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Existence of traveling wave fronts of delayed Fisher-type equations with degenerate nonlinearities

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A B S T R A C T

In this paper, two different kinds of degenerate n-degree Fisher-type equations with delays are considered. Due to the difference of the reaction terms, the existence of traveling front are proved by different methods. More precisely, when the reaction term satisfies the weak quasimonotonicity condition, for $c > 2$, the existence result is given by the super-sub solution method and the fixed point theorem. Then for $c_* < c \leqslant 2$, where c_* is the minimal speed of degenerate p-degree Fisher-type equations without delays, the existence result is proved by the perturbation method and the implicit function theory. For the other type reaction term, we apply the monotone iteration method and the super-sub solution method to obtain the existence conclusion.

1. Introduction and main results

In this paper, we focus on the existence of traveling wave fronts of the following two different types of degenerate $p-$degree Fisher-type equations with delays

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2} + u^p(x,t)(1-u(x,t-\tau)), \tag{1.1}$$

and

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2} + u^p(x,t-\tau)(1-u(x,t)), \tag{1.2}$$

where $p > 1$ is a number (no need to be integer).

When $\tau = 0$, Eqs. (1.1) and (1.2) are reduced to

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2} + u^p(x,t)(1-u(x,t)), \tag{1.3}$$

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which describes some isothermal autocatalytic chemical reactions introduced in [1,2]. In recent years, the existence and stability of traveling wave fronts of (1.3), including the continuous model and the lattice model, have been studied, such as [1–16] and references therein. From [2,9], we know for each $p > 1$, there exists a critical speed $c_*(p) > 0$, which is also described as the minimal speed, such that (1.3) admits an unique traveling front solution ϕ if and only if $c \geqslant c_*(p)$. The relationships between the value of $c_*(p)$ and the parameter p have been discussed in the existing studies. When $p = 2$, the minimum wave speed $c_*(2) = \frac{\sqrt{2}}{2}$ was first observed numerically in [10], and further confirmed by the phase-plane form of comparison theorems in [7]. For $p > 1$, in [9], the minimum wave speed $c_*(p)$ was proved to be continuous and monotonically decreasing on p, and tested asymptotically as

$$c_*(p) = \begin{cases} 2 - 2.33107(p-1)^{\frac{2}{3}} + o((p-1)^{\frac{2}{3}}), & p \to 1^+, \\ \frac{\sqrt{2}}{p} + o(\frac{\sqrt{2}}{p}), & p \to +\infty. \end{cases}$$

For the population dynamics, the models are related to the matured age of species, the so-called time-delay. The existence of traveling wave fronts for such time-delayed equations has been one of hot research spots as we know. The methods adopted for proofs are various, for example, the super-sub solution method combining the fixed point theorem [17,18], the monotone iteration method combining the super-sub solution method [19,20], the phase plane techniques [21], the perturbation method combining the implicit function theory [22] for the case of delayed degenerate diffusion equations, and so on. Very recently, in [3], the existence of traveling wave fronts of (1.2) has been proved by the monotone semiflows theorem, and the other properties of such fronts have also been given. To our best knowledge, the study on the existence of traveling wave fronts of delayed diffusion equations with degenerate nonlinearities, such as (1.1) and (1.2), is quite incomplete. To prove the existence of traveling waves for (1.1) and (1.2), respectively, is the main purpose in this paper. Since the reaction terms in (1.1) and (1.2) are different, we will adopt different methods mentioned above to give the existence conclusions of (1.1) and (1.2) respectively.

Traveling wave fronts of (1.1) or (1.2) connecting 0 and 1, are defined as follows:

$$\phi''(\xi) - c\phi'(\xi) + \phi^p(\xi)(1 - \phi(\xi - c\tau)) = 0, \tag{1.4}$$

or

$$\phi''(\xi) - c\phi'(\xi) + \phi^p(\xi - c\tau)(1 - \phi(\xi)) = 0, \tag{1.5}$$

correspondingly, with

$$\lim_{\xi\to-\infty} \phi(\xi) = 0, \qquad \lim_{\xi\to+\infty} \phi(\xi) = 1, \tag{1.6}$$

where $' := \frac{d}{d\xi}$, $\xi = x + ct$ and c is the wave speed.

Our main results are as follows.

Theorem 1.1. *When $\tau > 0$ is small, (1.4) with (1.6) has an increasing solution for any $c > 2$. That is, (1.1) admits a traveling wave front ϕ connecting 0 and 1.*

Theorem 1.2. *(1.2) admits a strictly increasing traveling wave front ϕ connecting 0 and 1 for any $c > c^*(\tau)$, where $c^*(\tau)$ uniquely satisfies*

$$\inf_{\phi\in\Gamma_2} \sup_{\xi\in\mathbb{R}} \frac{\phi''(\xi) + \phi^p(\xi - c\tau)(1 - \phi(\xi))}{\phi'(\xi)} < c$$

and

$$\Gamma_2 = \left\{ \phi \in C^2(\mathbb{R},\mathbb{R}) : \begin{array}{l} (i) \lim_{\xi\to-\infty} \phi(\xi) = 0, \quad \lim_{\xi\to+\infty} \phi(\xi) = 1; \\ (ii)\ \phi'(\xi) > 0, \quad \xi \in \mathbb{R}. \end{array} \right\}.$$

Remark 1.1. When we revised this paper after receiving the review reports, we happened to note a new online publication [3], which is mainly to investigate the influence of delay and degeneracy on the propagation threshold. The existence of traveling wave fronts (1.2) was simply claimed to be obtained in [3] (see Lemma 1.2 therein) by the theory of the monotone semiflow but without a detailed proof. It seems to us that the proof of the existence for such traveling waves is not straightforward. Here, as an independent study, we indeed prove it by a different approach, the so-called monotonic iteration technique combining with the super-sub solution method. In addition, in [3], the critical speed is positive and less than or equal to the constant $\sqrt{1/K(n)}$, where $K(n)$ is a strictly increasing function of n. While in our Theorem 1.2, we give a new up-bound of the critical speed.

Remark 1.2. The nonexistence problem is also interesting and important, while, we focus on the existence results and could not solve the nonexistence problem in this paper. This problem will be our further study.

From the above results on the minimal speed $c_*(p)$, we see that $0 < c_*(p) < 2$. It is interesting for us to investigate the existence of traveling wave fronts of (1.1) for $c_*(p) < c < 2$, which is given in the following theorem.

Theorem 1.3. *Assume $\tau > 0$ is small. Then (1.1) admits a strictly increasing traveling wave front ϕ connecting 0 and 1 for each $c(\tau) > c_*(p)$, where $c_*(p)$ is the minimal speed of (1.3).*

Remark 1.3. From Theorem 1.1, when the delay τ is small, we can prove the existence of traveling wave fronts of (1.1) for the speed c is larger than 2, which does not depend on the delay τ. And we can also see that 2 is an upper bound of the minimal speed. While from Theorem 1.3, we obtain the strict monotonicity of traveling wave front.

The rest of the paper is organized as follows. In the next section, we give the basic assumptions, and introduce the abstract existence theorems shown in [17–20]. These theorems will be applied to prove Theorems 1.1–1.3. In Section 3, due to the different kinds of nonlinearities of (1.4) and (1.5), we prove main results by the super-sub solution method combining the fixed pointed theorem, or the monotone iteration method combining the super-sub solution method, or the phase analysis method, respectively.

2. Preliminaries

In this section, we introduce two lemmas about the existence of traveling wave fronts of delayed diffusion equations with different kinds of nonlinearities. Firstly, for convenience, we let $\phi(s) := \phi(\xi)(s) = \phi(\xi + s)$, where $s \in [-c\tau, 0]$, and introduce the following wave equation

$$\phi''(\xi) - c\phi'(\xi) + f(\phi(\xi)(0), \phi(\xi)(s)) = 0, \tag{2.1}$$

where $s \in [-c\tau, 0]$. Then we give some assumptions on f.
(A1) $f(0, \tilde{0}) = f(1, \tilde{1}) = 0$, where $\tilde{\phi} : [-c\tau, 0] \to \mathbb{R}$ is the constant function with value 0 or 1 for all $\xi \in \mathbb{R}$;
(A2) There exists a positive constant L such that

$$|f(\phi(0), \phi(s)) - f(\psi(0), \psi(s))| \leqslant L \sup_{s\in[-c\tau,0]} |\phi(s) - \psi(s)|$$

for $\phi,\ \psi \in C([-c\tau, 0], \mathbb{R})$ with $0 \leqslant \phi(s),\ \psi(s) \leqslant 1$, $s \in [-c\tau, 0]$;
(A3) $f(\hat{\phi}, \check{\phi})$ is increasing in $\check{\phi}$, for $0 \leqslant \hat{\phi},\ \check{\phi} \leqslant 1$. And there is a $\beta \geqslant 0$ such that

$$f(\phi(0), \phi(s)) - f(\psi(0), \psi(s)) + \beta[\phi(0) - \psi(0)] \geqslant 0,$$

for $\phi,\ \psi \in C([-c\tau, 0], \mathbb{R})$ with $0 \leqslant \phi(s),\ \psi(s) \leqslant 1$, $s \in [-c\tau, 0]$; or

(A4) There is a $\beta \geqslant 0$ such that

$$f(\phi(0),\phi(s)) - f(\psi(0),\psi(s)) + \beta[\phi(0)-\psi(0)] \geqslant 0,$$

for $\phi,\ \psi \in C([-c\tau,0],\mathbb{R})$ with (i) $0 \leqslant \psi(s) \leqslant \phi(s) \leqslant 1$ for $s \in [-c\tau,0]$ and (ii) $e^{\beta s}[\phi(s)-\psi(s)]$ increasing in $s \in [-c\tau,0]$.

In the following, we will use the super-sub solution method to discuss the existence of traveling wave fronts. Thus, we introduce the super-sub solution here.

Definition 2.1. A continuous function $\overline{\phi}$: $\mathbb{R} \to \mathbb{R}$ is called a supersolution of (2.1), if $\overline{\phi}'$ and $\overline{\phi}''$ exist almost everywhere in $\mathbb{R}$ and they are essentially bounded on $\mathbb{R}$ and if $\overline{\phi}$ satisfies

$$F(\overline{\phi}) := \overline{\phi}''(\xi) - c\overline{\phi}'(\xi) + f(\overline{\phi}(0),\overline{\phi}(s)) \leqslant 0, \quad a.e. \ \ in \ \ \mathbb{R}.$$

A subsolution is defined in a similar way by reversing the inequality in the above inequality.

From Theorem 3.1 in [17], Theorems 2.1 and 2.2 in [18], Theorems 3.6 and 4.5 in [19] and Theorem 1.2 in [20], we can conclude the following existence lemmas.

Lemma 2.1. *Assume that (A1), (A2) and (A4) hold. We assume that the supersolution $\overline{\phi}$ and the subsolution $\underline{\phi}$ satisfy*
(H1) $0 \leqslant \underline{\phi} \leqslant \overline{\phi} \leqslant 1$, $\xi \in \mathbb{R}$;
(H2) $\lim_{\xi\to-\infty} \underline{\phi}(\xi) = 0$, $\lim_{\xi\to+\infty} \overline{\phi}(\xi) = 1$;
(H3) The set

$$\Gamma_1(\underline{\phi},\overline{\phi}) = \left\{ \phi \in C(\mathbb{R},\mathbb{R}) : \begin{array}{l} (i)\ \phi \text{ is increasing in } \mathbb{R} \text{ and } \underline{\phi} \leqslant \phi \leqslant \overline{\phi}; \\ (ii)\ e^{\beta\xi}[\overline{\phi}(\xi)-\phi(\xi)] \text{ and } e^{\beta\xi}[\phi(\xi)-\underline{\phi}(\xi)] \text{ are} \\ \quad \text{increasing in } \xi \in \mathbb{R}; \\ (iii)\ e^{\beta\xi}[\phi(\xi+s)-\phi(\xi)] \text{ is increasing in} \\ \quad \xi \in \mathbb{R} \text{ for every } s>0. \end{array} \right\}$$

is non-empty.
(H4) $f(\phi(0),\tilde{\phi}(s)) \neq 0$ *for* $\phi \in (0, \inf_{\xi\in\mathbb{R}} \overline{\phi}(\xi)] \cup [\sup_{\xi\in\mathbb{R}} \underline{\phi}(\xi), 1)$.
Then (2.1) with (1.6) has an increasing solution.

Lemma 2.2. *Assume that (A1), (A2) and (A3) hold. Suppose*

$$\inf_{\phi\in\Gamma_2} \sup_{\xi\in\mathbb{R}} \frac{\phi''(\xi) + f(\phi(0),\phi(-c_0\tau))}{\phi'(\xi)} < \infty$$

for some $c_0 \geqslant 0$, *where*

$$\Gamma_2 = \left\{ \phi \in C^2(\mathbb{R},\mathbb{R}) : \begin{array}{l} (i)\ \lim_{\xi\to-\infty}\phi(\xi)=0, \quad \lim_{\xi\to+\infty}\phi(\xi)=1; \\ (ii)\ \phi'(\xi)>0, \quad \xi\in\mathbb{R}. \end{array} \right\}.$$

Then for any $c > c^*(\tau)$, *there exists a strictly increasing solution of (2.1) with (1.6), where* $c^*(\tau) \geqslant 0$ *uniquely satisfies*

$$\inf_{\phi\in\Gamma_2} \sup_{\xi\in\mathbb{R}} \frac{\phi''(\xi) + f(\phi(0),\phi(-c\tau))}{\phi'(\xi)} < c$$

for any $c > c^*(\tau)$.

3. Proof of main results

In this section, we firstly apply the existence results given in Section 2 to (1.1) and (1.2) to finish the proofs of Theorems 1.1–1.2. Now, we begin to prove Theorem 1.1.

Proof of Theorem 1.1. From (1.4), $f(\phi(0),\phi(s)) = \phi^p(0)(1-\phi(-c\tau))$. Obviously, f satisfies (A1) and (A2). Then we will prove f satisfies (A4). From (i) and (ii) in (A4)

$$\begin{aligned} f(\phi(0),\phi(s)) - f(\psi(0),\psi(s)) =& \phi^p(0)(1-\phi(-c\tau)) - \psi^p(0)(1-\psi(-c\tau)) \\ =&(1-\phi(-c\tau))(\phi^p(0)-\psi^p(0)) + \psi^p(0)(\psi(-c\tau)-\phi(-c\tau)) \\ \geqslant& -e^{\beta c\tau}(\phi(0)-\psi(0)). \end{aligned}$$

If $\beta > 1$ and τ is small enough, then

$$f(\phi(0),\phi(s)) - f(\psi(0),\psi(s)) + \beta[\phi(0)-\psi(0)] \geqslant (\beta - e^{\beta c\tau})(\phi(0)-\psi(0)) \geqslant 0,$$

which implies f satisfying (A4).

Then in order to apply Lemma 2.1 to obtain the existence results, we need to construct a pair of super-sub solutions that satisfy (H1)-(H4). Inspired by the ways to construct subsolutions in [23], let

$$\underline{\phi}(\xi) = \begin{cases} 0, & \xi \leqslant \xi_1, \\ \frac{1}{2} - k_1 e^{-d\xi}, & \xi > \xi_1, \end{cases}$$

where $\xi_1 = -\frac{1}{d}\ln\frac{1}{2k_1}$, $k_1 < \frac{1}{2}$ and $d > 0$ is small enough determined later. Obviously, $\xi_1 < 0$.

When $\xi \leqslant \xi_1$, $\underline{\phi}(\xi) = 0$ and $\underline{\phi}(\xi - c\tau) = 0$. Thus $F(\underline{\phi}) = 0$. When $\xi_1 < \xi \leqslant \xi_1 + c\tau$, $\underline{\phi}(\xi) = \frac{1}{2} - k_1 e^{-d\xi}$ and $\underline{\phi}(\xi - c\tau) = 0$. Then

$$F(\underline{\phi}) = -k_1 e^{-d\xi} d^2 - ck_1 e^{-d\xi} d + (\frac{1}{2} - k_1 e^{-d\xi})^p := g_1(d).$$

Since $g_1(0) = (\frac{1}{2} - k_1)^p > 0$, then for small $d > 0$, $g_1(d) \geqslant 0$ for $\xi \in (\xi_1, \xi_1 + c\tau]$. Thus $F(\underline{\phi}) \geqslant 0$. When $\xi > \xi_1 + c\tau$, $\underline{\phi}(\xi) = \frac{1}{2} - k_1 e^{-d\xi}$ and $\underline{\phi}(\xi - c\tau) = \frac{1}{2} - k_1 e^{-d\xi} e^{dc\tau}$. Hence

$$\begin{aligned} F(\underline{\phi}) =& -k_1 e^{-d\xi} d^2 - ck_1 e^{-d\xi} d + (\frac{1}{2} + k_1 e^{-d\xi} e^{dc\tau})(\frac{1}{2} - k_1 e^{-d\xi})^p \\ \geqslant& [-k_1 d^2 - ck_1 d + (\frac{1}{2} - k_1 e^{-d(\xi_1 + c\tau)})^p k_1 e^{dc\tau}] e^{-d\xi} \\ :=& g_2(d) e^{-d\xi} \end{aligned}$$

where $g_2(d) = -k_1 d^2 - ck_1 d + (\frac{1}{2} - k_1 e^{-d(\xi_1 + c\tau)})^p k_1 e^{dc\tau}$. Repeating the similar argument, since $g_2(0) = (\frac{1}{2} - k_1)^p k_1 > 0$, then for small $d > 0$, $F(\underline{\phi}) \geqslant 0$. Therefore $\underline{\phi}$ is a subsolution to (1.4).

We continue to give the supersolution. From Proposition 5.1.2 in [19], for $c > 2$ and small τ, the function $\overline{\phi} = \frac{e^{\lambda_1 \xi}}{e^{\lambda_1 \xi} + k_2}$, where $0 < k_2 < e^{\lambda_1 \xi_1}$ and $\lambda_1 = \frac{c - \sqrt{c^2 - 4}}{2} > 0$, satisfies

$$\overline{\phi}''(\xi) - c\overline{\phi}'(\xi) + \overline{\phi}(\xi)(1 - \overline{\phi}(\xi - c\tau)) \leqslant 0.$$

Hence, by noting $0 \leqslant \overline{\phi} \leqslant 1$, we have

$$F(\overline{\phi}) = \overline{\phi}''(\xi) - c\overline{\phi}'(\xi) + \overline{\phi}^p(\xi)(1 - \overline{\phi}(\xi - c\tau)) \leqslant \overline{\phi}''(\xi) - c\overline{\phi}'(\xi) + \overline{\phi}(\xi)(1 - \overline{\phi}(\xi - c\tau)) \leqslant 0.$$

Therefore $\overline{\phi}$ is a supersolution to (1.4).

In the end, we check $\overline{\phi}$ and $\underline{\phi}$ given above are satisfying (H1)-(H4). Obviously $\lim_{\xi\to-\infty}\underline{\phi}=0$ and $\lim_{\xi\to+\infty}\overline{\phi}=1$, thus, (H2) is fulfilled. When $\xi\leqslant\xi_1$, $\overline{\phi}-\underline{\phi}=\overline{\phi}\geqslant 0$. When $\xi>\xi_1$, by recalling $0<k_2<e^{\lambda_1\xi_1}$,

$$\overline{\phi}-\underline{\phi}\geqslant\frac{e^{\lambda_1\xi_1}-k_2}{2(e^{\lambda_1\xi}+k_2)}\geqslant 0.$$

Therefore, by recalling $\underline{\phi}\geqslant 0$ and $\overline{\phi}\leqslant 1$, we see that (H1) is fulfilled. We further prove $\Gamma(\underline{\phi},\overline{\phi})$ is non-empty. It is sufficient for us to prove $\overline{\phi}\in\Gamma(\underline{\phi},\overline{\phi})$. By combining the fact $\overline{\phi}'=\frac{\lambda_1 k_2 e^{\lambda_1\xi}}{(e^{\lambda_1\xi}+k_2)^2}\geqslant 0$ with the proof of (H1), (i) in (H3) is fulfilled. Since $e^{\beta\xi}[\overline{\phi}(\xi)-\overline{\phi}(\xi)]=0$, then $e^{\beta\xi}[\overline{\phi}(\xi)-\overline{\phi}(\xi)]=0$ is increasing. When $\xi\leqslant\xi_1$, $e^{\beta\xi}[\overline{\phi}(\xi)-\underline{\phi}(\xi)]=e^{\beta\xi}\overline{\phi}(\xi)$, which is obviously increasing. When $\xi>\xi_1$, by recalling $k_2<e^{\lambda_1\xi_1}$,

$$\{e^{\beta\xi}[\overline{\phi}(\xi)-\underline{\phi}(\xi)]\}'=e^{\beta\xi}\frac{\beta(e^{2\lambda_1\xi}-k_2^2)+2k_2\lambda_1e^{\lambda_1\xi}+2(\beta-d)k_1(e^{\lambda_1\xi}+k_2)^2e^{-d\xi}}{2(e^{\lambda_1\xi}+k_2)^2}>0$$

if d is small. In conclusion, (ii) in (H3) is fulfilled. Repeating the proof in [17], if $\beta\geqslant\lambda_1$, then for every $s>0$, $e^{\beta\xi}[\phi(\xi+s)-\phi(\xi)]=e^{\beta\xi}[\overline{\phi}(\xi+s)-\overline{\phi}(\xi)]$ is increasing in $\xi\in\mathbb{R}$. In a word, $\Gamma(\underline{\phi},\overline{\phi})$ is non-empty. It is easy to verify that $f(\phi(0),\tilde{\phi}(s))\neq 0$ for $\phi\in(0,\inf_{\xi\in\mathbb{R}}\overline{\phi}(\xi)]\cup[\sup_{\xi\in\mathbb{R}}\underline{\phi}(\xi),1)=(\frac{1}{2},1)$, which implies that (H4) is fulfilled. Then from Lemma 2.1, we deduce the conclusion.

With the aid of Lemma 2.2, we continue to complete the proof of Theorem 1.2.

Proof of Theorem 1.2. For (1.5), now, $f(\phi(0),\phi(s))=\phi^p(-c\tau)(1-\phi(0))$. Since $0\leqslant\phi(-c\tau)$, $\phi(0)\leqslant 1$, then obviously, $f(\hat{\phi},\check{\phi})$ is increasing in $\check{\phi}$. Moreover, if $0\leqslant\psi(-c\tau)\leqslant\phi(-c\tau)\leqslant 1$, $0\leqslant\psi(0)\leqslant\phi(0)\leqslant 1$, and $\beta\geqslant 1$, then

$$\begin{aligned}f(\phi(0),\phi(s))-f(\psi(0),\psi(s))=&(\phi^p(-c\tau)-\psi^p(-c\tau))(1-\phi(0))-\psi^p(-c\tau)(\phi(0)-\psi(0))\\ \geqslant&-\psi^p(-c\tau)(\phi(0)-\psi(0))\\ \geqslant&-\beta(\phi(0)-\psi(0)).\end{aligned}$$

Since the traveling wave front to (1.3) with $c=c_*(p)$ belongs to Γ_2, then, from [2,9], taking $c_0=c_*(p)$ gives

$$\inf_{\phi\in\Gamma_2}\sup_{\xi\in\mathbb{R}}\frac{\phi''(\xi)+\phi^p(\xi-c_0\tau)(1-\phi(\xi))}{\phi'(\xi)}\leqslant\inf_{\phi\in\Gamma_2}\sup_{\xi\in\mathbb{R}}\frac{\phi''(\xi)+\phi^p(\xi)(1-\phi(\xi))}{\phi'(\xi)}\leqslant c_*(p)<\infty.$$

Thus from Lemma 2.2, we can prove the results in Theorem 1.2.

In Theorem 1.1, due to the way of constructing the supersolution, we need the speed c is larger than 2 to guarantee the existence of traveling wave fronts of (1.1). While when $\tau=0$, by recalling the content regarding to the minimal speed $c_*(p)$ in Introduction, we know $c_*(p)<2$ for $p>1$. It is natural for us to consider the existence of traveling wave fronts for $c_*(p)<c<2$. Inspired by [22], we will apply the perturbation method and the implicit function theory to discuss this problem.

Proof of Theorem 1.3. Let $\psi(\xi)=\phi'(\xi)$, then a strictly increasing solution $\phi(\xi)$ to (1.4) with (1.6) is equivalent to the solution to

$$\begin{cases}\phi'(\xi)=\psi(\xi),\\ \psi'(\xi)=c\psi(\xi)-\phi^p(\xi)(1-\phi(\xi-c\tau)),\\ \phi'(\xi)>0,\quad\xi\in\mathbb{R},\end{cases}\tag{3.1}$$

connecting $(0,0)$ and $(1,0)$. Repeating the similar proof of Proposition 2.1 in [24], we conclude that for some fixed speed $c>0$, $\phi(\xi)$ is a strictly increasing solution to (1.4) with (1.6) if and only if $\psi(\phi)>0$ for any $\phi\in(0,1)$ is a solution of

$$\begin{cases}\frac{d\psi}{d\phi}=c-\frac{\phi^p(1-\phi_{c\tau})}{\psi},\\ \psi(0^+)=0,\quad\psi(1^-)=0,\end{cases}\tag{3.2}$$

where $\phi_{c\tau} = \phi(\xi - c\tau)$ for short. We denote the trajectory of (3.1) starting from the point $(0, 0)$ by $\psi = T_0(\phi, c, \tau)$ with $T_0(0, c, \tau) = 0$, which is also the solution to

$$\begin{cases} \frac{d\psi}{d\phi} = c - \frac{\phi^p(1-\phi_{c\tau})}{\psi}, \\ \psi(0^+) = 0. \end{cases}$$

While the trajectory of (3.1) starting from the point $(1, 0)$ is denoted by $\psi = T_1(\phi, c, \tau)$ with $T_1(1, c, \tau) = 0$, which is also the solution to

$$\begin{cases} \frac{d\psi}{d\phi} = c - \frac{\phi^p(1-\phi_{c\tau})}{\psi}, \\ \psi(1^-) = 0. \end{cases}$$

When $\tau = 0$, from [2,9], for any $c \geqslant c_*(p)$, there exist trajectories connecting $(0, 0)$ and $(1, 0)$. Thus for small $\tau > 0$, trajectories must cross the line $\phi = \frac{1}{2}$.

For any fixed speed $c_1 \geqslant c_*(p)$, the unperturbed problem corresponds to the solution $\psi = T(\phi, c_1)$, $\phi \in [0, 1]$. Hence, when $\tau = 0$, $T_0(\phi, c_1, \tau) = T_1(\phi, c_1, \tau) := T(\phi, c_1)$, with $c_1 \geqslant c_*(p)$ and $\phi \in [0, 1]$. Let

$$\Phi(c, \tau) = T_0(\frac{1}{2}, c, \tau) - T_1(\frac{1}{2}, c, \tau).$$

Obviously $\Phi(c_1, 0) = 0$ and

$$\frac{\partial \Phi(c_1, 0)}{\partial c} = \frac{\partial T_0(\frac{1}{2}, c_1, 0)}{\partial c} - \frac{\partial T_1(\frac{1}{2}, c_1, 0)}{\partial c}.$$

Moreover, let $h_0(\phi) = \frac{\partial T_0(\phi, c_1, 0)}{\partial c}$ and $h_1(\phi) = \frac{\partial T_1(\phi, c_1, 0)}{\partial c}$. From (3.2), since

$$\frac{dT_0}{d\phi} = c - \frac{\phi^p(1-\phi_{c\tau})}{T_0},$$

then $T_0(\phi, c, \tau) = c\phi - \int_0^\phi \frac{s^p(1-s_{c\tau})}{T_0(s,c,\tau)} ds$. Thus

$$h_0(\phi) = \frac{\partial T_0(\phi, c_1, 0)}{\partial c} = \phi + \int_0^\phi \frac{s^p(1-s)}{T^2(s, c_1)} h_0(s) ds.$$

Directly computing gives $h_0(0) = 0$ and $\frac{dh_0(\phi)}{d\phi} = 1 + \frac{\phi^p(1-\phi)}{T^2(\phi,c_1)} h_0(\phi)$. Therefore

$$h_0(\phi) = \int_0^\phi e^{\int_t^\phi \frac{s^p(1-s)}{T^2(s,c_1)} ds} dt.$$

Specially,

$$h_0(\frac{1}{2}) = \int_0^{\frac{1}{2}} e^{\int_t^{\frac{1}{2}} \frac{s^p(1-s)}{T^2(s,c_1)} ds} dt.$$

Repeating the similar argument, we have

$$h_1(\frac{1}{2}) = \int_1^{\frac{1}{2}} e^{\int_t^{\frac{1}{2}} \frac{s^p(1-s)}{T^2(s,c_1)} ds} dt.$$

Thus $h_0(\frac{1}{2}) - h_1(\frac{1}{2}) > 0$, which implies $\frac{\partial \Phi(c_1,0)}{\partial c} \neq 0$. By using the implicit function theorem, for sufficiently small τ, $\Phi(c(\tau), \tau) = 0$. Therefore, we can complete the proof of Theorem 1.3.

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