Midterm Exam

Math 264: Advanced Calculus for Engineers Fall Semester 2018

Instructor: *Prof. Ming Mei*

Time: 4:05pm – 5:25pm, October 30, 2018

Student Name:___________________________________ **Student ID:** ____________________________

INSTRUCTIONS

1. If you are not registered in this section, your grade will NOT count.
2. This is a closed book exam, calculators and cell phones are NOT permitted.
3. Make sure you READ CAREFULLY the question before embarking on the solution.
4. SHOW ALL YOUR WORK.

<u>Questions</u> **[**10 points each, 50 points in total**]**

1. Let $\boldsymbol{F}(x,y) = y\,\mathbf{i} + x\,\mathbf{j}$ be a vector field.
   a). Find its field lines and sketch these lines.
   b). Sketch the vector field
   c). Show it to be conservative by finding its potential curves, and sketch these curves
   d). Calculate $\int_C \boldsymbol{F}\cdot d\boldsymbol{r}$, where C is the curve given by $\boldsymbol{r}(t) = \cos\frac{\pi\sqrt{1+3\sin t}}{2}\,\mathbf{i} + e^t\,\mathbf{j}$ for $0 \le t \le \frac{\pi}{2}$.

2. Evaluate $\iint_S y\,dS$, where S is the surface $z = x + y^2$, $0 \le x \le 1$, $0 \le y \le 2$.

3. Evaluate $\oint_C (\sin x + 3y^2)dx + \left(2x - e^{-y^2}\right)dy$, where C is the boundary of the half-disk $x^2 + y^2 \le a^2$, $y \ge 0$, oriented counterclockwise.

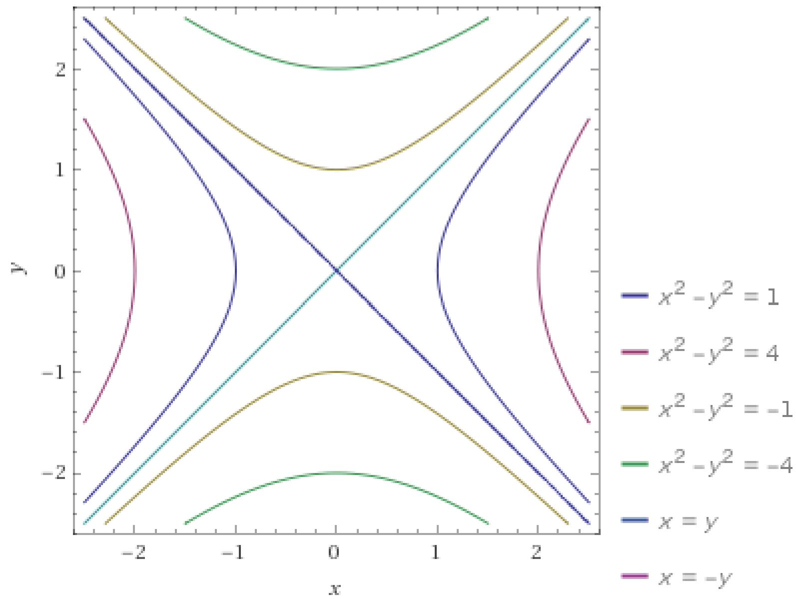
4. Evaluate $\iint_S \text{curl}\,\boldsymbol{F}\cdot d\boldsymbol{S}$, where $\boldsymbol{F}(x,y,z) = xz\,\mathbf{i} + yz\,\mathbf{j} + xy\,\mathbf{k}$ and S is the part of sphere $x^2 + y^2 + z^2 = 4$ that lies inside the cylinder $x^2 + y^2 = 1$ and above the xy-plane.

5. Evaluate $\iint_S \boldsymbol{F}\cdot d\boldsymbol{S}$, where $\boldsymbol{F}(x,y,z) = xy\,\mathbf{i} + (y^2 + e^{xz^2})\,\mathbf{j} + \sin(xy)\,\mathbf{k}$ and S is the surface of the region D bounded by the parabolic cylinder $z = 1 - x^2$ and the planes $z = 0$, $y = 0$, and $y + z = 2$.

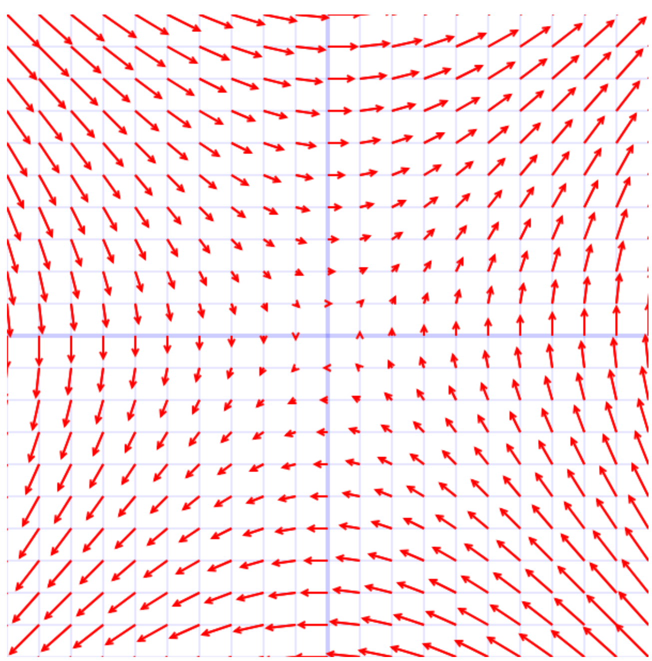
Solutions

1. **a).**

From $\frac{dx}{F_1} = \frac{dy}{F_2}$, we have $\frac{dx}{-y} = \frac{dy}{x}$, namely $xdx = ydy$. So $\int xdx = \int ydy$, which yields the following filed lines: $x^2 - y^2 = C$ for arbitrary constant C (positive, or negative, or zero).



b). vector field



c). In order to prove the vector field to be conservative, we are looking for a smooth function $\phi(x, y)$ such that $\nabla\phi = \boldsymbol{F}(x, y)$, namely,

$$\frac{\partial\phi}{\partial x} = y, \qquad \frac{\partial\phi}{\partial y} = x. \qquad (1)$$

Integrating the first equation with respect to x gives

$$\phi(x,y) = \int y dx = yx + C_1(y).$$

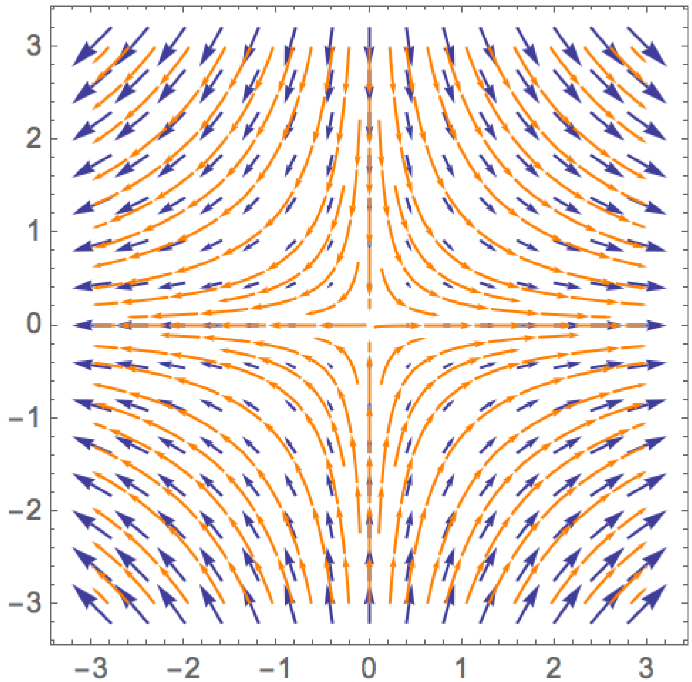
Differentiating the above equation with respect to y, we have

$$\frac{\partial \phi}{\partial y} = x + \frac{dC_1(y)}{dy}.$$

Comparing it with the second equation of (1), we have $C_1(y) = C\ (constant)$. Then the potential curves are

$$\phi(x,y) = xy + C.$$

So, the vector field is conservative.



d). Since the vector field is conservative, the line integral is independent of the path. So,

$$\int_c \boldsymbol{F} \cdot d\boldsymbol{r} = \int_c \nabla\phi \cdot d\boldsymbol{r} = \phi(x,y)\_\{t = \tfrac{\pi}{2}\} - \phi(x,y)|\_\{t = 0\} = \cos\pi\, e^{\pi/2} - \cos\tfrac{\pi}{2} \cdot 1 = -e^{\pi/2}.$$

2.

$$\iint_S y\, dS = \int_0^1 \int_0^2 y\sqrt{1 + (z_x)^2 + (z_y)^2}\ dydx = \int_0^1 \int_0^2 y\sqrt{2 + 4y^2}\ dydx$$

[by substituting $u = 2 + 4y^2$]

$$= \frac{1}{8}\int_0^1 \int_2^{18} \sqrt{u}\ dudx = 39\sqrt{2}.$$

3. By Green's Theorem, we have

$$\oint_C (\sin x + 3y^2)\, dx + (2x - e^{-y^2})dy = \iint_D \left[\frac{\partial(2x - e^{-y^2})}{\partial x} - \frac{\partial(\sin x + 3y^2)}{\partial y}\right] dxdy$$

$$= \iint_D [2 - 6y]\, dxdy \qquad [\,polar\ coordinate\!: x = r\sin\theta, \quad y = r\cos\theta]$$

$$= \int_0^{\pi} \int_0^{a} (2 - 6r\sin\theta) r\, dr\, d\theta = \pi a^2 - 4a^3.$$

4. Intersection curve of the sphere $x^2 + y^2 + z^2 = 4$ and the cylinder $x^2 + y^2 = 1$ above *xy*-plane is:

$$x^2 + y^2 = 1, \;\; z = \sqrt{3},$$

which can be denoted in the vector form in the polar coordinate: $\boldsymbol{r(\theta)} = < \cos\theta, \sin\theta, \sqrt{3} >$, and its derivative is : $d\boldsymbol{r} = < -\sin\theta, \cos\theta, 0 >$. By Stokes's theorem, we have

$$\iint_S Curl\, \boldsymbol{F} \cdot d\boldsymbol{S}$$

$$= \oint_{\boldsymbol{C}} \boldsymbol{F} \cdot \boldsymbol{dr}$$

$$= \oint_{\boldsymbol{C}} < \sqrt{3}\cos\theta, \sqrt{3}\sin\theta, \sin\theta\cos\theta > \cdot < -\sin\theta, \cos\theta, 0 > d\theta = \int_0^{2\pi} 0\, d\theta = 0.$$

5. Region D is specified as

$$D = \{(x, y, z) |\, -1 \le x \le 1, \;\; 0 \le z \le 1 - x^2, \; 0 \le y \le 2 - z\}.$$

By the Divergence Theorem (Gaussian Theorem), we have

$$\iint_S \boldsymbol{F} \cdot \boldsymbol{dS} = \iiint_D div\, \boldsymbol{F}\, dV = \iiint_D 3y\, dV = 3\int_{-1}^{1}\int_0^{1-x^2}\int_0^{2-z} y\, dy\, dz\, dx = \frac{184}{35}.$$