solutions of degenerate diffusion equations exhibiting sharp edges can give a realistic description of a moving cohesive swarm of individuals with sharp edges, as observed in fish schooling [7] and birds flocking [25].

When $m = 1$, the equation (1.1) reduces to the random diffusion case. The monotonicity of critical wave speed c^* with respect to time delay r was proved by Wei-Wu-Mei [33,34] and Trofimchuk in [30]. Further, Gomez and Trofimchuk [15] established efficient existence criteria for monotone traveling fronts by a variant of the Hale-Lin functional-analytic approach. For the reaction-diffusion equation with nonlocal delay, Wang-Li-Ruan [32] showed that the delay can slow the critical wave speed and the nonlocality can increase the speed. While for the case $m > 1$, (1.1) is degenerate in diffusion. Note that the critical wave speed is nonlinear speed determinacy due to the degenerate diffusion. Recently, the authors in [37] proved that the linear wave speed c_0 determined by the linearizing of (1.1), i.e. characteristic equation, equals to zero; while the critical wave speed of (1.1) is nonlinear selected such that $c^* > c_0 = 0$. The dependence of critical wave speed c^* on time delay can not be calculated by analyzing the characteristic equations directly. Via the variational approach, the authors in [38] further proved that

$$c^*(m, D, r) < c^*(m, D, 0), \quad \forall r > 0, \tag{1.2}$$

which says that the time delay slows down the critical wave speed. But it is not clear how the dependence of the critical speed on different time delays.

This paper can be regarded as a continuity of our previous studies [37,38]. The goal of the present paper is to prove the strictly monotone dependence of the critical wave speed on time delay. Our main results show that time delay r monotonically slows down the minimal admissible wave speed of traveling waves. A new phase transform approach is proposed for the proof.

Theorem 1.1. *The critical wave speed $c^*(m, D, r)$ is strictly monotone decreasing with respect to the time delay r, i.e.,*

$$c^*(m, D, r_2) < c^*(m, D, r_1), \quad \forall r_2 > r_1. \tag{1.3}$$

Here, we focus on the sharp type traveling waves for this degenerate diffusion equations with time delay. It is shown in [38] that there exists a unique critical wave speed $c^* = c^*(m, D, r) > 0$ such that (1.1) admits a unique (up to shift) sharp traveling wave $\phi_{c^*}(x + c^*t)$ satisfying:

(i) $\phi_{c^*}(\xi)$ is monotone increasing, $0 \leq \phi_{c^*}(\xi) \leq K$, $\phi_{c^*}(-\infty) = 0$, $\phi_{c^*}(+\infty) = K$;

(ii) $\phi_{c^*} \in C(\mathbb{R})$, $\phi_{c^*}^m \in W_{\mathrm{loc}}^{1,2}$, and ϕ_{c^*} satisfies the following differential equation in the weak sense

$$c^*\phi_{c^*}'(\xi) = D(\phi_{c^*}^m(\xi))'' - d(\phi_{c^*}(\xi)) + b(\phi_{c^*}(\xi - c^* r)), \quad \xi \in \mathbb{R}; \tag{1.4}$$

(iii) $\phi_{c^*}(\xi)$ is semi-compact, i.e., there exists $\xi_0 \in \mathbb{R}$, such that $\phi_{c^*}(\xi) \equiv 0$ for $\xi \le \xi_0$ and $\phi_{c^*}(\xi) > 0$ for $\xi > \xi_0$.

Moreover, the traveling waves $\phi_c(x + ct)$ exist if and only if the corresponding wave speed $c \ge c^*$, and for $c > c^*$ the traveling waves are not semi-compact, instead they are positive for all $\xi \in \mathbb{R}$. Note that the critical wave speed c^* is also the asymptotic propagation speed as proved in [36]. The monotone dependence of the critical wave speed implies the monotone dependence of the propagation speed on the time delay, which coincides with the model backgrounds.

For the classical Fisher-KPP equations (i.e. $r = 0$ and $m = 1$), there exists a critical wave speed $c^* = 2\sqrt{b'(0) - d'(0)} > 0$ for all the traveling waves connecting the two constant equilibria [12,19]. Reaction-diffusion models with delayed response are a widely studied topic, we refer to the works of So-Wu-Zou [28], Thieme-Zhao [29], Liang-Zhao [21], Faria-Huang-Wu [11], Gomez-Trofimchuk [14,15], Alfaro-Ducrot-Giletti [1], Li-Ruan-Wang [20], Ma [22], Mei-Lin-Lin-So [24], Chern-Mei-Zhang-Yang [9] and the references therein. When $m > 1$, $r = 0$, (1.1) is usually called the Porous Medium Equation with logistic source and has been widely studied in the literature, we refer the reader to the book [31,35] for general references on this topic and to [2,6,10,13,23,27] for the study of traveling wave solutions. When $r > 0, m > 1$, there has, however, been relatively rare researches on the traveling wave solutions of degenerate diffusion equations with time delay. In the recent paper [17], Huang-Jin-Mei-Yin proved the existence and stability of traveling waves for (1.1) with small delays and sufficiently fast wave speeds. In [37,38], the authors proved that (1.1) admits a unique critical sharp type (semi-compactly supported) traveling wave $\phi(x + c^*t)$ corresponding to the critical wave speed $c^* = c^*(m, r)$. Moreover, it was proved in [39] there exist sharp-oscillatory traveling waves for some special wave speed and time delay. For the study of traveling waves for more general cases of degenerate doubly nonlinear diffusion equations, we refer the readers to [3–5].

Different from the linear diffusion case where the critical wave speed can be determined by the characteristic equations directly, the critical wave speed for delayed degenerate diffusion equations via variation approach satisfies

$$\begin{aligned} c^*(m, D, r) =& 2\sqrt{D}\int_0^K \sqrt{-ms^{m-1}\hat{g}(s)\hat{g}'(s)(b(s) - d(s))}\mathrm{d}s \\ & - \int_0^K \hat{g}(\phi)\frac{Dm\phi^{m-1}(b(\phi) - b(\tilde{\phi}_{cr}(\phi)))}{\tilde{\psi}}\mathrm{d}\phi, \end{aligned} \tag{1.5}$$

from which it follows that $c^*(m, D, r) < c^*(m, D, 0)$ for any $r > 0$, however, we are not able to derive the monotone dependence (1.3) according to the above variational characteristic expression. Here, $\hat{g}$ is a special function in $\mathscr{D}$, $\mathscr{D}$ is the subset of $C^1[0, K]$ with functions $g(K) = 0$, $\int_0^K g(s)\mathrm{d}s = 1$, $g'(s) < 0$, and more specifically, $\tilde{\psi}$ is the phase function depending on all the parameters, $\tilde{\phi}_{cr}(\phi)$ is an interpretation of the delayed term $\phi(\xi - cr)$ in the generalized phase plane. Alternatively, we are able to come back to the phase transform method and formulate the comparison of local properties for waves with different speeds and time delays, and finally, the comparison of critical wave speeds is derived by the combination of global properties of sharp waves.

We address here how the current work is different from previous works:

• The similar conclusions that time delay slows down the critical wave speed are already known for linear diffusion ($m = 1$) case, see for example the works by Zou [40], Li-Ruan-Wang [20], and also Wang-Li-Ruan [32]. However, all these critical speeds c^* are determined by the characteristic equation $\Delta_c(\lambda)$, which is the linearized problem near the zero equilibrium, see Lemma 3.5 and Lemma 3.14 in [20], Lemma 2.2 in [32], Remark 2.2 in [40]. As shown by the works [37], [38], and [39], for degenerate diffusion ($m > 1$) case, in which the linearization of the second order differential term Δu^m vanishes near $u = 0$, the characteristic equation does not tell us any information on the critical wave speed, and further to some extent it indicates the difficulties caused by degenerate diffusion. In fact, the critical wave speed c^* is determined in a more complicated fashion, and the verification of monotone reducing mechanism in this paper requires new framework.

• The previous work [38] on the critical wave speed c^* for degenerate diffusion ($m > 1$) case formulated a variational inequality (1.5) and further deduced the comparison of critical speeds between delayed case and non-delayed case. However, it fails to compare the critical speeds for different time delays. Thanks to the delicate structure of delayed degenerate diffusion equation such that the degeneracy (the support of the critical wave is semi-compact) and the time delay (the evolution of Cauchy problem can be regarded as propagating step by step), we can describe the sharp wave by its local property and global property. Together with the precise comparison of different waves via a phase transform method, we then show the monotone reducing mechanism for all delays.

Our proofs of monotone dependence of critical wave speed on time delay rely on a new phase transform method. The classical phase plane analysis does not work now due to the infinite dimension of phase spaces in the time-delayed cases. By a delicate phase transform, we establish the comparison of local properties for waves with different speeds and time delays, and analyze the global properties of sharp waves to get the monotone dependence of critical wave speed on the time delay. In the linear diffusion case, monotone dependence of critical wave speed on time delay can be determined by the characteristic equation; however, due to the increased complexity of degenerate diffusion equations, the waves speed is nonlinearly selected for degenerate diffusion equations, we are unable to establish the dependence of wave speed on time delay by a simple characteristic equation. To overcome these difficulties, we give a detailed examination of the construction of sharp traveling waves and precisely analyze the local property and global property of critical sharp waves.

2. Proof of the main result

The idea of the proof lies in the examination of the construction of sharp traveling waves based on the local property and global property. To be more specifically, the unique traveling wave $\phi_{c^*}(x + c^*t)$ and the unique critical wave speed c^* for degenerate diffusion equation (1.1) have the following properties besides the differential equation (1.4):

(i) local property: there exists $\xi_0 \in \mathbb{R}$, such that $\phi_{c^*}(\xi) \equiv 0$ for $\xi \le \xi_0$ and $\phi_{c^*}(\xi) > 0$ for $\xi > \xi_0$;

(ii) global property: $\phi_{c^*}(\xi)$ is monotone increasing and $\phi_{c^*}(+\infty) = K$.

Without loss of generality, we may always shift ξ_0 to 0 in the following. The local property (i) together with the time-delayed structure of (1.4) implies that locally for $\xi \in (0, c^*r)$, $\phi_{c^*}(\xi)$ satisfies

$$\begin{cases} c^*\phi'_{c^*}(\xi) = D(\phi^m_{c^*}(\xi))'' - d(\phi_{c^*}(\xi)), \quad \xi \in (0, c^*r), \\ \phi_{c^*}(0) = 0, \quad \phi_{c^*}(\xi) > 0, \ \xi \in (0, c^*r). \end{cases} \tag{2.1}$$

As proved in [38], the problem (2.1) has singularity and the solutions are not unique if without the condition $\phi_{c^*}(\xi) > 0$ for $\xi \in (0, c^*r)$. Actually, $\phi_{c^*}(\xi)$ is the unique maximal solution and

$$\phi_{c^*}(\xi) = \Big(\frac{(m-1)c^*}{Dm}\xi\Big)^{\frac{1}{m-1}} + o(\xi^{\frac{1}{m-1}}), \quad \xi \to 0^+. \tag{2.2}$$

The local property (i) and the local equation (2.1) do not tell us the value of c^*. In fact, for any indefinite speed $c > 0$, the local equation (2.1) admits a maximal solution $\phi_c(x + ct)$ on $(0, cr)$ and then on $(cr, 2cr)$, according to the time-delayed structure again such that $\phi_c(\xi - cr)$ is already known, we can solve the traveling wave equation (1.4) locally as

$$\begin{cases} c\phi'_c(\xi) = D(\phi^m_c(\xi))'' - d(\phi_c(\xi)) + b(\phi_c(\xi - cr)), \quad \xi \in (cr, 2cr), \\ \phi_c(cr) \text{ and } \phi'_c(cr) \text{ are determined from the left hand side.} \end{cases} \tag{2.3}$$

Similarly, on $(2cr, 3cr)$, $(3cr, 4cr)$, $\dots$, we can solve and extend the traveling wave equation (1.4) step by step until $\phi_c(\xi)$ decays to zero or blows up. The local solution $\phi_c(\xi)$ will also be denoted as $\phi_{c,r}(\xi)$ in order to emphasize the time delay.

Among all the local solutions $\{\phi_c(\xi)\}$ corresponding to different wave speeds $c > 0$, the sharp wave $\phi_{c^*}(\xi)$ is the unique one such that it satisfies the global property (ii); meanwhile, $c^* > 0$ is the unique speed such that $\phi_{c^*}(\xi)$ exists globally and $\phi_{c^*}(+\infty) = K$. Therefore, the combination of the local property (i) and the global property (ii) gives rise to the construction of the sharp traveling wave $\phi_{c^*}(\xi)$. Henceforth, we compare the local solutions corresponding to different speeds or time delays step by step and formulate the comparison of critical wave speeds.

For the sake of simplicity, we denote

$$c_1 := c^*(m, D, r_1) \quad \text{and} \quad c_2 := c^*(m, D, r_2)$$

for $r_2 > r_1 > 0$, and further

$$\phi_1(\xi) := \phi_{c_1}(\xi) = \phi_{c^*(m,D,r_1)}(\xi), \quad \phi_2(\xi) := \phi_{c_2}(\xi) = \phi_{c^*(m,D,r_2)}(\xi),$$

such that $\phi_{c_i}(x + c_i t)$ is the unique sharp traveling wave corresponding to the unique critical wave speed $c^*(m, D, r_i)$ for time delay r_i with $i = 1, 2$.

In order to prove that $c_2 < c_1$ for $r_2 > r_1$, it suffices to show that, for the time delay r_2, the local solution with speed c_1 constructed step by step as in the procedures (2.1), (2.3) and so on, denoted by $\phi_{c_1,r_2}(\xi)$, exceeds K as ξ tends to positive infinity. We state the following auxiliary lemma.

Lemma 2.1. *Let $\phi_{c_1,r_2}(\xi)$ be the local solution corresponding to the speed $c_1 = c^*(m, D, r_1)$ and time delay r_2. If $\phi_{c_1,r_2}(+\infty) > K$, or equivalently (note that ϕ_{c_1,r_2} is nondecreasing), there exists $\xi_1 > 0$ such that $\phi_{c_1,r_2}(\xi_1) > K$, then $c^*(m, D, r_2) < c^*(m, D, r_1)$.*

Proof. For the time delay r_2, the local solution $\phi_{c_2,r_2}(\xi)$ is the unique one such that $\phi_{c_2,r_2}(\xi)$ is strictly monotone increasing on $[0, +\infty)$ and $\phi_{c_2,r_2}(+\infty) = K$ since $c_2 = c^*(m, D, r_2)$ is the unique critical wave speed. If the local solution $\phi_{c_1,r_2}(\xi)$ with speed c_1 exceeds K as ξ tends to positive infinity, then the monotone dependence of the local solutions $\phi_{c,r_2}(\xi)$ on different speeds c (Lemma 3.6 of [38]) tells us

$$\phi_{c_1,r_2}(\xi) > \phi_{c_2,r_2}(\xi) \Leftrightarrow c_1 > c_2.$$

The proof is completed. □

Lemma 2.1 shows that the comparison of the different behaviors of local solutions with different speeds for the same time delay implies the comparison of wave speeds. On the other hand, we compare the local solutions with different delays for the same speed, which is the key observation of the proof for the monotone dependence property.

Lemma 2.2. *Let $\phi_{c_1,r_1}(\xi)$ and $\phi_{c_1,r_2}(\xi)$ be the local solutions corresponding to the time delays $r_1 < r_2$ respectively but with the same speed $c_1 = c^*(m, D, r_1)$. Then $\phi_{c_1,r_2}(\xi) \geq \phi_{c_1,r_1}(\xi)$ for $\xi \in (0, \hat{\xi})$, where, $(0, \hat{\xi})$ is the maximal existence interval of $\phi_{c_1,r_2}(\xi)$, and $\phi_{c_1,r_1}(\xi)$ is the sharp traveling wave for time delay r_1 that exists globally. Moreover, the inequality is strict, $\phi_{c_1,r_2}(\xi) > \phi_{c_1,r_1}(\xi)$ for $\xi \in (c_1 r_1, \hat{\xi})$.*

Proof. According to the construction of local solutions as in (2.1), $\phi_{c_1,r_1}(\xi)$ and $\phi_{c_1,r_2}(\xi)$ satisfy

$$\begin{cases} c_1 \phi'_{c_1,r_1}(\xi) = D(\phi^m_{c_1,r_1}(\xi))'' - d(\phi_{c_1,r_1}(\xi)), & \xi \in (0, c_1 r_1], \\ \phi_{c_1,r_1}(0) = 0, \quad \phi_{c_1,r_1}(\xi) > 0, & \xi \in (0, c_1 r_1], \end{cases} \tag{2.4}$$

and

$$\begin{cases} c_1 \phi'_{c_1,r_2}(\xi) = D(\phi^m_{c_1,r_2}(\xi))'' - d(\phi_{c_1,r_2}(\xi)), & \xi \in (0, c_1 r_2], \\ \phi_{c_1,r_2}(0) = 0, \quad \phi_{c_1,r_2}(\xi) > 0, & \xi \in (0, c_1 r_2], \end{cases} \tag{2.5}$$

respectively. Note that $r_2 > r_1$ such that $c_1 r_2 > c_1 r_1$. Then in $(c_1 r_1, 2c_1 r_1)$, according to (2.3), $\phi_{c_1,r_1}(\xi)$ satisfies

$$\begin{cases} c_1 \phi'_{c_1,r_1}(\xi) = D(\phi^m_{c_1,r_1}(\xi))'' - d(\phi_{c_1,r_1}(\xi)) + b(\phi_{c_1,r_1}(\xi - c_1 r_1)), \ \xi \in (c_1 r_1, 2c_1 r_1), \\ \phi_{c_1,r_1}(c_1 r_1) \text{ and } \phi'_{c_1,r_1}(c_1 r_1) \text{ are given by (2.4).} \end{cases} \tag{2.6}$$

While in $(c_1 r_2, 2c_1 r_2)$, $\phi_{c_1,r_2}(\xi)$ satisfies

$$\begin{cases} c_1 \phi'_{c_1,r_2}(\xi) = D(\phi^m_{c_1,r_2}(\xi))'' - d(\phi_{c_1,r_2}(\xi)) + b(\phi_{c_1,r_2}(\xi - c_1 r_2)), \ \xi \in (c_1 r_2, 2c_1 r_2), \\ \phi_{c_1,r_2}(c_1 r_2) \text{ and } \phi'_{c_1,r_2}(c_1 r_2) \text{ are given by (2.5).} \end{cases} \tag{2.7}$$

The problems (2.4) and (2.5) imply that both $\phi_{c_1,r_2}(\xi)$ and $\phi_{c_1,r_1}(\xi)$ are the maximal solution of the same equation on the joint interval $(0, c_1r_1)$, such that

$$\phi_{c_1,r_2}(\xi) \equiv \phi_{c_1,r_1}(\xi), \quad \xi \in (0, c_1r_1).$$

We assert that

$$\phi_{c_1,r_2}(\xi) > \phi_{c_1,r_1}(\xi), \quad \xi \in (c_1r_1, \min\{c_1r_2, 2c_1r_1\}).$$

Actually, on $(c_1r_1, \min\{c_1r_2, 2c_1r_1\})$, $\phi_{c_1,r_1}(\xi)$ satisfies (2.6) while $\phi_{c_1,r_2}(\xi)$ satisfies (2.5) with the same conditions at c_1r_1:

$$\phi_{c_1,r_1}(c_1r_1) = \phi_{c_1,r_2}(c_1r_1), \quad \phi'_{c_1,r_1}(c_1r_1) = \phi'_{c_1,r_2}(c_1r_1), \tag{2.8}$$

and the only difference is the reaction term $b(\phi_{c_1,r_1}(\xi - c_1r_1))$ in (2.6). Note that $\phi_{c_1,r_1}(\xi - c_1r_1) > 0$ since $\xi - c_1r_1 > 0$ for $\xi \in (c_1r_1, \min\{c_1r_2, 2c_1r_1\})$, and then $b(\phi_{c_1,r_1}(\xi - c_1r_1)) > 0$, which behaves as a damping term for the second order differential equation (2.6). The phase plane analysis of (2.5) and the generalized phase plane analysis of (2.6) implies that

$$\phi_{c_1,r_2}(\xi) > \phi_{c_1,r_1}(\xi) \text{ for } \xi \in (c_1r_1, \min\{c_1r_2, 2c_1r_1\}). \tag{2.9}$$

Here, the generalized phase plane means that we need to rewrite the time-delayed term $b(\phi_{c_1,r_1}(\xi - c_1r_1))$ into a function of ϕ_{c_1,r_1} and the phase function $\tilde{\psi}_{c_1,r_1}(\phi_{c_1,r_1})$, which is the function

$$\psi_{c_1,r_1}(\xi) := D(\phi^m_{c_1,r_1}(\xi))' = Dm\phi^{m-1}_{c_1,r_1}(\xi) \cdot \phi'_{c_1,r_1}(\xi) \tag{2.10}$$

interpreted as a function of ϕ_{c_1,r_1} in the phase plane such that

$$\tilde{\psi}_{c_1,r_1}(\phi_{c_1,r_1}) := \psi_{c_1,r_1}(\xi).$$

Furthermore, the key ingredient is to rewrite the time-delayed term into

$$\phi_{c_1,r_1;c_1r_1}(\phi_{c_1,r_1}; \tilde{\psi}_{c_1,r_1}) := \phi_{c_1,r_1}(\xi - c_1r_1) = \inf\left\{\theta > 0;\ \int_\theta^{\phi_{c_1,r_1}} \frac{Dms^{m-1}}{\tilde{\psi}_{c_1,r_1}(s)} \mathrm{d}s \le c_1r_1\right\}. \tag{2.11}$$

We explain more details about the expression (2.11): (i) if $\phi_{c_1,r_1}(\xi - c_1r_1) > 0$, then $\phi_{c_1,r_1}(\xi) > \phi_{c_1,r_1}(\xi - c_1r_1)$ and

$$c_1r_1 = \xi - (\xi - c_1r_1) = \int_{\phi_{c_1,r_1}(\xi - c_1r_1)}^{\phi_{c_1,r_1}(\xi)} \frac{\mathrm{d}\xi}{\mathrm{d}\phi} \mathrm{d}\phi = \int_{\phi_{c_1,r_1}(\xi - c_1r_1)}^{\phi_{c_1,r_1}(\xi)} \frac{Dms^{m-1}}{\tilde{\psi}_{c_1,r_1}(s)} \mathrm{d}s,$$

such that

$$\phi_{c_1,r_1}(\xi - c_1r_1) = \min\left\{\theta > 0;\ \int_\theta^{\phi_{c_1,r_1}} \frac{Dms^{m-1}}{\tilde{\psi}_{c_1,r_1}(s)}\mathrm{d}s \le c_1r_1\right\};$$

(ii) if $\phi_{c_1,r_1}(\xi - c_1r_1) = 0$, we only focus on the case $\phi_{c_1,r_1}(\xi) > 0$, then there exists a $\tau_0 \in (0, c_1r_1]$ such that $\phi_{c_1,r_1}(\hat{\xi}) > 0$ for all $\hat{\xi} \in (\xi - \tau_0, \xi]$ and $\phi_{c_1,r_1}(\hat{\xi}) = 0$ for all $\hat{\xi} \in [\xi - c_1r_1, \xi - \tau_0]$, i.e., $\xi - \tau_0$ is the edge of the support. Therefore,

$$\tau_0 = \xi - (\xi - \tau_0) = \int_{\phi_{c_1,r_1}(\xi-\tau_0)}^{\phi_{c_1,r_1}(\xi)} \frac{\mathrm{d}\xi}{\mathrm{d}\phi}\mathrm{d}\phi = \int_0^{\phi_{c_1,r_1}(\xi)} \frac{Dms^{m-1}}{\tilde{\psi}_{c_1,r_1}(s)}\mathrm{d}s,$$

and $\int_\theta^{\phi_{c_1,r_1}} \frac{Dms^{m-1}}{\tilde{\psi}_{c_1,r_1}(s)}\mathrm{d}s \le c_1r_1$ for all $\theta > 0$, hence

$$\inf\left\{\theta > 0;\ \int_\theta^{\phi_{c_1,r_1}} \frac{Dms^{m-1}}{\tilde{\psi}_{c_1,r_1}(s)}\mathrm{d}s \le c_1r_1\right\} = 0 = \phi_{c_1,r_1}(\xi - c_1r_1).$$

The notations ψ_{c_1,r_2}, $\tilde{\psi}_{c_1,r_2}$, and $\phi_{c_1,r_2;c_1r_2}(\phi_{c_1,r_2}; \tilde{\psi}_{c_1,r_2})$ are defined similarly. Here we note that the infimum in (2.11) is attained at a unique $\phi_{c_1,r_1;c_1r_1} > 0$ if $\phi_{c_1,r_1}(\xi) > \phi_{c_1,r_1}(c_1r_1)$ such that $\phi_{c_1,r_1}(\xi - c_1r_1) > 0$; while we have $\phi_{c_1,r_1;c_1r_1} = 0$ if $\phi_{c_1,r_1}(\xi) \le \phi_{c_1,r_1}(c_1r_1)$ and then $\phi_{c_1,r_1}(\xi - c_1r_1) = 0$. Hence the interpretation (2.11) is valid in all cases.

For $\xi \in (\min\{c_1r_2, 2c_1r_1\}, \hat{\xi})$, we employ the above generalized phase plane analysis method. According to the first step comparison (2.9) and the equality (2.8), we see that locally at a right neighborhood of c_1r_1, denoted by $(c_1r_1, \hat{\xi}_0)$, there holds

$$\tilde{\psi}_{c_1,r_2}(\phi) > \tilde{\psi}_{c_1,r_1}(\phi). \tag{2.12}$$

Without loss of generality, we assume that $(c_1r_1, \hat{\xi}_0)$ is the maximal interval with the above comparison property. We assert that within the interval $(c_1r_1, \hat{\xi}_0)$

$$\phi_{c_1,r_2;c_1r_2}(\phi_{c_1,r_2}; \tilde{\psi}_{c_1,r_2}) < \phi_{c_1,r_1;c_1r_1}(\phi_{c_1,r_1}; \tilde{\psi}_{c_1,r_1}) \tag{2.13}$$

at where $\phi_{c_1,r_2} = \phi_{c_1,r_1}$. In other words, if for some $\xi_1, \xi_2 \in (c_1r_1, \hat{\xi}_0)$ such that $\phi_{c_1,r_2}(\xi_2) = \phi_{c_1,r_1}(\xi_1)$, then

$$\phi_{c_1,r_2}(\xi_2 - c_1r_2) < \phi_{c_1,r_1}(\xi_1 - c_1r_1).$$

Therefore, (2.13) provides the comparison of the time-delayed terms between solutions corresponding to different delays.

The proof of (2.13) lies in the interpretation (2.11) such that

$$\int_{\phi_{c_1,r_1;c_1r_1}}^{\phi_{c_1,r_1}} \frac{Dms^{m-1}}{\tilde{\psi}_{c_1,r_1}(s)}\mathrm{d}s = c_1r_1,$$

and

$$\int_{\phi_{c_1,r_2;c_1r_2}}^{\phi_{c_1,r_2}} \frac{Dms^{m-1}}{\tilde{\psi}_{c_1,r_2}(s)}\mathrm{d}s = c_1r_2.$$

Observe that $r_2 > r_1$ and $\tilde{\psi}_{c_1,r_2}(s) \geq \tilde{\psi}_{c_1,r_1}(s)$, then $\phi_{c_1,r_2;c_1r_2} < \phi_{c_1,r_1;c_1r_1}$ if $\phi_{c_1,r_2} = \phi_{c_1,r_1}$. The comparison of time-delayed terms (2.13) plays an essential role in the following proof.

According to the definition (2.10) and the interpretation (2.11), $(\phi_{c_1,r_1}, \psi_{c_1,r_1})$ satisfies the following dynamic system

$$\begin{cases} \dfrac{\mathrm{d}\phi_{c_1,r_1}}{\mathrm{d}\xi} = \dfrac{\psi_{c_1,r_1}}{Dm\phi_{c_1,r_1}^{m-1}}, \\ \dfrac{\mathrm{d}\psi_{c_1,r_1}}{\mathrm{d}\xi} = c_1\dfrac{\psi_{c_1,r_1}}{Dm\phi_{c_1,r_1}^{m-1}} - \Big(b(\phi_{c_1,r_1;c_1r_1}) - d(\phi_{c_1,r_1})\Big). \end{cases} \tag{2.14}$$

Next, we rewrite the system (2.14) such that $\tilde{\psi}_{c_1,r_1}(\phi_{c_1,r_1}) := \psi_{c_1,r_1}(\xi)$ satisfies the following differential equation

$$\frac{\mathrm{d}\tilde{\psi}_{c_1,r_1}}{\mathrm{d}\phi} = c_1 - \frac{Dm\phi^{m-1}(b(\phi_{c_1,r_1;c_1r_1}) - d(\phi))}{\tilde{\psi}_{c_1,r_1}}. \tag{2.15}$$

Similarly, we have

$$\frac{\mathrm{d}\tilde{\psi}_{c_1,r_2}}{\mathrm{d}\phi} = c_1 - \frac{Dm\phi^{m-1}(b(\phi_{c_1,r_2;c_1r_2}) - d(\phi))}{\tilde{\psi}_{c_1,r_2}}. \tag{2.16}$$

Observing the comparison of time-delayed terms in (2.13), according to (2.16), we obtain

$$\frac{\mathrm{d}\tilde{\psi}_{c_1,r_2}}{\mathrm{d}\phi} > c_1 - \frac{Dm\phi^{m-1}(b(\phi_{c_1,r_1;c_1r_1}) - d(\phi))}{\tilde{\psi}_{c_1,r_2}}. \tag{2.17}$$

Thus, (2.15) and (2.17) imply that

$$\frac{\mathrm{d}\tilde{\psi}_{c_1,r_2}}{\mathrm{d}\phi} - \frac{\mathrm{d}\tilde{\psi}_{c_1,r_1}}{\mathrm{d}\phi} > \Big(Dm\phi^{m-1}(b(\phi_{c_1,r_1;c_1r_1}) - d(\phi))\Big)\cdot\frac{\tilde{\psi}_{c_1,r_2} - \tilde{\psi}_{c_1,r_1}}{\tilde{\psi}_{c_1,r_2}\tilde{\psi}_{c_1,r_1}}. \tag{2.18}$$

Finally, since ϕ_{c_1,r_1} is the unique sharp traveling wave for time delay r_1 with the unique speed $c_1 = c^*(m, D, r_1)$, we see that ϕ_{c_1,r_1} is strictly monotone increasing and then $\tilde{\psi}_{c_1,r_1}(\phi) > 0$ for all $\phi \in (0, K)$. Therefore, (2.12) tells us that $\tilde{\psi}_{c_1,r_2}(\phi) > \tilde{\psi}_{c_1,r_1}(\phi) > 0$. Then locally at any closed subinterval of $(0, K)$, both $\tilde{\psi}_{c_1,r_2}$ and $\tilde{\psi}_{c_1,r_1}$ have positive infimum. Also note that $Dm\phi^{m-1}(b(\phi_{c_1,r_1;c_1r_1}) - d(\phi))$ in (2.18) is uniformly bounded and $\tilde{\psi}_{c_1,r_2}(\phi) > \tilde{\psi}_{c_1,r_1}(\phi)$ locally near c_1r_1 as in (2.12), it follows from (2.18) that $\tilde{\psi}_{c_1,r_2}(\phi) > \tilde{\psi}_{c_1,r_1}(\phi)$ holds globally on $(c_1r_1, \hat{\xi}_0)$ such that $(0, \hat{\xi}_0)$ actually is the maximal existence interval $(0, \hat{\xi})$. The proof is completed. □

To finish the proof of Theorem 1.1, we enhance the comparison results in Lemma 2.2 as follows.

Lemma 2.3. *Under the conditions in Lemma 2.2, there also holds that the difference* $\phi_{c_1,r_2}(\xi) - \phi_{c_1,r_1}(\xi)$ *is increasing for* $\xi \in (0, \hat{\xi})$.

Proof. The conclusion follows from the proof of Lemma 2.2 with a further explanation. Actually, we have already proved that (2.12) is valid on $(c_1 r_1, \hat{\xi}_0)$ and then $\tilde{\psi}_{c_1,r_2}(\phi) \geq \tilde{\psi}_{c_1,r_1}(\phi)$ holds on all $(0, \hat{\xi}_0)$ in the proof of Lemma 2.2. It says that at any points that $\phi_{c_1,r_2}(\xi_2) = \phi_{c_1,r_1}(\xi_1) =: \phi$, the derivatives $\phi'_{c_1,r_2}(\xi_2) \geq \phi'_{c_1,r_1}(\xi_1)$ according to the comparison $\tilde{\psi}_{c_1,r_2}(\phi) \geq \tilde{\psi}_{c_1,r_1}(\phi)$ in the phase plane. Therefore, the difference $\phi_{c_1,r_2}(\xi) - \phi_{c_1,r_1}(\xi)$ is increasing for $\xi \in (0, \hat{\xi})$. □

Proof of Theorem 1.1. Note that $\phi_{c_1,r_1}(\xi)$ is the unique sharp traveling wave such that $\phi_{c_1,r_1}(+\infty) = K$. Lemma 2.2 implies that $\phi_{c_1,r_2}(+\infty) > \phi_{c_1,r_1}(+\infty) = K$ if $\phi_{c_1,r_2}(\xi)$ exists globally, otherwise $\phi_{c_1,r_2}(\xi)$ blows up in finite time and there exists $\xi_1 > 0$ such that $\phi_{c_1,r_2}(\xi_1) > K$. Note that Lemma 2.3 precludes the case that $\phi_{c_1,r_2}(\xi) > \phi_{c_1,r_1}(\xi)$ on $(c_1 r_1, +\infty)$ but $\phi_{c_1,r_2}(+\infty) = \phi_{c_1,r_1}(+\infty) = K$. According to Lemma 2.1, $c^*(m, D, r_2) < c^*(m, D, r_1)$ for $r_2 > r_1$. The proof is completed. □

Data availability

No data was used for the research described in the article.

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References

[1] M. Alfaro, A. Ducrot, T. Giletti, Travelling waves for a non-monotone bistable equation with delay: existence and oscillations, Proc. Lond. Math. Soc. 116 (2018) 729–759.

[2] D.G. Aronson, Density-dependent interaction-diffusion systems, in: Proc. Adv. Seminar on Dynamics and Modeling of Reactive System, Academic Press, New York, 1980.

[3] A. Audrito, Bistable reaction equations with doubly nonlinear diffusion, Discrete Contin. Dyn. Syst. 39 (2019) 2977–3015.

[4] A. Audrito, J.L. Vázquez, The Fisher-KPP problem with doubly nonlinear diffusion, J. Differ. Equ. 263 (2017) 7647–7708.

[5] A. Audrito, J.L. Vázquez, Travelling wave behaviour arising in nonlinear diffusion problems posed in tubular domains, J. Differ. Equ. 269 (2020) 2664–2696.

[6] R.D. Benguria, M.C. Depassier, A variational principle for the asymptotic speed of fronts of the density dependent diffusion-reaction equation, Phys. Rev. E 52 (1995) 3285–3287.

[7] C.M. Breder, Equations descriptive of fish schools and other animal aggregations, Ecology 35 (1954) 361–370.

[8] E.A. Carl, Population control in Arctic ground squirrels, Ecology 52 (1971) 395–413.

[9] I-L. Chern, M. Mei, Q. Zhang, X. Yang, Stability of non-monotone critical traveling waves for reaction-diffusion equations with time-delay, J. Differ. Equ. 259 (2015) 1503–1541.
[10] A. de Pablo, J.L. Vázquez, Travelling waves and finite propagation in a reaction-diffusion equation, J. Differ. Equ. 93 (1991) 19–61.
[11] T. Faria, W. Huang, J. Wu, Traveling waves for delayed reaction-diffusion equations with non-local response, Proc. R. Soc. Lond. A 462 (2006) 229–261.
[12] R.A. Fisher, The wave of advance of advantageous genes, Ann. Hum. Genet. 7 (1937) 353–369.
[13] B.H. Gilding, R. Kersner, A Fisher/KPP-type equation with density-dependent diffusion and convection: travelling-wave solutions, J. Phys. A 38 (2005) 3367–3379.
[14] A. Gomez, S. Trofimchuk, Monotone traveling wavefronts of the KPP-Fisher delayed equation, J. Differ. Equ. 250 (2011) 1767–1787.
[15] A. Gomez, S. Trofimchuk, Monotone traveling wavefronts of the KPP-Fisher delayed equation, J. Lond. Math. Soc. 89 (2014) 47–68.
[16] W.S.C. Gurney, R.M. Nisbet, The regulation of inhomogeneous population, J. Theor. Biol. 52 (1975) 441–457.
[17] R. Huang, C.H. Jin, M. Mei, J.X. Yin, Existence and stability of traveling waves for degenerate reaction-diffusion equation with time delay, J. Nonlinear Sci. 28 (2018) 1011–1042.
[18] K. Kobayashi, On the semilinear heat equation with time-lag, Hiroshima Math. J. 7 (1977) 459–472.
[19] A. Kolmogorov, I. Petrovskii, N. Piscounov, Étude de l'équation de la diffusion avec croissance de la quantite de matière et son application à un problème biologique, Bull. Univ. Moskow, Ser. Internat., Sec. A 1 (1937) 1–25.
[20] W.-T. Li, S. Ruan, Z.-C. Wang, On the diffusive Nicholson's blowflies equation with nonlocal delay, J. Nonlinear Sci. 17 (2007) 505–525.
[21] X. Liang, X.-Q. Zhao, Asymptotic speeds of spread and traveling waves for monotone semiflows with applications, Commun. Pure Appl. Math. 60 (2007) 1–40.
[22] S. Ma, Traveling waves for non-local delayed diffusion equations via auxiliary equations, J. Differ. Equ. 237 (2007) 259–277.
[23] G.S. Medvedev, K. Ono, P.J. Holmes, Travelling wave solutions of the degenerate Kolmogorov-Petrovski-Piskunov equation, Eur. J. Appl. Math. 14 (2003) 343–367.
[24] M. Mei, C.K. Lin, C.T. Lin, J.W.-H. So, Traveling wavefronts for time-delayed reaction-diffusion equation: (i) local nonlinearity, J. Differ. Equ. 247 (2009) 495–510.
[25] R.S. Miller, W. Stephen, Spatial relationships in flocks of sandhill cranes, Ecology 47 (1966) 323–327.
[26] J.D. Murray, Mathematical Biology I: An Introduction, Springer, New York, USA, 2002.
[27] F. Sánchez-Garduño, P.K. Maini, M.E. Kappos, A shooting argument approach to a sharp-type solution for nonlinear degenerate Fisher-KPP equations, IMA J. Appl. Math. 57 (1996) 211–221.
[28] J.W.-H. So, J. Wu, X. Zou, A reaction-diffusion model for a single species with age structure. I Travelling wavefronts on unbounded domains, Proc. R. Soc. Lond. A 457 (2001) 1841–1853.
[29] H.R. Thieme, X.-Q. Zhao, Asymptotic speeds of spread and traveling waves for integral equations and delayed reaction-diffusion models, J. Differ. Equ. 195 (2003) 430–470.
[30] E. Trofimchuk, S. Trofimchuk, Admissible wavefront speeds for a single species reaction-diffusion equation with delay, Discrete Contin. Dyn. Syst. 20 (2008) 407–423.
[31] J.L. Vázquez, The Porous Medium Equation: Mathematical Theory, Oxford Univ. Press, 2006.
[32] Z.-C. Wang, W.-T. Li, S.G. Ruan, Traveling fronts in monostable equations with nonlocal delayed effects, J. Dyn. Differ. Equ. 20 (2008) 573–607.
[33] D. Wei, J.-Y. Wu, M. Mei, Remark on critical speed of traveling wavefronts for Nicholson's blowflies equation with diffusion, Acta Math. Sci. 30B (2010) 1561–1566.
[34] J.-Y. Wu, D. Wei, M. Mei, Analysis on the critical speed of traveling waves, Appl. Math. Lett. 20 (2007) 712–718.
[35] Z. Wu, J. Zhao, J. Yin, H. Li, Nonlinear Diffusion Equations, World Scientific Publishing Co. Put. Ltd., 2001.
[36] T.Y. Xu, S.M. Ji, M. Mei, J.X. Yin, Propagation speed of degenerate diffusion equations with time delay, preprint, arXiv:2011.14813, 2021.
[37] T.Y. Xu, S.M. Ji, M. Mei, J.X. Yin, Traveling waves for time-delayed reaction diffusion equations with degenerate diffusion, J. Differ. Equ. 265 (2018) 4442–4485.
[38] T.Y. Xu, S.M. Ji, M. Mei, J.X. Yin, Variational approach of critical sharp front speeds in degenerate diffusion model with time delay, Nonlinearity 33 (2020) 4013–4029.
[39] T.Y. Xu, S.M. Ji, M. Mei, J.X. Yin, Sharp oscillatory traveling waves of structured population dynamics model with degenerate diffusion, J. Differ. Equ. 269 (2020) 8882–8917.
[40] X. Zou, Delay induced traveling wave fronts in reaction diffusion equations of KPP-Fisher type, J. Comput. Appl. Math. 146 (2002) 309–321.