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Stability of steady-state for 3-D hydrodynamic model of unipolar semiconductor with Ohmic contact boundary in hollow ball

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Abstract

The existence of stationary subsonic solutions and their stability for 3-D hydrodynamic model of unipolar semiconductors with the Ohmic contact boundary have been open for long time due to some technical reason, as we know. In this paper, we consider 3-D radial solutions to the system in a hollow ball, and prove that the 3-D radial subsonic stationary solutions uniquely exist and are asymptotically stable, when the initial perturbations around the subsonic steady-state are small enough. Different from the existing studies on the radial solutions for fluid dynamics where the inner boundary of the hollow ball must be far away from the singular origin, here we may allow the chosen inner boundary arbitrarily close to the singular origin and reveal the relationship between the inner boundary and the large time behavior of the radial solution. This partially answers the open question of the stability of stationary waves subjected to the Ohmic contact boundary conditions in the multiple dimensional space. We also prove the existence of non-flat stationary subsonic solution, which essentially improve and develop the previous studies in this subject. The proof is based on the technical energy estimates in certain weighted Sobolev spaces, where the weight functions are artfully selected to be the distance of the targeted spatial location and the singular point.

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1. Introduction

Proposed first by Blötekjær [4], the dynamic motion of the charged fluid particles such as electrons in semiconductor devices and the charged ions in plasma is modeled as the so-called hydrodynamic system [19,25], which is represented mathematically by Euler-Poisson equations:

$$\begin{cases} \rho_t + \bigtriangledown \cdot (\rho \vec{u}) = 0, \\ (\rho \vec{u})_t + \bigtriangledown \cdot (\rho \vec{u} \otimes \vec{u}) + \bigtriangledown [P(\rho)] - \rho \bigtriangledown \Phi = -\dfrac{\rho \vec{u}}{\tau}, \\ \Delta \Phi = \rho - D(\vec{x}). \end{cases} \tag{1.1}$$

Here, $\vec{x} = (x_1, x_2, x_3) \in R^3$, $\rho = \rho(\vec{x}, t) > 0$ is the electronic density, $\vec{u} = (u_1, u_2, u_3)(\vec{x}, t)$ is the electronic velocity at location $\vec{x}$ and time t, $\Phi(\vec{x}, t)$ is the electrostatic potential, $\tau > 0$ is the relaxation time (without loss of generality we assume $\tau = 1$ throughout of the paper), and $P(\rho)$ is the pressure function satisfying

$$P \in C^3(0, +\infty), \text{ with } s^2 P'(s) > 0 \text{ strictly increasing for } s > 0. \tag{1.2}$$

$D(\vec{x}) > 0$ is the doping profile standing for the density of impurities in semiconductor devices.

The main interest of the paper is to investigate the existence and uniqueness of the solutions to 3-D hydrodynamic system (1.1) as well as their convergence to the corresponding steady-state subsonic solutions, subjected to the following initial-boundary-value problem

$$(\rho, \vec{u})|_{t=0} = (\rho_0, \vec{u}_0)(\vec{x}), \quad \vec{x} \in \Omega, \tag{1.3}$$

$$\rho|_{\partial\Omega} = \rho_1(\vec{x}, t) > 0, \quad \vec{x} \in \partial\Omega, t > 0, \tag{1.4}$$

$$\Phi|_{\partial\Omega} = \Phi_1(\vec{x}, t), \quad \vec{x} \in \partial\Omega, t > 0, \tag{1.5}$$

where, $\Omega \subset R^3$ is a bounded domain with smooth boundary $\partial\Omega$, and the boundary condition (1.4) is physically called the *Ohmic contact boundary*, which is in a general form. Here $\rho_0(\vec{x}) = \rho_1(\vec{x}, 0)$ for $\vec{x} \in \partial\Omega$ is the compatibility condition.

In 1-D case, when the boundary is completely subsonic, Degond and Markowich [5] first proved the existence of subsonic steady-state solution. The uniqueness of solution was obtained with a very strong subsonic background, namely, $|u| \ll 1$. See also the significant development on subsonic steady-state solutions contributed in [6,8,18]. When the boundary is sonic/supersonic, or the doping profile is non-subsonic, the corresponding steady-state equations may possess supersonic/shock-transonic/C^1-transonic stationary solutions [1,2,9,21–24,27,28]. Particularly, regarding the time-dependent hydrodynamic system with subsonic background (subsonic contact boundary, subsonic initial data and subsonic doping profile), Li-Markowich-Mei [20] first showed that the 1-D Euler-Poisson system (1.1) possesses a unique subsonic solution which time-asymptotically converges to the corresponding subsonic steady-state solution. The convergence results in the case of non-flat doping profile were then improved by Nishibata-Suzuki [26] and Guo-Strauss [12]. For the Cauchy problems, the convergence of time-dependent subsonic solutions to the corresponding subsonic stationary waves or diffusion-waves in the switch-on case were intensively studied in [7,13–15].

In n-D case, the relevant studies are quite limited as we know. Guo-Strauss [12] first considered the 3-D case with the insulation boundary condition, where the steady-state can be constructed by the standard monotone elliptic equations, and further proved the stability of steady-state of semiconductor, but the 3-D case with the Ohmic contact boundary conditions was open, because the existence of corresponding 3-D stationary solutions in a general bounded domain is still unknown, of course, it is nothing to talk about their stability. While, in the full space R^n, Huang-Mei-Wang-Yu [16] studied the n-D Cauchy problem, and showed the time-exponentially convergence of n-D subsonic solutions to the planar stationary wave, which are the solutions to the corresponding 1-D porous media equations. See also the n-D case for Euler-Poisson system in [3,10,13].

Since the 3-D case with the physical contact boundary conditions in the general bounded domain $\Omega \subset R^3$ is open, naturally, the first attempt for us is to consider a special domain like a hollow ball, namely we look for the radial solutions for 3-D hydrodynamic system of semiconductors (1.1).

Let us denote

$$\begin{aligned}
&r = |\vec{x}| = \sqrt{x_1^2 + x_2^2 + x_3^2},\\
&\rho(\vec{x}, t) = \rho(r, t),\\
&\vec{u}(\vec{x}, t) = (u_1, u_2, u_3)(\vec{x}, t) = \Big(\frac{u(r,t)x_1}{r}, \frac{u(r,t)x_2}{r}, \frac{u(r,t)x_3}{r}\Big),\\
&\Phi(\vec{x}, t) = \Phi(r, t),\\
&\quad D(\vec{x}) = D(r),\\
&j(r,t) := \rho(r,t)u(r,t), \quad \text{the current density of electrons,}
\end{aligned}$$

then the system (1.1) is reduced to

$$\begin{cases}
\rho_t + j_r + \dfrac{2j}{r} = 0, & \text{(a)}\\
j_t + \Big(\dfrac{j^2}{\rho} + P(\rho)\Big)_r + \dfrac{2j^2}{\rho r} - \rho\Phi_r + j = 0, & \text{(b)}\\
\Phi_{rr} + \dfrac{2\Phi_r}{r} = \rho - D(r). & \text{(c)}
\end{cases} \tag{1.6}$$

From the above system, it is clear that $r = 0$ is the singular point, so the targeted domain should be a hollow ball $\Omega = [\epsilon_0, 1]$ for $\epsilon_0 > 0$, and the subjected initial value and the contact boundary conditions are

$$(\rho, j)|_{t=0} = (\rho_0, j_0)(r), \quad r \in [\epsilon_0, 1], \tag{1.7}$$

$$\rho(t, \epsilon_0) = \rho_L > 0, \quad \rho(t, 1) = \rho_R > 0, \tag{1.8}$$

$$\Phi(t, \epsilon_0) = 0, \quad \Phi(t, 1) = \Phi_R > 0. \tag{1.9}$$

Here ρ_L, ρ_R and Φ_R are positive constants. In addition, we assume that the compatibility conditions hold:

$$\rho_0(\epsilon_0)=\rho_L, \quad \rho_0(1)=\rho_R, \quad \Big(j_{0r}+\frac{2j_0}{\epsilon_0}\Big)(\epsilon_0)=(j_{0r}+2j_0)(1)=0. \tag{1.10}$$

In what follows, we concentrate ourselves to the IBVP (1.6)-(1.9), and prove the global existence and uniqueness of the above radial solutions $(\rho, j, \Phi)(t,r)$, as well as the time-exponential convergence to the corresponding stationary subsonic solutions $(\tilde{\rho}, \tilde{j}, \tilde{\Phi})(r)$ given by

$$\begin{cases} \tilde{j}_r+\dfrac{2\tilde{j}}{r}=0, \text{ namely, } \tilde{j}=\dfrac{const.}{r^2}, \\ \Big(\dfrac{\tilde{j}^2}{\tilde{\rho}}+P(\tilde{\rho})\Big)_r+\dfrac{2\tilde{j}^2}{\tilde{\rho}r}-\tilde{\rho}\tilde{\Phi}_r+\tilde{j}=0, \\ \tilde{\Phi}_{rr}+\dfrac{2\tilde{\Phi}_r}{r}=\tilde{\rho}-D(r), \end{cases} \tag{1.11}$$

with the contact boundary conditions

$$\begin{cases} \tilde{\rho}(\epsilon_0)=\rho_L, \quad \tilde{\rho}(1)=\rho_R, \\ \tilde{\Phi}(\epsilon_0)=0, \quad \tilde{\Phi}(1)=\Phi_R. \end{cases} \tag{1.12}$$

Here are some technical features of the paper. Different from the existing studies on the radial solutions for fluid dynamics where the inner boundary $r=\epsilon_0$ for the hollow ball is needed to be far from the singular point $r=0$, in this paper we may allow the chosen inner boundary $r=\epsilon_0$ arbitrarily close to such a singular origin $r=0$ and reveal the relationship between the inner boundary and the large time behavior of the radial solution. This is the first technical point in our paper. The second technical point is that, in order to treat such a singularity when ϵ_0 is sufficiently close to 0, artfully the working solution space will be designed as a weighted Sobolev space with the weight functions as the proportion of distance between the targeted location and the singular origin, namely, the weight functions are r, $\epsilon_0 r$ and $\epsilon_0^2 r$. The third point is that we may allow the doping profile $D(r)$ to be non-flat, namely, $|D'(r)|\not\ll 1$, while, such a smallness was often requested in the previous studies. With this help, we show another new result that the steady-state solutions can be non-flat, namely, the derivatives of steady-state solutions can be large. This is also different from the existing studies with $|\partial_r\tilde{\rho}|\ll 1$. The last but a crucial technique is the artful selection for the weight function $\tilde{h}(r)$ in the first order energy estimates of the *a priori* estimates in section 3. This idea is inspired by [12] but developed with some significance because of the singularity.

By the terminology from gas dynamics, we call $c:=\sqrt{P'(\rho)}$ the sound speed. So, the hydrodynamic system (1.6) is said to be subsonic, if

$$\text{fluid velocity: } u=\frac{j}{\rho}<\sqrt{P'(\rho)}: \text{ sound speed.}$$

We are going to look for the global solution to (1.6)-(1.9) satisfying, for $t>0$,

$$\inf\Big(P'(\rho)-\frac{j^2}{\rho^2}\Big)>c_1>0, \tag{1.13}$$

$$\inf \rho > 0 \tag{1.14}$$

for some positive constant c_1. Throughout the paper, we assume that the initial data and the boundary values satisfy the subsonic conditions (1.13) and (1.14).

Notations. In this paper, we denote the generic positive constants by C, independent of ϵ_0. We also denote the norm of $L^2(\Omega)$ by $\|f\|$, and the norm of H^k by $\|f\|_k$, where, without confusion, the derivatives are simply denoted by $\partial_r f = f_r$ and $\partial_r^2 f = f_{rr}$. A weighted Sobolev space $H_r^k(\Omega)$ with the weight function $w(r) = r$, is defined by $f \in H_r^k(\Omega)$, where $r\partial_r^l f \in L^2(\Omega)$ for $l = 0, 1, \cdots, k$, with the norm

$$\|f\|_{H_r^k(\Omega)} = \Big(\sum_{l=0}^{k} \int_{\Omega} |r \cdot \partial_r^l f|^2 dr\Big)^{\frac{1}{2}}.$$

For given $T > 0$, the solution spaces without/with the weight function are defined by

$$\text{(non-weighted space): } \chi_k([0,T];\Omega) = \{f \mid \partial_t^{k-l}\partial_r^l f \in L^2(\Omega), \|\partial_t^{k-l} f(t)\|_{H^l(\Omega)} \in C^0[0,T], \quad \text{for} \quad 0 \le l \le k\}$$

equipped with the norm

$$\|f\|_{\chi_k([0,T];\Omega)} = \max_{0\le t\le T} \sum_{l=0}^{k} \|\partial_t^{k-l} f(t)\|_{H^l(\Omega)},$$

and

$$\text{(weighted space): } \chi_{k,r}([0,T];\Omega) = \{f \mid \partial_t^{k-l}\partial_r^l f \in L_r^2(\Omega), \|\partial_t^{k-l} f(t)\|_{H_r^l(\Omega)} \in C^0[0,T], \quad \text{for} \quad 0 \le l \le k\}$$

equipped with norms

$$\|f\|_{\chi_{k,r}([0,T];\Omega)} = \max_{0\le t\le T} \sum_{l=0}^{k} \|\partial_t^{k-l} f(t)\|_{H_r^l(\Omega)}.$$

Generally, we denote the norm of $C^0(\Omega)$ by $|f|_0$.

For convenience, we introduce the vector-valued function $\vec{s}(r) = \begin{pmatrix} s_1(r) \\ s_2(r) \end{pmatrix}$. Here $\vec{s}(r) \in H^k(\Omega)$ is defined by $s_1(r) \in H^k(\Omega)$ and $s_2(r) \in H^k(\Omega)$ with the norm

$$\|\vec{s}\|_{H^k(\Omega)} := \|s_1\|_{H^k(\Omega)} + \|s_2\|_{H^k(\Omega)}.$$

In the same way, we define $\vec{s}(r) \in H_r^k(\Omega)$ with the norm

$$\|\vec{s}\|_{H_r^k(\Omega)} := \|s_1\|_{H_r^k(\Omega)} + \|s_2\|_{H_r^k(\Omega)}.$$

Now we are going to state our main results.

Theorem 1.1 *(Existence of 3-D radial steady-state). Let $0 < \epsilon_0 \ll 1$ be arbitrarily given, and define $A(r) := \rho_L + \frac{\rho_R - \rho_L}{1-\epsilon_0}(r - \epsilon_0)$. Assume that $|\rho_L - \rho_R| + |\Phi_R| \le C\epsilon_0^\alpha$ with $\alpha > 2$, and that $D(r)$ satisfies $0 < \tilde{c} \le D(r)$ and $\max\limits_{r\in[\epsilon_0,1]}\{r|A(r) - D(r)|\} \le C_1\epsilon_0$ with some positive constants $\tilde{c}$ and C_1. Then the stationary system (1.11)-(1.12) has a unique solution $(\tilde{\rho}, \tilde{j}, \tilde{\Phi})(r) \in [H^2(\Omega)]^3$, satisfying that, for some positive constants $\bar{C}_0$, C_2, C_3 and c_1',*

$$|\tilde{j}| \le \bar{C}_0\epsilon_0^{\alpha-1} \triangleq J_0, \quad C_- \le \tilde{\rho} \le C_+, \quad \|\tilde{\rho} - A\|_1 \le C_2, \quad \|r\tilde{\rho}_{rr}\| \le C_3, \tag{1.15}$$

and

$$\inf\Big(P'(\tilde{\rho}) - \frac{\tilde{j}^2}{\tilde{\rho}^2}\Big) > c_1' > 0, \tag{1.16}$$

where

$$\begin{aligned} &C_- = \min\{\rho_L, \rho_R, \frac{c_-}{2}\}, \quad C_+ = \max\{\rho_L, \rho_R, \frac{c_-}{2} + c_+\}, \\ &c_- = \min_{r\in[\epsilon_0,1]} D(r) > 0, \quad c_+ = \max_{r\in[\epsilon_0,1]} D(r). \end{aligned} \tag{1.17}$$

Theorem 1.2 *(Stability of steady-state). Suppose that the assumptions of Theorem 1.1 hold. Let $(\tilde{\rho}, \tilde{j}, \tilde{\phi})(r)$ be the solution to the steady state system of (1.11)-(1.12) obtained in Theorem 1.1. Assume that the initial perturbations around the steady-state in the weighted space are small:*

$$\Big\|\begin{pmatrix}\rho_0 - \tilde{\rho}\\ j_0 - \tilde{j}\end{pmatrix}\Big\|_{L^2_r} + \epsilon_0\Big\|\partial_r\begin{pmatrix}\rho_0 - \tilde{\rho}\\ j_0 - \tilde{j}\end{pmatrix}\Big\|_{L^2_r} + \epsilon_0^2\Big\|\partial_r^2\begin{pmatrix}\rho_0 - \tilde{\rho}\\ j_0 - \tilde{j}\end{pmatrix}\Big\|_{L^2_r} + \|\Phi_r(0) - \tilde{\Phi}_r\|_{L^2_r} \le C_4\epsilon_0^\gamma \tag{1.18}$$

for any $\gamma \ge \frac{5}{2}$ and some positive constant C_4, where

$$\Phi_r(0) - \tilde{\Phi}_r = r^{-2}\Big[\int_{\epsilon_0}^{r} s^2(\rho_0 - \tilde{\rho})(s)ds - \frac{\epsilon_0}{1-\epsilon_0}\int_{\epsilon_0}^{1} r^{-2}\Big(\int_{\epsilon_0}^{r} s^2(\rho_0 - \tilde{\rho})(s)ds\Big)dr\Big]. \tag{1.19}$$

Then the Euler-Poisson system (1.6)-(1.9) has a unique solution $(\rho, j, \Phi)(t, r) \in [\chi_{2,r}([0, \infty); \Omega)]^3$ satisfying the condition (1.13)-(1.14). Moreover, it holds that

$$\sum_{0\le l\le 2}\epsilon_0^l\Big\|\partial^l\begin{pmatrix}\rho - \tilde{\rho}\\ j - \tilde{j}\end{pmatrix}(t)\Big\|_{L^2_r} + \|\partial_r(\Phi - \tilde{\Phi})(t)\|_{L^2_r} \le C\epsilon_0^\gamma e^{-\frac{c' t}{2}}, \quad \forall t \in [0, +\infty), \tag{1.20}$$

for some positive constant C, independent of ϵ_0, where we denote a derivative in both r and t of order l by ∂^l.

Remark 1.1. 1. In Theorems 1.1 and 1.2, we allow the stationary solution to be non-flat, namely, $|\partial_r\tilde{\rho}| \not\ll 1$. This is totally different from the existing studies in [5,7,8,12,15,16,20,26].

2. The constants C_i are independent of ϵ_0, where ϵ_0 can be arbitrarily taken close to 0. This is different from that of coefficients depending on ϵ_0 in [17], where ϵ_0 is the inner boundary of

$\Omega = [\epsilon_0, 1]$. Thus, we partially answer the open question in [12] on the existence and stability of subsonic solutions for 3-D hydrodynamic system of semiconductor with the Ohmic contact boundary conditions in a bounded domain specified as a hollow ball by Theorems 1.1 and 1.2.

3. When $\epsilon_0 \to 0^+$, Theorem 1.1 still guarantees the existence of the non-trivial stationary solutions with $\tilde{j} = 0$, $\tilde{\rho} \neq$ constant, and $\tilde{\Phi} \neq$ constant. However, Theorem 1.2 does not work out the stability of the stationary waves, and leaves the question still open.

4. For the case of $\epsilon_0 \geq C_0$, the similar results can be derived directly from Theorem 1.1 and 1.2 or by the same way shown in [12].

The paper is organized as follows. In section 2, we will show, by the linearized iteration scheme and the weighted energy method, the existence and uniqueness of steady solution to (1.6)-(1.9). Then, in section 3, by the weighted energy method and technical "energy" selection we will establish the *a priori* energy estimate of the solutions $(\sigma, \eta, \phi)(t, r)$ to (3.2)-(3.3). The *a priori* estimates, together with the local existence and continuity arguments, yield the global existence and uniqueness of (3.2)-(3.3), as well as the time-exponential convergence to the corresponding stationary subsonic solutions $(\tilde{\rho}, \tilde{j}, \tilde{\Phi})(r)$.

2. The steady solution

In this section, we consider the BVP of steady system

$$\begin{cases} \tilde{j}_r + \dfrac{2\tilde{j}}{r} = 0, & \text{(a)} \\ \left(\dfrac{\tilde{j}^2}{\tilde{\rho}} + P(\tilde{\rho})\right)_r + \dfrac{2\tilde{j}^2}{\tilde{\rho} r} - \tilde{\rho}\tilde{\Phi}_r + \tilde{j} = 0, & \text{(b)} \\ \tilde{\Phi}_{rr} + \dfrac{2\tilde{\Phi}_r}{r} = \tilde{\rho} - D(r), & \text{(c)} \end{cases} \tag{2.1}$$

with the contact boundary conditions

$$\begin{cases} \tilde{\rho}(\epsilon_0) = \rho_L, \quad \tilde{\rho}(1) = \rho_R, & \text{(a)} \\ \tilde{\Phi}(\epsilon_0) = 0, \quad \tilde{\Phi}(1) = \Phi_R. & \text{(b)} \end{cases} \tag{2.2}$$

And we will show the existence and uniqueness of solution $(\tilde{\rho}, \tilde{j}, \tilde{\Phi})(r)$ to (2.1)-(2.2) under the subsonic condition

$$\inf_{r \in [\epsilon_0, 1]} \tilde{\rho} > C_- > 0, \qquad \inf_{r \in [\epsilon_0, 1]} \left(P'(\tilde{\rho}) - \frac{\tilde{j}^2}{\tilde{\rho}^2}\right) > c_1' > 0. \tag{2.3}$$

Moreover, we may reduce (2.1)(b) to

$$F(\tilde{\rho}, \tilde{j})_r - \tilde{\Phi}_r + \frac{\tilde{j}}{\tilde{\rho}} = 0, \tag{2.4}$$

where $F(\tilde{\rho}, \tilde{j}) = h(\tilde{\rho}) + \frac{\tilde{j}^2}{2\tilde{\rho}^2}$ and $h(\tilde{\rho})$ is defined by $h'(s) = \frac{P'(s)}{s}$.

Our proof starts with the observation that $\tilde{j}$ and $\tilde{\Phi}$ have explicit expression on $\tilde{\rho}$ in Lemma 2.1.

Lemma 2.1. *Suppose that* $0 < \epsilon_0 \ll 1$ *and* $|\rho_L - \rho_R| + |\Phi_R| < C\epsilon_0^\alpha$ *with* $\alpha > 2$. *For any steady-state solution* $(\tilde{\rho}, \tilde{j}, \tilde{\Phi})(r)$ *of* (2.1)-(2.2) *satisfying*

$$C_- \le \tilde{\rho} \le C_+ \quad and \quad \tilde{j} \le J_0 := \bar{C}_0\epsilon_0^{\alpha-1}, \tag{2.5}$$

there holds that

$$\tilde{j}(r) = M_0[\tilde{\rho}]r^{-2} =: \tilde{J}[\tilde{\rho}](r),$$
$$\tilde{\Phi}(r) = \int_{\epsilon_0}^{r} s^{-2}\Big[\int_{\epsilon_0}^{s} \tau^2(\tilde{\rho}(\tau) - D(\tau))d\tau + \frac{\epsilon_0}{1-\epsilon_0}(\Phi_R - A[\tilde{\rho}])\Big]ds =: \tilde{\Psi}[\tilde{\rho}](r), \tag{2.6}$$

where

$$\mathbb{A} := \frac{1}{2\rho_R^2} - \frac{1}{2\rho_L^2\epsilon_0^4}, \quad \mathbb{B}[\tilde{\rho}] := \int_{\epsilon_0}^{1} \frac{1}{\tilde{\rho}(r)r^2}dr, \quad \mathbb{C} := h(\rho_R) - h(\rho_L) - \Phi_R, \tag{2.7}$$
$$M_0[\tilde{\rho}] := \frac{-2\mathbb{C}}{\mathbb{B}[\tilde{\rho}] + \sqrt{(\mathbb{B}[\tilde{\rho}])^2 - 4\mathbb{A}\mathbb{C}}} = const., \tag{2.8}$$

and

$$A[\tilde{\rho}] := \int_{\epsilon_0}^{1} r^{-2}\Big(\int_{\epsilon_0}^{r} s^2(\tilde{\rho}(s) - D(s))ds\Big)dr. \tag{2.9}$$

Proof. Multiplying (2.1)(c) by r^2 and integrating it over $[\epsilon_0, r]$, we get

$$\tilde{\Phi}_r(r) = r^{-2}\Big[\epsilon_0^2\tilde{\Phi}_r(\epsilon_0) + \int_{\epsilon_0}^{r} s^2(\tilde{\rho}(s) - D(s))ds\Big]. \tag{2.10}$$

To specify the value of $\tilde{\Phi}_r(\epsilon_0)$, we integrate (2.10) over $[\epsilon_0, 1]$, with the help of (2.2)(b), to get

$$\Phi_R = \epsilon_0(1-\epsilon_0)\tilde{\Phi}_r(\epsilon_0) + \int_{\epsilon_0}^{1} r^{-2}\Big(\int_{\epsilon_0}^{r} s^2(\tilde{\rho}(s) - D(s))ds\Big)dr,$$

namely,

$$\tilde{\Phi}_r(\epsilon_0) = \frac{1}{\epsilon_0(1-\epsilon_0)}\Big(\Phi_R - A[\tilde{\rho}]\Big), \tag{2.11}$$

where $A[\tilde{\rho}]$ is defined as in (2.9). Then, substituting (2.11) into (2.10) and integrating it over $[\epsilon_0, r]$ again gives (2.6).

Note that $\tilde{j}(r) = r^{-2}M_0[\tilde{\rho}]$ holds from (2.1)(a), where $M_0[\tilde{\rho}]$ is a constant. To specify the value of $M_0[\tilde{\rho}]$, we integrate (2.4) over $[\epsilon_0, 1]$ to have

$$F(\rho_R, M_0[\tilde{\rho}]) - F(\rho_L, M_0[\tilde{\rho}]\epsilon_0^{-2}) - \Phi_R + \int_{\epsilon_0}^{1} \frac{\tilde{j}}{\tilde{\rho}} dr = 0, \tag{2.12}$$

that is,

$$\mathbb{A}(M_0[\tilde{\rho}])^2 + \mathbb{B}[\tilde{\rho}]M_0[\tilde{\rho}] + \mathbb{C} = 0, \tag{2.13}$$

where $\mathbb{A}, \mathbb{B}[\tilde{\rho}], \mathbb{C}, M_0[\tilde{\rho}]$ are given in (2.7) and (2.8).

Under the conditions that $0 < \epsilon_0 \ll 1$ and $|\rho_L - \rho_R| + |\Phi_R| < C\epsilon_0^\alpha$ with $\alpha > 2$, we claim that

$$\hat{c}_1\epsilon_0^{-4} \le |\mathbb{A}| \le \hat{C}_1\epsilon_0^{-4}, \quad \hat{c}_2\epsilon_0^{-1} \le \mathbb{B}[\tilde{\rho}] \le \hat{C}_2\epsilon_0^{-1}, \quad \text{and} \quad |\mathbb{C}| \le \hat{C}_3\epsilon_0^\alpha \tag{2.14}$$

for some positive constants $\hat{c}_i (i = 1, 2)$ and $\hat{C}_j (j = 1, 2, 3)$ with $\hat{c}_i \le \hat{C}_i$.

Indeed, there exists a positive constant $\tilde{\delta}_1$ such that if $0 < \epsilon_0 < \tilde{\delta}_1$, then it holds that

$$|\mathbb{A}| = \frac{1}{2\rho_L^2\epsilon_0^4} - \frac{1}{2\rho_R^2} < \frac{1}{2\rho_L^2\epsilon_0^4},$$

and

$$|\mathbb{A}| = \frac{1}{2\rho_L^2\epsilon_0^4} - \frac{1}{2\rho_R^2} = \frac{1}{4\rho_L^2\epsilon_0^4} + \frac{1}{4\rho_L^2\epsilon_0^4} - \frac{1}{2\rho_R^2} > \frac{1}{4\rho_L^2\epsilon_0^4}.$$

On the other hand, with $C_- \le \tilde{\rho} \le C_+$, we have

$$\frac{1-\epsilon_0}{C_+\epsilon_0} = \frac{1}{C_+}\int_{\epsilon_0}^{1} \frac{1}{r^2} dr \le \mathbb{B}[\tilde{\rho}] = \int_{\epsilon_0}^{1} \frac{1}{\tilde{\rho}(r)r^2} dr \le \frac{1}{C_-}\int_{\epsilon_0}^{1} \frac{1}{r^2} dr = \frac{1-\epsilon_0}{C_-\epsilon_0}. \tag{2.15}$$

In addition,

$$|\mathbb{C}| = |h(\rho_R) - h(\rho_L) - \Phi_R| \le |h'(\theta)(\rho_R - \rho_L)| + |\Phi_R| \le \hat{C}_3\epsilon_0^\alpha.$$

Thus, the claim (2.14) holds, which further indicates, in view of the smallness of ϵ_0 and $\alpha > 2$, that

$$(\mathbb{B}[\tilde{\rho}])^2 - 4\mathbb{A}\mathbb{C} > (\hat{c}_2)^2\epsilon_0^{-2} - 4\hat{C}_1\hat{C}_3\epsilon_0^{-4+\alpha} > \frac{(\hat{c}_2)^2}{2}\epsilon_0^{-2}. \tag{2.16}$$

Therefore, (2.13) gives the two possible cases as follows,

$$\text{(i)}\quad M_0[\tilde\rho]=\frac{-\mathbb{B}[\tilde\rho]-\sqrt{(\mathbb{B}[\tilde\rho])^2-4\mathbb{A}\mathbb{C}}}{2\mathbb{A}},$$
$$\text{(ii)}\quad M_0[\tilde\rho]=\frac{-\mathbb{B}[\tilde\rho]+\sqrt{(\mathbb{B}[\tilde\rho])^2-4\mathbb{A}\mathbb{C}}}{2\mathbb{A}}.$$

For case (i), we have

$$|M_0[\tilde\rho]|=\frac{\mathbb{B}[\tilde\rho]+\sqrt{(\mathbb{B}[\tilde\rho])^2-4\mathbb{A}\mathbb{C}}}{2|\mathbb{A}|}\ge\frac{\epsilon_0^4}{2\hat C_1}(\mathbb{B}[\tilde\rho]+\sqrt{(\mathbb{B}[\tilde\rho])^2-4\mathbb{A}\mathbb{C}}).$$

Recall that

$$\mathbb{B}[\tilde\rho]\le\mathbb{B}[\tilde\rho]+\sqrt{(\mathbb{B}[\tilde\rho])^2-4\mathbb{A}\mathbb{C}}\le 2\mathbb{B}[\tilde\rho],$$

we get

$$|M_0[\tilde\rho]|\ge\frac{\epsilon_0^4}{2\hat C_1}\mathbb{B}[\tilde\rho]\ge\frac{\hat c_2\epsilon_0^3}{2\hat C_1},$$

and consider the value of $\tilde j(r)$ at the point $r=\epsilon_0$:

$$|\tilde j(\epsilon_0)|=|M_0[\tilde\rho]|\epsilon_0^{-2}\ge\frac{\hat c_2\epsilon_0^3}{2\hat C_1}\epsilon_0^{-2}\ge\frac{\hat c_2\epsilon_0}{2\hat C_1}.$$

This is a contradiction to the condition $|\tilde j(r)|<\bar C_0\epsilon_0^{\alpha-1}$, $r\in[\epsilon_0,1]$ with $\alpha>2$, for some positive constant $\bar C_0$ as $0<\epsilon_0\ll 1$.

For case (ii), i.e.,

$$M_0[\tilde\rho]=\frac{-\mathbb{B}[\tilde\rho]+\sqrt{(\mathbb{B}[\tilde\rho])^2-4\mathbb{A}\mathbb{C}}}{2\mathbb{A}}=\frac{-2\mathbb{C}}{\mathbb{B}[\tilde\rho]+\sqrt{(\mathbb{B}[\tilde\rho])^2-4\mathbb{A}\mathbb{C}}},$$

so we have

$$|M_0[\tilde\rho]|=\frac{2|\mathbb{C}|}{\mathbb{B}[\tilde\rho]+\sqrt{(\mathbb{B}[\tilde\rho])^2-4\mathbb{A}\mathbb{C}}}\le\frac{2|\mathbb{C}|}{\mathbb{B}[\tilde\rho]}\le\frac{2\hat C_3}{\hat c_2}\epsilon_0^{\alpha+1}:=\bar C_0\epsilon_0^{\alpha+1},\tag{2.17}$$

where $\bar C_0:=\frac{2\hat C_3}{\hat c_2}$. Then,

$$|\tilde j(r)|=|M_0[\tilde\rho]|r^{-2}\le\bar C_0\epsilon_0^{\alpha-1}.$$

Consequently, the above analysis shows that $\tilde j$ can be uniquely expressed by

$$\tilde j(r)=M_0[\tilde\rho]r^{-2}=\frac{-2\mathbb{C}}{\mathbb{B}[\tilde\rho]+\sqrt{(\mathbb{B}[\tilde\rho])^2-4\mathbb{A}\mathbb{C}}}r^{-2}.$$

Thus, the proof is complete. □

Remark 2.1. The condition (2.5) implies that $(\tilde{\rho}, \tilde{j}, \tilde{\Phi})$ satisfies the condition (2.3) for some positive constant c_1'.

The Lemma 2.1 implies that the existence and uniqueness of solution $(\tilde{\rho}, \tilde{j}, \tilde{\Phi})(r)$ of (2.1)-(2.2) with the condition (2.5) is equivalent to that of the solution $\tilde{\rho}(r)$ with $C_- \le \tilde{\rho} \le C_+$ of the following BVP

$$\begin{cases} F(\tilde{\rho}, \tilde{J}[\tilde{\rho}])_r - \tilde{\Psi}_r[\tilde{\rho}] + \dfrac{\tilde{J}[\tilde{\rho}]}{\tilde{\rho}} = 0, \\ \tilde{\rho}(\epsilon_0) = \rho_L, \quad \tilde{\rho}(1) = \rho_R. \end{cases} \tag{2.18}$$

Thus, our next goal is to achieve the existence and uniqueness of solution to the BVP (2.18) with $C_- \le \tilde{\rho} \le C_+$.

To do this, we reduce (2.18) to the BVP of nonlinear elliptic equation as follows:

$$\begin{cases} \Big(\dfrac{\partial F}{\partial \tilde{\rho}}\tilde{\rho}_r\Big)_r + \dfrac{2}{r}\dfrac{\partial F}{\partial \tilde{\rho}}\tilde{\rho}_r - \tilde{\rho} = -\Big(\dfrac{\partial F}{\partial \tilde{J}[\tilde{\rho}]}\tilde{J}[\tilde{\rho}]_r\Big)_r - \dfrac{2}{r}\dfrac{\partial F}{\partial \tilde{J}[\tilde{\rho}]}\tilde{J}[\tilde{\rho}]_r + \dfrac{\tilde{J}[\tilde{\rho}]}{\tilde{\rho}^2}\tilde{\rho}_r - D, \\ \tilde{\rho}(\epsilon_0) = \rho_L, \quad \tilde{\rho}(1) = \rho_R. \end{cases} \tag{2.19}$$

To prove the existence of solution to (2.19), we introduce a subspace for the solution:

$$\mathfrak{A}_{C_2,C_3} = \Big\{ q \in H^2(\Omega) \,\Big|\, \|q - A\|_1 \le C_2,\ \|rq_{rr}\| \le C_3,\ C_- \le q \le C_+, \\ q(\epsilon_0) = \rho_L,\ q(1) = \rho_R \Big\},$$

equipped with the norm $\|\cdot\|_2$, where $A(r) = \rho_L + \frac{\rho_R - \rho_L}{1-\epsilon_0}(r - \epsilon_0)$, C_- and C_+ are given in Theorem 1.1, C_2 and C_3 are some constants to be specified. And naturally, we consider the BVP of linearized equation as follows:

$$\begin{cases} \Big(\frac{\partial F}{\partial q}(q, \tilde{J})\tilde{\rho}_r\Big)_r + \frac{2}{r}\frac{\partial F}{\partial q}(q, \tilde{J})\tilde{\rho}_r - \tilde{\rho} = -\Big(\frac{\partial F}{\partial \tilde{J}}(q, \tilde{J})\tilde{J}_r\Big)_r - \frac{2}{r}\frac{\partial F}{\partial \tilde{J}}\tilde{J}_r + \frac{\tilde{J}}{q^2}q_r - D, & \text{(a)} \\ \tilde{\rho}(\epsilon_0) = \rho_L, \quad \tilde{\rho}(1) = \rho_R & \text{(b)} \end{cases} \tag{2.20}$$

for given $q \in \mathfrak{A}_{C_2,C_3}$, where $\tilde{J} \triangleq \tilde{J}[q] = M_0[q]r^{-2}$ and $M_0[q]$ is given in (2.8). Furthermore, $\tilde{J}[q]$ has the following property.

Lemma 2.2. *Let $0 < \epsilon_0 \ll 1$, and let q, q_1, q_2 be such that $C_- \le q, q_1, q_2 \le C_+$, then it holds that*

$$\begin{cases} |\tilde{J}[q]|_0 \le J_0, \\ |\tilde{J}[q_1] - \tilde{J}[q_2]| \le C\epsilon_0^{\alpha - \frac{3}{2}}\|q_1 - q_2\|, \\ \|\tilde{J}[q_1] - \tilde{J}[q_2]\| \le C\epsilon_0^{\alpha-1}\|q_1 - q_2\|, \\ r|\tilde{J}[q_1] - \tilde{J}[q_2]| \le C\epsilon_0^{\alpha - \frac{1}{2}}\|q_1 - q_2\| + C\epsilon_0^{\alpha - \frac{3}{2}}\|rq_1 - rq_2\|, \\ \|r(\tilde{J}[q_1] - \tilde{J}[q_2])\| \le C\epsilon_0^{\alpha-1}\|rq_1 - rq_2\| + C\epsilon_0^{\alpha}\|q_1 - q_2\|. \end{cases} \tag{2.21}$$

Proof. Since $C_- \le q \le C_+$, we have $\frac{1-\epsilon_0}{C_+\epsilon_0} \le \mathbb{B}[q] \le \frac{1-\epsilon_0}{C_-\epsilon_0}$. In the same way as shown in (2.17), it is easy to verify that $M_0[q] \le \bar{C}_0\epsilon_0^{\alpha+1}$ and $|\tilde{J}[q]|_0 \le J_0$.

It follows from (2.15) and (2.16) that, for $0 < \epsilon_0 \ll 1$,

$$\frac{1}{2}\mathbb{B}[q] \le \sqrt{(\mathbb{B}[q])^2 - 4\mathbb{A}\mathbb{C}} \le 2\mathbb{B}[q],$$

which, together with (2.14), leads to

$$\begin{aligned}|\tilde{J}[q_1] - \tilde{J}[q_2]| &= \left|\frac{-2\mathbb{C}}{\mathbb{B}[q_1] + \sqrt{(\mathbb{B}[q_1])^2 - 4\mathbb{A}\mathbb{C}}} - \frac{-2\mathbb{C}}{\mathbb{B}[q_2] + \sqrt{(\mathbb{B}[q_2])^2 - 4\mathbb{A}\mathbb{C}}}\right| r^{-2} \\ &\le C\epsilon_0^{\alpha}(|\mathbb{B}[q_1] - \mathbb{B}[q_2]| + |\sqrt{(\mathbb{B}[q_1])^2 - 4\mathbb{A}\mathbb{C}} - \sqrt{(\mathbb{B}[q_2])^2 - 4\mathbb{A}\mathbb{C}}|) \\ &\le C\epsilon_0^{\alpha}|\mathbb{B}[q_1] - \mathbb{B}[q_2]| \\ &= C\epsilon_0^{\alpha-\frac{3}{2}}\|q_1 - q_2\| \end{aligned} \tag{2.22}$$

and

$$r|\tilde{J}[q_1] - \tilde{J}[q_2]| \le C\epsilon_0^{\alpha-\frac{1}{2}}\|q_1 - q_2\| + C\epsilon_0^{\alpha-\frac{3}{2}}\|rq_1 - rq_2\|.$$

Furthermore, we have

$$\|\tilde{J}[q_1] - \tilde{J}[q_2]\| \le \|C\epsilon_0^{\alpha+2}r^{-2}|\mathbb{B}[q_1] - \mathbb{B}[q_2]|\| \le C\epsilon_0^{\alpha-1}\|q_1 - q_2\|$$

and

$$\|r(\tilde{J}[q_1] - \tilde{J}[q_2])\| \le \|C\epsilon_0^{\alpha+2}r^{-1}|\mathbb{B}[q_1] - \mathbb{B}[q_2]|\| \le C\epsilon_0^{\alpha-1}\|rq_1 - rq_2\| + C\epsilon_0^{\alpha}\|q_1 - q_2\|.$$

Thus, the proof is complete. □

Remark 2.2. Let $0 < \epsilon_0 \ll 1$, for any q with $C_- \le q \le C_+$, then the pair of functions $(q, \tilde{J}[q])(r)$ satisfy

$$\frac{\partial F(q, \tilde{J}[q])}{\partial q} \ge \inf_{r\in\Omega}\left(P'(q) - \frac{\tilde{J}[q]^2}{q^2}\right) > c_1'$$

for some positive constant c_1', independent of q.

We are now in a position to show the existence of solution to (2.20).

Lemma 2.3. *Given $q \in \mathfrak{A}_{C_2,C_3}$, there exists a unique solution of (2.20) such that $\tilde{\rho} \in \mathfrak{A}_{C_2,C_3}$ for $0 < \epsilon_0 \ll 1$.*

Proof. For $q \in \mathfrak{A}_{C_2,C_3}$, (2.20) is strictly elliptic. Thus, from Theorem 9.15 of [11], there exists a unique solution $\tilde{\rho} \in W^{2,2}([\epsilon_0, 1])$ of (2.20). It remains to prove that $\tilde{\rho} \in \mathfrak{A}_{C_2,C_3}$ for $0 < \epsilon_0 \ll 1$.

Let $\chi(r) := \tilde{\rho}(r) - A(r)$, where $A(r)$ is given in Theorem 1.1. From the definition of $A(r)$ and the boundary condition (2.20)(b), we get $\chi(\epsilon_0) = \chi(1) = 0$. Then, multiplying (2.20)(a) by $-r^2\chi$ and integrating it over $[\epsilon_0, 1]$, we have

$$\int_{\epsilon_0}^{1} \Big[\Big(\frac{\partial F}{\partial q}(q, \tilde{J})\tilde{\rho}_r\Big)_r + \frac{2}{r}\frac{\partial F}{\partial q}(q, \tilde{J})\tilde{\rho}_r + \Big(\frac{\partial F}{\partial \tilde{J}}(q, \tilde{J})\tilde{J}_r\Big)_r + \frac{2}{r}\frac{\partial F}{\partial \tilde{J}}\tilde{J}_r\Big](-r^2\chi)dr$$
$$+ \int_{\epsilon_0}^{1} (-\tilde{\rho} + D)(-r^2\chi)dr - \int_{\epsilon_0}^{1} \frac{\tilde{J}}{q^2} q_r(-r^2\chi)dr = 0. \tag{2.23}$$

For the first integral in (2.23), in view of the boundary condition $\chi(\epsilon_0) = \chi(1) = 0$ and Lemma 2.2, by using integration by parts we get

$$\int_{\epsilon_0}^{1} \Big[\Big(\frac{\partial F}{\partial q}(q, \tilde{J})\tilde{\rho}_r\Big)_r + \frac{2}{r}\frac{\partial F}{\partial q}(q, \tilde{J})\tilde{\rho}_r + \Big(\frac{\partial F}{\partial \tilde{J}}(q, \tilde{J})\tilde{J}_r\Big)_r + \frac{2}{r}\frac{\partial F}{\partial \tilde{J}}\tilde{J}_r\Big](-r^2\chi)dr$$
$$= \int_{\epsilon_0}^{1} \Big(r^2\frac{\partial F}{\partial q}\tilde{\rho}_r + r^2\frac{\partial F}{\partial \tilde{J}}\tilde{J}_r\Big)\chi_r dr \geq \frac{c_1'}{2C_+}\int_{\epsilon_0}^{1} r^2\chi_r^2 dr - C|A_r|_0^2 - CJ_0^4. \tag{2.24}$$

Here we have used the fact $\tilde{J}_r = \frac{-2}{r}\tilde{J}$ and the result

$$\frac{\partial F}{\partial q} = \frac{1}{q}\Big(P'(q) - \frac{\tilde{J}^2}{q^2}\Big) \geq \frac{c_1'}{C_+} \tag{2.25}$$

from $F(q, \tilde{J}) = h(q) + \frac{\tilde{J}^2}{2q^2}$, where $h'(q) = \frac{P'(q)}{q}$, and c_1' is given in Remark 2.2.

Clearly, it holds that

$$\int_{\epsilon_0}^{1} (-\tilde{\rho} + D)(-r^2\chi)dr = \int_{\epsilon_0}^{1} r^2\chi^2 dr + \int_{\epsilon_0}^{1} (A - D)r^2\chi dr \geq \frac{1}{2}\int_{\epsilon_0}^{1} (r\chi)^2 dr - \frac{1}{2}C_1^2\epsilon_0^2. \tag{2.26}$$

On the other hand, with the definition of $\mathfrak{A}_{C_2,C_3}$ and Lemma 2.2, it is easy to see

$$\int_{\epsilon_0}^{1} \frac{\tilde{J}}{q^2} q_r(-r^2\chi)dr \leq \mu_1\|\chi\|_{L_r^2}^2 + \frac{C\epsilon_0^{2\alpha}}{\mu_1}(C_2^2 + |A_r|_0^2) \tag{2.27}$$

for a suitably small constant $\mu_1 > 0$.

Finally, substituting (2.24)-(2.27) into (2.23), we have

$$\|\chi\|_{L_r^2}^2 + \|\chi_r\|_{L_r^2}^2 \le C[|A_r|_0^2 + J_0^4 + C_1^2\epsilon_0^2 + \epsilon_0^{2\alpha}(C_2^2 + |A_r|_0^2)]$$
$$\le \hat{C}_4(\epsilon_0^{2\alpha}(1+C_2)^2 + C_1^2\epsilon_0^2). \tag{2.28}$$

Let

$$C_2 := 2\sqrt{\hat{C}_4}C_1 \tag{2.29}$$

and let ϵ_0 be small enough to satisfy

$$\frac{(1+C_2)\epsilon_0^{\alpha-1}}{C_1} \le 1, \tag{2.30}$$

then we get

$$\|\tilde{\rho} - A\|_1 = \|\chi\| + \|\chi_r\| \le 2\sqrt{\hat{C}_4}C_1 = C_2, \tag{2.31}$$

where C_1 is some positive constant to be determined.

To derive the estimate of $r\chi_{rr}$, we multiply (2.20)(a) by $r^2\chi_{rr}$ and take an integration of it over $[\epsilon_0, 1]$ to get

$$\int_{\epsilon_0}^{1}\Big(\frac{\partial F}{\partial q}(q,\tilde{J})\tilde{\rho}_r\Big)_r r^2\chi_{rr}dr + \int_{\epsilon_0}^{1}\frac{2}{r}\frac{\partial F}{\partial q}(q,\tilde{J})\tilde{\rho}_r r^2\chi_{rr}dr + \int_{\epsilon_0}^{1}(-\tilde{\rho}+D)r^2\chi_{rr}dr$$
$$+\int_{\epsilon_0}^{1}\Big[\Big(\frac{\partial F}{\partial \tilde{J}}(q,\tilde{J})\tilde{J}_r\Big)_r + \frac{2}{r}\frac{\partial F}{\partial \tilde{J}}\tilde{J}_r\Big]r^2\chi_{rr}dr + \int_{\epsilon_0}^{1}\frac{\tilde{J}}{q^2}q_r r^2\chi_{rr}dr = 0. \tag{2.32}$$

From the definition of $\mathfrak{A}_{C_2,C_3}$ and (2.31), it holds that

$$\|\tilde{\rho}_r\| \le C_2 + \|A_r\| \le C\epsilon_0^\alpha + C_2 \tag{2.33}$$

and

$$|rq_r|_0 \le 2(\|rq_r\| + \|q_r\| + \|rq_{rr}\|) \le 4(C\epsilon_0^\alpha + C_2 + C_3), \tag{2.34}$$

which yields, in view of (2.25), that

$$\int_{\epsilon_0}^{1}\Big(\frac{\partial F}{\partial q}(q,\tilde{J})\tilde{\rho}_r\Big)_r r^2\chi_{rr}dr$$
$$= \int_{\epsilon_0}^{1}\frac{\partial F}{\partial q}r^2\chi_{rr}^2dr + \int_{\epsilon_0}^{1}\Big(\frac{P''(q)}{q} - \frac{P'(q)}{q^2} + \frac{3\tilde{J}^2}{q^4}\Big)q_r\tilde{\rho}_r r^2\chi_{rr}dr + \int_{\epsilon_0}^{1}\frac{4\tilde{J}^2}{rq^3}\tilde{\rho}_r r^2\chi_{rr}dr$$

$$\geq \frac{1}{2}\int_{\epsilon_0}^{1} \frac{\partial F}{\partial q} r^2 \chi_{rr}^2 dr - C(|rq_r|_0^2 \|\tilde{\rho}_r\|^2 + J_0^4 \|\tilde{\rho}_r\|^2)$$

$$\geq \frac{1}{2}\int_{\epsilon_0}^{1} \frac{\partial F}{\partial q} r^2 \chi_{rr}^2 dr - (C(1+C_2+C_3)\epsilon_0^{\alpha} + C_2^2 + C_2C_3)^2 - C\epsilon_0^{4\alpha-4}(1+C_2)^2 \tag{2.35}$$

and

$$\begin{aligned}\int_{\epsilon_0}^{1} \frac{2}{r}\frac{\partial F}{\partial q}(q,\tilde{J})\tilde{\rho}_r r^2 \chi_{rr} dr &= \int_{\epsilon_0}^{1} \frac{2}{r}\frac{\partial F}{\partial q}(q,\tilde{J})\chi_r r^2 \chi_{rr} dr + \int_{\epsilon_0}^{1} \frac{2}{r}\frac{\partial F}{\partial q}(q,\tilde{J})A_r r^2 \chi_{rr} dr \\ &\leq \mu_1 \|\chi_{rr}\|_{L_r^2}^2 + \frac{C}{\mu_1}(1+C_2)^2,\end{aligned} \tag{2.36}$$

where μ_1 is given in (2.27).

For the third integral in (2.32), the constrains on $D(r)$

$$\max_{r\in[\epsilon_0,1]}\{r|A(r)-D(r)|\} \leq C_1\epsilon_0$$

given in Theorem 1.1 yields that

$$\begin{aligned}\int_{\epsilon_0}^{1}(-\tilde{\rho}+D)r^2\chi_{rr}dr &= \int_{\epsilon_0}^{1}(-\tilde{\rho}+A)r^2\chi_{rr}dr - \int_{\epsilon_0}^{1}(A-D)r^2\chi_{rr}dr \\ &\leq \mu_1\|\chi_{rr}\|_{L_r^2}^2 + \frac{C}{\mu_1}\|\chi\|_{L_r^2}^2 + \frac{C}{\mu_1}|r(A(r)-D(r)|_0^2 \\ &\leq \mu_1\|\chi_{rr}\|_{L_r^2}^2 + \frac{C}{\mu_1}(C_1+C_2)^2\epsilon_0^2.\end{aligned} \tag{2.37}$$

Moreover, with $\tilde{J}_r = \frac{-2\tilde{J}}{r}$ and $\tilde{J}_{rr} = \frac{6\tilde{J}}{r^2}$, we get

$$\begin{aligned}&\int_{\epsilon_0}^{1}\Big[\Big(\frac{\partial F}{\partial \tilde{J}}(q,\tilde{J})\tilde{J}_r\Big)_r + \frac{2}{r}\frac{\partial F}{\partial \tilde{J}}\tilde{J}_r\Big]r^2\chi_{rr}dr \\ &= \int_{\epsilon_0}^{1}\frac{4\tilde{J}^2}{q^3}q_r r\chi_{rr}dr + \int_{\epsilon_0}^{1}\frac{6\tilde{J}^2}{q^2}\chi_{rr}dr \\ &\leq \mu_1\|\chi_{rr}\|_{L_r^2}^2 + \frac{CJ_0^4}{\mu_1}\|q_r\|^2 + \frac{C}{\mu_1}J_0^4\epsilon_0^{-2} \\ &\leq \mu_1\|\chi_{rr}\|_{L_r^2}^2 + C(1+C_2)^2.\end{aligned} \tag{2.38}$$

Similarly, we have

$$\int_{\epsilon_0}^{1} \frac{\tilde{J}}{q^2} q_r r^2 \chi_{rr} dr \le \mu_1 \|\chi_{rr}\|^2_{L^2_r} + \frac{C}{\mu_1} \epsilon_0^{2\alpha} (1 + C_2)^2. \tag{2.39}$$

Finally, we substitute (2.35)-(2.39) into (2.32) and hence that

$$\|r\chi_{rr}\| \le \hat{C}_5[(1 + C_1 + C_2 + C_3)\epsilon_0 + 1 + C_2 + C_2^2 + C_2 C_3]. \tag{2.40}$$

Now, we wish to choose suitable C_1, C_2 and C_3 such that $\tilde{\rho} \in \mathfrak{A}_{C_2,C_3}$.

Indeed, for any $K_1 > 0$, $\tilde{C}_0 > 0$, let

$$\begin{aligned} K_2 &:= \hat{C}_5(2 + 2\sqrt{\hat{C}_4}\tilde{C}_0(1 + K_1) + 4\hat{C}_4\tilde{C}_0^2), \\ C_1 &:= \frac{K_1\tilde{C}_0}{1 + K_1 + K_2} \end{aligned}$$

and

$$C_3 := K_2.$$

Then, we deduce from (2.29) that

$$C_2 = 2\sqrt{\hat{C}_4} C_1 \le \frac{2\sqrt{\hat{C}_4}\tilde{C}_0 K_1}{1 + K_1 + K_2} \le 2\sqrt{\hat{C}_4}\tilde{C}_0$$

and

$$C_2 + C_2^2 + C_2 C_3 \le 2\sqrt{\hat{C}_4}\tilde{C}_0 + 4\hat{C}_4\tilde{C}_0^2 + \frac{2\sqrt{\hat{C}_4}\tilde{C}_0 K_1}{1 + K_1 + K_2} K_2 \le 2\sqrt{\hat{C}_4}\tilde{C}_0(1 + K_1) + 4\hat{C}_4\tilde{C}_0^2. \tag{2.41}$$

Therefore, we substitute (2.41) into (2.40) to get

$$\|r\tilde{\rho}_{rr}\| = \|r\chi_{rr}\| \le \hat{C}_5(2 + 2\sqrt{\hat{C}_4}\tilde{C}_0(1 + K_1) + 4\hat{C}_4\tilde{C}_0^2) = K_2 = C_3$$

provided that

$$(1 + C_1 + C_2 + C_3)\epsilon_0 < 1$$

for $\epsilon_0 \ll 1$.

Next we will show that $C_- \le \tilde{\rho} \le C_+$, where C_-, C_+ are given in (1.17). Define

$$D_1(r) := D(r) + \left(\frac{\partial F}{\partial \tilde{J}}(q, \tilde{J})\tilde{J}_r\right)_r + \frac{2}{r}\frac{\partial F}{\partial \tilde{J}}\tilde{J}_r - \frac{\tilde{J}}{q^2} q_r.$$

Then, we can rewrite (2.20)(a) as

$$\Big(\frac{\partial F}{\partial q}(q,\tilde{J})\Big)_r\tilde{\rho}_r+\frac{\partial F}{\partial q}(q,\tilde{J})\tilde{\rho}_{rr}+\frac{2}{r}\frac{\partial F}{\partial q}(q,\tilde{J})\tilde{\rho}_r-\tilde{\rho}=-D_1(r). \tag{2.42}$$

On one hand, the conditions $0<\tilde{c}\le D(r)$ and $\max\limits_{r\in[\epsilon_0,1]}\{r|A(r)-D(r)|\}<C_1\epsilon_0$ assure that $0<c_-\le D(r)\le c_+$ for some positive constants c_- and c_+. On the other hand, there holds that

$$\begin{aligned}&\Big|\Big(\frac{\partial F}{\partial \tilde{J}}(q,\tilde{J})\tilde{J}_r\Big)_r+\frac{2}{r}\frac{\partial F}{\partial \tilde{J}}\tilde{J}_r-\frac{\tilde{J}}{q^2}q_r\Big|\\&=\Big|\Big(\frac{\tilde{J}_r^2+\tilde{J}\tilde{J}_{rr}}{q^2}-\frac{2\tilde{J}\tilde{J}_rq_r}{q^3}\Big)\tilde{J}_r+\frac{2}{r}\frac{\tilde{J}\tilde{J}_r\tilde{J}_r}{q^2}-\frac{\tilde{J}\tilde{J}_r\tilde{J}_{rr}}{q^2}-\frac{\tilde{J}}{q^2}q_r\Big|\le C\epsilon_0^{\alpha-2}.\end{aligned}$$

Thus, $0<\frac{c_-}{2}\le D_1(r)\le\frac{c_-}{2}+c_+$ holds when $0<\epsilon_0\ll 1$ and $\alpha>2$.

Then, setting $\bar{\rho}_1=\tilde{\rho}-C_+$ we get

$$\Big(\frac{\partial F}{\partial q}(q,\tilde{J})\Big)_r\bar{\rho}_{1r}+\frac{\partial F}{\partial q}(q,\tilde{J})\bar{\rho}_{1rr}+\frac{2}{r}\frac{\partial F}{\partial q}(q,\tilde{J})\bar{\rho}_{1r}-\bar{\rho}_1=-D_1(r)+C_+. \tag{2.43}$$

We assume that $\bar{\rho}_1$ achieves the maximum value at point x_1.

We claim that $\bar{\rho}_1(x_1)\le 0$. If not, $\bar{\rho}_1(x_1)>0$, in view of the definition of C_+, which yields that $x_1\in(\epsilon_0,1)$. Then it follows that $\bar{\rho}_{1r}(x_1)=0$ and $\bar{\rho}_{1rr}(x_1)\le 0$, which leads to

$$\Big(\Big(\frac{\partial F}{\partial q}(q,\tilde{J})\Big)_r\bar{\rho}_{1r}+\frac{\partial F}{\partial q}(q,\tilde{J})\bar{\rho}_{1rr}+\frac{2}{r}\frac{\partial F}{\partial q}(q,\tilde{J})\bar{\rho}_{1r}-\bar{\rho}_1\Big)(x_1)<0. \tag{2.44}$$

However, the value of terms on the right-hand side of (2.43) at point x_1 is equal to $-D_1(x_1)+C_+\ge 0$, which contradicts (2.44). Thus, $\bar{\rho}_1(x_1)\le 0$ holds, which implies that $\tilde{\rho}(r)\le C_+$.

Similarly, by setting $\bar{\rho}_2=\tilde{\rho}-C_-$, we may show that $\tilde{\rho}(r)\ge C_-$. Thus, the proof is complete. □

Next, we turn to the BVP of nonlinear elliptic equation (2.19).

Lemma 2.4. *For $0<\epsilon_0\ll 1$, (2.19) has a solution $\tilde{\rho}\in\mathfrak{A}_{C_2,C_3}$. Furthermore, the stationary system (2.1)-(2.2) has a pair of solution $(\tilde{\rho},\tilde{J}[\tilde{\rho}],\tilde{\Psi}[\tilde{\rho}])(r)$ with $C_-\le\tilde{\rho}\le C_+$ and $\tilde{J}[\tilde{\rho}]\le J_0$, where J_0 is given by (1.15).*

Proof. To do this, we first define a mapping $\tilde{S}:\mathfrak{A}_{C_2,C_3}\to\mathfrak{A}_{C_2,C_3}$ with $\tilde{\rho}(r)=\tilde{S}(q)$ given by (2.20). And, we claim that $\tilde{S}$ is continuous.

Indeed, given $q_1,q_2\in\mathfrak{A}_{C_2,C_3}$, and $\tilde{J}_1=\tilde{J}[q_1]$, $\tilde{J}_2=\tilde{J}[q_2]$, then $\tilde{\rho}_1=\tilde{S}(q_1)$, $\tilde{\rho}_2=\tilde{S}(q_2)$ satisfy

$$\begin{aligned}&\Big(r^2\frac{\partial F}{\partial q_1}(q_1,\tilde{J}_1)(\tilde{\rho}_{1r}-\tilde{\rho}_{2r})\Big)_r-r^2(\tilde{\rho}_1-\tilde{\rho}_2)\\&=-\Big(r^2\Big(\frac{\partial F}{\partial q_1}(q_1,\tilde{J}_1)-\frac{\partial F}{\partial q_2}(q_2,\tilde{J}_2)\Big)\tilde{\rho}_{2r}\Big)_r+f_1-f_2,\end{aligned} \tag{2.45}$$

where

$$f_i = -\Big(r^2 \frac{\partial F}{\partial \tilde{J}_i}(q_i, \tilde{J}_i)\tilde{J}_{ir}\Big)_r + \frac{r^2 \tilde{J}_i}{q_i^2} q_{ir} - D, \quad i = 1, 2.$$

Then, we multiply (2.45) by $-(\tilde{\rho}_1 - \tilde{\rho}_2)$ and take the integration over $[\epsilon_0, 1]$ by parts to get

$$\int_{\epsilon_0}^{1} r^2 \frac{\partial F}{\partial q_1}(q_1, \tilde{J}_1)(\tilde{\rho}_{1r} - \tilde{\rho}_{2r})^2 dr + \int_{\epsilon_0}^{1} r^2 \Big(\frac{\partial F}{\partial q_1}(q_1, \tilde{J}_1) - \frac{\partial F}{\partial q_2}(q_2, \tilde{J}_2)\Big)\tilde{\rho}_{2r}(\tilde{\rho}_1 - \tilde{\rho}_2)_r dr$$
$$+ \int_{\epsilon_0}^{1} r^2 (\tilde{\rho}_1 - \tilde{\rho}_2)^2 dr = -\int_{\epsilon_0}^{1} (f_1 - f_2)(\tilde{\rho}_1 - \tilde{\rho}_2) dr. \tag{2.46}$$

Notice that

$$\Big|\frac{\partial F}{\partial q_1}(q_1, \tilde{J}_1) - \frac{\partial F}{\partial q_2}(q_2, \tilde{J}_2)\Big| = \Big|\frac{P'(q_1)}{q_1} - \frac{\tilde{J}_1^2}{q_1^3} - \frac{P'(q_2)}{q_2} + \frac{\tilde{J}_2^2}{q_2^3}\Big| \le C(|q_1 - q_2| + |\tilde{J}_1 - \tilde{J}_2|),$$

together with Lemma 2.2, we have

$$\begin{aligned}
&\Big|\int_{\epsilon_0}^{1} r^2 \Big(\frac{\partial F}{\partial q_1}(q_1, \tilde{J}_1) - \frac{\partial F}{\partial q_2}(q_2, \tilde{J}_2)\Big)\tilde{\rho}_{2r}(\tilde{\rho}_{1r} - \tilde{\rho}_{2r}) dr\Big| \\
&\le \mu_2 \| r\tilde{\rho}_{1r} - r\tilde{\rho}_{2r}\|^2 + \frac{C}{\mu_2}\Big|\frac{\partial F}{\partial q_1} - \frac{\partial F}{\partial q_2}\Big|_0^2 \| r\tilde{\rho}_{2r}\|^2 \\
&\le \mu_2 \| r\tilde{\rho}_{1r} - r\tilde{\rho}_{2r}\|^2 + \frac{C\epsilon_0^2}{\mu_2}\| q_1 - q_2\|_1^2
\end{aligned} \tag{2.47}$$

for a suitably small constant $\mu_2 > 0$.

On the other hand, an easy computation shows that

$$\begin{aligned}
&-\int_{\epsilon_0}^{1} (f_1 - f_2)(\tilde{\rho}_1 - \tilde{\rho}_2) dr \\
&= -\int_{\epsilon_0}^{1} \Big[\Big(-r^2 \frac{\partial F}{\partial \tilde{J}_1}(q_1, \tilde{J}_1)\tilde{J}_{1r} + r^2 \frac{\partial F}{\partial \tilde{J}_2}(q_2, \tilde{J}_2)\tilde{J}_{2r}\Big)_r + \frac{r^2 \tilde{J}_1}{q_1^2} q_{1r} - \frac{r^2 \tilde{J}_2}{q_2^2} q_{2r}\Big](\tilde{\rho}_1 - \tilde{\rho}_2) dr \\
&= \underbrace{\int_{\epsilon_0}^{1} \Big(\frac{2r\tilde{J}_1^2}{q_1^2} - \frac{2r\tilde{J}_2^2}{q_2^2}\Big)(\tilde{\rho}_{1r} - \tilde{\rho}_{2r}) dr}_{\triangleq I_1} \underbrace{- \int_{\epsilon_0}^{1} \Big(\frac{r^2 \tilde{J}_1}{q_1^2} q_{1r} - \frac{r^2 \tilde{J}_2}{q_2^2} q_{2r}\Big)(\tilde{\rho}_1 - \tilde{\rho}_2) dr.}_{\triangleq I_2}
\end{aligned}$$

Then, it is easy to see that

$$\begin{aligned}|I_1| &= \Big|\int_{\epsilon_0}^{1}\Big[\frac{2(\tilde{J}_1+\tilde{J}_2)(\tilde{J}_1-\tilde{J}_2)}{q_1^2}+2\tilde{J}_2^2\Big(\frac{1}{q_1^2}-\frac{1}{q_2^2}\Big)\Big](r\tilde{\rho}_{1r}-r\tilde{\rho}_{2r})dr\Big|\\ &\le \mu_2\|r\tilde{\rho}_{1r}-r\tilde{\rho}_{2r}\|^2+\frac{C}{\mu_2}(\epsilon_0^{2\alpha-2}\|\tilde{J}_1-\tilde{J}_2\|^2+\epsilon_0^{4\alpha-4}\|q_1-q_2\|^2)\\ &\le \mu_2\|r\tilde{\rho}_{1r}-r\tilde{\rho}_{2r}\|^2+\frac{C}{\mu_2}\epsilon_0^{2\alpha}\|q_1-q_2\|^2\end{aligned}$$

and

$$\begin{aligned}|I_2| &= \Big|\int_{\epsilon_0}^{1}\Big[-\frac{(r\tilde{J}_1-r\tilde{J}_2)q_{1r}+r\tilde{J}_2(q_{1r}-q_{2r})}{q_1^2}-r\tilde{J}_2q_{2r}\Big(\frac{1}{q_1^2}-\frac{1}{q_2^2}\Big)\Big](r\tilde{\rho}_1-r\tilde{\rho}_2)dr\Big|\\ &\le \mu_2\|r\tilde{\rho}_1-r\tilde{\rho}_2\|^2+\frac{C}{\mu_2}\epsilon_0^{2\alpha}\|q_1-q_2\|_1^2,\end{aligned}$$

which show that

$$\Big|-\int_{\epsilon_0}^{1}(f_1-f_2)(\tilde{\rho}_1-\tilde{\rho}_2)dr\Big| \le \mu_2\|r\tilde{\rho}_1-r\tilde{\rho}_2\|^2+\mu_2\|r\tilde{\rho}_{1r}-r\tilde{\rho}_{2r}\|^2+\frac{C}{\mu_2}\epsilon_0^{2\alpha}\|q_1-q_2\|_1^2. \tag{2.48}$$

Substituting (2.47)-(2.48) into (2.46), together with $\frac{\partial F}{\partial q_1}=\frac{1}{q_1}\Big(P'(q_1)-\frac{\tilde{J}_1^2}{q_1^2}\Big)\ge\frac{c_1'}{C_+}$, yields that

$$\|r\tilde{\rho}_1-r\tilde{\rho}_2\|^2+\|r\tilde{\rho}_{1r}-r\tilde{\rho}_{2r}\|^2\le C\epsilon_0^2\|q_1-q_2\|_1^2. \tag{2.49}$$

Now, to estimate the $\|\tilde{\rho}_{1rr}-\tilde{\rho}_{2rr}\|$, we multiply (2.45) by $\tilde{\rho}_{1rr}-\tilde{\rho}_{2rr}$ again and integrate it over $[\epsilon_0,1]$ to get

$$\begin{aligned}&\int_{\epsilon_0}^{1}\Big(r^2\frac{\partial F}{\partial q_1}(q_1,\tilde{J}_1)(\tilde{\rho}_{1r}-\tilde{\rho}_{2r})\Big)_r(\tilde{\rho}_{1rr}-\tilde{\rho}_{2rr})dr+\int_{\epsilon_0}^{1}-r^2(\tilde{\rho}_1-\tilde{\rho}_2)(\tilde{\rho}_{1rr}-\tilde{\rho}_{2rr})dr\\ &\quad+\int_{\epsilon_0}^{1}\Big(r^2\Big(\frac{\partial F}{\partial q_1}(q_1,\tilde{J}_1)-\frac{\partial F}{\partial q_2}(q_2,\tilde{J}_2)\Big)\tilde{\rho}_{2r}\Big)_r(\tilde{\rho}_{1rr}-\tilde{\rho}_{2rr})dr\\ &=\int_{\epsilon_0}^{1}(f_1-f_2)(\tilde{\rho}_{1rr}-\tilde{\rho}_{2rr})dr.\end{aligned} \tag{2.50}$$

A straightforward computation shows that

$$\begin{aligned}
&\int_{\epsilon_0}^{1} \Big(r^2 \frac{\partial F}{\partial q_1}(q_1, \tilde{J}_1)(\tilde{\rho}_{1r} - \tilde{\rho}_{2r})\Big)_r (\tilde{\rho}_{1rr} - \tilde{\rho}_{2rr}) dr \\
&= \int_{\epsilon_0}^{1} 2r \frac{\partial F}{\partial q_1}(\tilde{\rho}_{1r} - \tilde{\rho}_{2r})(\tilde{\rho}_{1rr} - \tilde{\rho}_{2rr}) dr + \int_{\epsilon_0}^{1} r^2 \frac{\partial F}{\partial q_1}(\tilde{\rho}_{1rr} - \tilde{\rho}_{2rr})^2 dr \\
&\quad \underbrace{+ \int_{\epsilon_0}^{1} r^2 \Big(\frac{\partial^2 F}{\partial q_1^2} q_{1r} + \frac{\partial^2 F}{\partial q_1 \partial \tilde{J}_1} \tilde{J}_{1r}\Big)(\tilde{\rho}_{1r} - \tilde{\rho}_{2r})(\tilde{\rho}_{1rr} - \tilde{\rho}_{2rr}) dr\,.}_{\triangleq I_3}
\end{aligned}$$

Using the fact of (2.34), we obtain

$$\begin{aligned}
|I_3| &= \Big| \int_{\epsilon_0}^{1} r^2 \Big(\frac{P''(q_1) q_{1r}}{q_1} - \frac{P'(q_1) q_{1r}}{q_1^2} + \frac{3\tilde{J}_1^2 q_{1r}}{q_1^4} - \frac{2\tilde{J}_1 \tilde{J}_{1r}}{q_1^3} \Big)(\tilde{\rho}_{1r} - \tilde{\rho}_{2r})(\tilde{\rho}_{1rr} - \tilde{\rho}_{2rr}) dr \Big| \\
&\le \mu_3 \|r\tilde{\rho}_{1rr} - r\tilde{\rho}_{2rr}\|^2 + \frac{C}{\mu_3}\Big(|q_{1r}|_0^2 + \Big|\frac{\tilde{J}_1}{r}\Big|_0^2 \Big) \|r\tilde{\rho}_{1r} - r\tilde{\rho}_{2r}\|^2 \\
&\le \mu_3 \|r\tilde{\rho}_{1rr} - r\tilde{\rho}_{2rr}\|^2 + \frac{C}{\mu_3} \|q_1 - q_2\|_1^2,
\end{aligned}$$

together with (2.49), which leads to

$$\begin{aligned}
&\Big| \int_{\epsilon_0}^{1} \Big(r^2 \frac{\partial F}{\partial q_1}(q_1, \tilde{J}_1)(\tilde{\rho}_{1r} - \tilde{\rho}_{2r})\Big)_r (\tilde{\rho}_{1rr} - \tilde{\rho}_{2rr}) dr \Big| \\
&\ge \frac{1}{2} \int_{\epsilon_0}^{1} \frac{\partial F}{\partial q_1}(r\tilde{\rho}_{1rr} - r\tilde{\rho}_{2rr})^2 dr - C\|q_1 - q_2\|_1^2. \qquad (2.51)
\end{aligned}$$

For the second integration of (2.50), it follows from (2.49) that

$$\Big| \int_{\epsilon_0}^{1} -r^2(\tilde{\rho}_1 - \tilde{\rho}_2)(\tilde{\rho}_{1rr} - \tilde{\rho}_{2rr}) dr \Big| \le \mu_3 \|r\tilde{\rho}_{1rr} - r\tilde{\rho}_{2rr}\|^2 + \frac{C}{\mu_3}\epsilon_0^2 \|q_1 - q_2\|_1^2 \qquad (2.52)$$

for a suitably small constant $\mu_3 > 0$ to be specified.

For the third integration of (2.50), it holds that

$$
\begin{aligned}
&\int_{\epsilon_0}^{1}\Big(r^2\Big(\frac{\partial F}{\partial q_1}(q_1,\tilde J_1)-\frac{\partial F}{\partial q_2}(q_2,\tilde J_2)\Big)\tilde\rho_{2r}\Big)_r(\tilde\rho_{1rr}-\tilde\rho_{2rr})dr\\
&=\int_{\epsilon_0}^{1}2r\Big(\frac{\partial F}{\partial q_1}-\frac{\partial F}{\partial q_2}\Big)\tilde\rho_{2r}(\tilde\rho_{1rr}-\tilde\rho_{2rr})dr+\int_{\epsilon_0}^{1}r^2\Big(\frac{\partial F}{\partial q_1}-\frac{\partial F}{\partial q_2}\Big)\tilde\rho_{2rr}(\tilde\rho_{1rr}-\tilde\rho_{2rr})dr\\
&\quad+\underbrace{\int_{\epsilon_0}^{1}r^2\Big(\frac{\partial F}{\partial q_1}-\frac{\partial F}{\partial q_2}\Big)_r\tilde\rho_{2r}(\tilde\rho_{1rr}-\tilde\rho_{2rr})dr\,.}_{\triangleq I_4}
\end{aligned}
$$

Then, we deduce easily from (2.34) that

$$
\begin{aligned}
|I_4|\le&\Big|\int_{\epsilon_0}^{1}r^2\Big(\frac{P''(q_1)q_{1r}}{q_1}-\frac{P'(q_1)q_{1r}}{q_1^2}-\frac{P''(q_2)q_{2r}}{q_2}+\frac{P'(q_2)q_{2r}}{q_2^2}\Big)\tilde\rho_{2r}(\tilde\rho_{1rr}-\tilde\rho_{2rr})dr\\
&+\int_{\epsilon_0}^{1}r^2\Big(\frac{3\tilde J_1^2q_{1r}}{q_1^4}-\frac{2\tilde J_1\tilde J_{1r}}{q_1^3}-\frac{3\tilde J_2^2q_{2r}}{q_2^4}+\frac{2\tilde J_2\tilde J_{2r}}{q_2^3}\Big)\tilde\rho_{2r}(\tilde\rho_{1rr}-\tilde\rho_{2rr})dr\Big|\\
\le&\,\mu_3\|r\tilde\rho_{1rr}-r\tilde\rho_{2rr}\|^2+\frac{C}{\mu_3}\|q_1-q_2\|_1^2,
\end{aligned}
$$

which implies that

$$
\begin{aligned}
&\Big|\int_{\epsilon_0}^{1}\Big(r^2\Big(\frac{\partial F}{\partial q_1}(q_1,\tilde J_1)-\frac{\partial F}{\partial q_2}(q_2,\tilde J_2)\Big)\tilde\rho_{2r}\Big)_r(\tilde\rho_{1rr}-\tilde\rho_{2rr})dr\Big|\\
&\le\mu_3\|r\tilde\rho_{1rr}-r\tilde\rho_{2rr}\|^2+\frac{C}{\mu_3}\Big(\Big|\frac{\partial F}{\partial q_1}-\frac{\partial F}{\partial q_2}\Big|_0^2\|\tilde\rho_{2r}\|^2+\Big|\frac{\partial F}{\partial q_1}-\frac{\partial F}{\partial q_2}\Big|_0^2\|r\tilde\rho_{2rr}\|^2\Big)\\
&\quad+\frac{C}{\mu_3}\|q_1-q_2\|_1^2\le\mu_3\|r\tilde\rho_{1rr}-r\tilde\rho_{2rr}\|^2+\frac{C}{\mu_3}\|q_1-q_2\|_1^2.
\end{aligned}
\tag{2.53}
$$

For the last integration of (2.50), we use Lemma 2.2 to obtain

$$
\begin{aligned}
&\Big|\int_{\epsilon_0}^{1}(f_1-f_2)(\tilde\rho_{1rr}-\tilde\rho_{2rr})dr\Big|\\
&=\Big|\int_{\epsilon_0}^{1}\Big(-r^2\frac{\partial F}{\partial\tilde J_1}\tilde J_{1r}+r^2\frac{\partial F}{\partial\tilde J_2}\tilde J_{2r}\Big)_r(\tilde\rho_{1rr}-\tilde\rho_{2rr})dr
\end{aligned}
$$

$$+\int_{\epsilon_0}^{1}\Big(\frac{r^2\tilde{J}_1}{q_1^2}q_{1r}-\frac{r^2\tilde{J}_2}{q_2^2}q_{2r}\Big)(\tilde{\rho}_{1rr}-\tilde{\rho}_{2rr})dr\Big|$$

$$=\Big|\int_{\epsilon_0}^{1}\Big(-\frac{4r\tilde{J}_1^2q_{1r}}{q_1^3}+\frac{4r\tilde{J}_2^2q_{2r}}{q_2^3}-\frac{6\tilde{J}_1^2}{q_1^2}+\frac{6\tilde{J}_2^2}{q_2^2}+\frac{r^2\tilde{J}_1}{q_1^2}q_{1r}-\frac{r^2\tilde{J}_2}{q_2^2}q_{2r}\Big)(\tilde{\rho}_{1rr}-\tilde{\rho}_{2rr})dr\Big|$$

$$\le\Big(\mu_3+\Big(\frac{J_0}{\epsilon_0}\Big)^2\Big)\|r\tilde{\rho}_{1rr}-r\tilde{\rho}_{2rr}\|^2+\frac{C}{\mu_3}\|q_1-q_2\|_1^2. \tag{2.54}$$

Finally, we substitute (2.51)-(2.54) into (2.50) to get

$$\|r\tilde{\rho}_{1rr}-r\tilde{\rho}_{2rr}\|^2\le C\|q_1-q_2\|_1^2 \tag{2.55}$$

for ϵ_0 and μ_3 satisfy $C\epsilon_0^{\alpha-2}+5\mu_3<\frac{c_1'}{2C_+}$, which can be reached by the smallness of ϵ_0 and μ_3.

Accordingly, (2.49) and (2.55) imply that

$$\|\tilde{\rho}_1-\tilde{\rho}_2\|_2\le C\epsilon_0^{-1}\|q_1-q_2\|_1,$$

which shows that $\tilde{S}$ is a continuous mapping in $\mathfrak{A}_{C_2,C_3}$ for given $\epsilon_0\ll 1$.

Having checked this claim, we can now return to the proof of Lemma 2.4. In fact, since $H^2([\epsilon_0,1])$ is a compact embedding into $C^1([\epsilon_0,1])$, then $\mathfrak{A}_{C_2,C_3}$ is a compact convex set of $C^1([\epsilon_0,1])$. Together with the Schauder fixed point theorem, there exists a fixed point $\tilde{\rho}\in\mathfrak{A}_{C_2,C_3}$ such that $\tilde{S}(\tilde{\rho})=\tilde{\rho}$. That is, $\tilde{\rho}$ is a solution of (2.19) in $\mathfrak{A}_{C_2,C_3}$.

Finally, thanks to Lemma 2.1, we see at once that $(\tilde{\rho},\tilde{J}[\tilde{\rho}],\tilde{\Psi}[\tilde{\rho}])(r)$ is a pair of solution of (2.1)-(2.2) with $\tilde{\rho}\in\mathfrak{A}_{C_2,C_3}$ and $\tilde{J}[\tilde{\rho}]\le J_0$. Thus, the proof is complete. □

Based on the Lemma 2.1 and Lemma 2.4, we will complete the proof of Theorem 1.1.

Proof of Theorem 1.1. Thanks to Lemma 2.1 and Lemma 2.4, it suffices to show the uniqueness of solution of (2.19) with $C_-\le\tilde{\rho}\le C_+$.

Let $\tilde{\rho}^{(1)}$, $\tilde{\rho}^{(2)}$ be two solutions of (2.19) with $C_-\le\tilde{\rho}^{(i)}\le C_+$. Then, $\tilde{\rho}^{(1)}$, $\tilde{\rho}^{(2)}$ satisfy

$$(r^2(F(\tilde{\rho}^{(1)},\tilde{J}^{(1)})-F(\tilde{\rho}^{(2)},\tilde{J}^{(1)}))_r)_r+(r^2(F(\tilde{\rho}^{(2)},\tilde{J}^{(1)})-F(\tilde{\rho}^{(2)},\tilde{J}^{(2)}))_r)_r-r^2(\tilde{\rho}^{(1)}-\tilde{\rho}^{(2)})$$
$$+r^2\tilde{J}^{(1)}\Big(\frac{1}{\tilde{\rho}^{(1)}}-\frac{1}{\tilde{\rho}^{(2)}}\Big)_r-(r^2\tilde{J}^{(1)}-r^2\tilde{J}^{(2)})\Big(\frac{1}{\tilde{\rho}^{(2)}}\Big)_r=0, \tag{2.56}$$

where $\tilde{J}^{(i)}=\tilde{J}[\tilde{\rho}^{(i)}]$, $i=1,2$. Setting $\zeta=\tilde{\rho}^{(1)}-\tilde{\rho}^{(2)}$, then we rewrite (2.56) as

$$(r^2(k(r)\zeta)_r)_r+\Big(r^2\Big(\frac{2(\tilde{J}^{(2)})^2-2(\tilde{J}^{(1)})^2}{r(\tilde{\rho}^{(2)})^2}-\frac{\tilde{\rho}_r^{(2)}}{(\tilde{\rho}^{(2)})^3}[(\tilde{J}^{(1)})^2-(\tilde{J}^{(2)})^2]\Big)\Big)_r$$
$$-r^2\zeta+r^2\tilde{J}^{(1)}\Big(g(r)\zeta\Big)_r-(r^2\tilde{J}^{(1)}-r^2\tilde{J}^{(2)})\Big(\frac{1}{\tilde{\rho}^{(2)}}\Big)_r=0, \tag{2.57}$$

where

$$k(r) = \int_0^1 \Big(\frac{P'(\tilde{\rho}^{(2)} + v(\tilde{\rho}^{(1)} - \tilde{\rho}^{(2)}))}{\tilde{\rho}^{(2)} + v(\tilde{\rho}^{(1)} - \tilde{\rho}^{(2)})} - \frac{(\tilde{J}^{(1)})^2}{(\tilde{\rho}^{(2)} + v(\tilde{\rho}^{(1)} - \tilde{\rho}^{(2)}))^3} \Big) dv \geq \frac{c_1'}{C_+} > 0 \tag{2.58}$$

and

$$g(r) = -\int_0^1 \Big(\frac{1}{(\tilde{\rho}^{(2)} + v(\tilde{\rho}^{(1)} - \tilde{\rho}^{(2)}))^2} \Big) dv.$$

We regard (2.57) as a new work system and multiply it by $k(r)\zeta$. Consequently,

$$\|r(k(r)\zeta)_r\|^2 + \|\sqrt{k(r)}r\zeta\|^2 \leq 0 \tag{2.59}$$

follows from the classical energy method, where the boundary condition $\zeta(\epsilon_0) = \zeta(1) = 0$ and the smallness of ϵ_0 have been used. Therefore, we conclude from (2.59) that $\zeta = 0$, namely, $\tilde{\rho}^{(1)} = \tilde{\rho}^{(2)}$ for $0 < \epsilon_0 \ll 1$. Thus, the proof is complete. □

3. Stability of steady-state

In order to obtain the stability of solution $(\rho, j, \Phi)(t, r)$ of (1.6)-(1.9), we consider the perturbation equations around the steady-state solution. Denote

$$\sigma := \rho - \tilde{\rho}, \quad \eta := j - \tilde{j}, \quad \phi := \Phi - \tilde{\Phi}, \quad \mathbb{U} := \begin{pmatrix} \sigma \\ \eta \end{pmatrix}, \tag{3.1}$$

where $(\tilde{\rho}, \tilde{j}, \tilde{\Phi})(r)$ is the solution of (2.1)-(2.2) given in Theorem 1.1. Then $(\sigma, \eta, \phi)(t, r)$ satisfies the perturbation systems

$$\begin{cases} \sigma_t + \eta_r + \frac{2\eta}{r} = 0, & \text{(a)} \\ \eta_t + \Big(\frac{j^2}{\rho} - \frac{\tilde{j}^2}{\tilde{\rho}}\Big)_r + \frac{2}{r}\Big(\frac{j^2}{\rho} - \frac{\tilde{j}^2}{\tilde{\rho}}\Big) + (P(\rho) - P(\tilde{\rho}))_r - \tilde{\rho}\phi_r - \sigma\tilde{\Phi}_r + \eta = \sigma\phi_r, & \text{(b)} \\ \phi_{rr} + \frac{2\phi_r}{r} = \sigma, & \text{(c)} \end{cases} \tag{3.2}$$

with the initial and boundary value

$$\sigma(t, \epsilon_0) = \sigma(t, 1) = \phi(t, \epsilon_0) = \phi(t, 1) = 0, \quad (\sigma, \eta)(0, r) = (\rho_0 - \tilde{\rho}, j_0 - \tilde{j})(r). \tag{3.3}$$

By multiplying (3.2)(c) by r^2 and integrating it over $[\epsilon_0, r]$ with respect to r, we get

$$r^2\phi_r(t, r) = \int_{\epsilon_0}^r s^2\sigma(t, s)ds + c_3(t) \tag{3.4}$$

for some function $c_3(t)$. Dividing (3.4) by r^2 and integrating it over $[\epsilon_0, 1]$ again, with the boundary condition $\phi|_{\partial\Omega} = 0$, we have

$$c_3(t) = \frac{-\epsilon_0}{1-\epsilon_0} \int_{\epsilon_0}^{1} \Big(r^{-2} \int_{\epsilon_0}^{r} s^2 \sigma(t,s) ds \Big) dr.$$

Then, we get the explicit formulas of $\phi_r(t,r)$ on $\sigma(t,r)$

$$r^2 \phi_r(t,r) = \int_{\epsilon_0}^{r} s^2 \sigma(t,s) ds - \frac{\epsilon_0}{1-\epsilon_0} \int_{\epsilon_0}^{1} r^{-2} \Big(\int_{\epsilon_0}^{r} s^2 \sigma(t,s) ds \Big) dr. \tag{3.5}$$

Thus, setting

$$v := r^2 \sigma, \ \ w := r^2 \eta,$$

we may rewrite (3.2)(b) as

$$w_t + \Big(P'(\rho) - \frac{j^2}{\rho^2} \Big) v_r + \frac{2j}{\rho} w_r + \Big(P''(\tilde\rho)\tilde\rho_r - \frac{2P'(\tilde\rho)}{r} - \tilde\Phi_r \Big) v + w - r^2 \tilde\rho \phi_r = r^2 \sigma \phi_r - R_1 - R_2, \tag{3.6}$$

where

$$\begin{aligned} R_1(t,r) := & -\frac{2r(2\tilde j + \eta)\eta}{\rho} + \frac{2r\tilde j^2 \sigma}{\rho \tilde\rho} + \frac{2r\sigma(\tilde j^2 + 2\tilde j \eta + \eta^2)}{\rho^2} \\ & - \frac{r^2(2\tilde j + \eta)\eta \tilde\rho_r}{\rho^2} + \frac{r^2 \tilde j^2 \tilde\rho_r (2\tilde\rho + \sigma)\sigma}{\rho^2 \tilde\rho^2} \end{aligned} \tag{3.7}$$

and

$$R_2(t,r) := 2(P'(\tilde\rho) - P'(\rho)) r\sigma + r^2 (P'(\rho) - P'(\tilde\rho) - P''(\tilde\rho)\sigma)\tilde\rho_r. \tag{3.8}$$

Furthermore, the problem of (3.2)-(3.3) is equal to that of the following matrix system:

$$\mathbb{V}_t + \mathcal{A}\mathbb{V}_r + \mathcal{M}\mathbb{V} + \mathcal{L} = \mathcal{N}, \tag{3.9}$$

with the initial value and the boundary condition

$$v(t,\epsilon_0) = v(t,1) = 0, \quad (v,\, w)(0,r) = (r^2(\rho_0 - \tilde\rho),\, r^2(j_0 - \tilde j))(r), \tag{3.10}$$

where

$$\mathbb{V} := \begin{pmatrix} v \\ w \end{pmatrix} = r^2 \mathbb{U}, \quad \mathcal{A} := \begin{pmatrix} 0 & 1 \\ a(\rho, j) & \frac{2j}{\rho} \end{pmatrix}, \quad \mathcal{M} := \begin{pmatrix} 0 & 0 \\ k_1(r) & 1 \end{pmatrix}, \quad \mathcal{L} := \begin{pmatrix} 0 \\ -r^2 \tilde\rho \phi_r \end{pmatrix},$$

$$\mathcal{N}(t,r) := \begin{pmatrix} 0 \\ r^2 \sigma \phi_r(t,r) - R_1(t,r) - R_2(t,r) \end{pmatrix}, \tag{3.11}$$

and

$$a(\rho, j) := P'(\rho) - \frac{j^2}{\rho^2}, \quad k_1(r) := P''(\tilde{\rho})\tilde{\rho}_r - \frac{2P'(\tilde{\rho})}{r} - \tilde{\Phi}_r. \tag{3.12}$$

By standard theory on symmetric hyperbolic system and the weighted energy estimates, we get the local existence of solution of (3.9)-(3.10), namely, the local existence of solution of (3.2)-(3.3). For the proofs we refer the reader to [26] and omit here.

Theorem 3.1 *(Local existence of perturbation equations). For an arbitrarily given $0 < \epsilon_0 \ll 1$, let $(\tilde{\rho}, \tilde{j}, \tilde{\phi})(r)$ be the solution of steady-state system (2.1)-(2.2) in Theorem 1.1, and assume that*

$$\left\| \begin{pmatrix} \rho_0 - \tilde{\rho} \\ j_0 - \tilde{j} \end{pmatrix} \right\|_{L_r^2} + \epsilon_0 \left\| \partial_r \begin{pmatrix} \rho_0 - \tilde{\rho} \\ j_0 - \tilde{j} \end{pmatrix} \right\|_{L_r^2} + \epsilon_0^2 \left\| \partial_r^2 \begin{pmatrix} \rho_0 - \tilde{\rho} \\ j_0 - \tilde{j} \end{pmatrix} \right\|_{L_r^2} \le \hat{C}\epsilon_0^k$$

for $k > \frac{3}{2}$ and some positive constant $\hat{C}$. Then, there exist some positive constants $t_0 = t_0(\hat{C}, \epsilon_0)$, such that (3.2)-(3.3) has a unique local solution $(\sigma, \eta)(t, r) \in [\chi_{2,r}([0, t_0]; \Omega)]^2$ satisfying (1.13)-(1.14). Moreover, there holds that

$$\left\| \begin{pmatrix} \sigma \\ \eta \end{pmatrix}(t) \right\|_{L_r^2(\Omega)} + \epsilon_0 \left\| \begin{pmatrix} \partial\sigma \\ \partial\eta \end{pmatrix}(t) \right\|_{L_r^2(\Omega)} + \epsilon_0^2 \left\| \begin{pmatrix} \partial^2\sigma \\ \partial^2\eta \end{pmatrix}(t) \right\|_{L_r^2(\Omega)} \le C'\epsilon_0^k, \quad \forall t \in [0, t_0], \tag{3.13}$$

for some positive constant $C' = C'(\hat{C})$, where ∂^l means a derivative in both r and t of order $l\,(l = 1, 2)$.

The remainder of this section will be devoted to the proof of Theorem 1.2. By Theorem 3.1 and (3.5), we know that the problem (3.2)-(3.3) with (1.18) has a unique local solution $(\sigma, \eta, \phi)(t, r)$ with (3.13) for $t \in [0, t_0(C_4, \epsilon_0)]$. Thus, by the continuity theory, it suffices to establish the *a priori* estimates. Denote

$$n(t) := \|\mathbb{U}(t)\|_{L_r^2} + \epsilon_0\|\mathbb{U}_r(t)\|_{L_r^2} + \epsilon_0^2\|\mathbb{U}_{rr}(t)\|_{L_r^2} + \|\phi_r(t)\|_{L_r^2} \tag{3.14}$$

and

$$N^*(\tau) := \sup_{0 \le s \le \tau} n(s). \tag{3.15}$$

Let $t^* \in (0, \infty]$ be the maximal time of existence of the classical solution. We claim that the following Theorem holds.

Theorem 3.2 *(The a priori estimates). Suppose that the initial perturbation satisfies (1.18) and $t^* \in (0, \infty]$ is the maximal time of existence of the classical solution $(\sigma, \eta, \phi)(t, r)$ to (3.2)-(3.3). Then, there exist some positive constant $C_5 > C_4$ such that, $\forall\, t \in (0, t^*)$,*

$$N^*(t) \le C_5\epsilon_0^\gamma \tag{3.16}$$

with $\gamma \ge \frac{5}{2}$, where $N^*(t)$ is defined as in (3.15).

Let

$$T=\sup\{t<t^*;\ N^*(\tau)\le 2C_5\epsilon_0^{\gamma},\ \forall\tau\in[0,t]\},\tag{3.17}$$

where $C_5>C_4$ is a positive constant to be determined. It is obvious from (1.18) that $T\in(0,t^*]$. Next, we will use the following eight lemmas to prove Theorem 3.2, where we require $t\in[0,T)$ in Lemma 3.1-Lemma 3.6 of this section.

3.1. Basic energy estimates

Lemma 3.1. *For $t\in[0,T)$, there holds that*

$$\begin{aligned}
&\frac{d}{dt}\Big[\int_{\epsilon_0}^{1}\Big(\frac{r^2(\tilde{j}\sigma-\tilde{\rho}\eta)^2}{2\rho\tilde{\rho}^2}+r^2(G(\rho)-G(\tilde{\rho})-G'(\tilde{\rho})\sigma)+\frac{r^2\phi_r^2}{2}\Big)(t,r)dr-\nu_1\int_{\epsilon_0}^{1}\frac{\eta r^2\phi_r}{\tilde{\rho}}(t,r)dr\Big]\\
&+\int_{\epsilon_0}^{1}\frac{r^2(\tilde{j}\sigma-\tilde{\rho}\eta)^2}{\rho\tilde{\rho}^2}(t,r)dr+\frac{\nu_1c_1}{2}\|\sigma(t)\|^2_{L^2_r}+\frac{3\nu_1}{4}\|\phi_r(t)\|^2_{L^2_r}\\
&+\frac{\nu_1\epsilon_0}{1-\epsilon_0}\int_{\epsilon_0}^{1}\eta(t,r)dr\int_{\epsilon_0}^{1}\frac{\eta}{\tilde{\rho}}(t,r)dr-\nu_1C_6\|\eta(t)\|^2_{L^2_r}\\
&\le C\Big(\frac{|\mathbb{U}(t)|_0}{\epsilon_0}+\frac{J_0}{\epsilon_0}\Big)(\|\sigma(t)\|^2_{L^2_r}+\|\phi_r(t)\|^2_{L^2_r}+\|\eta(t)\|^2_{L^2_r}+\|\epsilon_0r\sigma_rt\|^2+\epsilon_0^2\|\eta_r(t)\|^2_{L^2_r})
\end{aligned}\tag{3.18}$$

for some positive constants ν_1, C_6 and C, where T is defined as in (3.17) and ν_1 will be specified later.

Proof. To prove this Lemma, we introduce

$$\begin{aligned}
F(\varepsilon,t,r):=&\,r^2(\tilde{j}+\varepsilon\eta)_t+\Big(\frac{r^2(\tilde{j}+\varepsilon\eta)^2}{\tilde{\rho}+\varepsilon\sigma}\Big)_r+r^2P'(\tilde{\rho}+\varepsilon\sigma)(\tilde{\rho}+\varepsilon\sigma)_r\\
&-r^2(\tilde{\rho}+\varepsilon\sigma)(\tilde{\Phi}_r+\varepsilon\phi_r)+r^2(\tilde{j}+\varepsilon\eta),
\end{aligned}\tag{3.19}$$

$$Z(\varepsilon,t,r):=F(\varepsilon,t,r)\frac{(\tilde{j}+\varepsilon\eta)}{\tilde{\rho}+\varepsilon\sigma}.\tag{3.20}$$

Then we consider $Z(1,t,r)-Z(0,t,r)-Z_\varepsilon(0,t,r)$ and integrate it over $[\epsilon_0,1]$ with respect to r. Following from (1.6)(b) and (2.1)(b) that $F(1,t,r)=F(0,t,r)=0$, we have

$$\begin{aligned}
&\int_{\epsilon_0}^{1}(Z(1,t,r)-Z(0,t,r)-Z_\varepsilon(0,t,r))dr\\
&=\int_{\epsilon_0}^{1}\Big[F(1,t,r)\frac{j}{\rho}-F(0,t,r)\frac{\tilde{j}}{\tilde{\rho}}-F_\varepsilon(0,t,r)\frac{\tilde{j}}{\tilde{\rho}}-F(0,t,r)\Big(\frac{\tilde{j}+\varepsilon\eta}{\tilde{\rho}+\varepsilon\sigma}\Big)_\varepsilon\Big|_{\varepsilon=0}\Big](t,r)dr
\end{aligned}$$

$$= \int_{\epsilon_0}^{1} [F(1,t,r) - F(0,t,r) - F_\varepsilon(0,t,r)] \frac{\tilde{j}}{\tilde{\rho}} dr$$

$$= \int_{\epsilon_0}^{1} F_{\varepsilon\varepsilon}(\theta,t,r) \frac{\tilde{j}}{\tilde{\rho}} dr, \tag{3.21}$$

where $0 \le \theta \le 1$ is a constant.

From the Sobolev inequality, there holds that, for $t \ge 0$,

$$|\mathbb{U}(t)|_0 \le \sqrt{2}\|\mathbb{U}\|^{\frac{1}{2}}\|\mathbb{U}_r\|^{\frac{1}{2}} \le \sqrt{2}\epsilon_0^{-1}\|\mathbb{U}\|_{L_r^2}^{\frac{1}{2}}\|\mathbb{U}_r\|_{L_r^2}^{\frac{1}{2}} \le \sqrt{2}\epsilon_0^{-\frac{3}{2}}(\|\mathbb{U}\|_{L_r^2} + \epsilon_0\|\mathbb{U}_r\|_{L_r^2}), \tag{3.22}$$

$$\begin{aligned} |\mathbb{U}_r(t)|_0 &\le \sqrt{2}\|\mathbb{U}_r\|^{\frac{1}{2}}\|\mathbb{U}_{rr}\|^{\frac{1}{2}} \le \sqrt{2}\epsilon_0^{-1}\|\mathbb{U}_r\|_{L_r^2}^{\frac{1}{2}}\|\mathbb{U}_{rr}\|_{L_r^2}^{\frac{1}{2}} \\ &\le \sqrt{2}\epsilon_0^{-\frac{5}{2}}(\epsilon_0\|\mathbb{U}_r\|_{L_r^2} + \epsilon_0^2\|\mathbb{U}_{rr}\|_{L_r^2}), \end{aligned} \tag{3.23}$$

$$\begin{aligned} |\mathbb{U}_t(t)|_0 &\le \sqrt{2}\|\mathbb{U}_t\|^{\frac{1}{2}}\|\mathbb{U}_{tr}\|^{\frac{1}{2}} \le \sqrt{2}\epsilon_0^{-1}\|\mathbb{U}_t\|_{L_r^2}^{\frac{1}{2}}\|\mathbb{U}_{tr}\|_{L_r^2}^{\frac{1}{2}} \\ &\le \sqrt{2}\epsilon_0^{-\frac{5}{2}}(\epsilon_0\|\mathbb{U}_t\|_{L_r^2} + \epsilon_0^2\|\mathbb{U}_{tr}\|_{L_r^2}). \end{aligned} \tag{3.24}$$

Furthermore, if $t \in [0,T]$, we hence from the definition of T and the smallness of ϵ_0 that

$$|\mathbb{U}(t)|_0 + \epsilon_0|\mathbb{U}_r(t)|_0 \le 2\sqrt{2}C_5\epsilon_0^{\gamma-\frac{3}{2}} \ll 1. \tag{3.25}$$

Thus, the straightforward computations, together with Theorem 1.1 and (3.25), show that,

$$|\epsilon_0 F_{\varepsilon\varepsilon}(\varepsilon,t,r)| \le C(|r\phi_r|^2 + |r\sigma|^2 + |r\eta|^2 + |\epsilon_0 r\sigma_r|^2 + |\epsilon_0 r\eta_r|^2), \quad \forall t \in [0,T),$$

which yields that

$$\begin{aligned} &\Big|\int_{\epsilon_0}^{1} (Z(1,t,r) - Z(0,t,r) - Z_\varepsilon(0,t,r))dr\Big| \\ &\le C\frac{J_0}{\epsilon_0}\int_{\epsilon_0}^{1} |\epsilon_0 F_{\varepsilon\varepsilon}(\theta,t,r)|dr \\ &\le C\frac{J_0}{\epsilon_0}(\|\phi_r\|_{L_r^2}^2 + \|\sigma\|_{L_r^2}^2 + \|\eta\|_{L_r^2}^2 + \|\epsilon_0\sigma_r\|_{L_r^2}^2 + \|\epsilon_0\eta_r\|_{L_r^2}^2) \end{aligned} \tag{3.26}$$

for $t \in [0,T)$, where C is independent of C_5 and ϵ_0.

On the other hand, inspired by the methods mentioned in [12], we may rewrite $Z(1,t,r) - Z(0,t,r) - Z_\varepsilon(0,t,r)$ into another expression.

In fact, (3.2)(a) and (3.2)(c) imply that

$$(r^2\phi_r)_{tr} + (r^2\eta)_r = 0 \tag{3.27}$$

holds. Then, from the fact $r^2\tilde{j} = M_0[\tilde{\rho}] = constant$ and (3.27), we have

$$(r^2(\tilde{\rho} + \varepsilon\sigma))_t = \varepsilon(r^2\sigma)_t = -\varepsilon(r^2\eta)_r = -(r^2(\tilde{j} + \varepsilon\eta))_r \tag{3.28}$$

and

$$(r^2(\tilde{\Phi}_r + \varepsilon\phi_r))_{tr} + (r^2(\tilde{j} + \varepsilon\eta))_r = 0, \tag{3.29}$$

which gives

$$(r^2(\tilde{\Phi}_r + \varepsilon\phi_r))_t + r^2(\tilde{j} + \varepsilon\eta) = \beta_\varepsilon(t)$$

for some function $\beta_\varepsilon(t)$. By integrating it over $[\epsilon_0, 1]$ and using the boundary condition $\phi(t, \epsilon_0) = \phi(t, 1) = 0$ we hence that

$$\beta_\varepsilon(t) = M_0[\tilde{\rho}] + \frac{\varepsilon}{1-\epsilon_0}\int_{\epsilon_0}^{1} (r^2\eta - 2r\phi_t)dr,$$

which leads to

$$-r^2(\tilde{j} + \varepsilon\eta) = (r^2(\tilde{\Phi}_r + \varepsilon\phi_r))_t - \frac{1}{1-\epsilon_0}\int_{\epsilon_0}^{1} \varepsilon(r^2\eta - 2r\phi_t)dr - M_0[\tilde{\rho}]. \tag{3.30}$$

Based on (3.28)-(3.30), a straightforward but tedious computation shows that the each term of $Z(1, t, r) - Z(0, t, r) - Z_\varepsilon(0, t, r)$ can be rewritten as

$$\Big(\frac{r^2 j^2}{2\rho} - \frac{r^2\tilde{j}^2}{2\tilde{\rho}} - \frac{r^2\tilde{j}\eta}{\tilde{\rho}} + \frac{r^2\tilde{j}^2\sigma}{2\tilde{\rho}^2}\Big)_t = \Big(\frac{(r\tilde{j}\sigma - r\tilde{\rho}\eta)^2}{2\rho\tilde{\rho}^2}\Big)_t, \tag{3.31}$$

$$\Big(\frac{r^2}{2}\Phi_r^2 - \frac{r^2}{2}\tilde{\Phi}_r^2 - r^2\tilde{\Phi}_r\phi_r\Big)_t = \Big(\frac{r^2}{2}\phi_r^2\Big)_t, \tag{3.32}$$

$$\Big(\frac{r^2 j^3}{2\rho^2} - \frac{r^2\tilde{j}^3}{2\tilde{\rho}^2} - r^2\Big(\frac{3\tilde{j}^2\eta}{2\tilde{\rho}^2} - \frac{\tilde{j}^3\sigma}{\tilde{\rho}^3}\Big)\Big)_r = \Big(\frac{3r^2\tilde{j}\eta^2}{2\tilde{\rho}^2} + \frac{r^2\eta^3}{2\tilde{\rho}^2} + \frac{r^2\tilde{j}^3\sigma}{\tilde{\rho}^3} - \frac{r^2 j^3}{2\rho^2\tilde{\rho}^2}(2\tilde{\rho}\sigma + \sigma^2)\Big)_r, \tag{3.33}$$

$$(r^2G'(\rho)j - r^2G'(\tilde{\rho})\tilde{j} - r^2(G''(\tilde{\rho})\sigma\tilde{j} + G'(\tilde{\rho})\eta))_r = (r^2(G'(\rho) - G'(\tilde{\rho}))j - r^2G''(\tilde{\rho})\sigma\tilde{j})_r, \tag{3.34}$$

and

$$-\frac{\Phi_r}{1-\epsilon_0}\int_{\epsilon_0}^{1}(r^2\eta-2r\phi_t)dr-\Phi_r M_0[\tilde{\rho}]+\tilde{\Phi}_r M_0[\tilde{\rho}]+\frac{\tilde{\Phi}_r}{1-\epsilon_0}\int_{\epsilon_0}^{1}(r^2\eta-2r\phi_t)dr+\phi_r M_0[\tilde{\rho}]$$

$$=-\frac{\phi_r}{1-\epsilon_0}\int_{\epsilon_0}^{1}(r^2\eta-2r\phi_t)dr, \tag{3.35}$$

where $G''(\rho)$ is defined by $G''(\rho)=\frac{P'(\rho)}{\rho}$.

After integrating (3.31)-(3.35) over $[\epsilon_0,1]$, with $\sigma(\epsilon_0)=\sigma(1)=\phi(\epsilon_0)=\phi(1)=0$, we have

$$\begin{aligned}&\int_{\epsilon_0}^{1}(Z(1,t,r)-Z(0,t,r)-Z_\varepsilon(0,t,r))dr\\&=\frac{d}{dt}\Big[\int_{\epsilon_0}^{1}\Big(\frac{r^2(\tilde{j}\sigma-\tilde{\rho}\eta)^2}{2\rho\tilde{\rho}^2}+r^2(G(\rho)-G(\tilde{\rho})-G'(\tilde{\rho})\sigma)+\frac{r^2\phi_r^2}{2}\Big)dr\Big]\\&\quad+\int_{\epsilon_0}^{1}\frac{r^2(\tilde{j}\sigma-\tilde{\rho}\eta)^2}{\rho\tilde{\rho}^2}dr+\Big(\frac{3r^2\tilde{j}\eta^2}{2\tilde{\rho}^2}+\frac{r^2\eta^3}{2\tilde{\rho}^2}\Big)\Big|_{\epsilon_0}^{1}.\end{aligned} \tag{3.36}$$

Following Theorem 1.1 that

$$|r\tilde{\rho}_r|_0\le 2(\|r\tilde{\rho}_r\|+\|\tilde{\rho}_r\|+\|r\tilde{\rho}_{rr}\|)\le 4(C_2+C_3),\quad\text{and}\quad |\tilde{j}_r|_0=\Big|\frac{-2\tilde{j}}{r}\Big|_0\le C\frac{J_0}{\epsilon_0}\le C, \tag{3.37}$$

which yields that

$$\begin{aligned}\frac{3r^2\tilde{j}\eta^2}{2\tilde{\rho}^2}\Big|_{\epsilon_0}^{1}&=\int_{\epsilon_0}^{1}\Big(\frac{3r\tilde{j}\eta^2}{\tilde{\rho}^2}+\frac{3r^2\tilde{j}_r\eta^2}{2\tilde{\rho}^2}+\frac{3r^2\tilde{j}\eta\eta_r}{\tilde{\rho}^2}-\frac{3r^2\tilde{j}\eta^2\tilde{\rho}_r}{\tilde{\rho}^3}\Big)dr\\&\le C\frac{J_0}{\epsilon_0}(\|\eta(t)\|_{L_r^2}^2+\epsilon_0^2\|\eta_r(t)\|_{L_r^2}^2)\end{aligned}$$

and

$$\frac{r^2\eta^3}{2\tilde{\rho}^2}\Big|_{\epsilon_0}^{1}=\int_{\epsilon_0}\Big(\frac{r\eta^3}{\tilde{\rho}^2}+\frac{3r^2\eta^2\eta_r}{2\tilde{\rho}^2}-\frac{r^2\eta^3\tilde{\rho}_r}{\tilde{\rho}^3}\Big)dr\le C\frac{|\eta|_0}{\epsilon_0}(\|\eta(t)\|_{L_r^2}^2+\epsilon_0^2\|\eta_r(t)\|_{L_r^2}^2).$$

Therefore, it follows from (3.26) and (3.36) that

$$\frac{d}{dt}\Big[\int_{\epsilon_0}^{1}\Big(\frac{r^2(\tilde{j}\sigma-\tilde{\rho}\eta)^2}{2\rho\tilde{\rho}^2}+r^2(G(\rho)-G(\tilde{\rho})-G'(\tilde{\rho})\sigma)+\frac{r^2\phi_r^2}{2}\Big)(t,r)dr\Big]$$

$$+\int_{\epsilon_0}^{1} \frac{r^2(\tilde{j}\sigma-\tilde{\rho}\eta)^2}{\rho\tilde{\rho}^2}(t,r)dr$$
$$\le C\Big(\frac{J_0}{\epsilon_0}+\frac{|\eta|_0}{\epsilon_0}\Big)(\|\phi_r(t)\|^2_{L^2_r}+\|\sigma(t)\|^2_{L^2_r}+\|\eta(t)\|^2_{L^2_r}+\epsilon_0^2\|\sigma_r(t)\|^2_{L^2_r}+\epsilon_0^2\|\eta_r(t)\|^2_{L^2_r}) \tag{3.38}$$

for $t\in[0,T)$, where C is independent of C_5 and ϵ_0.

Compared (3.38) with (3.18), we have to estimate the term $\int_{\epsilon_0}^1(\sigma^2+\phi_r^2)dr$. To the end, we divide (1.6)(b) and (2.1)(b) by $\tilde{\rho}$ and make difference of the resultant equations to get

$$\frac{\eta_t}{\tilde{\rho}}+\frac{(P(\rho)-P(\tilde{\rho}))_r}{\tilde{\rho}}-\frac{\rho}{\tilde{\rho}}\Phi_r+\tilde{\Phi}_r+\frac{\eta}{\tilde{\rho}}=-\frac{1}{\tilde{\rho}}\Big[\Big(\frac{j^2}{\rho}\Big)_r-\Big(\frac{\tilde{j}^2}{\tilde{\rho}}\Big)_r\Big]-\frac{1}{\tilde{\rho}}\Big(\frac{2j^2}{\rho r}-\frac{2\tilde{j}^2}{\tilde{\rho}r}\Big). \tag{3.39}$$

On one hand, $\tilde{\Phi}_r=\frac{1}{\tilde{\rho}}\Big[\Big(\frac{\tilde{j}^2}{\tilde{\rho}}+P(\tilde{\rho})\Big)_r+\frac{2\tilde{j}^2}{\tilde{\rho}r}+\tilde{j}\Big]$ (see (2.1)(b)) gives

$$\begin{aligned}-\frac{\rho}{\tilde{\rho}}\Phi_r+\tilde{\Phi}_r&=-\phi_r-\frac{\sigma}{\tilde{\rho}}\phi_r-\frac{\sigma}{\tilde{\rho}}\tilde{\Phi}_r\\&=-\phi_r-\frac{\sigma}{\tilde{\rho}}\phi_r-\frac{\sigma P'(\tilde{\rho})\tilde{\rho}_r}{\tilde{\rho}^2}+\frac{\tilde{j}^2\tilde{\rho}_r\sigma}{\tilde{\rho}^4}+\frac{2\tilde{j}^2\sigma}{\tilde{\rho}^3r}-\frac{\sigma\tilde{j}}{\tilde{\rho}^2}.\end{aligned} \tag{3.40}$$

On the other hand, it is easy to see

$$\begin{aligned}\frac{(P(\rho)-P(\tilde{\rho}))_r}{\tilde{\rho}}&=\Big(\frac{P(\rho)-P(\tilde{\rho})}{\tilde{\rho}}\Big)_r+\frac{(P(\rho)-P(\tilde{\rho}))\tilde{\rho}_r}{\tilde{\rho}^2}\\&=\Big(\frac{P(\rho)-P(\tilde{\rho})}{\tilde{\rho}}\Big)_r+\frac{P'(\tilde{\rho})\sigma\tilde{\rho}_r}{\tilde{\rho}^2}+\frac{(P(\rho)-P(\tilde{\rho})-P'(\tilde{\rho})\sigma)\tilde{\rho}_r}{\tilde{\rho}^2}.\end{aligned} \tag{3.41}$$

Thus, substituting (3.40)-(3.41) into (3.39) leads to

$$\Big(\frac{P(\rho)-P(\tilde{\rho})}{\tilde{\rho}}\Big)_r-\phi_r=-\frac{\eta+\eta_t}{\tilde{\rho}}+I_{10}+I_{11}, \tag{3.42}$$

where

$$I_{10}:=-\frac{(P(\rho)-P(\tilde{\rho})-P'(\tilde{\rho})\sigma)\tilde{\rho}_r}{\tilde{\rho}^2}+\frac{\sigma}{\tilde{\rho}}\phi_r-\frac{\tilde{j}^2\tilde{\rho}_r\sigma}{\tilde{\rho}^4}-\frac{2\tilde{j}^2\sigma}{\tilde{\rho}^3r}+\frac{\sigma\tilde{j}}{\tilde{\rho}^2}$$

and

$$I_{11}:=-\frac{1}{\tilde{\rho}}\Big[\Big(\frac{j^2}{\rho}\Big)_r-\Big(\frac{\tilde{j}^2}{\tilde{\rho}}\Big)_r\Big]-\frac{1}{\tilde{\rho}}\Big(\frac{2j^2}{\rho r}-\frac{2\tilde{j}^2}{\tilde{\rho}r}\Big).$$

Multiplying (3.42) by $-r^2\phi_r$ and integrating it over $[\epsilon_0,1]$ by parts, together with (3.2)(c), we get

$$\int_{\epsilon_0}^{1}\Big(\frac{P(\rho)-P(\tilde\rho)}{\tilde\rho}\Big)r^2\sigma dr+\int_{\epsilon_0}^{1}(r\phi_r)^2dr=\int_{\epsilon_0}^{1}\frac{\eta+\eta_t}{\tilde\rho}r^2\phi_r dr-\int_{\epsilon_0}^{1}(I_{10}+I_{11})r^2\phi_r dr. \quad (3.43)$$

From (1.2), the first term on the left-hand side of (3.43) can be treated as follows

$$\int_{\epsilon_0}^{1}\Big(\frac{P(\rho)-P(\tilde\rho)}{\tilde\rho}\Big)r^2\sigma dr=\int_{\epsilon_0}^{1}\Big(\int_0^1\frac{P'(\tilde\rho+s\sigma)}{\tilde\rho}ds\Big)r^2\sigma^2dr\ge c_0\|\sigma(t)\|^2_{L^2_r} \quad (3.44)$$

for some positive constant c_0. With the smallness of $|\mathbb{U}(t,\cdot)|_0$ for $t\in[0,T)$ (see (3.25)) and the boundedness of $|r\tilde\rho_r|_0\le C$ (see (3.37)), we get

$$\begin{aligned}&\Big|\int_{\epsilon_0}^{1}(I_{10}+I_{11})r^2\phi_r dr\Big|\\ &\le C\Big(\frac{J_0}{\epsilon_0}+\frac{|\mathbb{U}(t)|_0}{\epsilon_0}\Big)(\|\phi_r(t)\|^2_{L^2_r}+\|\sigma(t)\|^2_{L^2_r}+\|\eta(t)\|^2_{L^2_r}+\epsilon_0^2\|\sigma_r(t)\|^2_{L^2_r}+\epsilon_0^2\|\eta_r(t)\|^2_{L^2_r}).\end{aligned} \quad (3.45)$$

In addition, differentiating (3.5) with respect to t and r gives

$$\begin{aligned}r^2\phi_{tr}&=\int_{\epsilon_0}^{r}s^2\sigma_t(t,s)ds-\frac{\epsilon_0}{1-\epsilon_0}\int_{\epsilon_0}^{1}\Big(r^{-2}\int_{\epsilon_0}^{r}s^2\sigma_t(t,s)ds\Big)dr\\ &=-(r^2\eta(t,r)-\epsilon_0^2\eta(t,\epsilon_0))+\frac{\epsilon_0}{1-\epsilon_0}\int_{\epsilon_0}^{1}r^{-2}(r^2\eta(t,r)-\epsilon_0^2\eta(t,\epsilon_0))dr\\ &=-r^2\eta(t,r)+\frac{\epsilon_0}{1-\epsilon_0}\int_{\epsilon_0}^{1}\eta(t,r)dr,\end{aligned} \quad (3.46)$$

which yields that

$$\begin{aligned}&\int_{\epsilon_0}^{1}\frac{\eta_t+\eta}{\tilde\rho}r^2\phi_r dr\\ &=\int_{\epsilon_0}^{1}\Big[\Big(\frac{\eta r^2\phi_r}{\tilde\rho}\Big)_t-\frac{1}{\tilde\rho}\eta r^2\phi_{tr}\Big]dr+\int_{\epsilon_0}^{1}\frac{r^2\eta\phi_r}{\tilde\rho}dr\\ &\le\frac{d}{dt}\Big(\int_{\epsilon_0}^{1}\frac{\eta r^2\phi_r}{\tilde\rho}(t,r)dr\Big)+\frac14\|\phi_r(t)\|^2_{L^2_r}+C_6\|\eta(t)\|^2_{L^2_r}\end{aligned}$$

$$-\frac{\epsilon_0}{1-\epsilon_0}\int_{\epsilon_0}^{1}\eta(t,r)dr\int_{\epsilon_0}^{1}\frac{\eta}{\tilde{\rho}}(t,r)dr \tag{3.47}$$

for some positive constant C_6.

Then, substituting (3.44)-(3.47) into (3.43), we have

$$\begin{aligned}
&c_0\|\sigma(t)\|^2_{L^2_r}+\frac{3}{4}\|\phi_r(t)\|^2_{L^2_r}+\frac{\epsilon_0}{1-\epsilon_0}\int_{\epsilon_0}^{1}\eta(t,r)dr\int_{\epsilon_0}^{1}\frac{\eta}{\tilde{\rho}}(t,r)dr\\
&\le\frac{d}{dt}\Big(\int_{\epsilon_0}^{1}\frac{\eta r^2\phi_r}{\tilde{\rho}}(t,r)dr\Big)+C_6\|\eta(t)\|^2_{L^2_r}+C\Big(\frac{J_0}{\epsilon_0}+\frac{|\mathbb{U}(t)|_0}{\epsilon_0}\Big)\\
&\quad\times(\|\sigma(t)\|^2_{L^2_r}+\|\phi_r(t)\|^2_{L^2_r}+\|\eta(t)\|^2_{L^2_r}+\epsilon_0^2\|\sigma_r(t)\|^2_{L^2_r}+\epsilon_0^2\|\eta_r(t)\|^2_{L^2_r}).
\end{aligned}\tag{3.48}$$

Finally, by taking the step as (3.38) $+\,\nu_1$(3.48), we conclude that (3.18) holds. Thus, the proof is complete. □

Before establishing the first order and higher order energy estimates, we deal with the nonlinear terms in Lemma 3.2 in advance.

Lemma 3.2. *For $0<\epsilon_0\ll 1$, there holds that, for $t\in[0,T)$,*

$$|\mathcal{N}(t,r)|\le C(J_0+|\mathbb{U}(t)|_0)(|r\mathbb{U}(t,r)|+|r\phi_r(t,r)|), \tag{3.49}$$

$$\begin{aligned}
|\mathcal{N}_r(t,r)|\le\;& C\Big(\frac{J_0}{\epsilon_0}+\frac{|\mathbb{U}(t)|_0+|r\phi_r(t)|_0}{\epsilon_0}\Big)(|r\mathbb{U}(t,r)|+|\epsilon_0 r\mathbb{U}_r(t,r)|)\\
&+C(J_0+|\mathbb{U}(t)|_0)|r\mathbb{U}(t)|_0|r\tilde{\rho}_{rr}(r)|,
\end{aligned}\tag{3.50}$$

$$\begin{aligned}
|\mathcal{N}_{tr}(t,r)|\le\;& C\Big(\frac{J_0}{\epsilon_0}+\frac{|\mathbb{U}(t)|_0+|r\phi_r(t)|_0}{\epsilon_0}+|\mathbb{U}_r(t)|_0\Big)(|r\mathbb{U}_t(t,r)|+|\epsilon_0 r\mathbb{U}_{tr}(t,r)|\\
&+|r\sigma_r(t,r)|)+C(J_0+|\mathbb{U}(t)|_0)|r\mathbb{U}_t(t)|_0|r\tilde{\rho}_{rr}(r)|,
\end{aligned}\tag{3.51}$$

where C is independent of C_5 and ϵ_0, $\mathcal{N}(t,r)$ and T are defined in (3.11) and (3.17) respectively.

Proof. For $0<\epsilon_0\ll 1$, by using (3.7) and (3.8), we have

$$|R_1(t,r)|+|R_2(t,r)|\le C(J_0+|\mathbb{U}(t)|_0)|r\mathbb{U}(t,r)|.$$

Then, (3.49) follows from (3.11).

Similarly, differentiating (3.7) and (3.8) with respect to r and further differentiating the resultant equation with respect to t, by a straightforward but tedious computation, we derive these estimates (3.50)-(3.51) presented in Lemma 3.2 for $|\mathbb{U}(t)|_0+\epsilon_0|\mathbb{U}_r(t)|_0$ small (see (3.25)) and for $|r\tilde{\rho}_r|_0\le C$ bounded (see (3.37)). Thus, the proof is complete. □

3.2. First order energy estimates

By observation, we get the relations between $\mathbb{U}_t$ and $\mathbb{U}_r$ in Lemma 3.3.

Lemma 3.3. *For $0 < \epsilon_0 \ll 1$, there holds that, for $t \in [0, T)$,*

$$\begin{aligned}
&|\mathbb{U}_t(t)|_0 \le C\Big(|\phi_r(t)|_0 + \frac{|\mathbb{U}(t)|_0}{\epsilon_0} + |\mathbb{U}_r(t)|_0\Big),\\
&|r\mathbb{U}_t(t)|_0 \le C(|r\phi_r(t)|_0 + |\mathbb{U}(t)|_0 + |r\mathbb{U}_r(t)|_0),\\
&\|\mathbb{U}_t(t)\|_{L^2_r} \le C\Big(\|\phi_r(t)\|_{L^2_r} + \frac{\|\mathbb{U}(t)\|_{L^2_r}}{\epsilon_0} + \|\mathbb{U}_r(t)\|_{L^2_r}\Big),\\
&\|\mathbb{U}_r(t)\|_{L^2_r} \le C\Big(\|\phi_r(t)\|_{L^2_r} + \frac{\|\mathbb{U}(t)\|_{L^2_r}}{\epsilon_0} + \|\mathbb{U}_t(t)\|_{L^2_r}\Big).
\end{aligned} \tag{3.52}$$

Proof. Firstly, (3.2)(a) gives

$$\sigma_t = -\eta_r - \frac{2\eta}{r},$$

which implies that

$$\begin{aligned}
&|\sigma_t(t)|_0 \le C\Big(|\eta_r(t)|_0 + \frac{|\eta(t)|_0}{\epsilon_0}\Big), \quad |r\sigma_t(t)|_0 \le C(|r\eta_r(t)|_0 + |\eta(t)|_0),\\
&\|\sigma_t(t)\|_{L^2_r} \le C(\|\eta_r(t)\|_{L^2_r} + \|\eta(t)\|) \le C\Big(\|\eta_r(t)\|_{L^2_r} + \frac{\|\eta(t)\|_{L^2_r}}{\epsilon_0}\Big),
\end{aligned}$$

and

$$\|\eta_r(t)\|_{L^2_r} \le C(\|\sigma_t(t)\|_{L^2_r} + \|\eta(t)\|) \le C\Big(\|\sigma_t(t)\|_{L^2_r} + \frac{\|\eta(t)\|_{L^2_r}}{\epsilon_0}\Big). \tag{3.53}$$

Secondly, (2.1)(b) gives

$$\tilde{\Phi}_r = \frac{1}{\tilde{\rho}}\Big[\Big(\frac{\tilde{j}^2}{\tilde{\rho}} + P(\tilde{\rho})\Big)_r + \frac{2\tilde{j}^2}{\tilde{\rho}r} + \tilde{j}\Big],$$

which, together with (3.37), yields that

$$|r\tilde{\Phi}_r(r)|_0 \le C \quad \text{and} \quad \|\tilde{\Phi}_r\| \le C. \tag{3.54}$$

Then from (3.2)(b), we get

$$\begin{aligned}
\eta_t =& -\frac{2\eta(\tilde{j}_r + \eta_r)}{\rho} - \frac{2\tilde{j}\eta_r}{\rho} - 2\tilde{j}\tilde{j}_r\Big(\frac{1}{\rho} - \frac{1}{\tilde{\rho}}\Big) + \frac{(\eta + 2\tilde{j})\eta\rho_r + \tilde{j}^2\sigma_r}{\rho^2} + \tilde{j}^2\tilde{\rho}_r\Big(\frac{1}{\rho^2} - \frac{1}{\tilde{\rho}^2}\Big)\\
&- P'(\rho)\sigma_r - (P'(\rho) - P'(\tilde{\rho}))\tilde{\rho}_r - \frac{2}{r}\Big(\frac{(\eta + 2\tilde{j})\eta}{\rho} + \tilde{j}^2\Big(\frac{1}{\rho} - \frac{1}{\tilde{\rho}}\Big)\Big) + \tilde{\rho}\phi_r + \sigma\tilde{\Phi}_r\\
&- \eta + \sigma\phi_r.
\end{aligned} \tag{3.55}$$

Thus, using the boundedness of $|r\tilde{\rho}_r| + |\tilde{j}_r| + \left|\frac{\tilde{j}}{r}\right| \le C$ (see (3.37)), the smallness of $|\mathbb{U}(t,\cdot)|_0$ for $t \in [0, T)$ (see (3.25)), we derive from (3.54) that

$$|\eta_t(t)|_0 \le C\Big(|\phi_r(t)|_0 + \frac{|\mathbb{U}(t)|_0}{\epsilon_0} + |\mathbb{U}_r(t)|_0\Big), \quad |r\eta_t(t)|_0 \le C\Big(|r\phi_r(t)|_0 + |\mathbb{U}(t)|_0 + |r\mathbb{U}_r(t)|_0\Big)$$

and

$$\|\eta_t(t)\|_{L_r^2} \le C\Big(\frac{\|\mathbb{U}(t)\|_{L_r^2}}{\epsilon_0} + \|\mathbb{U}_r(t)\|_{L_r^2} + \|\phi_r(t)\|_{L_r^2}\Big)$$

for $t \in [0, T)$.

On the other hand, we can rewrite (3.55) as

$$\begin{aligned}\Big(P'(\rho) - \frac{j^2}{\rho^2}\Big)\sigma_r = &-\Big(P'(\rho) - \frac{j^2}{\rho^2} - \Big(P'(\tilde{\rho}) - \frac{\tilde{j}^2}{\tilde{\rho}^2}\Big)\Big)\tilde{\rho}_r - \eta_t - \Big(\frac{2jj_r}{\rho} - \frac{2\tilde{j}\tilde{j}_r}{\tilde{\rho}}\Big)\\ &-\frac{2}{r}\Big(\frac{j^2}{\rho} - \frac{\tilde{j}^2}{\tilde{\rho}}\Big) + \tilde{\rho}\phi_r + \sigma\tilde{\Phi}_r - \eta + \sigma\phi_r.\end{aligned}$$

We claim that for $0 < \epsilon_0 \ll 1$, there exists a positive constant c_1 such that the subsonic condition

$$\inf_{r\in\Omega}\Big(P'(\rho) - \frac{j^2}{\rho^2}\Big) > c_1 > 0, \quad t \in [0, T) \tag{3.56}$$

holds, which is clear from the smallness of $\tilde{j}$ and $|\mathbb{U}(t)|_0$.

Thus, by using the estimates (3.53) and (3.56) we get, for $t \in [0, T)$, that

$$\begin{aligned}\sqrt{c_1}\|\sigma_r(t)\|_{L_r^2} &\le \Big\|\sqrt{P'(\rho) - \frac{j^2}{\rho^2}}\, r\sigma_r(t)\Big\|\\ &\le C\Big(\frac{\|\sigma(t)\|_{L_r^2} + \|\eta(t)\|_{L_r^2}}{\epsilon_0} + \|\eta_t(t)\|_{L_r^2} + \|\sigma_t(t)\|_{L_r^2} + \|\phi_r(t)\|_{L_r^2}\Big).\end{aligned}$$

The proof is complete. □

As shown in (3.12), $a(\rho, j) := P'(\rho) - \frac{j^2}{\rho^2}$, we naturally denote $a(\tilde{\rho}, \tilde{j}) := P'(\tilde{\rho}) - \frac{\tilde{j}^2}{\tilde{\rho}^2}$. Now, we look for a diagonal matrix $\mathcal{D} = \begin{pmatrix} \tilde{s} & 0 \\ 0 & \tilde{h} \end{pmatrix}$, where $\tilde{s} = a(\tilde{\rho}, \tilde{j})\tilde{h}(r)$, and $\tilde{h} = \tilde{h}(r)$ is a weight function and will be technically specified later. It will act as an approximate symmetrizer of (3.9).

We multiply (3.9) by matrix $\mathcal{D}$ and get the following system:

$$\mathcal{D}\mathbb{V}_t + A\mathbb{V}_r + M\mathbb{V} + L = N, \tag{3.57}$$

where

$$A = \mathcal{D}\mathcal{A}, \quad M = \mathcal{D}\mathcal{M}, \quad L = \mathcal{D}\mathcal{L}, \quad N = \mathcal{D}\mathcal{N},$$

and $\mathcal{A}, \mathcal{M}, \mathcal{L}, \mathcal{N}$ are given in (3.11).

Differentiating (3.57) with respect to r and making use of (3.57), we obtain

$$\mathcal{D}\mathbb{V}_{tr} + A\mathbb{V}_{rr} + (A_r - \mathcal{D}_r\mathcal{A} + M)\mathbb{V}_r + (M_r - \mathcal{D}_r\mathcal{M})\mathbb{V} + L_r - \mathcal{D}_r\mathcal{L} = N_r - \mathcal{D}_r\mathcal{N}. \tag{3.58}$$

Lemma 3.4 *(First order energy estimates). For $0 < \epsilon_0 \ll 1$, and for $t \in [0, T)$, there holds that*

$$\begin{aligned}
&\frac{d}{dt}\Big(\int_{\epsilon_0}^{1} \Big[\frac{\tilde{s}}{2}\epsilon_0^2(v(t,r))^2 + \frac{\tilde{h}}{2}\epsilon_0^2(w_r(t,r))^2\Big]dr - \int_{\epsilon_0}^{1} \frac{|b(\mathbb{U})|}{2}\tilde{h}\epsilon_0^2(v_r(t,r))^2 dr \\
&\quad - \int_{\epsilon_0}^{1} \frac{(\frac{2P'(\tilde{\rho})}{r} + \tilde{\Phi}_r - P''(\tilde{\rho})\tilde{\rho}_r)_r\tilde{h}\epsilon_0^2(v(t,r))^2}{2}dr - \int_{\epsilon_0}^{1} \nu_2\tilde{h}\epsilon_0^2 w_r v(t,r)dr\Big) \\
&\quad + \Big(\frac{1}{2} - \nu_2 C_7\Big)\|\epsilon_0\sqrt{\tilde{h}}w_r(t)\|^2 + \frac{\nu_2 c_1}{2}\|\epsilon_0\sqrt{\tilde{h}}v_r(t)\|^2 \\
&\le C\Big(\frac{J_0}{\epsilon_0} + \frac{|r\phi_r(t)|_0 + |\mathbb{U}(t)|_0}{\epsilon_0} + |\mathbb{U}_r(t)|_0 + |\mathbb{U}_t(t)|_0\Big) \\
&\quad \times (\|\mathbb{U}(t)\|_{L_r^2}^2 + \epsilon_0^2\|\mathbb{U}_r(t)\|_{L_r^2}^2 + \|\epsilon_0\sqrt{\tilde{h}}\mathbb{V}_r(t)\|^2) + C(\|\phi_r(t)\|_{L_r^2}^2 + \|\sigma(t)\|_{L_r^2}^2)
\end{aligned} \tag{3.59}$$

for some positive constants ν_2, C_7 and C, independent of C_5, T and ϵ_0, T is defined as in (3.17), where ν_2 will be specified later. Here $\mathbb{V} = r^2\mathbb{U} = r^2\begin{pmatrix}\sigma \\ \eta\end{pmatrix}$ and

$$\tilde{h}(r) = h_0 r^{-2} e^{\int_{\epsilon_0}^{r} \frac{P''(\tilde{\rho})\tilde{\rho}_r - \tilde{\Phi}_r}{P'(\tilde{\rho})}(s)ds} \tag{3.60}$$

for any positive constant h_0.

Proof. Multiplying (3.58) by $\mathbb{V}_r$ and integrating it over $[\epsilon_0, 1]$, we have

$$\begin{aligned}
&\int_{\epsilon_0}^{1} \mathcal{D}\mathbb{V}_{tr}\cdot\mathbb{V}_r dr + \int_{\epsilon_0}^{1} A\mathbb{V}_{rr}\cdot\mathbb{V}_r dr + \int_{\epsilon_0}^{1} (A_r - \mathcal{D}_r\mathcal{A} + M)\mathbb{V}_r\cdot\mathbb{V}_r dr \\
&+ \int_{\epsilon_0}^{1} (M_r - \mathcal{D}_r\mathcal{M})\mathbb{V}\cdot\mathbb{V}_r dr + \int_{\epsilon_0}^{1} (L_r - \mathcal{D}_r\mathcal{L})\cdot\mathbb{V}_r dr \\
&= \int_{\epsilon_0}^{1} (N_r - \mathcal{D}_r\mathcal{N})\cdot\mathbb{V}_r dr,
\end{aligned} \tag{3.61}$$

while

$$\int_{\epsilon_0}^{1} \mathcal{D}\mathbb{V}_{tr}\cdot\mathbb{V}_r dr = \int_{\epsilon_0}^{1} \begin{pmatrix} \tilde{s} & 0 \\ 0 & \tilde{h} \end{pmatrix} \begin{pmatrix} v_{tr} \\ w_{tr} \end{pmatrix} \cdot \begin{pmatrix} v_r \\ w_r \end{pmatrix} dr$$
$$= \frac{d}{dt}\Big(\int_{\epsilon_0}^{1} \Big[\frac{\tilde{s}}{2}(v_r(t,r))^2 + \frac{\tilde{h}}{2}(w_r(t,r))^2 \Big] dr \Big). \tag{3.62}$$

We decompose matrix A as

$$A = \frac{A + A^T}{2} + \frac{A - A^T}{2} =: A_{symm} + A_{skew}, \tag{3.63}$$

where A^T is the transpose of A. Then, for the second term in (3.61), by the symmetry of A_{symm} and skew symmetry of A_{skew}, we have

$$\int_{\epsilon_0}^{1} A\mathbb{V}_{rr}\cdot\mathbb{V}_r dr = \int_{\epsilon_0}^{1} \Big[\frac{1}{2}((A\mathbb{V}_r\cdot\mathbb{V}_r)_r - A_r\mathbb{V}_r\cdot\mathbb{V}_r) + A_{skew}\mathbb{V}_{rr}\cdot\mathbb{V}_r \Big] dr. \tag{3.64}$$

By the boundary condition $w_r(\epsilon_0) = -v_t(\epsilon_0) = w_r(1) = -v_t(1) = 0$ (see (3.2)(a) and (3.10)), we have

$$\int_{\epsilon_0}^{1} (A\mathbb{V}_r\cdot\mathbb{V}_r)_r dr = (A\mathbb{V}_r\cdot\mathbb{V}_r)\Big|_{\epsilon_0}^{1} = \Big[\tilde{s}w_r v_r + (a(\rho,j)\tilde{h}v_r + \frac{2j}{\rho}\tilde{h}w_r)w_r \Big]\Big|_{\epsilon_0}^{1} = 0.$$

Now, we collect all like terms of the type $\mathbb{V}_r\cdot\mathbb{V}_r$ in (3.61) and (3.64) and get the following,

$$\int_{\epsilon_0}^{1} (A_r - \mathcal{D}_r\mathcal{A} + M - \frac{1}{2}A_r)\mathbb{V}_r\cdot\mathbb{V}_r dr$$
$$= \int_{\epsilon_0}^{1} \Big(\frac{\mathcal{D}\mathcal{A}_r - \mathcal{D}_r\mathcal{A}}{2} + M \Big)\mathbb{V}_r\cdot\mathbb{V}_r dr$$
$$= \int_{\epsilon_0}^{1} (q_{11}(v_r)^2 + (q_{12} + q_{21})v_r w_r + q_{22}(w_r)^2) dr, \tag{3.65}$$

where

$$q_{12} = -\frac{1}{2}\tilde{s}_r = -\frac{1}{2}[a(\tilde{\rho}, \tilde{j})_r \tilde{h} + a(\tilde{\rho}, \tilde{j})\tilde{h}_r],$$
$$q_{21} = \frac{\tilde{h}a(\rho, j)_r - \tilde{h}_r a(\rho, j)}{2} + k_1(r)\tilde{h},$$
$$q_{22} = \frac{\tilde{h}\left(\frac{2j}{\rho}\right)_r - \tilde{h}_r \frac{2j}{\rho}}{2} + \tilde{h}.$$

For the coefficient of cross-term $v_r w_r$ in (3.65), we have

$$\begin{aligned} q_{12} + q_{21} &= \frac{\tilde{h}}{2}[a(\rho, j)_r - a(\tilde{\rho}, \tilde{j})_r] - \frac{\tilde{h}_r}{2}[a(\rho, j) + a(\tilde{\rho}, \tilde{j})] + k_1(r)\tilde{h} \\ &= -P'(\tilde{\rho})\tilde{h}_r + k_1(r)\tilde{h} + \frac{\tilde{j}^2}{\tilde{\rho}^2}\tilde{h}_r + \frac{\tilde{h}}{2}[a(\rho, j)_r - a(\tilde{\rho}, \tilde{j})_r] - \frac{\tilde{h}_r}{2}[a(\rho, j) - a(\tilde{\rho}, \tilde{j})], \end{aligned}$$

where $k_1(r) = P''(\tilde{\rho})\tilde{\rho}_r - \frac{2P'(\tilde{\rho})}{r} - \tilde{\Phi}_r$ is given in (3.57). Thus, we may choose $\tilde{h} > 0$ as follows:

$$\tilde{h} = h_0 r^{-2} \exp\Big(\int_{\epsilon_0}^{r} \frac{P''(\tilde{\rho})\tilde{\rho}_r - \tilde{\Phi}_r}{P'(\tilde{\rho})}(s)ds\Big)$$

for a positive constant h_0, such that

$$P'(\tilde{\rho})\tilde{h}_r = k_1(r)\tilde{h},$$

which yields that

$$q_{12} + q_{21} = \frac{\tilde{j}^2}{\tilde{\rho}^2}\tilde{h}_r + \frac{\tilde{h}}{2}b(\mathbb{U})_r - \frac{\tilde{h}_r}{2}b(\mathbb{U}), \tag{3.66}$$

where

$$b(\mathbb{U}) := a(\rho, j) - a(\tilde{\rho}, \tilde{j}). \tag{3.67}$$

Furthermore, the boundedness of $r\tilde{\rho}_r, r\tilde{j}, r\tilde{\Phi}_r$ in $C^0(\Omega)$ (see (3.37) and (3.54)) and the smallness of $|\mathbb{U}(t, \cdot)|_0$ for $t \in [0, T)$ (see (3.25)) ensure that

$$|b(\mathbb{U})(t, r)| = \Big|P'(\rho) - \frac{j^2}{\rho^2} - P'(\tilde{\rho}) + \frac{\tilde{j}^2}{\tilde{\rho}^2}\Big| \le C|\mathbb{U}(t)|_0 \le C\epsilon_0^{\frac{1}{2}} \ll 1, \tag{3.68}$$

$$\begin{aligned} |b(\mathbb{U})_r(t, r)| &= \Big|P''(\rho)\rho_r - \frac{2jj_r}{\rho^2} + \frac{2j^2\rho_r}{\rho^3} - \Big(P''(\tilde{\rho})\tilde{\rho}_r - \frac{2\tilde{j}\tilde{j}_r}{\tilde{\rho}^2} + \frac{2\tilde{j}^2\tilde{\rho}_r}{\tilde{\rho}^3}\Big)\Big| \\ &\le C\Big(\frac{|\mathbb{U}(t)|_0}{\epsilon_0} + |\mathbb{U}_r(t)|_0\Big), \end{aligned} \tag{3.69}$$

and

$$q_{22} = \frac{\tilde{h}\left(\frac{2j}{\rho}\right)_r - \tilde{h}_r \frac{2j}{\rho}}{2} + \tilde{h} \geq \tilde{h} - C\Big(|\mathbb{U}_r(t)|_0 + \frac{|\mathbb{U}(t)|_0}{\epsilon_0} + \frac{J_0}{\epsilon_0}\Big)\tilde{h} \tag{3.70}$$

for $t \in [0, T)$, where C is independent of C_5 and ϵ_0.

Therefore, substituting (3.66)-(3.70) into (3.65) gives

$$\begin{aligned}
&\int_{\epsilon_0}^{1} (A_r - \mathcal{D}_r\mathcal{A} + M - \frac{1}{2}A_r)\mathbb{V}_r \cdot \mathbb{V}_r dr \\
&\geq \int_{\epsilon_0}^{1} \Big[- C\Big(|\mathbb{U}_r(t)|_0 + \frac{|\mathbb{U}(t)|_0}{\epsilon_0} + \frac{J_0}{\epsilon_0}\Big)\tilde{h}|v_r w_r| \\
&\quad + \Big(\tilde{h} - C\Big(|\mathbb{U}_r(t)|_0 + \frac{|\mathbb{U}(t)|_0}{\epsilon_0} + \frac{J_0}{\epsilon_0}\Big)\tilde{h}\Big)(w_r)^2\Big]dr \\
&\geq \|\sqrt{\tilde{h}}w_r(t)\|^2 - C\Big(|\mathbb{U}_r(t)|_0 + \frac{|\mathbb{U}(t)|_0}{\epsilon_0} + \frac{J_0}{\epsilon_0}\Big)\|\sqrt{\tilde{h}}\mathbb{V}_r(t)\|^2, \quad \forall t \in [0, T),
\end{aligned} \tag{3.71}$$

where we have used the fact that $|\tilde{h}_r| \leq Cr^{-1}\tilde{h}$ from $|k_1(r)| \leq \frac{C}{r}$.

On the other hand, from the definition of matrix A_{skew} (3.63) and the equality $w_{rr} = -v_{tr}$ (see (3.2)(a)), we get

$$\begin{aligned}
A_{skew}\mathbb{V}_{rr} \cdot \mathbb{V}_r &= \frac{1}{2}(b(\mathbb{U})v_{tr}v_r\tilde{h} - b(\mathbb{U})v_{rr}v_t\tilde{h}) \\
&= \Big(\frac{\tilde{h}b(\mathbb{U})}{2}(v_r)^2\Big)_t - \frac{\tilde{h}}{2}b(\mathbb{U})_t(v_r)^2 - \Big(\frac{\tilde{h}}{2}b(\mathbb{U})v_r v_t\Big)_r \\
&\quad - \frac{\tilde{h}_r}{2}b(\mathbb{U})v_r w_r - \frac{\tilde{h}}{2}b(\mathbb{U})_r v_r w_r.
\end{aligned} \tag{3.72}$$

Similar with (3.68), from the definition of $b(\mathbb{U})$ given in (3.67), we get

$$|b(\mathbb{U})_t| = \Big|P''(\rho)\rho_t - \frac{2jj_t}{\rho^2} + \frac{2j^2\rho_t}{\rho^3}\Big| \leq C|\mathbb{U}_t(t)|_0,$$

which yields, together with (3.68) and the boundary condition of $v_t(\epsilon_0) = v_t(1) = 0$ (see (3.10)), that

$$\begin{aligned}
&\Big|\int_{\epsilon_0}^{1} A_{skew}\mathbb{V}_{rr} \cdot \mathbb{V}_r dr\Big| \\
&\leq \frac{d}{dt}\Big(\int_{\epsilon_0}^{1} \frac{b(\mathbb{U})}{2}\tilde{h}(r)(v_r(t,r))^2 dr\Big) + C\Big(\frac{|\mathbb{U}(t)|_0}{\epsilon_0} + |\mathbb{U}_r(t)|_0 + |\mathbb{U}_t(t)|_0\Big) \\
&\quad \times (\|\sqrt{\tilde{h}}v_r(t)\|^2 + \|\sqrt{\tilde{h}}w_r(t)\|^2),
\end{aligned} \tag{3.73}$$

where $|\tilde{h}_r|$ is replaced by $Cr^{-1}\tilde{h}$.

For the fourth term of the left-hand side in (3.61), we use the fact $w_r = -v_t$ (see (3.2)(a)) to get

$$\int_{\epsilon_0}^{1} (M_r - \mathcal{D}_r\mathcal{M})\mathbb{V}\cdot\mathbb{V}_r dr = -\int_{\epsilon_0}^{1} k_{1r}(r)\tilde{h}vv_t dr$$
$$= -\frac{d}{dt}\Big(\int_{\epsilon_0}^{1} \frac{k_{1r}(r)\tilde{h}(r)(v(t,r))^2}{2}dr\Big), \tag{3.74}$$

where $k_{1r}(r)$ is the derivative of $k_1(r)$ given in (3.57).

Next, from (3.54) and Theorem 1.1, we notice that

$$\Big|\int_{\epsilon_0}^{r} \frac{\tilde{\Phi}_r - P''(\tilde{\rho})\tilde{\rho}_r}{P'(\tilde{\rho})}dr\Big| \le C(\|\tilde{\Phi}_r\| + \|\tilde{\rho}_r\|) \le c_2$$

holds for some positive constant c_2, independent of ϵ_0. Thus, it follows that

$$h_0 e^{-c_2} r^{-2} < \tilde{h}(r) < h_0 e^{c_2} r^{-2}. \tag{3.75}$$

Therefore, with the boundedness of $r\tilde{\rho}_r$ (see (3.37)) in $C^0(\Omega)$ and the definition of $\mathcal{L}$ given in (3.11), in view of $(r^2\phi_r)_r = r^2\sigma$ (see (3.2)(c)), we can deal with the fifth term of the left-hand side in (3.61) as follows:

$$\Big|\int_{\epsilon_0}^{1} (L_r - \mathcal{D}_r\mathcal{L})\cdot\begin{pmatrix} v_r \\ w_r \end{pmatrix} dr\Big| = \Big| -\int_{\epsilon_0}^{1} (\tilde{\rho}_r r^2\phi_r + \tilde{\rho}r^2\sigma)\tilde{h}w_r dr\Big|$$
$$\le C\int_{\epsilon_0}^{1} |\sqrt{\tilde{h}}r\phi_r||\sqrt{\tilde{h}}w_r|dr + C\int_{\epsilon_0}^{1} |\sqrt{\tilde{h}}w_r||r^2\sqrt{\tilde{h}}\sigma|dr$$
$$\le \frac{1}{4}\|\sqrt{\tilde{h}}w_r(t)\|^2 + C(\|\phi_r(t)\|^2 + \|\sigma(t)\|^2_{L^2_r}), \tag{3.76}$$

where, in the last step we have used (3.75).

Finally, using Lemma 3.2 and (3.75) again, we get the following for the last term of the left-hand side in (3.61),

$$\Big|\int_{\epsilon_0}^{1} (N_r - \mathcal{D}_r\mathcal{N})\cdot\begin{pmatrix} v_r \\ w_r \end{pmatrix} dr\Big|$$
$$\le C\Big|\int_{\epsilon_0}^{1} \tilde{h}\mathcal{N}_r\cdot\mathbb{V}_r dr\Big|$$

$$
\le C\Big(\frac{J_0}{\epsilon_0}+\frac{|\mathbb{U}(t)|_0+|r\phi_r(t)|_0}{\epsilon_0}\Big)(\|\mathbb{U}(t)\|^2+\|r\mathbb{U}_r(t)\|^2+\|\sqrt{\tilde h}\mathbb{V}_r(t)\|^2)
$$

$$
+\underbrace{C(J_0+|\mathbb{U}(t)|_0)|r\mathbb{U}(t)|_0\int_{\epsilon_0}^{1}\tilde h|r\tilde\rho_{rr}||\mathbb{V}_r|dr.}_{\triangleq I_{12}} \tag{3.77}
$$

In view of $\|r\tilde\rho_{rr}\|\le C$ (see Theorem 1.1) and (3.75), we have

$$
\begin{aligned}
I_{12} &\le C\Big(\frac{(J_0+|\mathbb{U}(t)|_0)|r\mathbb{U}(t)|_0}{\epsilon_0}\Big)\int_{\epsilon_0}^{1}|r\tilde\rho_{rr}||\sqrt{\tilde h}\mathbb{V}_r|dr\\
&\le C\Big(\frac{J_0+|\mathbb{U}(t)|_0}{\epsilon_0}\Big)|r\mathbb{U}(t)|_0\|\sqrt{\tilde h}\mathbb{V}_r(t)\|\\
&\le C\Big(\frac{J_0+|\mathbb{U}(t)|_0}{\epsilon_0}\Big)(\|\mathbb{U}(t)\|^2+\|r\mathbb{U}_r(t)\|^2+\|\sqrt{\tilde h}\mathbb{V}_r(t)\|^2),
\end{aligned}
$$

which leads to

$$
\begin{aligned}
&\Big|\int_{\epsilon_0}^{1}(N_r-\mathcal{D}_r\mathcal{N})\cdot\begin{pmatrix} v_r\\ w_r\end{pmatrix}dr\Big|\\
&\le C\Big(\frac{J_0}{\epsilon_0}+\frac{|\mathbb{U}(t)|_0+|r\phi_r(t)|_0}{\epsilon_0}\Big)(\|\mathbb{U}(t)\|^2+\|r\mathbb{U}_r(t)\|^2+\|\sqrt{\tilde h}\mathbb{V}_r(t)\|^2).
\end{aligned} \tag{3.78}
$$

Therefore, substituting (3.61), (3.64), (3.71), (3.73)-(3.78) into (3.61), together with Lemma 3.3, implies that, for $t\in[0,T)$,

$$
\begin{aligned}
&\frac{d}{dt}\Big[\int_{\epsilon_0}^{1}\Big(\frac{\tilde s}{2}(v_r(t,r))^2+\frac{\tilde h}{2}(w_r(t,r))^2\Big)dr-\int_{\epsilon_0}^{1}\frac{|b(\mathbb{U})|}{2}\tilde h(r)(v_r(t,r))^2dr\Big]+\frac{1}{2}\|\sqrt{\tilde h}w_r(t)\|^2\\
&\le -\frac{d}{dt}\Big(\int_{\epsilon_0}^{1}\frac{k_{1r}(r)\tilde h(r)(v(t,r))^2}{2}dr\Big)+C(\|\phi_r(t)\|^2+\|\sigma(t)\|^2_{L^2_r})\\
&\quad+C\Big(\frac{J_0}{\epsilon_0}+|\mathbb{U}_r(t)|_0+\frac{|\mathbb{U}(t)|_0+|r\phi_r(t)|_0}{\epsilon_0}\Big)(\|\mathbb{U}(t)\|^2+\|\mathbb{U}_r(t)\|^2_{L^2_r}+\|\sqrt{\tilde h}\mathbb{V}_r(t)\|^2).
\end{aligned} \tag{3.79}
$$

To the proof end, we still need additional estimates for

$$
\int_{\epsilon_0}^{1}\tilde h(r)(v_r(t,r))^2dr.
$$

Multiplying (3.58) by $-\begin{pmatrix} 0 \\ v \end{pmatrix}$ and integrating it over $[\epsilon_0, 1]$ gives

$$\begin{aligned}
&\int_{\epsilon_0}^{1} \mathcal{D}\mathbb{V}_{tr} \cdot \Big(-\begin{pmatrix} 0 \\ v \end{pmatrix}\Big) dr + \int_{\epsilon_0}^{1} [A\mathbb{V}_{rr} + (A_r - \mathcal{D}_r\mathcal{A} + M)\mathbb{V}_r] \cdot \Big(-\begin{pmatrix} 0 \\ v \end{pmatrix}\Big) dr \\
&\quad + \int_{\epsilon_0}^{1} (M_r - \mathcal{D}_r\mathcal{M})\mathbb{V} \cdot \Big(-\begin{pmatrix} 0 \\ v \end{pmatrix}\Big) dr + \int_{\epsilon_0}^{1} (L_r - \mathcal{D}_r\mathcal{L}) \cdot \Big(-\begin{pmatrix} 0 \\ v \end{pmatrix}\Big) dr \\
&= \int_{\epsilon_0}^{1} (N_r - \mathcal{D}_r\mathcal{N}) \cdot \Big(-\begin{pmatrix} 0 \\ v \end{pmatrix}\Big) dr. \qquad (3.80)
\end{aligned}$$

With the fact $w_r = -v_t$ (see (3.2)(a)) and $a(\rho, j) = P'(\rho) - \frac{j^2}{\rho^2} > c_1$ (see (3.56)), we get

$$-\int_{\epsilon_0}^{1} \mathcal{D}\mathbb{V}_{tr} \cdot \begin{pmatrix} 0 \\ v \end{pmatrix} dr = -\frac{d}{dt}\Big(\int_{\epsilon_0}^{1} \tilde{h} w_r v dr\Big) - \int_{\epsilon_0}^{1} \tilde{h}(r)(w_r(t,r))^2 dr \qquad (3.81)$$

and

$$\begin{aligned}
&-\int_{\epsilon_0}^{1} A\mathbb{V}_{rr} \cdot \begin{pmatrix} 0 \\ v \end{pmatrix} dr - \int_{\epsilon_0}^{1} (A_r - \mathcal{D}_r\mathcal{A} + M)\mathbb{V}_r \cdot \begin{pmatrix} 0 \\ v \end{pmatrix} dr \\
&= \int_{\epsilon_0}^{1} \Big(a(\rho, j)\tilde{h} v_r + \frac{2j}{\rho}\tilde{h} w_r\Big) v_r dr + \int_{\epsilon_0}^{1} (\mathcal{D}_r\mathcal{A} - M)\mathbb{V}_r \cdot \begin{pmatrix} 0 \\ v \end{pmatrix} dr \\
&\geq \frac{c_1}{2}\|\sqrt{\tilde{h}} v_r\|^2 - C\|\sqrt{\tilde{h}} w_r\|^2 - \int_{\epsilon_0}^{1} (\mathcal{D}_r\mathcal{A} - M)\mathbb{V}_r \cdot \begin{pmatrix} 0 \\ v \end{pmatrix} dr.
\end{aligned}$$

Recall that $|rk_1(r)| \leq C$ and $|\tilde{h}_r| \leq Cr^{-1}\tilde{h}$. Together with (3.75), we have

$$\begin{aligned}
&\int_{\epsilon_0}^{1} (\mathcal{D}_r\mathcal{A} - M)\mathbb{V}_r \cdot \begin{pmatrix} 0 \\ v \end{pmatrix} dr \\
&= \int_{\epsilon_0}^{1} \Big[(a(\rho, j)\tilde{h}_r - k_1(r)\tilde{h}) v_r + \Big(\frac{2j}{\rho}\tilde{h}_r - \tilde{h}\Big) w_r\Big] v dr \\
&\leq \frac{c_1}{4}\|\sqrt{\tilde{h}} v_r(t)\|^2 + C\|\sigma(t)\|^2 + C\|\sqrt{\tilde{h}} w_r(t)\|^2.
\end{aligned}$$

Thus, it follows that

$$
\begin{aligned}
&-\int_{\epsilon_0}^{1} A\mathbb{V}_{rr}\cdot\begin{pmatrix}0\\ v\end{pmatrix}dr-\int_{\epsilon_0}^{1}(A_r-\mathcal{D}_r\mathcal{A}+M)\mathbb{V}_r\cdot\begin{pmatrix}0\\ v\end{pmatrix}dr\\
&\geq \frac{c_1}{4}\|\sqrt{\tilde{h}}v_r(t)\|^2-C\|\sigma(t)\|^2-C\|\sqrt{\tilde{h}}w_r(t)\|^2. \qquad (3.82)
\end{aligned}
$$

Next we consider the third term of the left-hand side in (3.80). Differentiating $k_1(r)=P''(\tilde{\rho})\tilde{\rho}_r-\frac{2P'(\tilde{\rho})}{r}-\tilde{\Phi}_r$ (see (3.12)) with respect to r, and by straightforward computations we get

$$
|k_{1r}(r)|\leq C\Big(\frac{1}{r^2}+|\tilde{\rho}_{rr}|\Big), \qquad (3.83)
$$

which, together with the inequality $\|r\tilde{\rho}_{rr}\|\leq C$ in Theorem 1.1 and (3.75), implies that

$$
\begin{aligned}
\Big|-\int_{\epsilon_0}^{1}(M_r-\mathcal{D}_r\mathcal{M})\mathbb{V}\cdot\begin{pmatrix}0\\ v\end{pmatrix}dr\Big| &= \Big|\int_{\epsilon_0}^{1}k_{1r}\tilde{h}(r)(v(t,r))^2dr\Big|\\
&\leq \Big|\int_{\epsilon_0}^{1}\tilde{h}\Big(\frac{1}{r^2}+|\tilde{\rho}_{rr}|\Big)(v(t,r))^2dr\Big|\\
&\leq C\|\sigma\|^2+C\|\sigma\|(\|\sigma\|+\|r\sigma_r\|)\\
&\leq \frac{c_1}{8}\|\sqrt{\tilde{h}}v_r(t)\|^2+C\|\sigma(t)\|^2. \qquad (3.84)
\end{aligned}
$$

Here in last step we have used $\|r\sigma_r\|\leq C(\|\sqrt{\tilde{h}}v_r(t)\|+\|\sigma(t)\|)$.

For the last two terms of the left-hand side in (3.80), we carry out the same argument as in proof (3.76) and (3.78) to prove

$$
\begin{aligned}
\Big|-\int_{\epsilon_0}^{1}(L_r-\mathcal{D}_r\mathcal{L})\cdot\begin{pmatrix}0\\ v\end{pmatrix}dr\Big| &= \Big|\int_{\epsilon_0}^{1}(\tilde{\rho}_r r^2\phi_r+\tilde{\rho}r^2\sigma)\tilde{h}v\,dr\Big|\\
&\leq C\Big(\|\phi_r(t)\|^2_{L^2_r}+\|\sigma(t)\|^2\Big) \qquad (3.85)
\end{aligned}
$$

and

$$
\begin{aligned}
&\Big|-\int_{\epsilon_0}^{1}(N_r-\mathcal{D}_r\mathcal{N})\cdot\begin{pmatrix}0\\ v\end{pmatrix}dr\Big|\\
&\leq C\Big(\frac{J_0}{\epsilon_0}+\frac{|\mathbb{U}(t)|_0+|r\phi_r(t)|_0}{\epsilon_0}\Big)(\|\mathbb{U}(t)\|^2+\|\mathbb{U}_r(t)\|^2_{L^2_r}). \qquad (3.86)
\end{aligned}
$$

Then, substituting (3.81), (3.82), (3.84)-(3.86) into (3.80), we get

$$
\begin{aligned}
&\frac{c_1}{8}\|\sqrt{\tilde{h}}v_r(t)\|^2 \\
&\le \frac{d}{dt}\Big(\int_{\epsilon_0}^{1} \tilde{h}w_r v dr\Big) + C_7\|\sqrt{\tilde{h}}w_r\|^2 + C(\|\sigma(t)\|^2 + \|\phi_r(t)\|^2_{L^2_r}) \\
&\quad + C\Big(\frac{J_0}{\epsilon_0} + \frac{|\mathbb{U}(t)|_0 + |r\phi_r(t)|_0}{\epsilon_0}\Big)(\|\mathbb{U}(t)\|^2 + \|\mathbb{U}_r(t)\|^2_{L^2_r}). \qquad (3.87)
\end{aligned}
$$

And by taking the step as $\epsilon_0^2[(3.79) + \nu_2(3.87)]$, we verify that (3.59) holds. Thus, the proof is complete. $\square$

3.3. Second order energy estimates

Similar with Lemma 3.3, we can get the following relations between $\mathbb{U}_{tr}$ and $\mathbb{U}_{rr}$.

Lemma 3.5. *For $0 < \epsilon_0 \ll 1$ and for $t \in [0, T)$, it holds that*

$$\epsilon_0^2\|\mathbb{U}_{rr}(t)\|_{L^2_r} \le C(\epsilon_0^2\|\mathbb{U}_{tr}(t)\|_{L^2_r} + \epsilon_0\|\mathbb{U}_r(t)\|_{L^2_r} + \|\mathbb{U}(t)\|_{L^2_r} + \|\phi_r(t)\|_{L^2_r}), \qquad (3.88)$$

$$\epsilon_0^2\|\mathbb{U}_{tr}(t)\|_{L^2_r} \le C(\epsilon_0^2\|\mathbb{U}_{rr}(t)\|_{L^2_r} + \epsilon_0\|\mathbb{U}_r(t)\|_{L^2_r} + \|\mathbb{U}(t)\|_{L^2_r} + \|\phi_r(t)\|_{L^2_r}), \qquad (3.89)$$

where $\mathbb{U} = \begin{pmatrix} \sigma \\ \eta \end{pmatrix}$, T is defined as in (3.17), and C is independent of C_5, T and ϵ_0.

Proof. Differentiating (3.9) with respect to r, we have

$$\mathbb{V}_{tr} + \mathcal{A}_r\mathbb{V}_r + \mathcal{A}\mathbb{V}_{rr} + \mathcal{M}_r\mathbb{V} + \mathcal{M}\mathbb{V}_r + \mathcal{L}_r = \mathcal{N}_r.$$

Since $\mathbb{V} = r^2\mathbb{U}$, we replace $\mathbb{V}$ by $\mathbb{U}$ to get

$$
\begin{aligned}
-r\mathcal{A}\mathbb{U}_{rr} &= r\mathbb{U}_{tr} + \Big(\frac{2\mathcal{A}}{r} + \mathcal{A}_r + \mathcal{M}\Big)r\mathbb{U}_r + \Big(\frac{-2\mathcal{A}}{r} + \mathcal{A}_r + \mathcal{M} + r\mathcal{M}_r\Big)\mathbb{U} \\
&\quad + \Big(\frac{\mathcal{L}_r}{r} - \frac{2\mathcal{L}}{r^2}\Big) + \Big(\frac{\mathcal{N}_r}{r} - \frac{2\mathcal{N}}{r^2}\Big). \qquad (3.90)
\end{aligned}
$$

In the same way as in proof (3.25), it follows that, for $t \in [0, T)$,

$$|r\mathbb{U}_r(t)|_0 \le \sqrt{2}\|r\mathbb{U}_r(t)\|^{\frac{1}{2}}(\|\mathbb{U}_r(t)\| + \|r\mathbb{U}_{rr})(t)\|)^{\frac{1}{2}} \le \epsilon_0^{\frac{1}{2}} \ll 1, \qquad (3.91)$$

which leads to $|r\rho_r|_0 + |rj_r|_0 \le C$ due to the boundedness of $|r\tilde{\rho}_r|_0 + |r\tilde{j}_r|_0$ as shown in (3.37), where C is independent of C_5, T and ϵ_0.

Consequently, it is easy to verify that

$$\Big|\Big|\epsilon_0^2\Big(\frac{2\mathcal{A}}{r} + \mathcal{A}_r + \mathcal{M}\Big)r\mathbb{U}_r(t)\Big|\Big| \le C\epsilon_0||r\mathbb{U}_r(t)||$$

and

$$\Big\|\epsilon_0^2\Big(\frac{\mathcal{L}_r}{r}-\frac{2\mathcal{L}}{r^2}\Big)(t)\Big\|=\Big\|\epsilon_0^2\Big(\frac{\tilde{\rho}_r r^2\phi_r+\tilde{\rho}r^2\sigma}{r}+\frac{r^2\tilde{\rho}\phi_r}{r^2}\Big)\Big\|\le C(\|\phi_r(t)\|_{L_r^2}+\|\sigma(t)\|_{L_r^2}).$$

In addition, using the inequality (3.83), we have

$$\|r\mathcal{M}_r\epsilon_0^2\mathbb{U}\|=C\|rk_{1r}\epsilon_0^2\sigma\|\le C\Big\|\Big(\frac{1}{r}+|r\tilde{\rho}_{rr}|\Big)\epsilon_0^2\sigma\Big\|\le C(\|\sigma\|_{L_r^2}+\epsilon_0\|\sigma_r\|_{L_r^2}).$$

On the other hand, note that

$$\begin{aligned}|\phi_r(t)|_0&\le\sqrt{2}\|\phi_r(t)\|^{\frac12}\|r^{-1}\phi_r(t)+\sigma(t)\|^{\frac12}\\&\le\sqrt{2}\epsilon_0^{-\frac32}(\|\phi_r(t)\|_{L_r^2}+\|\sigma(t)\|_{L_r^2})\le\epsilon_0^{\frac12}\ll1,\quad\forall t\in[0,T)\end{aligned}\tag{3.92}$$

holds, together with the smallness of $|\mathbb{U}(t,\cdot)|_0$ (see (3.25)) and Lemma 3.2, which yields that

$$\begin{aligned}&\Big\|\epsilon_0^2\Big(\frac{\mathcal{N}_r}{r}-\frac{2\mathcal{N}}{r^2}\Big)(t)\Big\|\\&\le C\Big(J_0+\epsilon_0|\phi_r(t)|_0+|\mathbb{U}(t)|_0\Big)(\|\phi_r(t)\|_{L_r^2}+\|\mathbb{U}(t)\|_{L_r^2}+\epsilon_0\|\mathbb{U}_r(t)\|_{L_r^2})\\&\le C(\|\phi_r(t)\|_{L_r^2}+\|\mathbb{U}(t)\|_{L_r^2}+\epsilon_0\|\mathbb{U}_r(t)\|_{L_r^2})\end{aligned}$$

for $t\in[0,T)$.

Therefore, it follows from the above relations that

$$\|\epsilon_0^2 r\mathcal{A}\mathbb{U}_{rr}(t)\|\le C(\epsilon_0^4\|\mathbb{U}_{tr}(t)\|_{L_r^2}+\epsilon_0\|\mathbb{U}_r(t)\|_{L_r^2}+\|\mathbb{U}(t)\|_{L_r^2}+\|\phi_r(t)\|_{L_r^2}).\tag{3.93}$$

Recall that

$$\epsilon_0^2 r\mathcal{A}\mathbb{U}_{rr}=\begin{pmatrix}\epsilon_0^2 r\eta_{rr}\\ a(\rho,j)\epsilon_0^2 r\sigma_{rr}+\frac{2j}{\rho}\epsilon_0^2 r\eta_{rr}\end{pmatrix},$$

thus, in view of the inequality (3.56), we have

$$\epsilon_0^2\|\mathbb{U}_{rr}(t)\|_{L_r^2}\le C(\epsilon_0^4\|\mathbb{U}_{tr}(t)\|_{L_r^2}+\epsilon_0\|\mathbb{U}_r(t)\|_{L_r^2}+\|\mathbb{U}(t)\|_{L_r^2}+\|\phi_r(t)\|_{L_r^2}).$$

Similarly, (3.89) can be deduced from (3.90). Thus, the proof is complete. □

Lemma 3.6 *(Second order energy estimates). For $0<\epsilon_0\ll1$ and for $t\in[0,T)$, there holds that*

$$\begin{aligned}&\frac{d}{dt}\Big[\int_{\epsilon_0}^{1}\Big(\frac{\tilde{s}}{2}\epsilon_0^4(v_{tr})^2+\frac{\tilde{h}}{2}\epsilon_0^4(w_{tr})^2\Big)dr-\int_{\epsilon_0}^{1}\frac{b(\mathbb{U})}{2}\tilde{h}\epsilon_0^4(v_{tr})^2dr-\int_{\epsilon_0}^{1}\frac{k_{1r}\tilde{h}}{2}\epsilon_0^4(v_t)^2dr\Big]\\&+\Big(\frac12-C_8\nu_3\Big)\|\epsilon_0^2\sqrt{\tilde{h}}w_{tr}(t)\|^2+\frac{\nu_3c_1}{4}\|\epsilon_0^2\sqrt{\tilde{h}}v_{tr}(t)\|^2\end{aligned}$$

$$\le C\Big(\frac{J_0}{\epsilon_0}+|\mathbb{U}_r(t)|_0+|\mathbb{U}_t(t)|_0+\frac{|\mathbb{U}(t)|_0}{\epsilon_0}+|r\phi_r(t)|_0\Big)$$
$$\times\Big(\epsilon_0^2\|\mathbb{U}_t(t)\|^2_{L^2_r}+\epsilon_0^4\|\mathbb{U}_{tr}(t)\|_{L^2_r}+\epsilon_0^2\|\sigma_r(t)\|^2_{L^2_r}+\|\epsilon_0^2\sqrt{\tilde h}\mathbb{V}_{tr}(t)\|^2+\|\epsilon_0\sqrt{\tilde h}\mathbb{V}_r(t)\|^2$$
$$+\|\epsilon_0^2\sqrt{\tilde h}v_{rr}(t)\|^2\Big)+C\|\eta(t)\|^2_{L^2_r}+C_9\|\epsilon_0\sqrt{\tilde h}w_r(t)\|^2 \tag{3.94}$$

for some positive constant ν_3 to be specified. Here $\mathbb{V}=r^2\mathbb{U}=\begin{pmatrix} r^2\sigma\\ r^2\eta\end{pmatrix}$, C, C_8, C_9 are some positive constants independent of C_5, T and ϵ_0, and $k_1(r), \tilde h, b(\mathbb{U})$ are respectively given in (3.12)*,* (3.60) *and* (3.67)*.*

Proof. Differentiating (3.58) with respect to t, we get

$$[\mathcal{D}\partial_t+A\partial_r+(A_r-\mathcal{D}_r\mathcal{A}+M)]\mathbb{V}_{tr}+\mathcal{D}\mathcal{M}_r\mathbb{V}_t+A_t\mathbb{V}_{rr}+\mathcal{D}\mathcal{A}_{tr}\mathbb{V}_r+\mathcal{D}\mathcal{L}_{tr}=\mathcal{D}\mathcal{N}_{tr}, \tag{3.95}$$

where $\mathcal{D}=\begin{pmatrix}\tilde s & 0\\ 0 & \tilde h\end{pmatrix}$, $A=\mathcal{D}\mathcal{A}$, $M=\mathcal{D}\mathcal{M}$ and $\mathcal{A},\mathcal{M},\mathcal{L},\mathcal{N}$ are given in (3.11).

Multiplying (3.95) by $\mathbb{V}_{tr}$ and integrating it over $[\epsilon_0,1]$, we have

$$\int_{\epsilon_0}^1[(\mathcal{D}\partial_t+A\partial_r+(A_r-\mathcal{D}_r\mathcal{A}+M))\mathbb{V}_{tr}\cdot\mathbb{V}_{tr}+\mathcal{D}\mathcal{M}_r\mathbb{V}_t\cdot\mathbb{V}_{tr}]dr+\int_{\epsilon_0}^1 A_t\mathbb{V}_{rr}\cdot\mathbb{V}_{tr}dr$$
$$+\int_{\epsilon_0}^1\mathcal{D}\mathcal{A}_{tr}\mathbb{V}_r\cdot\mathbb{V}_{tr}dr+\int_{\epsilon_0}^1\mathcal{D}\mathcal{L}_{tr}\cdot\mathbb{V}_{tr}dr=\int_{\epsilon_0}^1\mathcal{D}\mathcal{N}_{tr}\cdot\mathbb{V}_{tr}dr. \tag{3.96}$$

In the same way as in proof of Lemma 3.4, then we get

$$\int_{\epsilon_0}^1[(\mathcal{D}\partial_t+A\partial_r+(A_r-\mathcal{D}_r\mathcal{A}+M))\mathbb{V}_{tr}\cdot\mathbb{V}_{tr}+\mathcal{D}\mathcal{M}_r\mathbb{V}_t\cdot\mathbb{V}_{tr}]dr$$
$$\ge\frac{d}{dt}\Big[\int_{\epsilon_0}^1\Big(\frac{\tilde s}{2}(v_{tr})^2+\frac{\tilde h}{2}(w_{tr})^2\Big)dr\Big]-\frac{d}{dt}\Big(\int_{\epsilon_0}^1\frac{b(\mathbb{U})}{2}\tilde h(r)(v_{tr})^2dr\Big)-\frac{d}{dt}\Big(\int_{\epsilon_0}^1\frac{k_{1r}\tilde h}{2}(v_t)^2dr\Big)$$
$$+\|\sqrt{\tilde{\tilde h}}w_{tr}(t)\|^2-C\Big(\frac{J_0}{\epsilon_0}+\frac{|\mathbb{U}(t)|_0}{\epsilon_0}+|\mathbb{U}_r(t)|_0\Big)(\|\sqrt{\tilde h}v_{tr}(t)\|^2+\|\sqrt{\tilde h}w_{tr}(t)\|^2). \tag{3.97}$$

Note that

$$|a(\rho,j)_t|+\Big|\Big(\frac{2j}{\rho}\Big)_t\Big|=\Big|P''(\rho)\rho_t-\frac{1jj_t}{\rho^2}+\frac{2j^2\rho_t}{\rho^3}\Big|+\Big|\frac{2j_t}{\rho}-\frac{2j\rho_t}{\rho^2}\Big|\le C|\mathbb{U}_t|_0, \tag{3.98}$$

which yields

$$\Big|\int_{\epsilon_0}^{1} A_t\mathbb{V}_{rr}\cdot\mathbb{V}_{tr}dr\Big| = \Big|\int_{\epsilon_0}^{1}\Big(a(\rho,j)_t\tilde{h}v_{rr}+\Big(\frac{2j}{\rho}\Big)_t\tilde{h}w_{rr}\Big)w_{tr}dr\Big|$$
$$\le C|\mathbb{U}_t(t)|_0(\|\sqrt{\tilde{h}}v_{rr}(t)\|^2+\|\sqrt{\tilde{h}}v_{tr}(t)\|^2+\|\sqrt{\tilde{h}}w_{tr}(t)\|^2). \quad (3.99)$$

For the third term of left-hand side in (3.96), it holds that

$$\int_{\epsilon_0}^{1}\mathcal{D}\mathcal{A}_{tr}\mathbb{V}_r\cdot\mathbb{V}_{tr}dr=\int_{\epsilon_0}^{1}a(\rho,j)_{tr}\tilde{h}v_rw_{tr}dr+\int_{\epsilon_0}^{1}\Big(\frac{2j}{\rho}\Big)_{tr}\tilde{h}w_rw_{tr}dr=:I_{13}+I_{14}. \quad (3.100)$$

It follows from Lemma 3.3 and the relations (3.25), (3.91)-(3.92) that

$$|r\mathbb{U}_t(t)|_0\le C(|r\phi_r(t)|_0+|\mathbb{U}(t)|_0+|r\mathbb{U}_r|_0)\le C\epsilon_0^{\frac{1}{2}}\ll 1,\quad \forall t\in[0,T), \quad (3.101)$$

together with the boundedness $|r\rho_r|_0+|rj_r|_0\le C$ (see (3.91)) and relations $\sigma_{tr}=v_{tr}-2r\sigma_t$, $\eta_{tr}=w_{tr}-2r\eta_t$, which yields that

$$|I_{13}|$$
$$=\Big|\int_{\epsilon_0}^{1}\Big(P'''(\rho)\rho_r\sigma_t+p''(\rho)\sigma_{tr}-\frac{2j_r\eta_t+2j\eta_{tr}}{\rho^2}+\frac{4j\eta_t\rho_r}{\rho^3}+\frac{4jj_r\sigma_t+2j^2\sigma_{tr}}{\rho^3}-\frac{6j^2\sigma_t\rho_r}{\rho^4}\Big)$$
$$\times\tilde{h}v_rw_{tr}dr\Big|\le C\Big(\frac{J_0}{\epsilon_0}+\frac{|\mathbb{U}|_0}{\epsilon_0}+|\mathbb{U}_r|_0+|\mathbb{U}_t|_0\Big)\Big(\Big\|\frac{\sqrt{\tilde{h}}v_r}{r}\Big\|^2+\|\sqrt{\tilde{h}}w_{tr}\|^2+\|\sqrt{\tilde{h}}v_{tr}\|^2\Big) \quad (3.102)$$

and

$$|I_{14}|=\Big|\int_{\epsilon_0}^{1}\Big(\frac{2\eta_{tr}}{\rho}-\frac{2\eta_t\rho_r}{\rho^2}-\frac{2j_r\sigma_t+2j\sigma_{tr}}{\rho^2}+\frac{4j\sigma_t\rho_r}{\rho^3}\Big)\tilde{h}w_rw_{tr}dr\Big|$$
$$\le C\Big(\frac{J_0}{\epsilon_0}+\frac{|\mathbb{U}|_0}{\epsilon_0}+|\mathbb{U}_t|_0+|\mathbb{U}_r|_0\Big)\Big(\Big\|\frac{\sqrt{\tilde{h}}w_r}{r}\Big\|^2+\|\sqrt{\tilde{h}}v_{tr}\|^2+\|\sqrt{\tilde{h}}w_{tr}\|^2\Big). \quad (3.103)$$

Thus, we may conclude that

$$\Big|\int_{\epsilon_0}^{1}\mathcal{D}\mathcal{A}_{tr}\mathbb{V}_r\cdot\mathbb{V}_{tr}dr\Big|$$
$$\le C\Big(\frac{J_0}{\epsilon_0}+\frac{|\mathbb{U}(t)|_0}{\epsilon_0}+|\mathbb{U}_r(t)|_0+|\mathbb{U}_t(t)|_0\Big)(\|\sqrt{\tilde{h}}\mathbb{V}_{tr}(t)\|^2+\|r^{-1}\sqrt{\tilde{h}}\mathbb{V}_r(t)\|^2). \quad (3.104)$$

For the last term of left hand-side in (3.96), we get

$$\int_{\epsilon_0}^{1} \mathcal{D}\mathcal{L}_{tr}\cdot \mathbb{V}_{tr}dr = -\int_{\epsilon_0}^{1} \tilde{h}\Big[\tilde{\rho}_r\Big(-r^2\eta + \frac{\epsilon_0}{1-\epsilon_0}\int_{\epsilon_0}^{1}\eta dr\Big) + \tilde{\rho}r^2\sigma_t\Big]w_{tr}dr$$
$$\leq \frac{\|\sqrt{\tilde{h}}w_{tr}(t)\|^2}{2} + C(\epsilon_0^{-4}\|\eta(t)\|_{L_r^2}^2 + \|\sigma_t(t)\|_{L_r^2}^2), \tag{3.105}$$

where we have used the boundedness of $r\tilde{\rho}_r$ (see (3.37)) in $C^0(\Omega)$ and (3.75).

Similarly, by the method applied in (3.78), the estimates of $\mathcal{N}_{tr}$ in Lemma 3.2 give

$$\Big|\int_{\epsilon_0}^{1} \mathcal{D}\mathcal{N}_{tr}\cdot \mathbb{V}_{tr}dr\Big| \leq C\int_{\epsilon_0}^{1} \tilde{h}|\mathcal{N}_{tr}\cdot \mathbb{V}_{tr}|dr$$
$$\leq C\Big(\frac{J_0}{\epsilon_0} + \frac{|\mathbb{U}(t)|_0 + |r\phi_r|_0}{\epsilon_0} + |\mathbb{U}_r(t)|_0\Big)$$
$$\times (\|\sqrt{\tilde{h}}\mathbb{V}_{tr}(t)\|^2 + \|\mathbb{U}_t(t)\|^2 + \|\mathbb{U}_{tr}(t)\|_{L_r^2}^2 + \|\sigma_r(t)\|^2). \tag{3.106}$$

Thus, substituting (3.104)-(3.106) into (3.96), we have

$$\frac{d}{dt}\Big[\int_{\epsilon_0}^{1} \Big(\frac{\tilde{s}}{2}(v_{tr})^2 + \frac{\tilde{h}}{2}(w_{tr})^2\Big)dr - \int_{\epsilon_0}^{1} \frac{b(\mathbb{U})}{2}\tilde{h}(r)(v_{tr})^2dr - \int_{\epsilon_0}^{1} \frac{k_{1r}\tilde{h}}{2}(v_t)^2dr\Big] + \frac{1}{2}\|\sqrt{\tilde{h}}w_{tr}(t)\|^2$$
$$\leq C\Big(\frac{J_0}{\epsilon_0} + \frac{|\mathbb{U}(t)|_0 + |r\phi_r|_0}{\epsilon_0} + |\mathbb{U}_r(t)|_0 + |\mathbb{U}_t(t)|_0\Big)$$
$$\times (\|\sqrt{\tilde{h}}\mathbb{V}_{tr}(t)\|^2 + \|\mathbb{U}_t(t)\|^2 + \|\mathbb{U}_{tr}(t)\|_{L_r^2}^2 + \|\sigma_r(t)\|^2 + \|\sqrt{\tilde{h}}v_{rr}(t)\|^2 + \|r^{-1}\sqrt{\tilde{h}}\mathbb{V}_r(t)\|^2)$$
$$+ C\epsilon_0^{-4}(\|\eta(t)\|_{L_r^2}^2 + \epsilon_0^2\|\sigma_t(t)\|_{L_r^2}^2). \tag{3.107}$$

To the proof end, we still need additional estimates for

$$\int_{\epsilon_0}^{1} \tilde{h}(r)(v_{tr})^2dr.$$

We multiply (3.95) by $-\begin{pmatrix} 0 \\ v_t \end{pmatrix}$ and integrating it over $[\epsilon_0, 1]$ and have

$$\int_{\epsilon_0}^{1} [\mathcal{D}\partial_t + A\partial_r + (A_r - \mathcal{D}_r\mathcal{A} + M)]\mathbb{V}_{tr}\cdot\Big(-\begin{pmatrix} 0 \\ v_t \end{pmatrix}\Big)dr + \int_{\epsilon_0}^{1} \mathcal{D}\mathcal{M}_r\mathbb{V}_t\cdot\Big(-\begin{pmatrix} 0 \\ v_t \end{pmatrix}\Big)dr$$

$$+\int_{\epsilon_0}^{1} A_t\mathbb{V}_{rr}\cdot\Big(-\begin{pmatrix}0\\ v_t\end{pmatrix}\Big)dr+\int_{\epsilon_0}^{1}\mathcal{D}\mathcal{A}_{tr}\mathbb{V}_r\cdot\Big(-\begin{pmatrix}0\\ v_t\end{pmatrix}\Big)dr+\int_{\epsilon_0}^{1}\mathcal{D}\mathcal{L}_{tr}\cdot\Big(-\begin{pmatrix}0\\ v_t\end{pmatrix}\Big)dr$$

$$=\int_{\epsilon_0}^{1}\mathcal{D}\mathcal{N}_{tr}\cdot\Big(-\begin{pmatrix}0\\ v_t\end{pmatrix}\Big)dr. \tag{3.108}$$

Similarly, in the same way as in (3.81) and (3.82), it holds that

$$\begin{aligned}&\int_{\epsilon_0}^{1}[\mathcal{D}\partial_t+A\partial_r+(A_r-\mathcal{D}_r\mathcal{A}+M)]\mathbb{V}_{tr}\cdot\Big(-\begin{pmatrix}0\\ v_t\end{pmatrix}\Big)dr\\ &\geq\frac{3c_1}{8}\|\sqrt{\tilde{h}}v_{tr}(t)\|^2-\frac{d}{dt}\Big(\int_{\epsilon_0}^{1}\tilde{h}w_{tr}w_rdr\Big)-C\|\sqrt{\tilde{h}}w_{tr}(t)\|^2\\ &\quad-C\epsilon_0^{-2}\|\sigma_t(t)\|^2_{L^2_r}-C\|\sqrt{\tilde{h}}w_r(t)\|^2.\end{aligned} \tag{3.109}$$

And from (3.75) and the estimates of $|rk_{1r}|$ shown in (3.83), together with $\|r\tilde{\rho}_{rr}\|\leq C$, we obtain

$$\begin{aligned}\Big|-\int_{\epsilon_0}^{1}\mathcal{D}\mathcal{M}_r\mathbb{V}_t\cdot\begin{pmatrix}0\\ v_t\end{pmatrix}dr\Big|&=\Big|\int_{\epsilon_0}^{1}k_{1r}\tilde{h}(r)(v_t)^2dr\Big|\\ &\leq C\int_{\epsilon_0}^{1}\Big(\frac{1}{r^2}+|\tilde{\rho}_{rr}|\Big)(r\sigma_t)^2dr\\ &\leq\frac{c_1}{8}\|\sqrt{\tilde{h}}v_{tr}(t)\|^2+C\epsilon_0^{-2}\|\sigma_t(t)\|^2_{L^2_r},\end{aligned} \tag{3.110}$$

where in last step we have used the fact $\|r\sigma_{tr}\|\leq C(\|\sigma_t\|+\|\sqrt{\tilde{h}}v_{tr}\|)$.

Moreover, note that $\|\sqrt{\tilde{h}}v_t\|\leq C\|r\sigma_t\|$. From (3.98), we get

$$\begin{aligned}-\int_{\epsilon_0}^{1}A_t\mathbb{V}_{rr}\cdot\begin{pmatrix}0\\ v_t\end{pmatrix}dr&=-\int_{\epsilon_0}^{1}\Big(a(\rho,j)_t\tilde{h}v_{rr}+\Big(\frac{2j}{\rho}\Big)_t\tilde{h}w_{rr}\Big)v_tdr\\ &\leq C|\mathbb{U}_t(t)|_0(\|\sqrt{\tilde{h}}v_{rr}(t)\|^2+\|\sqrt{\tilde{h}}v_{tr}(t)\|^2+\|r\sigma_t\|^2).\end{aligned} \tag{3.111}$$

Now, we have to deal with the fourth term of left-hand side in (3.108). It is easy to see

$$-\int_{\epsilon_0}^{1}\mathcal{D}\mathcal{A}_{tr}\mathbb{V}_r\cdot\begin{pmatrix}0\\ v_t\end{pmatrix}dr=-\int_{\epsilon_0}^{1}a(\rho,j)_{tr}\tilde{h}v_rv_tdr+\int_{\epsilon_0}^{1}\Big(\frac{2j}{\rho}\Big)_{tr}\tilde{h}w_rv_tdr=:I_{15}+I_{16}.$$

In the same fashion as in (3.102)-(3.103), we get

$$
\begin{aligned}
|I_{15}| = & \Big|\int_{\epsilon_0}^{1}\Big(P'''(\rho)\rho_r\sigma_t + p''(\rho)\sigma_{tr} - \frac{2j_r\eta_t + 2j\eta_{tr}}{\rho^2} + \frac{4j\eta_t\rho_r}{\rho^3} \\
& + \frac{4jj_r\sigma_t + 2j^2\sigma_{tr}}{\rho^3} - \frac{6j^2\sigma_t\rho_r}{\rho^4}\Big)\tilde{h}v_r v_t dr\Big| \\
& \le C\Big(\frac{J_0}{\epsilon_0} + \frac{|\mathbb{U}|_0}{\epsilon_0} + |\mathbb{U}_r|_0 + |\mathbb{U}_t|_0\Big)\Big(\Big\|\frac{\sqrt{\tilde{h}}v_r}{r}\Big\|^2 + \|\sqrt{\tilde{h}}w_{tr}\|^2 + \|\sqrt{\tilde{h}}v_{tr}\|^2 + \|\sqrt{\tilde{h}}v_t\|^2\Big)
\end{aligned}
$$

and

$$
\begin{aligned}
|I_{16}| = & \Big|\int_{\epsilon_0}^{1}\Big(\frac{2\eta_{tr}}{\rho} - \frac{2\eta_t\rho_r}{\rho^2} - \frac{2j_r\sigma_t + 2j\sigma_{tr}}{\rho^2} + \frac{4j\sigma_t\rho_r}{\rho^3}\Big)\tilde{h}w_r v_t dr\Big| \\
& \le C\Big(\frac{J_0}{\epsilon_0} + \frac{|\mathbb{U}|_0}{\epsilon_0} + |\mathbb{U}_t|_0 + |\eta_r|_0\Big) \\
& \quad \times\Big(\Big\|\frac{\sqrt{\tilde{h}}w_r}{r}(t)\Big\|^2 + \|\sigma_t(t)\|^2_{L^2_r} + \|\sqrt{\tilde{h}}v_{tr}(t)\|^2 + \|\sqrt{\tilde{h}}w_{tr}(t)\|^2\Big),
\end{aligned}
$$

which implies, in view of $\|\sqrt{\tilde{h}}v_t\| \le C\|r\sigma_t\|$, that

$$
\begin{aligned}
& \Big|-\int_{\epsilon_0}^{1}\mathcal{D}\mathcal{A}_{tr}\mathbb{V}_r\cdot\begin{pmatrix}0\\ v_t\end{pmatrix}dr\Big| \\
& \le C\Big(\frac{J_0}{\epsilon_0} + \frac{|\mathbb{U}(t)|_0}{\epsilon_0} + |\mathbb{U}_r(t)|_0 + |\mathbb{U}_t(t)|_0\Big)\Big(\Big\|\frac{\sqrt{\tilde{h}}\mathbb{V}_r}{r}(t)\Big\|^2 + \|\sqrt{\tilde{h}}\mathbb{V}_{tr}(t)\|^2 + \|\sigma_t(t)\|^2_{L^2_r}\Big).
\end{aligned}
\tag{3.112}
$$

Similarly as in (3.105) and (3.106), using the fact $\|\sqrt{\tilde{h}}v_t\| \le C\|r\sigma_t\|$ again, we have

$$
\begin{aligned}
& \Big|-\int_{\epsilon_0}^{1}\mathcal{D}\mathcal{L}_{tr}\cdot\begin{pmatrix}0\\ v_t\end{pmatrix}dr\Big| \\
& = \Big|\int_{\epsilon_0}^{1}\Big(\tilde{\rho}_r\Big(-r^2\eta + \frac{\epsilon_0}{1-\epsilon_0}\int_{\epsilon_0}^{1}\eta dr\Big) + \tilde{\rho}r^2\sigma_t\Big)\tilde{h}v_t dr\Big| \\
& \le C\epsilon_0^{-2}\|\eta\|^2 + C\|r\sigma_t\|^2
\end{aligned}
\tag{3.113}
$$

and

$$
\begin{aligned}
&\Big| - \int_{\epsilon_0}^{1} \mathcal{D}\mathcal{N}_{tr} \cdot \begin{pmatrix} 0 \\ v_t \end{pmatrix} dr \Big| \\
&\le C \int_{\epsilon_0}^{1} \tilde{h} |\mathcal{N}_{tr}| |v_t| dr \\
&\le C \Big(\frac{J_0}{\epsilon_0} + \frac{|\mathbb{U}(t)|_0 + |r\phi_r|_0}{\epsilon_0} + |\mathbb{U}_r(t)|_0 \Big) (\|\mathbb{U}_t(t)\|^2 + \|\mathbb{U}_{tr}(t)\|^2_{L^2_r} + \|\sigma_r(t)\|^2). \qquad (3.114)
\end{aligned}
$$

Substituting (3.109)-(3.114) into (3.108), we have

$$
\begin{aligned}
&\frac{c_1}{4} \|\sqrt{\tilde{h}} v_{tr}(t)\|^2 \\
&\le \frac{d}{dt} \Big(\int_{\epsilon_0}^{1} \tilde{h} w_{tr} w_r dr \Big) + C_8 \|\sqrt{\tilde{h}} w_{tr}(t)\|^2 + C \|\sqrt{\tilde{h}} w_r(t)\|^2 + C\epsilon_0^{-2} (\|\eta(t)\|^2 + \|\sigma_t(t)\|^2_{L^2_r}) \\
&\quad + C \Big(\frac{J_0}{\epsilon_0} + \frac{|\mathbb{U}(t)|_0 + |r\phi_r|_0}{\epsilon_0} + |\mathbb{U}_r(t)|_0 + |\mathbb{U}_t(t)|_0 \Big) \Big(\|\mathbb{U}_t(t)\|^2 + \|\mathbb{U}_{tr}(t)\|^2_{L^2_r} + \|\sigma_r(t)\|^2 \\
&\quad + \|\sigma_t(t)\|^2_{L^2_r} + \|\sqrt{\tilde{h}} \mathbb{V}_{tr}(t)\|^2 + \Big\| \frac{\sqrt{\tilde{h}} \mathbb{V}_r(t)}{r} \Big\|^2 + \|\sqrt{\tilde{h}} v_{rr}(t)\|^2 \Big) \qquad (3.115)
\end{aligned}
$$

for some positive constant C_8.

By taking the step as $\epsilon_0^4[(3.107) + \nu_3(3.115)]$, we deduce from the fact $\epsilon_0 \|\sigma_t(t)\|_{L^2_r} = \left\| \frac{\epsilon_0}{r} v_t \right\| = \left\| \frac{\epsilon_0}{r} w_r \right\| \le C \|\epsilon_0 \sqrt{\tilde{h}} w_r\|$ that (3.94) holds. Thus, the proof is complete. □

By straightforward computation, we get the following lemma.

Lemma 3.7. *For $t \in [0, T)$, there exist positive constants $\bar{c}_1, \bar{c}_2, \bar{c}_3, \bar{c}_4$, independent of ϵ_0 and t, such that*

$$
\bar{c}_1 (\|\mathbb{U}(t)\|^2_{L^2_r} + \epsilon_0^2 \|\mathbb{U}_r(t)\|^2_{L^2_r}) \le \|\mathbb{U}(t)\|^2_{L^2_r} + \|\epsilon_0 \sqrt{\tilde{h}} \mathbb{V}_r(t)\|^2 \le \bar{c}_2 (\|\mathbb{U}(t)\|^2_{L^2_r} + \epsilon_0^2 \|\mathbb{U}_r(t)\|^2_{L^2_r}),
$$

and

$$
\begin{aligned}
&\bar{c}_3 (\|\phi_r(t)\|^2_{L^2_r} + \|\mathbb{U}(t)\|^2_{L^2_r} + \epsilon_0^2 \|\mathbb{U}_r(t)\|^2_{L^2_r} + \epsilon_0^4 \|\mathbb{U}_{tr}(t)\|^2_{L^2_r}) \\
&\le \|\phi_r(t)\|^2_{L^2_r} + \|\mathbb{U}(t)\|^2_{L^2_r} + \epsilon_0^2 \|\mathbb{U}_r(t)\|^2_{L^2_r} + \epsilon_0^4 \|\sqrt{\tilde{h}} \mathbb{V}_{tr}(t)\|^2 \\
&\le \bar{c}_4 (\|\phi_r(t)\|^2_{L^2_r} + \|\mathbb{U}(t)\|^2_{L^2_r} + \epsilon_0^2 \|\mathbb{U}_r(t)\|^2_{L^2_r} + \epsilon_0^4 \|\mathbb{U}_{tr}(t)\|^2_{L^2_r}).
\end{aligned}
$$

Proof. We claim that there exist positive constants $\tilde{c}_1, \tilde{c}_2$ such that

$$
(r\mathbb{U})^2 + (\epsilon_0 r \mathbb{U}_r)^2 \le \tilde{c}_1 (r\mathbb{U})^2 + \tilde{c}_2 (\epsilon_0 \sqrt{\tilde{h}} \mathbb{V}_r)^2.
$$

To see this, compute

$$
\begin{aligned}
&\tilde{c}_1(r\mathbb{U})^2+\tilde{c}_2(\epsilon_0\sqrt{\tilde{h}}\mathbb{V}_r)^2-(r\mathbb{U})^2-(\epsilon_0 r\mathbb{U}_r)^2\\
&\geq \tilde{c}_1(r\mathbb{U})^2+\tilde{c}_2\frac{h_0}{e^{c_2}}r^{-2}\epsilon_0^2(2r\mathbb{U}+r^2\mathbb{U}_r)^2-(r\mathbb{U})^2-(\epsilon_0 r\mathbb{U}_r)^2\\
&\geq \Big(\tilde{c}_1-1-8\tilde{c}_2\frac{h_0}{e^{c_2}}\Big)(r\mathbb{U})^2+\Big(\frac{\tilde{c}_2}{2}\frac{h_0}{e^{c_2}}-1\Big)(\epsilon_0 r\mathbb{U}_r)^2+4\tilde{c}_2\frac{h_0}{e^{c_2}}(\epsilon_0\mathbb{U})^2\\
&>0,
\end{aligned}
$$

provided that

$$\tilde{c}_2\geq\frac{2e^{c_2}}{h_0},\quad \text{and}\quad \tilde{c}_1\geq 1+8\tilde{c}_2\frac{h_0}{e^{c_2}}.$$

Thus, it follows that

$$\|\mathbb{U}(t)\|_{L_r^2}^2+\epsilon_0^2\|\mathbb{U}_r(t)\|_{L_r^2}^2\leq\tilde{c}_1\|\mathbb{U}(t)\|_{L_r^2}^2+\tilde{c}_2\|\epsilon_0\sqrt{\tilde{h}}\mathbb{V}_r(t)\|^2\leq\frac{1}{\bar{c}_1}(\|\mathbb{U}(t)\|_{L_r^2}^2+\|\epsilon_0\sqrt{\tilde{h}}\mathbb{V}_r(t)\|^2). \tag{3.116}$$

Moreover, from (3.75), it holds that

$$
\begin{aligned}
(r\mathbb{U})^2+(\epsilon_0\sqrt{\tilde{h}}\mathbb{V}_r)^2&\leq(r\mathbb{U})^2+(\epsilon_0\sqrt{h_0e^{c_2}}r^{-1}(2r\mathbb{U}+r^2\mathbb{U}_r))^2\\
&\leq(1+8h_0e^{c_2})[(r\mathbb{U})^2+(\epsilon_0 r\mathbb{U}_r)^2],
\end{aligned}
$$

together with (3.116), which shows the first inequality in the lemma.

For the second one, we can do it in the same way. Thus, the proof is complete. □

Before we prove the Theorem 3.2, we give the following lemma.

Lemma 3.8. *Let $g(x)=x-x_0-dx^\beta$ for $x\geq 0$, where constants β, x_0, d satisfy $\beta>1, x_0>0$ and $d>0$. If $0<x_0<\dfrac{\beta-1}{\beta}\Big(\dfrac{1}{\beta d}\Big)^{\frac{1}{\beta-1}}$, then there exist $x_1^*>0$ and $x_2^*>0$ such that*

(1) $g(x)<0$ *for* $x\in[0,x_1^*)\cup(x_2^*,+\infty)$,
(2) $g(x)>0$ *for* $x\in(x_1^*,x_2^*)$.

Proof. Take the derivative of $g(x)$,

$$\frac{dg(x)}{dx}=1-d\beta x^{\beta-1},$$

and let $x_*=\Big(\dfrac{1}{d\beta}\Big)^{\frac{1}{\beta-1}}$. Then $\dfrac{dg(x)}{dx}>0$ for $x\in[0,x_*)$ and $\dfrac{dg(x)}{dx}<0$ for $x\in(x_*,\infty)$, which lead to the desired result (1) and (2) provided that

$$g(x_*) = \frac{\beta-1}{\beta}\Big(\frac{1}{\beta d}\Big)^{\frac{1}{\beta-1}} - x_0 > 0. \quad \square$$

Now, we complete the proof of Theorem 3.2 as follows.

Proof of Theorem 3.2. By taking the step as (3.18) + ν_5[(3.59) + ν_4(3.94)], we have

$$\begin{aligned}
&\frac{dE_1(t)}{dt} + F_1(t)\\
&\le C\Big(\frac{J_0}{\epsilon_0} + \frac{|\mathbb{U}(t)|_0 + |r\phi_r|_0}{\epsilon_0} + |\mathbb{U}_r(t)|_0 + |\mathbb{U}_t(t)|_0\Big)\Big(\|\phi_r\|^2_{L^2_r} + \|\mathbb{U}(t)\|^2_{L^2_r} + \epsilon_0^2\|\mathbb{U}_t(t)\|^2_{L^2_r}\\
&\quad + \epsilon_0^2\|\mathbb{U}_r(t)\|^2_{L^2_r} + \epsilon_0^4\|\mathbb{U}_{tr}(t)\|^2_{L^2_r} + \|\epsilon_0\sqrt{\tilde h}\mathbb{V}_r(t)\|^2 + \|\epsilon_0^2\sqrt{\tilde h}\mathbb{V}_{tr}(t)\|^2\\
&\quad + \|\epsilon_0^2\sqrt{\tilde h}v_{rr}(t)\|^2\Big), \quad \forall t \in [0,T),
\end{aligned} \tag{3.117}$$

where ν_4, ν_5 are some positive constants to be determined. Here we use the notations,

$$\begin{aligned}
E_1(t) :=& \int_{\epsilon_0}^1 \Big(\frac{r^2(\tilde j\sigma - \tilde\rho\eta)^2}{2\rho\tilde\rho^2} + r^2(G(\rho) - G(\tilde\rho) - G'(\tilde\rho)\sigma) + \frac{r^2\phi_r^2}{2} + \frac{\nu_5\tilde s}{2}\epsilon_0^2(v_r)^2 + \frac{\nu_5\tilde h}{2}\epsilon_0^2(w_r)^2\\
&+ \nu_5\nu_4\frac{\tilde s}{2}\epsilon_0^4(v_{tr})^2 + \nu_5\nu_4\frac{\tilde h}{2}\epsilon_0^4(w_{tr})^2\Big)dr - \nu_1\int_{\epsilon_0}^1 \frac{\eta r^2\phi_r}{\tilde\rho}dr - \int_{\epsilon_0}^1 \nu_5\frac{|b(\mathbb{U})|}{2}\tilde h\epsilon_0^2(v_r)^2dr\\
&- \int_{\epsilon_0}^1 \nu_5\frac{k_{1r}\tilde h\epsilon_0^2(v(t,r))^2}{2}dr - \int_{\epsilon_0}^1 \nu_5\nu_2\tilde h\epsilon_0^2 w_r v dr - \int_{\epsilon_0}^1 \nu_5\nu_4\frac{|b(\mathbb{U})|}{2}\tilde h\epsilon_0^4(v_{tr})^2dr\\
&- \int_{\epsilon_0}^1 \nu_5\nu_4\frac{k_{1r}\tilde h}{2}\epsilon_0^4(v_t)^2dr - \nu_5\nu_4\nu_3\int_{\epsilon_0}^1 \epsilon_0^4\tilde h w_{tr}w_r dr
\end{aligned}$$

and

$$\begin{aligned}
F_1(t) :=& \int_{\epsilon_0}^1 \frac{r^2(\tilde j\sigma - \tilde\rho\eta)^2}{\rho\tilde\rho^2}dr + \nu_1 c_0\|\sigma(t)\|^2_{L^2_r} + \frac{3\nu_1}{4}\|\phi_r(t)\|^2_{L^2_r} + \frac{\nu_1\epsilon_0}{1-\epsilon_0}\int_{\epsilon_0}^1 \eta dr \int_{\epsilon_0}^1 \frac{\eta}{\tilde\rho}dr\\
&- \nu_1 C_6\|\eta(t)\|^2_{L^2_r} + \nu_5(\frac{1}{2} - \nu_2 C_7 - C_9\nu_4)\|\epsilon_0\sqrt{\tilde h}w_r(t)\|^2 + \nu_5\frac{\nu_2 c_1}{8}\|\epsilon_0\sqrt{\tilde h}v_r(t)\|^2\\
&+ \nu_5\nu_4\Big(\frac{1}{2} - C_8\nu_3\Big)\|\epsilon_0^2\sqrt{\tilde h}w_{tr}(t)\|^2 + \frac{\nu_5\nu_4\nu_3 c_1}{4}\|\epsilon_0^2\sqrt{\tilde h}v_{tr}(t)\|^2\\
&- \nu_5 C_{10}(\|\phi_r(t)\|^2_{L^2_r} + \|\sigma(t)\|^2_{L^2_r} + \|\eta(t)\|^2_{L^2_r}).
\end{aligned}$$

Firstly, we use (3.75), and Lemma 3.3, Lemma 3.5, and Lemma 3.7 to derive that

$$\|\phi_r\|^2_{L^2_r} + \|\mathbb{U}(t)\|^2_{L^2_r} + \epsilon_0^2\|\mathbb{U}_t(t)\|^2_{L^2_r} + \epsilon_0^2\|\mathbb{U}_r(t)\|^2_{L^2_r} + \epsilon_0^4\|\mathbb{U}_{tr}(t)\|^2_{L^2_r} + \|\epsilon_0^2\sqrt{\tilde{h}}v_{rr}(t)\|$$
$$\le C(\|\phi_r(t)\|^2_{L^2_r} + \|\mathbb{U}(t)\|^2_{L^2_r} + \|\epsilon_0\sqrt{\tilde{h}}\mathbb{V}_r(t)\|^2 + \|\epsilon_0^2\sqrt{\tilde{h}}\mathbb{V}_{tr}(t)\|^2). \tag{3.118}$$

Thus, denote

$$E(t) := \|\epsilon_0^2\sqrt{\tilde{h}}\mathbb{V}_{tr}(t)\|^2 + \|\epsilon_0\sqrt{\tilde{h}}\mathbb{V}_r(t)\|^2 + \|\mathbb{U}(t)\|^2_{L^2_r} + \|\phi_r(t)\|^2_{L^2_r}.$$

And it follows from Lemma 3.5 and Lemma 3.7 that there exist positive constants $\bar{c}_5$ and $\bar{c}_6$ such that

$$\bar{c}_5 n^2(t) \le E(t) \le \bar{c}_6 n^2(t). \tag{3.119}$$

On the other hand, by Lemma 3.3 and by (3.22)-(3.23), we get

$$\frac{|\mathbb{U}(t)|_0 + |r\phi_r|_0}{\epsilon_0} + |\mathbb{U}_r(t)|_0 + |\mathbb{U}_t(t)|_0$$
$$\le C\epsilon_0^{-\frac{5}{2}}(\|\phi_r(t)\|_{L^2_r} + \|\mathbb{U}(t)\|_{L^2_r} + \epsilon_0\|\mathbb{U}_r(t)\|_{L^2_r} + \epsilon_0^2\|\mathbb{U}_{rr}(t)\|_{L^2_r})$$
$$= C\epsilon_0^{-\frac{5}{2}}n(t), \tag{3.120}$$

where we have used the (3.92).

Thus, by (3.119) and (3.120), we may estimate the right-hand side of in (3.117):

$$\frac{dE_1(t)}{dt} + F_1(t) \le C\Big(\frac{J_0}{\epsilon_0} + \epsilon_0^{-\frac{5}{2}}\sqrt{E(t)}\Big)E(t). \tag{3.121}$$

For the $E_1(t)$ and $F_1(t)$, we claim that there exist positive constants C_{11}, C_{12} and C_{13} such that

$$C_{11}E(t) \le E_1(t) \le C_{12}E(t), \tag{3.122}$$
$$F_1(t) \ge C_{13}E(t). \tag{3.123}$$

To see this, first we have

$$r^2G(\rho) - r^2G(\tilde{\rho}) - r^2G'(\tilde{\rho})\sigma = \frac{P'(\bar{\rho})}{2\bar{\rho}}(r\sigma)^2 \ge c_7(r\sigma)^2, \quad \bar{\rho} \text{ is between } \tilde{\rho} \text{ and } \rho.$$

Secondly, we handle the terms containing k_{1r} in $E_1(t)$ by the same method as in (3.84) and (3.110).

Finally, it follows from (3.75) and (3.68) that

$$E_1(t) \ge C_{14}[(1-\epsilon_0^{\alpha-1}-\nu_5-\nu_5\nu_2)\|\sigma(t)\|_{L_r^2}^2+(1-\epsilon_0^{\alpha-1}-\nu_1)\|\eta(t)\|_{L_r^2}^2+(1-\nu_1)\|\phi_r(t)\|_{L_r^2}^2$$
$$+\nu_5\Big(1-\epsilon_0^{\frac{1}{2}}-\frac{1}{2}\Big)\|\epsilon_0\sqrt{\tilde{h}}v_r(t)\|^2+\nu_5(1-\nu_2-\nu_3\nu_4\epsilon_0^2-\nu_4)\|\epsilon_0\sqrt{\tilde{h}}w_r(t)\|^2$$
$$+\nu_5\nu_4(1-\epsilon_0^{\frac{1}{2}}-\mu_8)\|\epsilon_0^2\sqrt{\tilde{h}}v_{tr}(t)\|^2+\nu_5\nu_4(1-\nu_3)\|\epsilon_0^2\sqrt{\tilde{h}}w_{tr}(t)\|^2]$$

for some constant C_{14}.

Likewise, it is easy to show that

$$F_1(t) \ge C_{15}\Big[(1-\epsilon_0^{\alpha-1}-\nu_1-\nu_5C_{10})\|\eta(t)\|_{L_r^2}^2+\Big(\frac{\nu_1c_1}{2}-\epsilon_0^{\alpha-1}-\nu_5C_{10}\Big)\|\sigma(t)\|_{L_r^2}^2$$
$$+\Big(\frac{3\nu_1}{4}-\nu_5C_{10}\Big)\|\phi_r(t)\|_{L_r^2}^2+\frac{\nu_5\nu_4\nu_3c_1}{4}\|\epsilon_0^2\sqrt{\tilde{h}}v_{tr}(t)\|^2+\frac{\nu_5\nu_2c_1}{8}\|\epsilon_0\sqrt{\tilde{h}}v_r(t)\|^2$$
$$+\nu_5\nu_4\Big(\frac{1}{2}-C_8\nu_3\Big)\|\epsilon_0^2\sqrt{\tilde{h}}w_{tr}(t)\|^2+\nu_5\Big(\frac{1}{2}-\nu_2C_7-\nu_4C_9\Big)\|\epsilon_0\sqrt{\tilde{h}}w_r(t)\|^2\Big]$$

for some constant C_{15}.

Now, we may choose some suitable positive constants $\nu_i (i=1,2,3,4,5)$ and μ_8 satisfying,

$$1-\epsilon_0^{\frac{1}{2}}-\mu_8 \ge \frac{1}{2}, \quad 1-\epsilon_0^{\frac{1}{2}}-\frac{1}{2} \ge \frac{1}{4}, \quad \frac{1}{2}-(1+C_8)\nu_3 \ge \frac{1}{4},$$
$$1-\epsilon_0^{\alpha-1}-\nu_1-\nu_5(1+\nu_2+C_{10}) \ge \frac{1}{2}, \quad \frac{1}{2}-(1+C_7)\nu_2-\nu_4(\nu_3\epsilon_0^2+1+C_9) \ge \frac{1}{4},$$
$$\frac{\nu_1c_1}{2}-\epsilon_0^{\alpha-1}-\nu_5C_{10}\Big(1+\frac{2c_1}{3}\Big) \ge \frac{\nu_1c_1}{4}$$

for ϵ_0 sufficiently small with $\alpha > 2$, which leads to

$$E_1(t) \ge C_{11}E(t) \quad \text{and} \quad F_1(t) \ge C_{13}E(t)$$

for some constants C_{11} and C_{13}.

On the other hand, we note that

$$E_1(t) \le C_{12}E(t)$$

for some positive constant C_{12}. Therefore, our claim (3.122) and (3.123) hold.

Accordingly, we can rewrite (3.117) as

$$\frac{dE_1(t)}{dt}+C_{16}E_1(t) \le C\Big(\epsilon_0^{\alpha-2}+\epsilon_0^{-\frac{5}{2}}\sqrt{E_1(t)}\Big)E_1(t)$$

with $\alpha > 2$. Furthermore, for $0 < \epsilon_0 \ll 1$, it holds that

$$\frac{dE_1(t)}{dt}+\frac{C_{16}}{2}E_1(t) \le C_{17}\epsilon_0^{-\frac{5}{2}}E_1(t)^{\frac{3}{2}}, \quad \forall t \in [0,T), \tag{3.124}$$

for some positive constant C_{17}, independent of C_5 and ϵ_0.

Now, let

$$M(t) = \sup_{\tau\in[0,t]} e^{c'\tau} E_1(\tau), \quad c' < \frac{C_{16}}{2}.$$

Then, we derive from (3.124) that

$$M(t) \le M(0) + C_{17}\epsilon_0^{-\frac{5}{2}} M^{\frac{3}{2}}(t) e^{(-\frac{C_{16}}{2}+c')t} \int_0^t e^{(\frac{C_{16}}{2}-c')\tau} d\tau \le M(0) + C_{17}\epsilon_0^{-\frac{5}{2}} M^{\frac{3}{2}}(t). \quad (3.125)$$

To get the upperbound of $M(t)$, we consider $g(x) = x - x_0 - dx^\beta$ mentioned in Lemma 3.8. Let $x = M(t)$, $x_0 = M(0) > 0$, $d = C_{17}\epsilon_0^{-\frac{5}{2}} > 0$ and $\beta = \frac{3}{2} > 1$. For $x_0 = M(0) < \frac{1}{3}\Big(\frac{2}{3C_{17}}\epsilon_0^\gamma\Big)^2$ and $x_* = \Big(\frac{2}{3C_{17}}\epsilon_0^\gamma\Big)^2$ with $\gamma \ge \frac{5}{2}$, there holds that $g(x_*) = \frac{1}{3}\Big(\frac{2}{3C_{17}}\epsilon_0^\gamma\Big)^2 - x_0 > 0$. By (3.125), Lemma 3.8 and by the continuity of $M(t)$ respect with to t, we get

$$M(t) < \Big(\frac{2\epsilon_0^\gamma}{3C_{17}}\Big)^2, \quad \forall t \in [0, T), \quad (3.126)$$

namely,

$$E_1(t) \le e^{-c't}\Big(\frac{2\epsilon_0^\gamma}{3C_{17}}\Big)^2, \quad \forall t \in [0, T). \quad (3.127)$$

Note that $C_{11}E(t) \le E_1(t) \le C_{12}E(t)$, in view of the equivalence between $n(t)$ and $E(t)$ (see (3.119)), we get the equivalence between $n(t)$ and $E_1(t)$, which implies that there exist some positive constants C_4 and C_5 with $C_4 < C_5$ such that if

$$n(0) = \epsilon_0^2\|\mathbb{U}_{rr}(0)\|_{L_r^2} + \epsilon_0\|\mathbb{U}_r(0)\|_{L_r^2} + \|\mathbb{U}(0)\|_{L_r^2} + \|\phi_r(0)\|_{L_r^2} \le C_4\epsilon_0^\gamma,$$

then it holds that

$$n(t) = \epsilon_0^2\|\mathbb{U}_{rr}(t)\|_{L_r^2} + \epsilon_0\|\mathbb{U}_r(t)\|_{L_r^2} + \|\mathbb{U}(t)\|_{L_r^2} + \|\phi_r(t)\|_{L_r^2} \le C_5 e^{-\frac{c't}{2}}\epsilon_0^\gamma, \quad \forall t \in [0, T). \quad (3.128)$$

That is,

$$N^*(T) \le C_5\epsilon_0^\gamma, \quad (3.129)$$

which implies, together with the definition of T, that

$$T = t^*.$$

The proof is complete. □

In the end, we prove the Theorem 1.2.

Proof of Theorem 1.2. It suffices to prove that $t^* = \infty$.

If $t^* < \infty$, using Theorem 3.2 we get $N^*(t^*) \le C_5\epsilon_0^\gamma$ with $\gamma \ge \frac{5}{2}$. Then, we regard t^* as the initial time and use Theorem 3.1 to draw a conclusion that there exists a $t_0(C_5, \epsilon_0) > 0$ such that (3.2)-(3.3) exists a unique solution $(\sigma, \eta, \phi)(t,r) \in [\chi_{2,r}([t^*, t^* + t_0(C_5, \epsilon_0)]; \Omega)]^3$. This is a contradiction to the definition of t^*, thus, $t^* = \infty$.

Therefore (1.6)-(1.7) has a unique solution $(\rho,\ j,\ \Phi)(t,r) \in [\chi_{2,r}([0,\infty); \Omega)]^3$ with $N^*(t) \le C_5\epsilon_0^\gamma$ for $t \ge 0$.

Moreover, let

$$m(t) := \|\mathbb{U}(t)\|_{L_r^2} + \epsilon_0\|\partial\mathbb{U}(t)\|_{L_r^2} + \epsilon_0^2\|\partial^2\mathbb{U}(t)\|_{L_r^2} + \|\phi_r(t)\|_{L_r^2}. \tag{3.130}$$

Differentiating (3.9) with respect to t, we get the following

$$\begin{aligned}
\epsilon_0^2\|\mathbb{U}_{tt}(t)\|_{L_r^2} &\le C(\epsilon_0^2\|\mathbb{U}_{tr}(t)\|_{L_r^2} + \epsilon_0\|\mathbb{U}_t(t)\|_{L_r^2} + \|\mathbb{U}(t)\|_{L_r^2} + \|\phi_r(t)\|_{L_r^2}),\\
\epsilon_0^2\|\mathbb{U}_{tr}(t)\|_{L_r^2} &\le C(\epsilon_0^2\|\mathbb{U}_{tt}(t)\|_{L_r^2} + \epsilon_0\|\mathbb{U}_t(t)\|_{L_r^2} + \|\mathbb{U}(t)\|_{L_r^2} + \|\phi_r(t)\|_{L_r^2}),
\end{aligned}$$

which, together with Lemma 3.3 and Lemma 3.5, gives the equivalence of $m(t)$ and $n(t)$.

Then, it follows from (3.128) that

$$m(t) \le Ce^{-\frac{c't}{2}}\epsilon_0^\gamma, \quad \forall t \in [0,\infty), \tag{3.131}$$

for some positive constant C, where $\gamma \ge \frac{5}{2}$. Thus, the proof is complete. □

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