McGill University
Faculty of Science

December 6, 2018
Final Examination

Math 264: Advanced Calculus for Engineers

December 6, 2018, 9:00 am – 12:00 pm

Examiner: Prof. Ming Mei **Associate Examiner: Prof. Rustum Choksi**

Student Name:________________________________ **Student ID:** ___________________________

INSTRUCTIONS

1. This is a **closed book exam, except** you are allowed **one double-sided 8.5 x 11 inches sheet of information**. Do not hand in this sheet of information.
2. Calculators and cell phones are **NOT** permitted.
3. Make sure you READ CAREFULLY the question before embarking on the solution.
4. Note the value of each question.
5. This exam consists of 12 pages (including the cover page). Please check that all pages are intact and provide all your answers on this exam.

Question	Q1	Q2	Q3	Q4	Q5	Q6	Q7	Q8	Q9	Q10	Total
Mark											
Out of	10	10	10	10	10	10	10	10	10	10	100

1. Let $\boldsymbol{F}(x, y) = (4x^3y^2 - 2xy^3)\mathbf{i} + (2x^4y - 3x^2y^2 + 4y^3)\mathbf{j}.$

   a). Show that $\boldsymbol{F}(x, y)$ is conservative by finding the potential curves.

   b). Calculate $\int_C \boldsymbol{F} \cdot \boldsymbol{dr}$, where C is the curve given by $\boldsymbol{r}(t) = (\mathrm{t} + \sin \pi \mathrm{t})\ \mathbf{i} + (2t + \cos \pi t)\ \mathbf{j}$ for $0 \leq t \leq 1$.

2. a). Find curl **F** and div **F**, if $\boldsymbol{F}(x, y, z) = e^{-x}\sin y\, \mathbf{i} + e^{-y}\sin z\, \mathbf{j} + e^{-z}\sin x\, \mathbf{k}$.

b). Show that there is no vector field **G** such that curl $\mathbf{G} = 2x\, \mathbf{i} + 3yz\, \mathbf{j} - xz^2\, \mathbf{k}$.

3. If f is a harmonic function, that is, $\nabla^2 f = f_{xx} + f_{yy} = 0$, show that the line integral $\int_C f_y dx - f_x dy$ is independent of path C in any simple region D.

4. Evaluate $\int_C \sqrt{1+x^3}\, dx + 2xy\, dy$, where C is the triangle with vertices (0,0), (1,0), (1,3).

5. Evaluate $\iint_S \boldsymbol{F} \cdot d\boldsymbol{S}$, where $\boldsymbol{F}(x, y, z) = x^2\,\mathbf{i} + xy\,\mathbf{j} + z\,\mathbf{k}$ and S is the part of paraboloid $z = x^2 + y^2$ below the plane $z = 1$ with upward orientation.

6. Evaluate $\iint_S$ curl $\boldsymbol{F} \cdot d\boldsymbol{S}$, where $\boldsymbol{F}(x, y, z) = x^2yz\,\mathbf{i} + yz^2\,\mathbf{j} + z^3e^{xy}\,\mathbf{k}$ and S is the part of sphere $x^2 + y^2 + z^2 = 5$ that lies above the plane $z = 1$ and S is oriented upward.

7. Evaluate $\iint_S \boldsymbol{F} \cdot d\boldsymbol{S}$, where $\boldsymbol{F}(x, y, z) = x^3\,\mathbf{i} + y^3\,\mathbf{j} + z^3\,\mathbf{k}$ and S is the surface of the solid bounded by the cylinder $x^2 + y^2 = 1$ and the planes $z = 0$, $z = 2$.

8. a). Find the eigenvalues and eigenvectors: $y'' + \lambda y = 0, y'(0) = 0,\ y'(L) = 0.$

8. b). Find the Fourier series for the function

$$f(x) = \begin{cases} x + 2, & -2 \le x < 0, \\ 2 - x, & 0 \le x < 2; \end{cases} \qquad f(x+4) = f(x).$$

9. Solve the following initial-boundary value problem

$$\begin{cases} u_t = 5u_{xx}, \quad 0 \le x \le \pi, \qquad t > 0, \\ u(0,t) = 10, \quad u(\pi,t) = 20, \quad t > 0, \\ u(x,0) = \cos 2x - \cos 4x, \quad x \in [0,\pi]. \end{cases}$$

10. Consider the initial-value problem to the wave equation

$$\begin{cases} u_{tt} = a^2 u_{xx}\,, & -\infty < x < \infty, \quad t > 0, \\ u(x,0) = f(x), & -\infty < x < \infty, \\ u_t(x,0) = 0, & -\infty < x < \infty, \end{cases}$$

which can be reduced to the form $u_{\xi\eta} = 0$ by the change of variables $\xi = x - at,\ \eta = x + at$.

a). Show that the solution can be written as

$$u(x,t) = \phi(\xi) + \psi(\eta) = \phi(x - at) + \psi(x + at),$$

where ϕ and ψ are the functions satisfying

$$\phi(x) + \psi(x) = f(x), \qquad -\phi'(x) + \psi'(x) = 0.$$

10 b). By solving ϕ and ψ in part a), thereby show the following D'Alembert formula:

$$u(x,t) = \frac{1}{2}[f(x - at) + f(x + at)].$$