Applied Mathematics Letters 154 (2024) 109075

Contents lists available at ScienceDirect

Applied Mathematics Letters

journal homepage: www.elsevier.com/locate/aml

Regular article

Stability of traveling wavefronts for advection–reaction–diffusion equation

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ARTICLE INFO

Keywords:
Traveling wavefronts
Stability
Advection–reaction–diffusion equation
Minimal wave speed

ABSTRACT

In this paper, we study an advection–reaction–diffusion equation, where the nonlinear advection has neither monotonicity nor variational structure. For all wavefronts with the speed $c > c_0$, where c_0 is the minimal wave speed, we use the technical weighted energy method to prove that these wavefronts are exponentially stable, when the initial perturbations are small in a weighted Sobolev space.

1. Introduction

In this paper, we focus on the stability of traveling wavefronts for the following an advection–reaction–diffusion equation in the form

$$\begin{cases} \frac{\partial u(x,t)}{\partial t} + \frac{\partial J(u(x,t))}{\partial x} = \frac{\partial^2 u(x,t)}{\partial x^2} + f(u(x,t)), & x \in \mathbb{R}, \quad t > 0 \\ u(x,0) = u_0(x), \quad x \in \mathbb{R}. \end{cases} \tag{1.1}$$

This model describes pattern formation and chemotaxis phenomena in biology, physiology, physics and chemistry, for example, see [1–4] and references. Here, $J(u)$ and $f(u)$ satisfy the following hypothesis

(**H1**) $f(0) = f(1) = 0, f'(0) > 0 > f'(1),\ f(u) > 0,$ for $u \in (0,1)$;
(**H2**) $f \in C^4(\mathbb{R},\mathbb{R}),\ f'(u) \le f'(0)$ for $u \in [0,1]$ and $f^{(i)}$ is bounded on $\mathbb{R}$ for $i = 1,2,\dots,4$;
(**H3**) $J \in C^5(\mathbb{R},\mathbb{R}),\ J''(u) \le 0$ for $u \in [0,1]$; and $J^{(k)}$ are bounded on $\mathbb{R}$ for $k = 1,2,\dots,5$.

When $J(u) = \frac{p}{2}u^2$ and $f(u) = u(1-u)$, the first equation in Eq. (1.1) is reduced to the Burgers–KPP–Fisher equation

$$\frac{\partial u(x,t)}{\partial t} + pu\frac{\partial (u(x,t))}{\partial x} = \frac{\partial^2 u(x,t)}{\partial x^2} + u(x,t)(1-u(x,t)), \quad x \in \mathbb{R}, \quad t > 0, \tag{1.2}$$

where p is a real physical constant. The model (1.2) was studied and the minimal speed was obtained based on the formal stability analysis and the existence of some exact solutions [5,6]. In fact, Eq. (1.1) covers plenty of classical models, which can be founded in the section of applications in [7] and references. Furthermore, traveling wavefronts for Eq. (1.1) were established, wherein they constructed an invariant region with a lower cure smoothly connecting two fixed points where a heteroclinic orbit exists [7].

As mentioned before, the primary aim of this paper is to examine the stability of traveling wavefronts in relation to Eq. (1.1). The lack of monotonicity in Eq. (1.1) precludes the application of the comparison principle [8]. Additionally, the nonlinear advection

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https://doi.org/10.1016/j.aml.2024.109075
Received 14 February 2024; Received in revised form 18 March 2024; Accepted 18 March 2024
Available online 22 March 2024

term in Eq. (1.1) presents considerable challenges when deriving energy estimates. In order to overcome these difficulties, we will construct the energy estimates in the weighted Sobolev space $H^2_w(\mathbb{R})$ with the weighted function $w(\xi) \geq 1$ (see (2.4)).

Throughout this paper, $C > 0$ denotes a generic constant, while $C_i > 0 (i = 0, 1, 2, \ldots)$ represents a specific constant. Let $L^2(\mathbb{R})$ is the space of the integrable functions, and $H^k(\mathbb{R})$ the Sobolev space. Let $T > 0$ be a number and $\mathcal{B}$ be a Banach space. We denote by $C([0,T], \mathcal{B})$ the space of the $\mathcal{B}$-valued continuous functions on $[0,T]$, $L^2([0,T], \mathcal{B})$ as the space of the $\mathcal{B}$-valued L^2-functions on $[0,T]$.

The rest of the paper is organized as follows. In the next section, we give the existence and uniqueness of traveling wavefronts shown in [7] and state the nonlinear stability theorem. In Section 3, we prove the nonlinear stability by a priori estimates.

2. Preliminaries and stability theorem

The traveling wavefronts for (1.1) connecting with 0 and 1 at far fields are the monotone solution, we mean a solution $u(x,t) = \phi(\xi)$, $\xi = x - ct$, where c is the wave speed and $\phi(\xi)$ satisfies

$$\begin{cases} \phi'' = -c\phi' + J'(\phi)\phi' - f(\phi), \\ \phi(-\infty) = 1, \quad \phi(+\infty) = 0. \end{cases} \tag{2.1}$$

Notice that, the existence and uniqueness of traveling wavefronts to (2.1) was shown by Ma-Ou by the constructing an invariant region [7].

Proposition 2.1 (*Existence of Traveling Wavefronts [7]*). *If $f(u) \leq f'(0)u$ and $J'(u) \leq J'(0)$ for $u \in [0,1]$, then for any $c > c_0$, (1.1) has a unique(up to a translation by a constant) decreasing traveling wave, where c_0 is the minimal speed and $\lambda_1 = \lambda_1(c) < 0$ and $\lambda_2 = \lambda_2(c) < 0$, as the negative roots of the corresponding characteristic equation to (1.1)*

$$\lambda_i^2 + \left(c - J'(0)\right)\lambda_i + f'(0) = 0, \; i = 1, 2; \tag{2.2}$$

satisfying

$$\lambda^2 + \left(c - J'(0)\right)\lambda + f'(0) < 0, \; \text{for } \lambda_1 < \lambda < \lambda_2. \tag{2.3}$$

Throughout this paper, we define a weight function as

$$w(\xi) = \begin{cases} e^{-2\lambda(\xi - \xi_0)}, & \xi \geq \xi_0, \\ 1, & \xi < \xi_0, \end{cases} \tag{2.4}$$

where $\xi_0 < 0$ and $|\xi_0| \gg 1$; and λ is a negative number satisfying $\lambda_1 < \lambda < \lambda_2$.

Now we can state our main theorem.

Theorem 2.1 (*Nonlinear Stability*). *Suppose (H1)-(H3) hold. For any given wavefront $\phi(x - ct)$ with the speed $c > c_0$ to (1.1), suppose that $u_0(x) - \phi(x) \in H^2_w(\mathbb{R})$. There exists some constants $\delta_0 > 0$, $\mu_1 = \mu_1(c, \lambda) > 0$, and $0 < \mu = \mu(c, \lambda) < \mu_1$, all independent of x, t, and $u(x,t)$, when the initial perturbation is small:*

$$\|u_0 - \phi\|^2_{H^2_w} \leq \delta_0^2, \tag{2.5}$$

then the solution u(x,t) of (1.1) satisfies

$$u(x,t) - \phi(x - ct) \in C\left([0,\infty); H^2_w(\mathbb{R})\right) \cap L^2\left([0,\infty); H^2_w(\mathbb{R})\right) \tag{2.6}$$

and

$$\sup_{x\in\mathbb{R}} |u(x,t) - \phi(x - ct)| \leqslant Ce^{-\mu t}, \quad t > 0. \tag{2.7}$$

Remark 2.1. The weighted Sobolev Space $H^2_w(\mathbb{R})$ with $w(\xi) \geq 1$ for $\xi \in \mathbb{R}$ implies $|v(t)|_{c^1} \leq \|\sqrt{w}v\|_{c^1} \leq |v(t)|_{H^2_w}$(see (3.11)), which is crucial in order to overcome the nonlinear advection term. The nonlinear stability theorem, Theorem 2.1 directly implies the exponential convergence of the solution $u(x,t)$ to the traveling wavefront $\phi(x - ct)$.

3. Proof of main theorem

The perturbed equation. In order to prove the stability of the traveling wavefronts to (1.1), we reformulate them to a perturbed equation. Let $\phi(x - ct) = \phi(\xi)$ be a given traveling wavefront with speed $c > c_0$, and

$$v(\xi, t) := u(x,t) - \phi(x - ct), \qquad v_0(\xi) := u_0(x) - \phi(x).$$

Then, from (1.1) and (2.1), $v(\xi, t)$ satisfies

$$\begin{cases} \frac{\partial v}{\partial t} + \left(J'(v + \phi) - c\right)\frac{\partial v}{\partial \xi} - \frac{\partial^2 v}{\partial \xi^2} - f'(\phi)v + J''(\phi)\phi'(\xi)v = Q(v), \\ v_0(\xi) = v(\xi, 0), \quad \xi \in \mathbb{R}, \end{cases} \tag{3.1}$$

where

$$Q(v) := f(v+\phi) - f(\phi) - f'(\phi)v - (J'(v+\phi) - J'(\phi) - J''(\phi)v)\phi'(\xi), \tag{3.2}$$

with $v = v(\xi,t)$, $\phi = \phi(\xi)$. We define the solution space as

$$X(0,T) = \left\{ v | v(\xi,t) \in C([0,T]; H^2_w) \cap L^2([0,T]; H^2_w) \right\} \tag{3.3}$$

with

$$M_v(T)^2 := \sup_{t\in[0,T]} \|v(t)\|^2_{H^2_w(R)}. \tag{3.4}$$

As we known, by using the continuity extension method [9,10], the global existence of $v(\xi,t)$ and its exponential decay estimate directly follow from the local existence result and a priori estimate given below. The proof for the local existence of the solution is standard, because it can be proved by the iteration technique [11,12]. Consequently, the a priori estimates of the solution holds paramount significance in corroborating Theorem 2.1.

Proposition 3.1 (*A Priori Estimates*). *Under the assumption in Theorem 2.1, let $v(\xi,t) \in X(0,T)$ be a local solution of (3.1) for a given constant $T > 0$. Then there exist positive constant $\delta_1 > 0$, $C_0 > 1$, and $\mu > 0$ independent of T and $v(\xi,t)$ such that, when $M_v(T) \le \delta_1$,*

$$\|v(t)\|^2_{H^2_w} + \int_0^t e^{-2\mu(t-s)} \|v(s)\|^2_{H^2_w} ds \le Ce^{-2\mu t} \left\|v_0(0)\right\|^2_{H^2_w}. \tag{3.5}$$

First of all, we will establish the energy estimates for $v(\xi,t)$ in the weighted Sobolev space $H^2_w(\mathbb{R})$ with some techniques.

Lemma 3.1. *Let $v(\xi,t) \in X(0,T)$. Then there exists a constant $\mu_1 > 0$, such that, for $0 < \mu < \mu_1$, it holds that*

$$\|v(t)\|^2_{L^2_w} + \int_0^t e^{-2\mu(t-s)} \|v(s)\|^2_{L^2_w} ds \le Ce^{-2\mu t} \left\|v_0(0)\right\|^2_{L^2_w}, \tag{3.6}$$

provided $M_v(T) \ll 1$.

Proof. Multiplying (3.1) by $e^{2\mu t} w(\xi) v(\xi,t)$, where $\mu > 0$, we get

$$\begin{aligned}
&\left\{ \frac{1}{2} \mathrm{e}^{2\mu t} w v^2 \right\}_t + \left\{ \frac{1}{2} \mathrm{e}^{2\mu t} \left(J'(v+\phi) - c \right) w v^2 - \mathrm{e}^{2\mu t} w v v_\xi \right\}_\xi + \mathrm{e}^{2\mu t} w v_\xi^2 + \mathrm{e}^{2\mu t} w' v v_\xi \\
&+ \left\{ -\frac{J'(v+\phi) - c}{2} \frac{w'}{w} - \mu - f'(\phi) + \frac{1}{2} J''(\phi)\phi'(\xi) \right\} \mathrm{e}^{2\mu t} w v^2 \\
&- \frac{1}{2} J''(v+\phi) \mathrm{e}^{2\mu t} w v_\xi v^2 + \frac{1}{2} \left(J''(\phi) - J''(v+\phi) \right) \phi'(\xi) w v^2 \\
&= \mathrm{e}^{2\mu t} w v Q(v).
\end{aligned} \tag{3.7}$$

By the Cauchy–Schwarz inequality, we have

$$\left| \mathrm{e}^{2\mu t} w' v v_\xi \right| = \mathrm{e}^{2\mu t} w \left| v_\xi \cdot \frac{w'}{w} v \right| \le \mathrm{e}^{2\mu t} w v_\xi^2 + \frac{1}{4} \left(\frac{w'}{w} \right)^2 \mathrm{e}^{2\mu t} w v^2. \tag{3.8}$$

Applying (3.8) to (3.7), we obtain

$$\begin{aligned}
&\left\{ \frac{1}{2} \mathrm{e}^{2\mu t} w v^2 \right\}_t + \left\{ \frac{1}{2} \mathrm{e}^{2\mu t} \left(J'(v+\phi) - c \right) w v^2 - \mathrm{e}^{2\mu t} w v v_\xi \right\}_\xi \\
&+ \left\{ \frac{c - J'(\phi)}{2} \frac{w'}{w} - \frac{1}{4} \left(\frac{w'}{w} \right)^2 - \mu - f'(\phi) + \frac{1}{2} J''(\phi)\phi'(\xi) \right\} \mathrm{e}^{2\mu t} w v^2 - \frac{1}{2} J''(v+\phi) \mathrm{e}^{2\mu t} w v_\xi v^2 \\
&+ \frac{1}{2} \left(J''(\phi) - J''(v+\phi) \right) \phi'(\xi) w v^2 + \frac{1}{2} \left(J'(\phi) - J'(v+\phi) \right) \frac{w'}{w} w v^2 \le \mathrm{e}^{2\mu t} w v Q(v).
\end{aligned} \tag{3.9}$$

Integrating (3.9) over $\mathbb{R} \times [0,t]$ with respect to ξ and t yields

$$\begin{aligned}
&e^{2\mu t} \|v(t)\|^2_{L^2_w} \\
&+ \int_0^t \int_R e^{2\mu s} \left\{ \left(c - J'(\phi) \right) \frac{w'(\xi)}{w(\xi)} - \frac{1}{2} \left(\frac{w'(\xi)}{w(\xi)} \right)^2 - 2\mu - 2f'(\phi) + J''(\phi)\phi'(\xi) \right\} w(\xi) v^2(\xi,s) d\xi ds \\
&- \int_0^t \int_R e^{2\mu s} J''(v+\phi) w(\xi) v_\xi(\xi,s) v^2(\xi,s) d\xi ds - \int_0^t \int_R e^{2\mu s} J^{(3)}(\bar\phi_1) \phi'(\xi) w(\xi) v(\xi,s) v^2(\xi,s) d\xi ds \\
&- \int_0^t \int_R e^{2\mu s} J^{(2)}(\bar\phi_2) \frac{w'(\xi)}{w(\xi)} w(\xi) v(\xi,s) v^2(\xi,s) d\xi ds \\
&\le \|v_0(0)\|^2_{L^2_w} + 2 \int_0^t \int_R e^{2\mu s} w(\xi) v(\xi,s) Q(v(\xi,s)) d\xi ds,
\end{aligned} \tag{3.10}$$

where $\bar\phi_1$, $\bar\phi_2$ between ϕ and $\phi + v$.

On the other hand, by the definition of $M_v(T)$(see (3.4)) and $v \in C([0,T);H_w^2)$, we obtain $\sqrt{w}v \in H^2$. Using the Sobolev inequality $H^2 \hookrightarrow C^1$, we get $\|\sqrt{w}v\|_{c^1} \le C\|v\|_{H_w^2}$. Due to $w(\xi) \ge 1$ for $\xi \in \mathbb{R}$, so we obtain

$$\|v\|_{c^1} \le \|\sqrt{w}v\|_{c^1} \le C\|v\|_{H_w^2}, \tag{3.11}$$

which implies

$$|v(\xi,t)| \le CM_v(t), \quad \left|v_\xi(\xi,t)\right| \le CM_v(t) \text{ for } t \in [0,T], \quad \xi \in R, \tag{3.12}$$

and using the Taylor expansion

$$|Q(v)| = f(v+\phi) - f(\phi) - f'(\phi)v - \left(J'(v+\phi) - J'(\phi) - J''(\phi)v\right) \le C|v|^2,$$

where $C > 0$ is independent of v, we can estimate the nonlinear term as

$$\left|\int_0^t \int_R^{2\mu s} w(\xi)v(\xi,s)Q(v(\xi,s))d\xi ds\right| \le CM_v(T)\int_0^t \int_R^{2\mu s} w(\xi)v^2(\xi,s)d\xi ds. \tag{3.13}$$

By (H3) and applying (3.12) and (3.13) to (3.10), we obtain

$$\begin{aligned} &e^{2\mu t}\|v(t)\|_{L_w^2}^2 + \int_0^t \int_R e^{2\mu s}\left(A_{\mu,w}(\xi) - CM_v(t)\right)w(\xi)v^2(\xi,s)d\xi ds \\ &\le \|v_0(0)\|_{L_w^2}^2, \end{aligned} \tag{3.14}$$

where

$$A_{\mu,w}(\xi) := \left(c - J'(\phi)\right)\frac{w'(\xi)}{w(\xi)} - \frac{1}{2}\left(\frac{w'(\xi)}{w(\xi)}\right)^2 - 2\mu - 2f'(\phi) + J''(\phi)\phi'(\xi).$$

By the definition of $w(\xi)$ in (2.4), when $\xi < \xi_0$, $f'(\phi) \to f'(1)$, then we have $f'(\phi) < \frac{1}{2}f'(1) < 0$. We obtain

$$A_{\mu,w}(\xi) = -2\mu - 2f'(\phi) + J''(\phi)\phi'(\xi) > -2\mu - f'(1) := C_1 \ge 0, \tag{3.15}$$

by selecting $0 < \mu < -\frac{1}{2}f'(1)$. When $\xi \ge \xi_0$, then we have

$$\begin{aligned} A_{\mu,w}(\xi) &= \left(c - J'(\phi)\right)(-2\lambda) - 2\lambda^2 - 2f'(\phi) - 2\mu + J''(\phi)\phi'(\xi) \\ &\ge \left(c - J'(0)\right)(-2\lambda) - 2\lambda^2 - 2f'(0) - 2\mu \\ &= -2\left(\lambda^2 + \left(c - J'(0)\right)\lambda + f'(0)\right) - 2\mu := C_2 > 0, \end{aligned} \tag{3.16}$$

by selecting $0 < \mu < \mu_1 := -\left(\lambda^2 + \left(c - J'(0)\right)\lambda + f'(0)\right)$, where

$$\lambda^2 + \left(c - J'(0)\right)\lambda + f'(0) < 0(\textit{ see } (2.3)).$$

Applying (3.15), (3.16) to (3.14) and letting $M_v(T) \ll 1$, then the proof of this Lemma is complete.

Lemma 3.2. *Let $v(\xi,t) \in X(0,T)$. Then it holds that*

$$\|v_\xi(t)\|_{L_w^2}^2 + \int_0^t e^{-2\mu(t-s)}\|v_\xi\|_{L_w^2}^2 ds \le Ce^{-2\mu t}\left\|v_0(0)\right\|_{H_w^1}^2, \tag{3.17}$$

provided $M_v(T) \ll 1$.

Proof. Differentiating (3.1) with respect to ξ and multiplying by $e^{2\mu t}w(\xi)v_\xi(\xi,t)$, then we have

$$\begin{aligned} &\left\{\frac{1}{2}e^{2\mu t}wv_\xi^2\right\}_t + \left\{\frac{1}{2}e^{2\mu t}\left(J'(v+\phi) - c\right)wv_\xi^2 - e^{2\mu t}wv_\xi v_{\xi\xi}\right\}_\xi \\ &+ \left\{\frac{c - J'(\phi)}{2}\frac{w'}{w} - \frac{1}{4}\left(\frac{w'}{w}\right)^2 - \mu - f'(\phi) + \frac{3}{2}J''(\phi)\phi'\right\}e^{2\mu t}wv_\xi^2 \\ &- \frac{1}{2}J''(v+\phi)e^{2\mu t}wv_\xi^3 + \frac{3}{2}\left(J''(v+\phi) - J''(\phi)\right)\phi' e^{2\mu t}wv_\xi^2 + \frac{1}{2}\left(J'(\phi) - J'(v+\phi)\right)\frac{w'}{w}e^{2\mu t}wv_\xi^2 \\ &+ \left(f'(\phi) - f'(v+\phi)\right)e^{2\mu t}wv_\xi^2 - C_4 e^{2\mu t}w|vv_\xi| \le C_3 e^{2\mu t}w|v_\xi|v^2. \end{aligned} \tag{3.18}$$

By the Cauchy–Schwarz inequality

$$w\left|vv_\xi\right| \le \eta wv_\xi^2 + \frac{1}{4\eta}wv^2, \tag{3.19}$$

where η is sufficiently small. Integrating (3.18) over $\mathbb{R} \times [0,t]$ with respect to ξ and t yields and applying Lemma 3.1 and (3.12), we obtain

$$\begin{aligned} &e^{2\mu t}\left\|v_\xi(t)\right\|_{L_w^2}^2 + \int_0^t \int_R e^{2\mu s}\left(B_{\mu,w}(\xi) - C_5 M_u(t) - \eta\right)w(\xi)v_\xi^2(\xi,s)d\xi ds \\ &\le C_6\left\|v_0(0)\right\|_{H_w^1}^2, \end{aligned} \tag{3.20}$$

where

$$B_{\mu,w}(\xi) := \left(c - J'(\phi)\right)\frac{w'(\xi)}{w(\xi)} - \frac{1}{2}\left(\frac{w'(\xi)}{w(\xi)}\right)^2 - 2\mu - 2f'(\phi) + 3J''(\phi)\phi'(\xi). \tag{3.21}$$

Similarly, we have $B_{\mu,w}(\xi) \geq C_6 > 0$. Letting $M_v(T) \ll 1$, then the proof of this Lemma is complete.

Similarly, by taking

$$\int_0^t \int_R \partial_\xi^2(3.1) \times \omega(\xi) v_{\xi\xi}(\xi,s)d\xi ds$$

applying Lemmas 3.1 and 3.2 and (3.12), we obtain the energy estimates for $v_{\xi\xi}$.

Lemma 3.3. *Let $v(\xi,t) \in X(0,T)$. Then it holds that*

$$\|v_{\xi\xi}(t)\|^2_{L^2_w} + \int_0^t e^{-2\mu(t-s)}\|v_{\xi\xi}\|^2_{L^2_w} ds \leq Ce^{-2\mu t}\left\|v_0(0)\right\|^2_{H^2_w}, \tag{3.22}$$

provided $M_v(T) \ll 1$.

Combining Lemmas 3.1–3.3, the proof of Proposition 3.1 is complete.

Data availability

No data was used for the research described in the article.

Acknowledgments

This work was done while the Second author was visiting McGill University. He would like to express his sincere thanks to McGill University for its hospitality. The research by RJX was supported in part by Anhui Natural Science Foundation, China (2108085MA04) and Quality Engineering Projects at the Provincial Level in Anhui Province, China (2022jyxm030). The research of MM was supported in part by NSERC, Canada grant RGPIN 2022-03374.

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