Available online at www.sciencedirect.com
ScienceDirect
Journal of Differential Equations 313 (2022) 336–381
Journal of Differential Equations
www.elsevier.com/locate/jde

Newtonian limit for the relativistic Euler-Poisson equations with vacuum

La-Su Mai [a], Ming Mei [b,c,*]

[a] School of Mathematics Sciences, Inner Mongolia University, Hohhot 010021, PR China
[b] Department of Mathematics, Champlain College Saint-Lambert, Saint-Lambert, Quebec, J4P 3P2, Canada
[c] Department of Mathematics and Statistics, McGill University, Montreal, Quebec, H3A 2K6, Canada

Received 28 October 2021; accepted 3 January 2022

Abstract

This paper is concerned with the Newtonian limit of the relativistic Euler-Poisson equation. Under conditions of the free boundary and vacuum, we prove the existence and uniqueness of local smooth solutions, which converge to the solutions of the classical Euler-Poisson equation at the rate of c^{-2}, where c is the speed of light. From the mathematical standpoint, we successfully overcome the strong nonlinearity caused by the Lorentz factor, the vacuum occurring on the moving boundary and the singularity at the center point by applying the weighted Sobolev space, respectively.

Keywords: Relativistic Euler-Poisson equation; Smooth solution; Local well-posedness; Non-relativistic limit; Free boundary

1. Introduction

In the early of 1960's, various investigations on the dynamical stability of gaseous masses, in the framework of the general theory of relativity, showed that the theory predicts, already in the post-Newtonian approximation, the phenomena which are qualitatively different from those to be

* Corresponding author at: Department of Mathematics, Champlain College Saint-Lambert, Saint-Lambert, Quebec, J4P 3P2, Canada.
E-mail addresses: lasu_mai@163.com (L.-S. Mai), ming.mei@mcgill.ca (M. Mei).

https://doi.org/10.1016/j.jde.2022.01.003

expected on the Newtonian theory, namely gaseous masses are predicted to become dynamically unstable much before the Schwarzschild limit is reached. Because of these results, in 1965, Chandrasekhar [3,4] was motivated to start a systematic investigation of the post-Newtonian effects of general relativity on the behaviors of hydrodynamic systems. It was then necessary to deduce the generalization of the standard Euler equations of Newtonian hydrodynamic which could consistently allow for all effects of order $\frac{1}{c^2}$, originating in the exact field equations of Einstein (also see Thorne [48]), where c is the light speed. Up to now, the post-Newtonian approximation schemes are one of the most important techniques in general relativity for calculating physical quantities in order to compare theoretical studies with experiment results. For example, in gravitational wave astronomy, the post-Newtonian expansions are used to calculate gravitational wave forms that are emitted during gravitational collapse [2]. As the limit of $c \to \infty$, the post-Newtonian approximation system reduces to the corresponding classical system of Newtonian hydrodynamics, which is called the Newtonian limit in physics and the non-relativistic limit in mathematics. For the Einstein equation, Oliynyk [42,43] considered the existence and Newtonian limit of local solution, and proved that the solution of Einstein equation converges to the solution of the classical Euler-Poisson equation describing the gravitational interaction. The others type post-Newtonian approximation and Newtonian limit analysis can be founded in [1,12,45], respectively. For the relativistic Euler and relativistic Euler-Poisson equation, the related non-relativistic limit results were obtained in [15,32,37–39,41,46,47], respectively.

In this paper, we are concerned with the existence and Newtonian limit of local smooth solutions with vacuum to the relativistic Euler-Poisson equation. If the speed of charged particles is very large but less than the speed of light and electric field effect is stronger than that of magnetic field, the motion of isentropic relativistic electro-fluid can be described by the following relativistic Euler-Poisson equations [5,16,17]:

$$\begin{cases} \partial_t \left(\dfrac{\rho}{\sqrt{1-|\mathbf{v}|^2/c^2}} \right) + \operatorname{div} \left(\dfrac{\rho \mathbf{v}}{\sqrt{1-|\mathbf{v}|^2/c^2}} \right) = 0, \\ \partial_t \left(\dfrac{n(\rho)c^2 + p(n(\rho))}{c^2 - |\mathbf{v}|^2} \mathbf{v} \right) + \operatorname{div} \left(\dfrac{n(\rho)c^2 + p(n(\rho))}{c^2 - |\mathbf{v}|^2} \mathbf{v} \otimes \mathbf{v} \right) + \nabla p(\rho) \\ = \dfrac{\rho \nabla \varphi}{\sqrt{1-|\mathbf{v}|^2/c^2}}, \\ \triangle \varphi = \dfrac{4\pi\rho}{\sqrt{1-|\mathbf{v}|^2/c^2}}, \end{cases} \tag{1.1}$$

where $n, \ \rho, \ \mathbf{v}, \ c$ and φ represent the proper density of charge, the mass-energy density of the current, the velocity of electric fluid, the speed of light and the electrostatic potential, respectively. The pressure $p(n)$ is given by

$$p(n) = [n(\rho)]^\gamma, \quad \text{for } \gamma \geq 1, \tag{1.2}$$

where $\gamma = 1$ is the isothermal (positive pressure) flow and $\gamma > 1$ is the polytropic gas. Here, the mass energy density of electric fluid $n(\rho)$ is the function of ρ satisfying

$$\frac{dn}{d\rho} = \frac{n(\rho) + n^\gamma(\rho)/c^2}{\rho}, \tag{1.3}$$

which describes the first law of thermal dynamic. From (1.3), we can derive the relationship between n and ρ as

$$n = \rho\Big(1 - \frac{\rho^{\gamma-1}}{c^2}\Big)^{\frac{1}{1-\gamma}}. \tag{1.4}$$

For simplicity, we assume

$$\frac{\rho^{\gamma-1}}{c^2} < 1. \tag{1.5}$$

Formally, in the non-relativistic limit (Newtonian limit) $c \to \infty$, the relativistic Euler-Poisson equation (1.1) can be written the following well-known Euler-Poisson equation [5,11,40,44]:

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho \mathbf{v}) = 0, \\ \partial_t (\rho \mathbf{v}) + \operatorname{div}(\rho \mathbf{v} \otimes \mathbf{v}) + \nabla p(\rho) = \rho \nabla \varphi, \\ \triangle \varphi = 4\pi \rho. \end{cases} \tag{1.6}$$

As we know, the free boundary value problem of fluids containing vacuum is one of the most important and difficult problems in the study of partial differential equations for fluid dynamics. In this case, the moving region of the fluid changes with time along the particle path, and the system describing the motions of fluids becomes a degenerate system at the free boundary. Clearly, it is necessary to determine the free boundary, while the solutions of system can be then determined [6–8,28,29,33,34].

The free boundary problem of the relativistic fluids has been also received more attention recently. Jang et al. [27] based on the frameworks of [7,8] and Hadzic et al. [22] based on the frameworks of [28,29] established the a priori estimates of smooth solutions (supposed to exist) to the free boundary value problem for the 3D relativistic system (1.1), respectively, but no existence results have been proved [22,27]. Due to the strong nonlinearity caused by the Lorentz effect, the existence of short time smooth solution to the free boundary problem for the relativistic Equation does not seem to be carried out straightforward as those made in [7,8,28,29] for compressible Euler equations. In [38] and [39], we obtained the well-posedness and the non-relativistic limit for the smooth solution of the free boundary problem of the cylindrically symmetric equations and the one dimensional relativistic Euler equations, respectively. To the best of our knowledge, the mathematical theory for the free boundary value problem with vacuum of the relativistic systems (1.1) still remains open. Regarding the well-posedness, large time behaviors, the stability and the instability for solutions of the free boundary problem of the Euler-Poisson system, where the Poisson equation describes the gravitational interaction, we refer the interesting readers to see [18–21,25,26,30–34,36], respectively. The other interesting studies related to MHD equations and Navier-Stoke-Maxwell equations can be referred to [9,10,13,14,23,24] and the references therein.

In this paper, we study the well-posedness and Newtonian limit of the local smooth solution for the free boundary value problem to the relativistic Euler-Poisson equation (1.1) with repulsive forces as the mass energy density connects with the vacuum continuously at the free boundary. We first derive the corresponding equations in spherically symmetric coordinates, and establish the uniform a priori estimates of smooth solution to the free boundary problem, then we construct the approximate solutions to show the well-posedness of the smooth solution to original problem. In particular, we show that the smooth solution of the spherically symmetric relativistic Euler-Poisson equations (1.1) converges to the solution of the smooth solutions for the classical Euler-Poisson equation (1.6) in the C^0-norm at the rate $1/c^2$ (refer to Theorem 2.1 for details).

There are three main difficulties for the study of the free boundary value problem of the relativistic Euler-Poisson equation (1.1). One is that if the physical mass energy density (electron density) of the relativistic Euler-Poisson equation (1.1) becomes a degenerate hyperbolic system, then the classical theory of Friedrich-Lax-Kato for quasilinear strictly hyperbolic system can not be directly applied to prove the short time existence of classical solutions. To overcome this obstacle, we need to establish the uniform a-priori estimates of the local smooth solutions with respect to c by using the Hardy's inequality in a certain weighted Sobolev space. The other difficulty is caused by the singularity at the center point $r = 0$. We will increase the spatial regularities of fluids velocity near the center point, and apply the Hardy inequalities to establish the desired estimates at the singular point. Besides above two mentioned difficulties, the third is that the relativistic Euler-Poisson equations (1.1) has the strong nonlinearity caused by the Lorentz effect which makes the a-priori estimates more complicated and tedious, compared to the similar results of the classical Euler-Poisson equations (1.6).

This paper is arranged as follows. We systematically analyze the well-poseness and the non-relativistic limit of local smooth solutions for the relativistic Euler-Poisson equation (1.1) with repulsive force from Section 2 to Section 8. In Section 2, we reformulate the working problem and state main results in Lagrangian coordinates. In Section 3, we make some a-priori assumptions and computations, which are crucial to establish the a-priori estimates of solutions. In Section 4 and Section 5, we mainly show the uniformly a-priori estimates of local smooth solutions independent of the speed of light c for large enough c and suitably small T. The energy estimates for the higher order time derivatives are obtained in Section 4 and the elliptic type estimates are established in Section 5, respectively. In Section 6, we prove the existence results by a particular degenerate parabolic regularization to the relativistic Euler-Poisson system (1.1). Furthermore, we study the uniqueness and the non-relativistic limits of solution obtained in Section 7 and Section 8, respectively.

Notations and Weighted Sobolev Spaces. Let $H^k(0,1)$ denote the usual Sobolev spaces with the norm $\|\cdot\|_k$, especially $\|\cdot\|_0 = \|\cdot\|$. For real number l, the Sobolev spaces $H^l(0,1)$ and the norm $\|\cdot\|_l$ are defined by interpolation. The function space $L^\infty(0,1)$ is simplified by L^∞. The notation M_0 denotes the generic constants independent of c. Let $d(x)$ be a distance function to boundary $\Gamma = \{0, 1\}$. For any $a > 0$ and nonnegative b, the weighted Sobolev space $H^{a,b}$ is given by $H^{a,b} := \left\{ d^{\frac{a}{2}} F \in L^2(0,1) : \int_0^1 d^a(x) |D_x^k F|^2 dx \leq \infty,\ 0 \leq k \leq b \right\}$ with the norm $\|F\|^2_{H^{a,b}} := \sum_{k=0}^b \int_0^1 d^a(x) |D_x^k F|^2 dx$. Then, it holds the following embedding: $H^{a,b}(0,1) \hookrightarrow H^{b-a/2}(0,1)$, with the estimate $\|F\|_{b-a/2} \leq C_0 \|F\|_{H^{a,b}}$. In particular, we have

$$\|F\|_0^2 \leq C_0 \int_0^1 d^2(x) \left(|F(x)|^2 + |F'(x)|^2 \right) dx, \tag{1.7}$$

$$\|F\|_{1/2}^2 \leq C_0 \int_0^1 d(x) \left(|F(x)|^2 + |F'(x)|^2 \right) dx. \tag{1.8}$$

2. Working problems and main results

In this section, we first describe the free boundary value problem of the spherically symmetric relativistic Euler-Poisson equation (1.1). Then, we transform the free boundary value problem of the spherically symmetric relativistic Euler-Poisson equation (2.2) and (2.4) into the initial boundary value problem (2.21) and (2.23) in Lagrangian coordinates. Finally, we state the main results in Theorem 2.1.

We introduce the spherically symmetric transformation as:

$$\rho(x,t)=\rho(r,t),\ \ \mathbf{v}(x,t)=\upsilon(r,t)\frac{\mathbf{x}}{r},\ r=|\mathbf{x}|=\sqrt{x_1^2+x_2^2+x_3^2}, \tag{2.1}$$

where $\rho(r,t)$ and $\upsilon(r,t)$ are the scalar functions, respectively. Thus, it is derived from (1.1):

$$\begin{cases} \partial_t(\dfrac{r^2\rho}{\sqrt{1-\upsilon^2/c^2}})+\partial_r(\dfrac{r^2\rho\upsilon}{\sqrt{1-\upsilon^2/c^2}})=0, \\ \dfrac{(n(\rho)+n^\gamma/(\rho)c^2)(1-\gamma n^{\gamma-1}(\rho)\frac{\upsilon^2}{c^4})}{(1-\upsilon^2/c^2)^2}(\upsilon_t+\upsilon\upsilon_r) \\ -\dfrac{\gamma n^{\gamma-1}(\rho)(n(\rho)+n^\gamma(\rho)/c^2)}{c^2(1-\upsilon^2/c^2)}\upsilon\upsilon_r-\dfrac{2\gamma n^{\gamma-1}(\rho)(n(\rho)+n^\gamma(\rho)/c^2)}{rc^2(1-\upsilon^2/c^2)}\upsilon^2+p_r \\ =\dfrac{\rho}{r^2\sqrt{1-\upsilon^2/c^2}}\displaystyle\int_0^r g(s,\ t)s^2ds, \end{cases} \tag{2.2}$$

where we have used

$$\phi_r=\frac{1}{r^2}\int_0^r g(s,\ t)s^2ds,\ g(x,t)=\frac{4\pi\rho}{\sqrt{1-\upsilon^2/c^2}}, \tag{2.3}$$

which follows from $(1.1)_3$ and (2.1) with the help of the condition $\phi_r(0,t)=0$ $(t\geq 0)$.

This system is supplemented with the following free boundary condition and initial data in $(0,R(t))\times[0,T]$:

$$\begin{cases} n>0,\ \text{in } [0,R(t)), \\ n(R(t),t)=0, \upsilon(0,t)=0, \\ \dfrac{dR(t)}{dt}=\upsilon(R(t),t),\ R(0)=1, \\ (n,\upsilon)(x,0)=(n_0,\upsilon_0),\ n_0(r)>0 \text{ in } [0,1), \\ -\infty<\partial_r p'(n_0)<0,\ \text{on } r=1, \end{cases} \tag{2.4}$$

where $(2.4)_5$ confirms that $n_0^{\gamma-1}$ is equivalent to the distance function $d(r)$ of the boundary near $r=1$, and also is very important to obtain the regularities of higher order spatial derivatives of velocity, which is called the physical vacuum condition for the compressible Euler equation [8,28,33]. In special relativity, the light speed $c > 0$ is the maximal speed. Therefore, we assume that

$$\|\upsilon_0\|^2_{L^\infty(0,1)} \ll c^2, \tag{2.5}$$

which implies there exists a positive constant m_0 such that

$$\Theta_0 = \sqrt{1 - \upsilon_0^2/c^2} \geq m_0 > 0. \tag{2.6}$$

Similarly, the sound speed $\sqrt{p'(\rho_0)}$ should satisfy

$$\sqrt{p'(\rho_0)} < c. \tag{2.7}$$

Corresponding to the limit as $c \to \infty$, we obtain the spherically symmetric form of the classical Euler-Poisson equation (1.6) as:

$$\begin{cases} \partial_t(r^2\rho) + \partial_r(r^2\rho\upsilon), \\ \rho(\upsilon_t + \upsilon\upsilon_r) + p_r = \dfrac{4\pi\rho}{r^2}\displaystyle\int_0^r \rho s^2 ds. \end{cases} \tag{2.8}$$

We define the Lagrangian variables $r(x,t)$, in order to fix $(0, R(t))$ into $(0,1)$, by

$$\partial_t r(x,t) = \upsilon(r(x,t),t) \ \text{ for } \ t > 0 \text{ and } \ r(x,0) = x \text{ in } x \in (0,1). \tag{2.9}$$

Then, it follows from $(2.2)_1$ that

$$\int_0^{r(x,t)} g(s,t)s^2 ds = \int_0^x g_0(y)y^2 dy = \int_0^x \frac{\rho_0(y)}{\sqrt{1 - \upsilon_0^2(y)/c^2}} y^2 dy. \tag{2.10}$$

Define the Lagrangian density and velocity by

$$f(x,t) = \rho(r(x,t),t) \ \text{ and } \ \upsilon(r(x,t),t) = u(x,t). \tag{2.11}$$

Then, it holds from $(2.2)_1$ that

$$f = \frac{1}{r_x}\left(\frac{x}{r}\right)^2 \frac{\rho_0\sqrt{1 - u^2/c^2}}{\sqrt{1 - u_0^2/c^2}}. \tag{2.12}$$

From (2.12) and (1.2), denoting the Lorentz factor Θ by $\Theta = \sqrt{1 - u^2/c^2}$, we can reduce the second equation of (2.2) to

$$\mathcal{W}(n(f),u)n(f)u_t - \frac{\mathcal{B}(n(f),u)}{c^2}n^\gamma(f)u\frac{u_x}{r_x} - \frac{2\mathcal{B}(n(f),u)}{c^2}n^\gamma(f)\frac{u^2}{x} + \frac{(n^\gamma)_x}{r_x}$$
$$= \frac{1}{r_x}(\frac{x}{r})^2\frac{\rho_0}{\Theta\Theta_0}\frac{1}{r^2}\int_0^x g_0(y)y^2dy, \tag{2.13}$$

where

$$\begin{cases} \mathcal{W}(n(f),u) = \frac{1+\frac{n^{\gamma-1}(f)}{c^2}}{\Theta^4}\Big(1-\frac{u^2}{c^2}\frac{\gamma n^{\gamma-1}(f)}{c^2}\Big), \\ \mathcal{B}(n(f),u) = \gamma\frac{x}{r}\frac{1}{\Theta^2}(1+\frac{n^{\gamma-1}(f)}{c^2}). \end{cases} \tag{2.14}$$

Noting (1.4), we have

$$\rho_0 = \frac{n_0}{(1+\frac{n_0^{\gamma-1}}{c^2})^{\frac{1}{\gamma-1}}}, \tag{2.15}$$

which implies

$$f = \frac{n_0}{(1+\frac{n_0^{\gamma-1}}{c^2})^{\frac{1}{\gamma-1}}}\frac{1}{\Theta_0}\frac{1}{r_x}(\frac{x}{r})^2\Theta. \tag{2.16}$$

It holds that, from (1.4),

$$n = \bar{a}(\gamma)\frac{n_0}{r_x}, \tag{2.17}$$

where $\bar{a}(\gamma) = \bar{a}(n_0, u_0, u, r_x, \frac{x}{r})$ is given by

$$\bar{a}(\gamma) := \frac{1}{\Big(1+\frac{n_0^{\gamma-1}}{c^2}\Big)^{\frac{1}{\gamma-1}}\Theta_0}\left(1-\frac{1}{c^2}\frac{n_0^{\gamma-1}}{\Big(1+\frac{n_0^{\gamma-1}}{c^2}\Big)\Theta_0^{\gamma-1}}\frac{\Theta^{\gamma-1}}{r_x^{\gamma-1}}(\frac{x}{r})^{2(\gamma-1)}\right)^{\frac{1}{1-\gamma}}\Theta, \tag{2.18}$$

which in combination with (2.13) yields

$$a_1(\gamma)\frac{x^2}{r^2}n_0u_t - a_2(\gamma)(\frac{x^2n_0}{r^2r_x})^\gamma uu_x - 2a_2(\gamma)r_x(\frac{x^2n_0}{r^2r_x})^\gamma u\frac{u}{x} + \Big((\bar{a}(\gamma)\frac{x^2n_0}{r^2r_x})^\gamma\Big)_x$$
$$= x(\frac{x}{r})^4\frac{n_0}{\Theta}\Phi_0(\gamma), \tag{2.19}$$

where $a_1(\gamma) = \mathcal{W}(n,u)\bar{a}(\gamma)$ and $a_2(\gamma) = \mathcal{B}(n,u)(\bar{a}(\gamma))^\gamma$ with $\mathcal{W}(n,u)$, $\mathcal{B}(n,u)$ and n given by (2.14) and (2.17), respectively, and $\Phi_0(\gamma)$ satisfies

$$\Phi_0(\gamma) = \frac{1}{(1+\frac{n_0^{\gamma-1}}{c^2})^{\frac{1}{\gamma-1}}\Theta_0}\frac{1}{x^3}\int_0^x g_0(y)y^2dy. \tag{2.20}$$

In this paper, we mainly analyze the case of $\gamma = 2$. Denoting $\bar{a} = \bar{a}(2), a_1 = a_1(2),\ a_2 = a_2(2)$ and $\Phi_0 = \Phi(2)$, the system (2.19) can be written as

$$\begin{aligned} & a_1 x\alpha_0(x)u_t - \frac{a_2x^2}{c^2r^2r_x^2}\alpha_0^2(x)uu_x - 2\frac{a_2x^2}{c^2r^2r_x}\alpha_0^2(x)u\frac{u}{x} \\ & \quad + \left(\bar{a}^2\alpha_0^2(x)\frac{x^2}{r^2r_x^2}\right)_x - 2\bar{a}^2\frac{\alpha_0^2(x)}{x}\frac{x^3}{r^3r_x} = x^2\alpha_0(x)\frac{x^2}{r^2\Theta}\Phi_0, \end{aligned} \tag{2.21}$$

where

$$\bar{a} := \frac{1}{\left(1+\frac{n_0}{c^2}\right)\Theta_0}\left(1 - \frac{1}{c^2}\frac{n_0}{\left(1+\frac{n_0}{c^2}\right)\Theta_0}\frac{\Theta}{r_x}(\frac{x}{r})^2\right)^{-1}\Theta \tag{2.22}$$

and $\alpha_0(x) = \rho_0 x$.

The initial and boundary conditions (2.4) for $\gamma = 2$ become

$$\begin{cases} n_0 > 0, \text{ in } I, \\ n_0 = 0, \text{ on } \Gamma = \{0,1\}, \\ u(0,t) = 0, \text{ on } \{x=0\}\times(0,T], \\ (r,u)(x,0) = (x, u_0(x)), \ x\in I, \\ 0 < |\frac{\partial n_0}{\partial x}| < \infty, \text{ on } \Gamma = \{0,1\}. \end{cases} \tag{2.23}$$

For the limit $c \to \infty$, when $\gamma = 2$, the system (2.21) is reduced to the Lagrangian form of (2.8) as

$$x\alpha_0(x)u_t + \left(\alpha_0^2(x)\frac{x^2}{r^2r_x^2}\right)_x - 2\frac{\alpha_0^2(x)}{x}\frac{x^3}{r^3r_x} = x^2\alpha_0^2(x)\frac{x^2}{r^2}\frac{1}{x^3}\int_0^x \rho_0(y)y^2dy. \tag{2.24}$$

Due to different singularities at the original point $x = 0$ and the boundary point $x = 1$, we introduce the interior and the boundary C^∞ cut-off functions $\xi(x)$, $\zeta(x)$ by

$$\xi(x) = 1 \quad \text{on } [0,\delta],\ \ \xi(x) = 0 \ \text{ on } [2\delta, 1],\ \ |\xi'(x)| \le \frac{C_0}{\delta}, \tag{2.25}$$

$$\zeta(x) = 1 \quad \text{on } [\delta,1],\ \ \zeta(x) = 0 \ \text{ on } [0, \frac{\delta}{2}],\ \ |\zeta'(x)| \le \frac{C_0}{\delta}, \tag{2.26}$$

where C_0 and δ are positive constants and δ will be determined later.

The higher-order energy functional $E(t)$ is defined by

$$
\begin{aligned}
E(t) :=& \sum_{s=0}^{4} \|\partial_t^s u(t)\|^2_{H^{2-\frac{s}{2}}(0,1)} + \sum_{s=0}^{2} \|\alpha_0 \partial_t^{2s} u(t)\|^2_{H^{3-s}(0,1)} + \|\alpha_0^{\frac{3}{2}} \partial_t \partial_x^3 u(t)\|_0^2 \\
&+ \sum_{s=0}^{1} \|\frac{\partial_t^{2s} u}{x}(t)\|^2_{H^{1-s}(0,1)} + \sum_{s=1}^{2} \|\frac{\partial_t^{5-2s} u}{x}(t)\|^2_{H^{s-1}(0,1)} + \|\alpha_0^{\frac{3}{2}} \partial_t^3 \partial_x^2 u(t)\|_0^2 \\
&+ \sum_{s=0}^{1} \|\xi \alpha_0 \partial_t^{2s+1} u(t)\|^2_{H^{3-s}(0,1)} + \sum_{s=1}^{2} \|\xi \partial_t^{5-2s} u(t)\|^2_{H^{s}(0,1)} + \|\sqrt{\alpha_0} \partial_t \partial_x^2 u(t)\|_0^2 \\
&+ \|\frac{\partial_t \partial_x u(t)}{\sqrt{\alpha_0}}\|_0^2 + \|\sqrt{\alpha_0} \partial_t^3 \partial_x u(t)\|_0^2,
\end{aligned}
\tag{2.27}
$$

with the following compatibility conditions for $1 \le k \le 5$:

$$
\partial_t^k u(x,0) = \partial_t^{k-1}\Big[\frac{a_2(x,0)}{a_1(x,0)} n_0 u_0 (u_{0x} + 2\frac{u_0}{x})\Big] + \partial_t^{k-1}\Big[\frac{1}{a_1(x,0) n_0}(a_0^2(x,0) n_0^2)_x - \frac{x}{\Theta_0}\Phi_0\Big],
\tag{2.28}
$$

which are derived from (2.19) with $\gamma = 2$.

Without the loss of generality, we denote by $\mathcal{P}$ the generic polynomial function of f. For simplicity, $\mathcal{P}_0 = \mathcal{P}(E(0))$. We also denote $r = r^c, u = u^c$ for the solution of the equation (2.21) in order to describe the non-relativistic limit.

The main result of this paper for the case of $\gamma = 2$ is stated as follows.

Theorem 2.1. *Suppose that the initial data $n_0(x) \in C^2([0,1])$, and u_0 satisfies (2.5)-(2.7), (2.15), (2.23) and (2.28), and*

$$
E(0) < +\infty.
$$

Then, there exist two positive constants c_0 and T_{c_0} such that for any $c \ge c_0$, the problem (2.21) and (2.23) has a unique smooth solution (r^c, u^c) in $[0,1] \times [0, T_{c_0}]$ satisfying

$$
\sup_{t \in [0, T_{c_0}]} E(t) \le 2\mathcal{P}(E(0)).
\tag{2.29}
$$

Moreover, there exists a unique smooth solution (r, u) to the problem (2.23) and (2.24) so that it holds

$$
\|u^c - u\|_{C^0} + \|r_x^c - r_x\|_{C^0} \le \mathcal{O}(c^{-2}),
\tag{2.30}
$$

as $c \to \infty$.

Remark 2.2. **(Convergence)** Our energy functional $E(t)$ contains

$$
\|u^c\|_{H^2}, \ \|\frac{u^c}{x}\|_{H^1}, \text{ and } \|u_t^c\|_{H^{3/2}}.
$$

Thus, there exist subsequence (r^c, u^c) converges to (r, u, v, ω) which satisfies the problem (2.23) and (2.24) in classical sense, with the help of (2.29) and the fundamental theorem of calculus. However, the C^0-norm is enough to describe the convergence rate.

Remark 2.3. Due to the physical vacuum condition $(2.23)_5$, the value of γ confirms the rate of degeneracy near the vacuum boundary $x = 1$, but it will not affect the rate of degeneracy near the original point $x = 0$, since $\rho_0 \sim (1-x)^{\frac{1}{\gamma-1}}$ as $x \to 1$. In fact, the rate of degeneracy is more strong for the smaller value of γ. Thus, we divide γ into the two cases $1 < \gamma < 2$ and $\gamma > 2$. In the spirit of idea in [35], we can prove the well-posedness and non-relativistic limits of local smooth solutions by the similar argument to the case for $\gamma = 2$.

3. Preliminaries

In this section, we show some computations, assumptions and the related estimates. It is assumed that the problem (2.21) and (2.23) has a smooth solution (r, u) in $[0, 1] \times [0, T]$ satisfying

$$\sup_{t\in[0,T]} \left\| (u_t, u_x, u_{tt}, \alpha_0 u_{xt}, \frac{u}{x}, \frac{u_t}{x})(t) \right\|_{L^\infty} \leq K, \tag{3.1}$$

where K is some positive constant which will be determined later. Without loss of generality, the existence time T is taken as $0 < T < 1$.

The following Lemma 3.1 states some useful estimates, which will be repeatedly applied in the establishment of the a-priori estimates later.

Lemma 3.1. *Suppose that (r, u) is a smooth solution to the problem (2.21) and (2.23) on $[0, 1] \times [0, T]$ with the property (3.1). Then, there exist a small time $0 < \overline{T} \leq T$, the positive constants $\overline{c}$ and C_* (only depending on $\|(\rho_0, u_0)\|_{L^\infty}$) such that, for any $t \in (0, \overline{T}]$ and $c \geq \overline{c}$, the following estimates hold*

$$0 < C_*^{-1} \leq \Theta, r_x, \ \frac{x}{r}, \ 1 - \frac{1}{c^2} \frac{n_0}{(1+\frac{n_0}{c^2})\Theta_0} \frac{\Theta}{r_x} (\frac{x}{r})^2 \leq C_* \ , \ \|u\|_{L^\infty} \leq 4\|u_0\|_{L^\infty}. \tag{3.2}$$

$$\|\alpha_0 r_{xx}\|_{L^\infty} \leq M_0(K+1), \quad \|\alpha_0 u_{xx}\|_{L^\infty} \leq M_0(K^2+K). \tag{3.3}$$

Proof. Using the fundamental theorem of calculus, we can easily obtain (3.2). Differentiating (2.21) with respect to t, it holds that

$$\begin{aligned}
& x\alpha_0(x)a_1 u_{tt} + \partial_t a_1 x\alpha_0(x)u_t - \partial_t \left(\frac{a_2 x^2}{c^2 r^2 r_x^2} \alpha_0^2(x) u u_x \right) \\
& - 2\partial_t \left(\frac{a_2 x^2}{c^2 r^2 r_x} \alpha_0^2(x) u \frac{u}{x} \right) - 2\left[\alpha_0^2(x) \frac{x^2}{r^2 r_x^2} a_3 \left(\frac{\Theta^2}{r_x} u_x + \frac{x}{r}\Theta^2 a_4 \frac{u}{x} + \frac{1}{c^2} u u_t \right) \right]_x \\
& + 2\frac{\alpha_0^2(x)}{x} \frac{x^3}{r^3 r_x} a_3 \left(\frac{a_4}{r_x}\Theta^2 u_x + \frac{x}{r}(3 + \frac{a_5}{c^2})\Theta^2 \frac{u}{x} + \frac{1}{c^2} u u_t \right) \\
& = x^2 \alpha_0(x) \partial_t \left(\frac{x^2}{r^2 \Theta} \right) \Phi_0,
\end{aligned} \tag{3.4}$$

where

$$a_3 = \frac{1}{\left(1+\frac{n_0}{c^2}\right)^2 \Theta_0^2 \left(1 - \frac{1}{c^2}\frac{n_0}{\left(1+\frac{n_0}{c^2}\right)\Theta_0}\frac{\Theta}{r_x}(\frac{x}{r})^2\right)^3},$$

$$a_4 = 1 + \frac{1}{c^2}\frac{n_0}{\left(1+\frac{n_0}{c^2}\right)\Theta_0}\frac{\Theta}{r_x}(\frac{x}{r})^2, \quad a_5 = \frac{n_0}{\left(1+\frac{n_0}{c^2}\right)\Theta_0}\frac{\Theta}{r_x}(\frac{x}{r})^2. \tag{3.5}$$

Then, the estimates in (3.3) follow form (2.19) and (3.4) by using (3.1), respectively. □

The following Lemma 3.2 contributes to the closeness of the priori estimates.

Lemma 3.2. *Suppose that* (r, u) *is a smooth solution to the problem* (2.21) *and* (2.23) *on* $[0, 1] \times [0, T]$ *with the property* (3.1). *If* $E(t) < \infty$ *defined by* (2.27) *for* $(x, t) \in [0, 1] \times [0, T]$, *then the following estimates hold for any* $1 < p < \infty$:

$$\begin{aligned}
&\left\| \left(\frac{u}{x}, u_x, \alpha_0 u_{xx}, \partial_t u, \frac{\partial_t u}{x}, \alpha_0\partial_t\partial_x u, \zeta\sqrt{\alpha_0}\partial_t\partial_x u, \partial_t^2 u, \alpha_0\partial_t^2\partial_x u, \alpha_0\partial_t^3 u\right)(t)\right\|_{L^\infty} \\
&\quad + \left\| \left(\alpha_0\partial_t^3\partial_x u, \alpha_0\partial_t\partial_x^2 u\right)(t)\right\|_{\frac{1}{2}} + \left\| \left(\partial_t\partial_x u, \alpha_0\partial_t\partial_x^2 u, \partial_t^3 u, \alpha_0\partial_t^3\partial_x u\right)(t)\right\|_{L^p(0,1)} \\
&\le C\sqrt{E(u)}.
\end{aligned} \tag{3.6}$$

Moreover, the fundamental theorem of calculus shows for any $1 < p < \infty$

$$\begin{aligned}
&\left\| \left(\frac{u}{x}, \alpha_0 u_x, \partial_t u, \alpha_0\partial_t\partial_x u, \alpha_0\partial_t^2\partial_x u, \alpha_0\partial_t^3 u\right)(t)\right\|_{L^\infty} \\
&\quad + \left\| \left(u_x, \alpha_0\partial_x^2 u, \partial_t^2 u, \alpha_0\partial_t\partial_x^2 u\right)(t)\right\|_{L^p(0,1)} \le \mathcal{P}_0 + C\int_0^t \sqrt{E(\tau)}d\tau.
\end{aligned} \tag{3.7}$$

Proof. Using the fundamental theorem of calculus, $H^1(0, 1) \hookrightarrow L^\infty(0, 1)$, $H^{\frac{1}{2}}(0, 1) \hookrightarrow L^p(0, 1)$ for $1 < p < \infty$, and noting the weighted norm estimates (1.7) and (1.8), we can easily prove the estimates (3.6) and (3.7). □

Because of the complicated structures of coefficients in (2.21), we give some important estimates of these coefficients in the following Lemma 3.3 in order to simplify our priori estimates. Before the statement, we define some useful functions $\mathcal{H}_{t,x}^{i,j}$ $(i = 0, 1, ..., 5,\ j = 0, 1, 2, 3)$, where i is the order of time derivatives and j is the order of spatial derivatives, as

$$\mathcal{H}_{t,x}^{1,0}(x,t) := |\partial_t u| + |\partial_x u| + |\frac{u}{x}|,$$

$$\mathcal{H}_{t,x}^{i,0}(x,t) := \sum_{\substack{\mu+\nu=i\\ \mu,\nu\ge 1}} \mathcal{H}_{t,x}^{\mu,0}(x,t)\mathcal{H}_{t,x}^{\nu,0}(x,t) + |\partial_t^{i-1}\partial_x u| + |\frac{\partial_t^{i-1} u}{x}|$$

$$+\sum_{\substack{\mu+\nu=i\\ \mu,\nu=0}}|\partial_t^{\mu}u\|\partial_t^{\nu}u|,\quad (i\geq 2)$$

$$\mathcal{H}_{t,x}^{0,1}(x,t):=|u_x|+|r_{xx}|+|n_{0x}|+|u_{0x}|+|(\frac{x}{r})_x|,$$

$$\mathcal{H}_{t,x}^{0,j}(x,t):=\sum_{\substack{\mu+\nu=j\\ \mu,\nu\geq 1}}\mathcal{H}_{t,x}^{0,\mu}(x,t)\mathcal{H}_{t,x}^{0,\nu}(x,t)+|\partial_x^{j+1}r|+|\partial_x^{j}n_0|+|\partial_x^{j}u_0|$$

$$+\sum_{\substack{\mu+\nu=j\\ \mu,\nu=0}}|\partial_x^{\mu}u\|\partial_x^{\nu}u|+|\partial_x^{j}(\frac{x}{r})|,\quad (j\geq 2)$$

$$\mathcal{H}_{t,x}^{1,1}(x,t):=\mathcal{H}_{t,x}^{0,1}(x,t)\mathcal{H}_{t,x}^{1,0}(x,t)+|u\|u_{xt}|+|u_{xx}|+|(\frac{u}{x})_x|,$$

$$\mathcal{H}_{t,x}^{i,1}(x,t):=\mathcal{H}_{t,x}^{0,1}(x,t)\mathcal{H}_{t,x}^{i,0}(x,t)+\sum_{\substack{\mu+\nu=i\\ \mu,\nu\geq 1}}\mathcal{H}_{t,x}^{\mu,1}(x,t)\mathcal{H}_{t,x}^{\nu,0}(x,t)+|\partial_t^{i-1}\partial_x^2u|$$

$$+\sum_{\substack{\mu+\nu=i\\ \mu,\nu=0}}|\partial_t^{\mu}\partial_xu\|\partial_t^{\nu}u|+|(\frac{\partial_t^{i-1}u}{x})_x|.$$

Lemma 3.3. *Let*

$$f(u,r_x,\frac{x}{r},n_0,u_0)\in C^{\infty}\,(\prod_{i=1}[l_i,m_i]),\ (-\infty<l_i<q_i<\infty,\ i=1,2,3,4,5),$$

and for any $1<p<\infty$*,*

$$|\partial_t^i\partial_x^j f(x,t)|\leq M_0\mathcal{H}_{t,x}^{i,j}(x,t), i=1,2,3,4,5,\ j=0,1.$$

Then, the following estimates hold

$$\begin{aligned}&\|\mathcal{H}_{t,x}^{1,0}(t)\|_{L^p(0,1)}+\|\mathcal{H}_{t,x}^{2,0}(t)\|_0+\|\alpha_0\mathcal{H}_{t,x}^{2,0}(t)\|_{L^\infty}+\|\alpha_0\mathcal{H}_{t,x}^{3,0}(t)\|_{L^p(0,1)}+\|\alpha_0\mathcal{H}_{t,x}^{4,0}(t)\|_0\\ &\leq M_0\left[\mathcal{P}_0+\int_0^t\|(\alpha_0\partial_t^4\partial_xu,\sqrt{x\alpha_0}\partial_t^5u,\alpha_0\frac{\partial_t^4u}{x})\|_0d\tau+(\mathcal{P}(K)+1)\int_0^t\sqrt{E(\tau)}d\tau\right],\end{aligned}\tag{3.8}$$

and

$$\begin{aligned}&\|\mathcal{H}_{t,x}^{1,0}(x,t)\|_{L^\infty}+\|\mathcal{H}_{t,x}^{2,0}(x,t)\|_{L^p(0,1)}+\|\alpha_0\mathcal{H}_{t,x}^{3,0}(x,t)\|_{L^\infty}+\|\mathcal{H}_{t,x}^{3,0}(x,t)\|_0\\ &+\|\alpha_0\mathcal{H}_{t,x}^{4,0}(x,t)\|_{L^p(0,1)}\leq M_0\left[\mathcal{P}(K)+(\mathcal{P}(K)+1)\sqrt{E(t)}\right],\end{aligned}\tag{3.9}$$

and

$$\|\sqrt{x\alpha_0}\mathcal{H}^{5,0}_{t,x}(x,t)\|_0$$
$$\leq M_0\|(\alpha_0\partial_t^4\partial_x u, \sqrt{x\alpha_0}\partial_t^5 u, \alpha_0\frac{\partial_t^4 u}{x})\|_0 + M_0\left[\mathcal{P}(K)+(\mathcal{P}(K)+1)\sqrt{E(t)}\right], \tag{3.10}$$

and

$$\|\left(\mathcal{H}^{0,1}_{t,x}, \alpha_0\mathcal{H}^{1,1}_{t,x}, \xi\alpha_0\mathcal{H}^{2,1}_{t,x}\right)(t)\|_0 + \|\left(\xi\mathcal{H}^{0,1}_{t,x}, \alpha_0\mathcal{H}^{0,1}_{t,x}\right)(t)\|_{L^\infty}$$
$$\leq M_0\left[\mathcal{P}_0 + (\mathcal{P}(K)+1)\int_0^t \sqrt{E(\tau)}d\tau\right], \tag{3.11}$$

and

$$\|\mathcal{H}^{0,1}_{t,x}(x,t)\|_{L^\infty} + \|\mathcal{H}^{1,1}_{t,x}(x,t)\|_0 \leq M_0\left[\mathcal{P}(K)+(\mathcal{P}(K)+1)\sqrt{E(t)}\right]. \tag{3.12}$$

Proof. By the chain rules, (1.7)-(1.8), (2.27), (3.6)-(3.7), the Sobolev embedding and the fundamental theorem of calculus, we can easily obtain (3.8)-(3.12). □

Remark 3.4. Since the coefficients $a_i (i = 1, 2, 3, 4, 5)$ in (2.21) have the structure of $f(u, r_x, \frac{x}{r}, n_0, u_0)$, the estimates in (3.12) show the related estimates of $a_i (i = 1, 2, 3, 4, 5)$, which simplify the complexity of our energy estimates.

4. Energy estimates

In this section, we establish the higher order energy estimates of local smooth solutions to the problem (2.21) and (2.23).

Before the statement of the main results of this section. We first derive the higher order time derivatives equation (4.1) as follows. Taking ∂_t^k over equation (3.4), we have

$$\begin{aligned}
&x\alpha_0(x)a_1\partial_t^{k+2}u - 2\left[\alpha_0^2(x)\frac{x^2}{r^2r_x^2}a_3\left(\frac{\Theta^2}{r_x}\partial_t^k\partial_x u + \frac{x}{r}\Theta^2 a_4\frac{\partial_t^k u}{x} + \frac{1}{c^2}u\partial_t^{k+1}u\right)\right]_x \\
&\quad + 2\frac{\alpha_0^2(x)}{x}\frac{x^3}{r^3r_x}a_3\left[\frac{a_4}{r_x}\Theta^2\partial_t^k\partial_x u + \frac{x}{r}(3+\frac{a_5}{c^2})\Theta^2\frac{\partial_t^k u}{x} + \frac{1}{c^2}u\partial_t^{k+1}u\right] \\
&\quad + \frac{\alpha_0^2(x)}{c^2}\frac{a_2x^2}{r^2r_x^2}u\partial_t^{k+1}\partial_x u - 2\frac{\alpha_0^2(x)}{c^2}\frac{a_2x^2}{r^2r_x}u\frac{\partial_t^{k+1}u}{x} + \sum_{i=1}^{k+1}C^i_{k+1}\partial_t^i a_1\partial_t^{k+2-i}u + \sum_{i=1}^{6}S_i^k \\
&= x^2\alpha_0(x)\partial_t^{k+1}(\frac{x^2}{r^2\Theta})\Phi_o,
\end{aligned} \tag{4.1}$$

where

$$S_1^k := -\frac{\alpha_0^2(x)}{c^2}\sum_{i=1}^{k+1}C^i_{k+1}\left[\partial_t^i(\frac{a_2x^2}{r^2r_x^2})\partial_t^{k+1-i}(uu_x) + \frac{a_2x^2}{r^2r_x^2}\partial_t^i u\partial_t^{k+1-i}\partial_x u\right],$$

$$S_2^k := -2\frac{\alpha_0^2(x)}{c^2}\sum_{i=1}^{k+1} C_{k+1}^i\left[\partial_t^i(\frac{a_2x^2}{r^2r_x})\partial_t^{k+1-i}(u\frac{u}{x}) + \frac{a_2x^2}{r^2r_x}\partial_t^i u\frac{\partial_t^{k+1-i}u}{x}\right],$$

$$S_3^k := -2\sum_{i=1}^{k} C_k^i\left[\alpha_0^2(x)\left(\partial_t^i(\frac{\Theta^2x^2}{r^2r_x^3}a_3)\partial_t^{k-i}\partial_x u + \partial_t^i(\frac{x^3a_3a_4}{r^3r_x^2}\Theta^2)\frac{\partial_t^{k-i}u}{x}\right)\right]_x,$$

$$S_4^k := -2\sum_{i=1}^{k} C_k^i\left[\frac{\alpha_0^2(x)}{c^2}\left(\partial_t^i(\frac{x^2}{r^2r_x^2}a_3)\partial_t^{k-i}(uu_t) + \frac{x^2a_3}{r^2r_x^2}\partial_t^i u\partial_t^{k+1-i}u\right)\right]_x,$$

$$S_5^k := 2\frac{\alpha_0^2(x)}{x}\sum_{i=1}^{k} C_k^i\left[\partial_t^i(\frac{x^3\Theta^2}{r^3r_x^2}a_3a_4)\partial_t^{k-i}\partial_x u + \partial_t^i(\frac{x^4}{r^4r_x}a_3(3+\frac{a_5}{c^2})\Theta^2)\frac{\partial_t^{k-i}u}{x}\right],$$

$$S_6^k := 2\frac{\alpha_0^2(x)}{c^2x}\left[\partial_t^i(\frac{x^3}{r^3r_x}a_3)\partial_t^{k-i}(uu_t) + \frac{x^3}{r^3r_x}a_3\partial_t^i u\partial_t^{k+1-i}u\right]. \tag{4.2}$$

Here, $C_k^i = \frac{k!}{(k-i)!i!}$

The main result of this section is stated as follows.

Lemma 4.1. *Suppose that* (r, u) *is a smooth solution to the problem* (2.21) *and* (2.23) *on* $[0, 1] \times [0, T]$ *with the property* (3.1)*. Then, there exists a small time* $0 < T_1 \le T$ *and a positive constant* $\bar{c}_1 \ge \bar{c}$ *(only depending on* $\|(\rho_0, u_0)\|_{L^\infty}$*) such that for any* $t \in (0, T_1]$ *and* $c \ge \bar{c}_1$*, it holds that*

$$\begin{aligned}
&\|(\sqrt{x\alpha_0}\partial_t^5 u, \alpha_0\partial_t^4\partial_x u, \alpha_0\frac{\partial_t^4 u}{x})(t)\|_0^2 \\
&\le \mathcal{P}_0 + M_0(P(K)+1)\int_0^t (E^2(\tau)+E(\tau))d\tau + M_0(P(K)+1)E(t)\int_0^t E(\tau)d\tau.
\end{aligned} \tag{4.3}$$

Proof. Taking $k = 4$ in (4.1), multiplying the resultant equation by $\partial_t^5 u$ and integrating over $(0, t) \times (0, 1)$, we have, by integration by parts,

$$\begin{aligned}
&\int_0^t x\alpha_0(x)a_1\frac{(\partial_t^5 u)^2}{2}dx\Big|_0^t + J + \sum_{i=1}^{5} C_5^i\int_0^t\int_0^1 x\alpha_0(x)\partial_t^i a_1\partial_t^{6-i}u\partial_t^5 u dxd\tau \\
&+ \int_0^1 \alpha_0^2(x)\frac{x^2\Theta^2}{r^2r_x}a_3\left[\frac{1}{r_x^2}(\partial_t^4\partial_x u)^2 + 2\frac{x}{r}\frac{1}{r_x}a_4\frac{\partial_t^4 u}{x}\partial_t^4\partial_x u + \frac{x^2}{r^2}(3+\frac{a_5}{c^2})(\frac{\partial_t^4 u}{x})^2\right]dx\Big|_0^t \\
&+ \sum_{i=1}^{6}\int_0^t\int_0^1 S_i^4\partial_t^5 u dxd\tau = \int_0^t\int_0^1 x^2\alpha_0(x)\partial_t^5(\frac{x^2}{r^2\Theta})\Phi_0\partial_t^5 u dxd\tau,
\end{aligned} \tag{4.4}$$

where J is bounded by

$$|J| \le M_0(P(K)+1)\int_0^t \|(\sqrt{x\alpha_0}\partial_t^5 u, \alpha_0\partial_t^4\partial_x u, \alpha_0\frac{\partial_t^4 u}{x})(\tau)\|_0^2 d\tau, \tag{4.5}$$

with the help of (3.1), (3.2), (3.8) and (3.10). Due to (3.2) and (3.5), there exist two positive constants $\bar{c}_0$ and $\bar{C}$ independent of c such that for any $c \ge \bar{c}_0$, it holds that

$$\begin{aligned}
&\int_0^1 \alpha_0^2(x)\frac{x^2\Theta^2}{r^2 r_x^2}a_3\left[\frac{1}{r_x^2}(\partial_t^4\partial_x u)^2 + 2\frac{x}{r}\frac{1}{r_x}a_4\frac{\partial_t^4 u}{x}\partial_t^4\partial_x u + \frac{x^2}{r^2}(3+\frac{a_5}{c^2})(\frac{\partial_t^4 u}{x})^2\right]dx \\
&\ge \bar{C}\int_0^1 \alpha_0^2(x)\left[(\partial_t^4\partial_x u)^2 + (\frac{\partial_t^4 u}{x})^2\right]dx.
\end{aligned} \tag{4.6}$$

A simple computation shows for the second term of (4.4),

$$\begin{aligned}
\sum_{i=1}^5 \mathrm{C}_5^i \int_0^t\int_0^1 x\alpha_0(x)\partial_t^i a_1\partial_t^{6-i}u\partial_t^5 u dx d\tau &= \int_0^t\int_0^1 x\alpha_0(x)\partial_t^5 a_1\partial_t u\partial_t^5 u dx d\tau \\
&+ \sum_{i=1}^4 \mathrm{C}_5^i \int_0^t\int_0^1 x\alpha_0(x)\partial_t^i a_1\partial_t^{6-i}u\partial_t^5 u dx d\tau.
\end{aligned} \tag{4.7}$$

By using (3.1), (3.2) and the chain rules, we derive from (3.5) that,

$$\begin{aligned}
|\partial_t^5 a_1| \le{}& M_0 n_0\left(|\partial_t^4\partial_x u| + \left|\frac{\partial_t^4 u}{x}\right| + |u||\partial_t^5 u|\right) + M_0 n_0 K\left(|\partial_t^3\partial_x u| + \left|\frac{\partial_t^3 u}{x}\right| + |u||\partial_t^4 u|\right) \\
&+ M_0 n_0(K+|u_{xt}|)\left(|\partial_t^2\partial_x u| + \left|\frac{\partial_t^2 u}{x}\right| + |\partial_t^3 u|\right) \\
&+ M_0 n_0 K^2\left(|\partial_t^2\partial_x u| + \left|\frac{\partial_t^2 u}{x}\right| + |\partial_t^3 u|\right) \\
&+ M_0 n_0\left[(|u_{xt}|^2+K^2)K + (|u_{xt}|+K)K^3 + K^5\right].
\end{aligned} \tag{4.8}$$

Due to (3.6) and $L^4 - L^4 - L^2$ Hölder inequalities, we have

$$\begin{aligned}
&\int_0^t\int_0^1 x\alpha_0(x)\partial_t^5 a_1\partial_t u\partial_t^5 u dx d\tau \\
&\le M_0(\mathcal{P}(K)+1)\int_0^t \|(\sqrt{x\alpha_0}\partial_t^5 u, \alpha_0\partial_t^4\partial_x u, \alpha_0\frac{\partial_t^4 u}{x})(\tau)\|_0^2 d\tau
\end{aligned}$$

$$+ M_0(\mathcal{P}(K)+1)\int_0^t E(\tau)d\tau. \tag{4.9}$$

Similarly, the other terms in (4.7) can be estimated and bounded by the right side of (4.9).

For the fifth term of (4.4), we only give the detail for the estimate of S_3^4, while the other terms can be similarly estimated. Thus, we can obtain that the fifth term of (4.4) is controlled by the right side of (4.3). From (4.2), we have

$$\begin{aligned}
&\int_0^t\int_0^1 S_3^4\partial_t^5 u dx d\tau \\
&= 2\sum_{i=1}^4 \mathrm{C}_4^i\int_0^1 \alpha_0^2(x)\left[\partial_t^i\big(\frac{x^2\Theta^2}{r^2r_x^3}a_3\big)\partial_t^{4-i}\partial_x u + \partial_t^i\big(\frac{x^3\Theta^2}{r^3r_x^2}a_3a_4\big)\frac{\partial_t^{4-i}u}{x}\right]\partial_t^4\partial_x u dx\Big|_0^t \\
&- 2\sum_{i=1}^4 \mathrm{C}_4^i\int_0^t\int_0^1 \alpha_0^2(x)\partial_t\left[\partial_t^i\big(\frac{x^2\Theta^2}{r^2r_x^3}a_3\big)\partial_t^{4-i}\partial_x u + \partial_t^i\big(\frac{x^3\Theta^2}{r^3r_x^2}a_3a_4\big)\frac{\partial_t^{4-i}u}{x}\right]\partial_t^4\partial_x u dx d\tau \\
&= I_0|_0^t - I_1.
\end{aligned}$$

Set $I_0=\sum_{i=1}^4 I_0^i$, we estimate it as follows. For I_0^1 and I_0^4, by using (3.8) and (3.9), we have, for any positive constant ε,

$$\begin{aligned}
I_0^1 &= \mathrm{C}_4^1\int_0^1 \alpha_0^2(x)\left[\partial_t\big(\frac{x^2\Theta^2}{r^2r_x^3}a_3\big)\partial_t^3\partial_x u + \partial_t\big(\frac{x^3\Theta^2}{r^3r_x^2}a_3a_4\big)\frac{\partial_t^3 u}{x}\right]\partial_t^4\partial_x u dx \\
&\le M_0\|\mathcal{H}_{t,x}^{1,0}\|_{L^4(0,1)}\left(\|\alpha_0\partial_t^3\partial_x u(0)\|_0^2 + \|\alpha_0\frac{\partial_t^3 u}{x}(0)\|_0^2\right)\|\alpha_0\partial_t^4\partial_x u\|_0 \\
&\quad + M_0\|\mathcal{H}_{t,x}^{1,0}\|_{L^4(0,1)}\int_0^t (\|\alpha_0\partial_t^4\partial_x u(\tau)\|_0 + \|\alpha_0\frac{\partial_t^4 u}{x}(\tau)\|_0)d\tau\|\alpha_0\partial_t^4\partial_x u\|_0^2 \\
&\le M_0\mathcal{P}_0 + M_0(\mathcal{P}(K)+1)\int_0^t E^2(\tau)d\tau + \varepsilon\|\alpha_0\partial_t^4\partial_x u\|_0,
\end{aligned}$$

and

$$\begin{aligned}
I_0^4 &= \int_0^1 \alpha_0^2(x)\left[\partial_t^4(\frac{x^2\Theta^2}{r^2r_x^3})u_x + \partial_t^4(\frac{x^3\Theta^2}{r^3r_x^2}a_3a_4)\frac{u}{x}\right]\partial_t^4\partial_x u dx \\
&\le C\|\alpha_0\mathcal{H}_{t,x}^{4,0}\|_0(\|u_x(0)\|_{L^\infty} + \|\frac{u(0)}{x}\|_{L^\infty})\|\alpha_0\partial_t^4\partial_x u\|_0
\end{aligned}$$

$$+ C\|\alpha_0\mathcal{H}^{4,0}_{t,x}\|_{L^4}\int_0^t(\|\partial_t\partial_x u(\tau)\|_{L^4(0,1)}+\|\frac{\partial_t u}{x}(\tau)\|_{\alpha_0})d\tau\|\alpha_0\partial_t^4\partial_x u\|_0$$

$$\leq M_0\mathcal{P}_0+M_0(\mathcal{P}(K)+1)\int_0^t E^2(\tau)d\tau+\varepsilon\|\alpha_0\partial_t^4\partial_x u\|_0^2.$$

Similarly, we can estimate I_0^2 and I_0^3 by

$$I_0^2,\ I_0^3\leq M_0\mathcal{P}_0+M_0(\mathcal{P}(K)+1)\int_0^t E^2(\tau)d\tau+\varepsilon\|\alpha_0\partial_t^4\partial_x u\|_0^2. \tag{4.10}$$

Similar to the analysis of I_0, we denote $I_1=\sum_{i=1}^4 I_1^i$ and obtain

$$I_1^1\leq M_0\int_0^t\|\mathcal{H}^{2,0}_{t,x}\|_{L^4(0,1)}(\|\alpha_0\partial_t^3\partial_x u(\tau)\|_{L^4(0,1)}+\|\alpha_0\frac{\partial_t^3 u}{x}(\tau)\|_{L^4(0,1)})\|\alpha_0\partial_t^4\partial_x u(\tau)\|d\tau$$

$$+ M_0\int_0^t\|\mathcal{H}^{1,0}_{t,x}\|_{L^\infty}(\|\alpha_0\partial_t^4\partial_x u(\tau)\|_0+\|\alpha_0\frac{\partial_t^4 u}{x}(\tau)\|_0)\|\alpha_0\partial_t^4\partial_x u(\tau)\|_0 d\tau$$

$$\leq M_0(\mathcal{P}(K)+1)\int_0^t(E(\tau)+E^2(\tau))d\tau,$$

and,

$$I_1^4\leq M_0\int_0^t\|\alpha_0\mathcal{H}^{4,0}_{t,x}(\tau)\|_0(\|u_x(\tau)\|_{L^\infty}+\|\frac{u}{x}(\tau)\|_{L^\infty})\|\alpha_0\partial_t^4\partial_x u(\tau)\|_0^2 d\tau$$

$$+ M_0\int_0^t\|\alpha_0\mathcal{H}^{4,0}_{t,x}(\tau)\|_{L^4(0,1)}(\|\alpha_0\partial_t\partial_x u(\tau)\|_{L^4(0,1)}$$

$$+\|\alpha_0\frac{\partial_t u}{x}(\tau)\|_{L^\infty})\|\alpha_0\partial_t^4\partial_x u(\tau)\|_0 d\tau$$

$$\leq M_0(\mathcal{P}(K)+1)\int_0^t\|(\sqrt{x\alpha_0}\partial_t^5 u,\alpha_0\partial_t^4\partial_x u,\alpha_0\frac{\partial_t^4 u}{x})(\tau)\|_0^2 d\tau$$

$$+ M_0(\mathcal{P}(K)+1)\int_0^t(E(\tau)+E^2(\tau))d\tau. \tag{4.11}$$

Similarly, we can also estimate I_1^2 and I_1^3 as

$$I_1^2,\quad I_1^3 \le M_0(\mathcal{P}(K)+1)\int_0^t \|(\sqrt{x\alpha_0}\partial_t^5 u, \alpha_0\partial_t^4\partial_x u, \alpha_0\frac{\partial_t^4 u}{x})(\tau)\|_0^2 d\tau$$
$$+ M_0(\mathcal{P}(K)+1)\int_0^t (E(\tau)+E^2(\tau))d\tau. \tag{4.12}$$

Finally, we estimate the last term of (4.4) as

$$\int_0^t\int_0^1 x^2\alpha_0(x)\partial_t^5(\frac{x^2}{r^2\Theta})\Phi_0\partial_t^5 u dx d\tau$$
$$\le M_0\int_0^t (\|\sqrt{x\alpha_0}\mathcal{H}_{t,x}^{5,0}(\tau)\|_0^2 + \|\sqrt{x\alpha_0}\partial_t^5 u(\tau)\|_0^2)d\tau$$
$$\le M_0(\mathcal{P}(K)+1)\int_0^t (E(\tau)+E^2(\tau))d\tau$$
$$+ M_0\int_0^t \|(\sqrt{x\alpha_0}\partial_t^5 u, \alpha_0\partial_t^4\partial_x u, \alpha_0\frac{\partial_t^4 u}{x})(\tau)\|_0^2 d\tau, \tag{4.13}$$

with the help of (3.10).

Substituting (4.6), (4.5), (4.10), (4.11) and (4.13) into (4.4), we can conclude (4.3). □

5. Elliptic type estimates

In this section, we establish the higher order spatial derivatives estimates of the local smooth solutions to the problem (2.21) and (2.23) on $[0,1]\times[0,T]$ under the assumption (3.1). Due to the different singularities of the original point $x=0$ and the boundary point $x=1$, we divide our estimates of each terms into the interior estimates and the boundary estimates. More precisely, we give the estimates for u, $\partial_t u$ in subsection 5.1 and the estimates $\partial_t^3 u$ in subsection 5.2, respectively. Finally, we give the estimates of $E(t)$ in subsection 5.3.

We rewrite the equation (4.1) as

$$\alpha_0(x)\partial_t^k\partial_x^2 u + (2-\frac{a_5}{c^2})\alpha_0'(x)\partial_t^k\partial_x u - (2+\frac{a_5}{c^2})\alpha_0'(x)\frac{\partial_t^k u}{x} = \sum_{i=1}^6 \mathfrak{P}_i^k, \tag{5.1}$$

where

$$\begin{aligned}
\mathfrak{P}_1^k &:= (1-\frac{x^2}{r^2r_x^3})\big(2\alpha_0'(x)\partial_t^k\partial_x u+\alpha_0(x)\partial_t^k\partial_x^2 u\big)\\
&\quad+(1-\frac{x^3}{r^3r_x^2})\big(2\alpha_0'(x)\frac{\partial_t^k u}{x}+\alpha_0(x)(\frac{\partial_t^k u}{x})_x\big),\\
\mathfrak{P}_2^k &:= \frac{\alpha_0(x)}{x}(\frac{x^3}{r^3r_x^2}-1)a_4\partial_t^k\partial_x u+\frac{\alpha_0(x)}{x}(\frac{x^4}{r^4r_x})(3+\frac{a_5}{c^2})\frac{\partial_t^k u}{x},\\
\mathfrak{P}_3^k &:= -\frac{\alpha_0(x)}{a_3\Theta^2}\big[(\frac{x^3a_3\Theta^2}{r^3r_x^2})_x\frac{\partial_t^k u}{x}+(\frac{x^2a_3\Theta^2}{r^2r_x^3})_x\partial_t^k\partial_x u\big],\\
\mathfrak{P}_4^k &:= \frac{1}{c^2}\frac{1}{\alpha_0(x)a_3\Theta^2}\big(\alpha_0^2(x)\frac{x^2a_3}{r^2r_x^2}u\partial_t^{k+1}u\big)_x+\frac{1}{c^2}\frac{\alpha_0(x)}{x}\frac{x^3}{r^3r_x}u\partial_t^{k+1}u,\\
\mathfrak{P}_5^k &:= -(4+\frac{a_5}{c^2})\big(\frac{\alpha_0(x)}{x}\big)_x\partial_t^k u-\frac{a_5}{c^2}\big(\frac{\alpha_0(x)}{x}\big)_x x\partial_t^k\partial_x u,\\
\mathfrak{P}_6^k &:= \frac{\alpha_0(x)}{2c^2}\frac{a_2}{a_3}\frac{1}{\Theta^2}\frac{x^2}{r^2r_x^2}u\partial_t^{k+1}\partial_x u+2\frac{\alpha_0(x)}{c^2}\frac{a_2}{a_3}\frac{1}{\Theta^2}\frac{x^2}{r^2r_x^2}u\frac{\partial_t^{k+1}u}{x}\\
&\quad-\frac12\frac{1}{a_3\Theta^2}\sum_{i=1}^{k+1}C_{k+1}^i\partial_t^i a_1x\partial_t^{k+2-i}u-\frac12\frac{1}{\alpha_0(x)a_3\Theta^2}\sum_{i=1}^{6}S_i^k\\
&\quad+\frac{x}{2a_3\Theta^2}\partial_t^{k+1}(\frac{x^2}{r^2\Theta})\Phi_0-\frac12\frac{1}{a_3\Theta^2}a_1x\partial_t^{k+2}u.
\end{aligned}\tag{5.2}$$

We first determine the constant δ in (2.25) and (2.26). Because $\rho(0)>0$ and $\alpha_0'(0)=\rho_0(0)>0$, then there exists a positive constant δ_0 such that for any $x\in(0,\delta_0)$,

$$\frac{\rho_0(0)}{2}\le\alpha_0'(x)\le\frac{3\rho_0(0)}{2}.\tag{5.3}$$

Then, we take δ as $0<2\delta\le\delta_0$.

5.1. Estimates for u and u_t

In this subsection, the following estimate will be established.

Lemma 5.1. *Suppose that (r,u) is a smooth solution to the problem (2.21) and (2.23) on $[0,1]\times[0,T]$ with the property (3.1). Then, there exists a small time $0<T_2\le T$ and a positive constant $\bar c_2\ge\bar c$ (only depending on $\|(\rho_0,u_0)\|_{L^\infty}$) such that for any $t\in(0,T_2]$ and $c\ge\bar c_2$, it holds that*

$$\begin{aligned}
&\|\xi\alpha_0x\partial_t\partial_x^3(\frac{u}{x})\|_0^2+\|\xi\alpha_0\partial_t\partial_x^2(\frac{u}{x})\|_0^2+\|\xi\alpha_0'\partial_t\partial_x(\frac{u}{x})\|_0^2\\
&\le M_0\mathcal{P}_0+M_0(\mathcal{P}(K)+1)\int_0^t(E^2(\tau)+E(\tau))d\tau+M_0(\mathcal{P}(K)+1)E(t)\int_0^t E(\tau)d\tau,
\end{aligned}\tag{5.4}$$

and

$$\|\zeta\alpha_0^{\frac{3}{2}}\partial_t\partial_x^3 u\|_0^2+\|\zeta\alpha_0^{\frac{1}{2}}\alpha_0'\partial_t\partial_x^2 u\|_0^2\le M_0\mathcal{P}_0+M_0(\mathcal{P}(K)+1)\int_0^t(E^2(\tau)+E(\tau))d\tau$$
$$+M_0(\mathcal{P}(K)+1)E(t)\int_0^t E(\tau)d\tau+M_0\|\zeta\alpha_0^{\frac{1}{2}}\partial_t^3\partial_x u\|_0^2, \tag{5.5}$$

and

$$\|\zeta\alpha_0\partial_x^3 u\|_0^2+\|\zeta\alpha_0'\partial_x^2 u\|_0^2\le M_0\mathcal{P}_0+M_0(\mathcal{P}(K)+1)\int_0^t(E^2(\tau)+E(\tau))d\tau$$
$$+M_0(\mathcal{P}(K)+1)E(t)\int_0^t E(\tau)d\tau+M_0\|\zeta\partial_t^2\partial_x u\|_0^2. \tag{5.6}$$

Proof. We divide the proof into the following two steps.

Step1. Interior estimates of u and $\partial_t u$.

In this step, we prove the interior estimate in (5.4). Using

$$\partial_t^k\partial_x^j u=x\partial_t^k\partial_x^j(\frac{u}{x})+j\partial_t^k\partial_x^{j-1}(\frac{u}{x}),$$

we have by taking ∂_x over (5.2)

$$\begin{aligned}
&\alpha_0(x)x\partial_t^k\partial_x^3(\frac{u}{x})+(6-\frac{a_5}{c^2})\alpha_0(x)\partial_t^k\partial_x^2(\frac{u}{x})+(4-3\frac{a_5}{c^2})\alpha_0'(x)\partial_t^k\partial_x(\frac{u}{x})\\
&=-\big(\frac{\alpha_0(x)}{x}\big)_x x^2\partial_t^k\partial_x^2(\frac{u}{x})+\frac{\partial_x a_5}{c^2}\alpha_0(x)\partial_t^k\partial_x(\frac{u}{x})+2\big(\frac{a_5}{c^2}\alpha_0'(x)\big)_x\frac{\partial_t^k u}{x}\\
&\quad-\big[(2-\frac{a_5}{c^2})x^2(\frac{\alpha_0(x)}{x})_x\partial_t^k\partial_x(\frac{u}{x})\big]_x+\sum_{i=1}^{6}\partial_x\mathfrak{P}_i^k.
\end{aligned}\tag{5.7}$$

Multiplying (5.7) by ξ and taking L^2-norm, we have for $k=1$,

$$\begin{aligned}
&\Big\|\xi\big[\alpha_0 x\partial_t\partial_x^3(\frac{u}{x})+(6-\frac{a_5}{c^2})\alpha_0\partial_t\partial_x^2(\frac{u}{x})+(4-3\frac{a_5}{c^2})\alpha_0'\partial_t\partial_x(\frac{u}{x})\big]\Big\|_0^2\\
&\le\Big\|\xi\big[-(\frac{\alpha_0}{x})_x x^2\partial_t\partial_x^2(\frac{u}{x})+\frac{\partial_x a_5}{c^2}\alpha_0\partial_t\partial_x(\frac{u}{x})+2(\frac{a_5}{c^2}\alpha_0')_x\frac{\partial_t u}{x}\big]\Big\|_0^2\\
&\quad+\Big\|\xi\big[-\big((2-\frac{a_5}{c^2})x^2(\frac{\alpha_0}{x})_x\partial_t^k\partial_x^2(\frac{u}{x})\big)_x+\sum_{i=1}^{6}\partial_x\mathfrak{P}_i^k\big]\Big\|_0^2.
\end{aligned}\tag{5.8}$$

A straightforward computation implies the left hand side of (5.8)

$$
\begin{aligned}
&\Big\|\xi\big[\alpha_0 x\partial_t\partial_x^3(\frac{u}{x})+(6-\frac{a_5}{c^2})\alpha_0\partial_t\partial_x^2(\frac{u}{x})+(4-3\frac{a_5}{c^2})\alpha_0'\partial_t\partial_x(\frac{u}{x})\big]\Big\|_0^2\\
&=\Big\|\xi\alpha_0 x\partial_t\partial_x^3(\frac{u}{x})\Big\|_0^2+\Big\|\xi(6-\frac{a_5}{c^2})\alpha_0\partial_t\partial_x^2(\frac{u}{x})\Big\|_0^2+\Big\|\xi(4-3\frac{a_5}{c^2})\alpha_0'\partial_t\partial_x(\frac{u}{x})\Big\|_0^2\\
&\quad+2\int_0^1\xi^2\alpha_0^2(x)x(6-\frac{a_5}{c^2}\alpha_0(x))\partial_t\partial_x^3(\frac{u}{x})\partial_t\partial_x^2(\frac{u}{x})dx\\
&\quad+2\int_0^1\xi^2\alpha_0(x)\alpha_0'(x)x(4-3\frac{a_5}{c^2})\partial_t\partial_x^3(\frac{u}{x})\partial_t\partial_x(\frac{u}{x})dx\\
&\quad+2\int_0^1\xi^2\alpha_0(x)\alpha_0'(x)(4-3\frac{a_5}{c^2})(6-\frac{a_5}{c^2}\alpha_0(x))\partial_t\partial_x^2(\frac{u}{x})\partial_t\partial_x(\frac{u}{x})dx.
\end{aligned}
\tag{5.9}
$$

By the integration by parts, it holds for the fourth term on the right side of (5.9)

$$
\begin{aligned}
&2\int_0^1\xi^2\alpha_0^2(x)x(6-\frac{a_5}{c^2})\partial_t\partial_x^3(\frac{u}{x})\partial_t\partial_x^2(\frac{u}{x})dx\\
&=-3\int_0^1\xi^2\alpha_0^2(x)(6-\frac{a_5}{c^2})\big(\partial_t\partial_x^2(\frac{u}{x})\big)^2dx\\
&-2\int_0^1\xi^2\alpha_0^2(x)(\frac{\alpha_0(x)}{x})_x x^2(6-\frac{a_5}{c^2}\alpha_0(x))\big(\partial_t\partial_x^2(\frac{u}{x})\big)^2dx\\
&-2\int_0^1\xi\xi_x\alpha_0^2(x)x(6-\frac{a_5}{c^2}\alpha_0(x))\big(\partial_t\partial_x^2(\frac{u}{x})\big)^2dx+\int_0^1\xi^2\alpha_0^2(x)x\frac{\partial_x a_5}{c^2}\big(\partial_t\partial_x^2(\frac{u}{x})\big)^2dx,
\end{aligned}
\tag{5.10}
$$

where the last three terms on the right hand side of (5.10) can be bounded by for any positive constant ε

$$
\begin{aligned}
&\varepsilon\int_0^1\xi^2\alpha_0^2(x)\big(\partial_t\partial_x^2(\frac{u}{x})\big)^2dx+M_0(\varepsilon,\delta)\mathcal{P}_0\\
&+M_0(\varepsilon,\delta)\int_0^t E(\tau)d\tau+M_0(\varepsilon)E(t)\int_0^t E(\tau)d\tau,
\end{aligned}
$$

with the help of (2.25), (5.3), the fundamental theorem of calculus,

$$|\partial_x a_5| \le M_0(|\rho_{0x}| + |u_{0x}| + |r_{xx}| + |(\frac{x}{r})_x|), \tag{5.11}$$

and

$$\alpha_0'(x)x = (\frac{\alpha_0(x)}{x})_x x^2 + \alpha_0(x).$$

Similarly, we have for the fifth and sixth terms on the right hand side of (5.9)

$$\begin{aligned}
&2\int_0^1 \xi^2\alpha_0(x)\alpha_0'(x)x(4-3\frac{a_5}{c^2})\partial_t\partial_x^3(\frac{u}{x})\partial_t\partial_x(\frac{u}{x})dx\\
&+2\int_0^1 \xi^2\alpha_0(x)\alpha_0'(x)(6-\frac{a_5}{c^2})(4-3\frac{a_5}{c^2})\partial_t\partial_x^2(\frac{u}{x})\partial_t\partial_x(\frac{u}{x})dx\\
&=-2\int_0^1 \xi^2\alpha_0^2(x)(4-3\frac{a_5}{c^2})\big(\partial_t\partial_x^2(\frac{u}{x})\big)^2dx\\
&+\int_0^1 \xi^2\alpha_0(x)\alpha_0'(x)(4-3\frac{a_5}{c^2})(8-2\frac{a_5}{c^2})\partial_t\partial_x^2(\frac{u}{x})\partial_t\partial_x(\frac{u}{x})dx\\
&-2\int_0^1 \xi^2\alpha_0(x)(\frac{\alpha_0(x)}{x})_x x^2(4-3\frac{a_5}{c^2})\big(\partial_t\partial_x^2(\frac{u}{x})\big)^2dx\\
&-2\int_0^1 \xi^2\alpha_0'(x)(\frac{\alpha_0(x)}{x})_x x^2(4-3\frac{a_5}{c^2})\partial_t\partial_x^2(\frac{u}{x})\partial_t\partial_x(\frac{u}{x})dx\\
&-2\int_0^1 \xi^2\alpha_0(x)\alpha_0''(x)x(4-3\frac{a_5}{c^2})\partial_t\partial_x^2(\frac{u}{x})\partial_t\partial_x(\frac{u}{x})dx\\
&-4\int_0^1 \xi\xi_x\alpha_0(x)\alpha_0'(x)x(4-3\frac{a_5}{c^2})\partial_t\partial_x^2(\frac{u}{x})\partial_t\partial_x(\frac{u}{x})dx\\
&+\frac{6}{c^2}\int_0^1 \xi^2\partial_x a_5\alpha_0(x)\alpha_0'(x)x\partial_t\partial_x^2(\frac{u}{x})\partial_t\partial_x(\frac{u}{x})dx.
\end{aligned} \tag{5.12}$$

The second term of the right hand side in (5.12) is

$$
\begin{aligned}
&\int_0^1 \xi^2\alpha_0(x)\alpha_0'(x)(4-3\frac{a_5}{c^2})(8-2\frac{a_5}{c^2})\partial_t\partial_x^2(\frac{u}{x})\partial_t\partial_x(\frac{u}{x})dx\\
&=-\int_0^1 \xi^2\big(\alpha_0'(x)\big)^2(4-3\frac{a_5}{c^2})(4-\frac{a_5}{c^2})\big(\partial_t\partial_x(\frac{u}{x})\big)^2dx\\
&-2\int_0^1 \xi\xi_x\alpha_0(x)\alpha_0'(x)(4-3\frac{a_5}{c^2})(4-\frac{a_5}{c^2})\big(\partial_t\partial_x(\frac{u}{x})\big)^2dx\\
&-\int_0^1 \xi^2\alpha_0(x)\alpha_0''(x)(4-3\frac{a_5}{c^2})(4-\frac{a_5}{c^2})\big(\partial_t\partial_x(\frac{u}{x})\big)^2dx\\
&+\frac{1}{c^2}\int_0^1 \xi^2\alpha_0(x)\alpha_0'(x)\partial_x a_5(16-6\frac{a_5}{c^2})\big(\partial_t\partial_x(\frac{u}{x})\big)^2dx. \qquad (5.13)
\end{aligned}
$$

Similar to the analysis of (5.10), it holds that for any positive constant ε

$$
\begin{aligned}
&2\int_0^1 \xi^2\alpha_0(x)\alpha_0'(x)(4-3\frac{a_5}{c^2})\partial_t\partial_x^3(\frac{u}{x})\partial_t\partial_x(\frac{u}{x})dx\\
&+2\int_0^1 \xi^2\alpha_0(x)\alpha_0'(x)(4-3\frac{a_5}{c^2})(6-\frac{a_5}{c^2}\alpha_0)\partial_t\partial_x^2(\frac{u}{x})\partial_t\partial_x(\frac{u}{x})dx\\
&\geq -2\int_0^1 \xi^2\alpha_0^2(x)(4-3\frac{a_5}{c^2})\big(\partial_t\partial_x^2(\frac{u}{x})\big)^2dx\\
&-\int_0^1 \xi^2\big(\alpha_0'(x)\big)^2(4-3\frac{a_5}{c^2})(4-\frac{a_5}{c^2})\big(\partial_t\partial_x(\frac{u}{x})\big)^2dx\\
&-\varepsilon\int_0^1 \xi^2\big(\alpha_0(x)\big)^2\big(\partial_t\partial_x^2(\frac{u}{x})\big)^2dx-\varepsilon\int_0^1 \xi^2\big(\alpha_0'(x)\big)^2\big(\partial_t\partial_x(\frac{u}{x})\big)^2dx\\
&-M_0(\varepsilon,\delta)\mathcal{P}_0-M_0(\varepsilon,\delta)\int_0^t E(\tau)d\tau-M_0(\varepsilon)E(t)\int_0^t E(\tau)d\tau. \qquad (5.14)
\end{aligned}
$$

In particular, we have used by (5.3)

$$-\int_0^1 \xi^2\alpha_0(x)\alpha_0''(x)(4-3\frac{a_5}{c^2})(4-\frac{a_5}{c^2})\big(\partial_t\partial_x(\frac{u}{x})\big)^2dx$$

$$=-\int_0^{\delta_0} \xi^2\alpha_0(x)\frac{\alpha_0''(x)}{\alpha_0'(x)}\alpha_0'(x)(4-3\frac{a_5}{c^2})(4-\frac{a_5}{c^2})\big(\partial_t\partial_x(\frac{u}{x})\big)^2dx$$

$$-\int_{\delta_0}^1 \xi^2\alpha_0(x)\alpha_0''(x)(4-3\frac{a_5}{c^2})(4-\frac{a_5}{c^2})\big(\partial_t\partial_x(\frac{u}{x})\big)^2dx$$

$$\le\varepsilon\int_0^1 \xi^2\big(\alpha_0'(x)\big)^2\big(\partial_t\partial_x(\frac{u}{x})\big)^2dx+M_0(\varepsilon)\mathcal{P}_0+M_0(\varepsilon)\int_0^t E(\tau)d\tau. \tag{5.15}$$

Substituting (5.10)-(5.15) into (5.9), it follows that

$$\Big\|\xi\big[\alpha_0x\partial_t\partial_x^3(\frac{u}{x})+(6-\frac{a_5}{c^2})\alpha_0\partial_t\partial_x^2(\frac{u}{x})+(4-3\frac{a_5}{c^2})\alpha_0'\partial_t\partial_x(\frac{u}{x})\big]\Big\|_0^2$$

$$\ge\big\|\xi\alpha_0x\partial_t\partial_x^3(\frac{u}{x})\big\|_0^2+\int_0^1\xi^2\alpha_0(x)\big(10-3\frac{a_5}{c^2}+\frac{a_5^2}{c^4}-\varepsilon\big)\big(\partial_t\partial_x^2(\frac{u}{x})\big)^2dx$$

$$-2\frac{a_5}{c^2}\int_0^1\xi^2\alpha_0'(x)(4-3\frac{a_5}{c^2})\big(\partial_t\partial_x(\frac{u}{x})\big)^2dx-\varepsilon\int_0^1\xi^2\alpha_0'(x)\big(\partial_t\partial_x(\frac{u}{x})\big)^2dx.$$

Then, we obtain, for the small enough ε and large enough c, that

$$\Big\|\xi\big[\alpha_0x\partial_t\partial_x^3(\frac{u}{x})+(6-\frac{a_5}{c^2})\alpha_0\partial_t\partial_x^2(\frac{u}{x})+(4-3\frac{a_5}{c^2})\alpha_0'\partial_t\partial_x(\frac{u}{x})\big]\Big\|_0^2$$

$$\ge\big\|\xi\alpha_0x\partial_t\partial_x^3(\frac{u}{x})\big\|_0^2+5\big\|\xi\alpha_0(x)\partial_t\partial_x^2(\frac{u}{x})\big\|_0^2-M_0(\varepsilon)\mathcal{P}_0$$

$$-M_0\int_0^t E(\tau)d\tau-M_0E(t)\int_0^t E(\tau)d\tau, \tag{5.16}$$

where we have used

$$\Big\|\xi\alpha_0'\partial_t\partial_x(\frac{u}{x})\Big\|_0^2\le C\Big[\Big\|\xi\alpha_0x\partial_t\partial_x^3(\frac{u}{x})\Big\|_0^2+\Big\|\xi\alpha_0(6-\frac{a_5}{c^2}\alpha_0)\partial_t\partial_x^2(\frac{u}{x})\Big\|_0^2\Big]$$

$$+\Big\|\xi\big[-(\frac{\alpha_0}{x})_xx^2\partial_t\partial_x^2(\frac{u}{x})+\frac{\partial_xa_5}{c^2}\alpha_0\partial_t\partial_x(\frac{u}{x})+2(\frac{a_5}{c^2}\alpha_0')_x\frac{\partial_tu}{x}\big]\Big\|_0^2$$

$$+\left\|\xi\big[-\big((2-\frac{a_5}{c^2})x^2(\frac{\alpha_0}{x})_x\partial_t^k\partial_x^2(\frac{u}{x})\big)_x+\sum_{i=1}^{6}\partial_x\mathfrak{P}_i^k\big]\right\|_0^2, \tag{5.17}$$

which follows from (5.9).

We turn to deal with the right side of (5.8). Due to (3.6) and (5.11), the first term of (5.8) can be controlled by

$$\begin{aligned}&\left\|\xi\big[-(\frac{\alpha_0}{x})_x x^2\partial_t\partial_x^2(\frac{u}{x})+\frac{\partial_x a_5}{c^2}\alpha_0\partial_t\partial_x(\frac{u}{x})+2(\frac{a_5}{c^2}\alpha_0')_x\frac{\partial_t u}{x}\big]\right\|_0^2\\&\leq M_0\mathcal{P}_0+M_0\int_0^t E(\tau)d\tau+M_0E(t)\int_0^t E(\tau)d\tau.\end{aligned} \tag{5.18}$$

The estimates for the second term of (5.8) are more complicated. In order to simplify the tedious estimates, we only show how to treat the most difficult term, while the other terms can be similarly estimated. For $\partial_x\mathfrak{P}_1^1$, we conclude from (5.2) that

$$\begin{aligned}\partial_x\mathfrak{P}_1^1=&(1-\frac{x^2}{r^2r_x^3})\Big[2\alpha_0'(x)\partial_t\partial_x^2u+\alpha_0(x)\partial_t\partial_x^3u\Big]\\&+(1-\frac{x^3}{r^3r_x^2})\Big[2\alpha_0'(x)\partial_t\partial_x(\frac{u}{x})+\alpha_0(x)\partial_t\partial_x^2(\frac{u}{x})\Big]\\&+2\Big[\alpha_0'(x)(1-\frac{x^3}{r^3r_x^2})\Big]_x\frac{\partial_t u}{x}+\Big[\alpha_0(x)(1-\frac{x^3}{r^3r_x^2})\Big]_x\partial_t\partial_x(\frac{u}{x}).\end{aligned}$$

Since

$$\left|1-\frac{x^2}{r^2r_x^3}\right|\leq M_0\int_0^t\Big(|\frac{u}{x}(\tau)|+|u_x(\tau)|\Big)d\tau,$$

it holds that

$$\begin{aligned}&\Big\|\Big(1-\frac{x^2}{r^2r_x^3}\Big)\cdot\Big(2\alpha_0'(x)\partial_t\partial_x^2u+\alpha_0(x)\partial_t\partial_x^3u\Big)\Big\|_0^2\\&\leq M_0\int_0^t\Big(\|\frac{u}{x}(\tau)\|_{L^\infty}^2+\|u_x(\tau)\|_{L^\infty}^2\Big)d\tau\Big(\|\xi\partial_t\partial_x^2u\|_0^2+\|\xi\partial_t\partial_x^3u\|_0^2\Big)\end{aligned} \tag{5.19}$$

and

$$\left\|\xi[\alpha_0(1-\frac{x^2}{r^2r_x^3})]_x\partial_t\partial_x^2u\right\|_0^2 \le M_0\|\xi\partial_t\partial_x^2u\|_0^2\int_0^t\left(\|\frac{u}{x}(\tau)\|_{L^\infty}^2+\|u_x(\tau)\|_{L^\infty}^2\right)d\tau$$
$$+M_0\|\xi\partial_t\partial_x^2u\|_0^2\left(\|\alpha_0(\frac{u}{x})_x\|_{L^\infty}^2+\int_0^t\|\alpha_0u_{xx}(\tau)\|_{L^\infty}^2d\tau\right). \tag{5.20}$$

There, it holds that

$$\|\xi\partial_x\mathfrak{P}_1^1\|_0^2\le M_0E(t)\int_0^t E(\tau)d\tau. \tag{5.21}$$

For $\partial_x\mathfrak{P}_3^1$, it concludes from (5.2) that

$$\begin{aligned}|\partial_x\mathfrak{P}_3^1|\le & M_0\alpha_0(x)\left(|n_{0x}|+|u_{0x}|+|u_x|+|r_{xx}|+|(\frac{x}{r})_x|\right)\left(|\partial_t\partial_x(\frac{u}{x})|+|\partial_t\partial_x^2u|\right)\\
&+M_0\alpha_0(x)\left(|n_{xx}|+|u_{0xx}|+|u_{xx}|+|r_{xxx}|+|(\frac{x}{r})_{xx}|\right)\left(|\frac{\partial_tu}{x}|+|\partial_t\partial_xu|\right)\\
&+M_0\alpha_0(x)\left(|n_{0x}|^2+|u_{0x}|^2+|u_x|^2+|r_{xx}|^2+|(\frac{x}{r})_x|^2\right)\left(|\frac{\partial_tu}{x}|+|\partial_t\partial_xu|\right)\\
&+M_0|\alpha_0''(x)|\left(|n_{0x}|+|u_{0x}|+|u_x|+|r_{xx}|+|(\frac{x}{r})_x|\right)\left(|\frac{\partial_tu}{x}|+|\partial_t\partial_xu|\right).\end{aligned} \tag{5.22}$$

We need the following estimates

$$\begin{aligned}&\left\|\xi\alpha_0u_{xx}\left(|\frac{\partial_tu}{x}|+|\partial_t\partial_xu|\right)\right\|\\
&\le M_0\left(\|\xi\alpha_0u_{xx}(0)\|_{L^\infty}^2+\int_0^t\|\xi\alpha_0\partial_t\partial_x^2u(\tau)\|_{L^\infty}^2d\tau\right)\int_0^t\|\frac{\partial_tu}{x}(\tau)\|_0^2d\tau\\
&\quad+M_0\left(\|\xi\alpha_0u_{xx}(0)\|_{L^\infty}^2+\int_0^t\|\xi\alpha_0\partial_t\partial_x^2u(\tau)\|_{L^\infty}^2d\tau\right)\int_0^t\|\xi\partial_t^2\partial_xu(\tau)\|_0^2d\tau\\
&\quad+M_0\left(\|\frac{\partial_tu}{x}(0)\|_{L^\infty}^2+\|\alpha_0\partial_t\partial_xu(0)\|_{L^\infty}^2\right)\int_0^t\|\partial_t\partial_x^2u(\tau)\|_0^2d\tau\\
&\le M_0\mathcal{P}_0+M_0\int_0^t E^2(\tau)d\tau,\end{aligned}$$

and

$$
\begin{aligned}
&\|\xi\alpha_0\left(|r_{xxx}|+|(\frac{x}{r})_{xx}|\right)\left(|\frac{\partial_t u}{x}|+|\partial_t\partial_x u|\right)\|_0^2\\
&\le\|\xi\alpha_0\int_0^t\left(|u_{xxx}(\tau)|+|(\frac{x^2}{r^2}\frac{u}{x})_{xx}|\right)d\tau\left(\frac{\partial_t u}{x}+|\partial_t\partial_x u|\right)\|_0^2\\
&\le\left(\|\frac{\partial_t u}{x}\|_{L^\infty}^2+\|\xi\partial_t\partial_x u\|_{L^\infty}^2\right)\int_0^t\left(\|\alpha_0 u_{xxx}(\tau)\|_0^2+\|\alpha_0\partial_x^2(\frac{x^2}{r^2}\frac{u}{x})(\tau)\|_0^2\right)d\tau\\
&\le M_0(\mathcal{P}(K)+E(t))\int_0^t E(\tau)d\tau.
\end{aligned}
$$

Thus, we obtain

$$
\|\partial_x\mathfrak{P}_3^1\|_0^2\le M_0\mathcal{P}_0+M_0(\mathcal{P}(K)+1)\int_0^t(E^2(\tau)+E(\tau))d\tau+M_0E(t)\int_0^t E(\tau)d\tau,
$$

where the other terms of $\|\partial_x\mathfrak{P}_3^1\|$ can be simplify estimated. For $\partial_x\mathfrak{P}_6^1$, we only show the estimate of $\|\xi\partial_x(\frac{1}{2}\frac{1}{a_3}\frac{1}{\Theta^2}S_3^1)\|_0^2$. In fact, it follows from (4.2)

$$
\begin{aligned}
\partial_x[\frac{1}{2}\frac{1}{a_3}\frac{1}{\Theta^2}S_3^1]=&-\partial_x\left\{\frac{1}{a_3\Theta^2}\alpha_0(x)[\partial_t(\frac{x^2\Theta^2}{r^2r_x^3}a_3)u_{xx}+\partial_t(\frac{x^3\Theta^2}{r^3r_x^2}a_3a_4)(\frac{u}{x})_x]\right\}\\
&-\partial_x\left\{\frac{1}{a_3}\frac{1}{\Theta^2}\alpha_0(x)[\partial_t\partial_x(\frac{x^2\Theta^2}{r^2r_x^3}a_3)u_x+\partial_t\partial_x(\frac{x^3\Theta^2}{r^3r_x^2}a_3a_4)\frac{u}{x}]\right\}\\
&-\partial_x\left\{\frac{1}{a_3}\frac{1}{\Theta^2}\alpha_0'(x)[\partial_t(\frac{x^2\Theta^2}{r^2r_x^3}a_3)u_x+\partial_t(\frac{x^3\Theta^2}{r^3r_x^2}a_3a_4)\frac{u}{x}]\right\}.
\end{aligned}
\tag{5.23}
$$

For the second term on the right side of (5.23), a simple computation implies

$$
\begin{aligned}
&-\partial_x\left\{\frac{1}{a_3\Theta^2}\alpha_0'(x)[\partial_t\partial_x(\frac{x^2\Theta^2}{r^2r_x^3}a_3)u_x+\partial_t\partial_x(\frac{x^3\Theta^2}{r^3r_x^2}a_3a_4)\frac{u}{x}]\right\}\\
&=-\partial_x(\frac{1}{a_3\Theta^2})\alpha_0(x)[\partial_t\partial_x(\frac{x^2\Theta^2}{r^2r_x^3}a_3)u_x+\partial_t\partial_x(\frac{x^3\Theta^2}{r^3r_x^2}a_3a_4)\frac{u}{x}]\\
&\quad-\frac{1}{a_3\Theta^2}\alpha_0'(x)[\partial_t\partial_x(\frac{x^2\Theta^2}{r^2r_x^3}a_3)u_x+\partial_t\partial_x(\frac{x^3\Theta^2}{r^3r_x^2}a_3a_4)\frac{u}{x}]\\
&\quad-\frac{1}{a_3\Theta^2}\alpha_0'(x)[\partial_t\partial_x(\frac{x^2\Theta^2}{r^2r_x^3}a_3)u_{xx}+\partial_t\partial_x(\frac{x^3\Theta^2}{r^3r_x^2}a_3a_4)(\frac{u}{x})_x]
\end{aligned}
$$

$$- \frac{1}{a_3\Theta^2}\alpha_0'(x)[\partial_t\partial_x^2(\frac{x^2\Theta^2}{r^2r_x^3}a_3)u_x + \partial_t\partial_x^2(\frac{x^3\Theta^2}{r^3r_x^2}a_3a_4)\frac{u}{x}]. \tag{5.24}$$

A straightforward computation gives

$$\begin{aligned}\left|\partial_t\partial_x^2(\frac{x^2\Theta^2}{r^2r_x^3}a_3)\right| \le & M_0\left(|u_{xxt}| + |u_{xxx}| + |\partial_t\partial_x^2(\frac{x}{r})| + |u_x||u_{xt}| + |r_{xx}||u_{xx}| + |(\frac{x}{r})_x\partial_t\partial_x(\frac{x}{r})|\right)\\ & + M_0\left(|n_{0xx}| + |u_{0xx}| + |u_{xx}| + |r_{xxx}| + |(\frac{x}{r})_{xx}|][|u_t| + |u_x| + |\frac{x}{u}|\right)\\ & + M_0\left(|n_{0x}|^2 + |u_{0x}|^2 + |u_x|^2 + |r_{xx}|^2 + |(\frac{x}{r})_x|^2\right)\left(|u_t| + |u_x| + |\frac{x}{u}|\right).\end{aligned} \tag{5.25}$$

Thus, the following estimates hold

$$\begin{aligned}\|\xi\alpha_0u_{xxx}(|u_x| + |\frac{u}{x}|)\|_0^2 \le & M_0\mathcal{P}_0 + M_0(\|u_x(0)\|_{L^\infty}^2 + \|\frac{u}{x}(0)\|_{L^\infty}^2)\int_0^t \|\xi\alpha_0\partial_t\partial_x^3u\|_0^2 d\tau\\ & + M_0\|\alpha_0u_{xxx}(0)\|_0^2\int_0^t (\|\xi\partial_t\partial_xu(\tau)\|_{L^\infty}^2 + \|\xi\frac{\partial_tu}{x}(\tau)\|_{L^\infty}^2)d\tau\\ & + M_0\int_0^t \|\alpha_0\partial_t\partial_x^3u(\tau)\|_0^2 d\tau\int_0^t (\|\alpha_0\partial_t\partial_xu(\tau)\|_{L^\infty}^2 + \|\frac{\partial_tu}{x}(\tau)\|_{L^\infty}^2)d\tau,\end{aligned} \tag{5.26}$$

and

$$\begin{aligned}\|\xi\alpha_0u_{xx}u_x^2\|_0^2 \le & M_0\mathcal{P}_0 + M_0\|\xi\alpha_0u_{xx}(0)\|_{L^\infty}^2\int_0^t \|\partial_t\partial_xu(\tau)\|_0^4 d\tau\\ & + M_0\|u_x(0)\|_{L^\infty}^2\int_0^t \|\xi\alpha_0\partial_t\partial_x^2u(\tau)\|_{L^\infty}^2 d\tau\\ & + M_0\int_0^t \|\xi\partial_t\partial_xu(\tau)\|_{L^\infty}^2 d\tau\int_0^t \|\alpha_0\partial_t\partial_x^2u(\tau)\|_0^2 d\tau,\end{aligned}$$

and

$$\|\xi\alpha_0(\frac{x}{r})_{xx}u_x^2\|_0^2 \le M_0\|\int_0^t \left((K+1)|\frac{u}{x}(\tau)| + |u_x(\tau)| + |\alpha_0(\frac{u}{x})_x(\tau)| + |\alpha_0u_x(\tau)|\right)d\tau\|_0^2$$

$$\times\ (|u_x(0)| + \int_0^t |\xi\partial_t\partial_x)^2\|_0^2 \le M_0\mathcal{P}_0 + M_0\int_0^t E^2(\tau)d\tau.$$

Thus, we can handle the main difficult term of (5.24) as

$$\begin{aligned}&\|\xi\alpha_0\partial_t\partial_x^2(\frac{x^2\Theta^2}{r^2r_x^3}a_3)u_x\|_0^2\\ &\le M_0\mathcal{P}_0 + M_0(\mathcal{P}(K)+1)\int_0^t E(\tau)d\tau + M_0\int_0^t E^2(\tau)d\tau.\end{aligned}\tag{5.27}$$

Similarly, we can estimate the other terms in $\|\xi\partial_x(\frac{1}{2a_3\Theta^2}S_3^1)\|_0^2$, which is bounded by the right side of (5.4). This completes the proof of (5.4).

Step 2. Boundary estimates of u and $\partial_t u$.

In this step, we prove the boundary estimate in (5.5), while the estimate in (5.6) can be similarly treated.

Taking ∂_x over (5.1), we obtain for $k=1$ that

$$\begin{aligned}&\alpha_0(x)\partial_t\partial_x^3u + (3-\frac{a_5}{c^2})\alpha_0'(x)\partial_t\partial_x^2u\\ &= -(2-\frac{a_5}{c^2})\alpha_0''(x)\partial_t\partial_x u + \frac{\partial_x a_5}{c^2}\alpha_0'(x)\partial_t\partial_x u + \big[(2+\frac{a_5}{c^2})\alpha_0'(x)\frac{\partial_t^k u}{x}\big]_x + \sum_{i=1}^6\partial_x\mathfrak{P}_i^k.\end{aligned}\tag{5.28}$$

Multiplying (5.28) by $\zeta\sqrt{\alpha_0(x)}$, it holds that

$$\begin{aligned}&\|\zeta\sqrt{\alpha_0}\Big(\alpha_0\partial_t\partial_x^3u + (3-\frac{a_5}{c^2})\alpha_0'\partial_t\partial_x^2u\Big)\|_0^2\\ &\le \|\zeta\sqrt{\alpha_0}(2-\frac{a_5}{c^2})\alpha_0''\partial_t\partial_x u\|_0^2 + \|\zeta\sqrt{\alpha_0}\frac{\partial_x a_5}{c^2}\alpha_0'\partial_t\partial_x u\|_0^2\\ &+ \Big\|\zeta\sqrt{\alpha_0}\big[(2+\frac{a_5}{c^2})\alpha_0'\frac{\partial_t^k u}{x}\big]_x\Big\|_0^2 + \sum_{i=1}^6\|\zeta\sqrt{\alpha_0}\partial_x\mathfrak{P}_i^k\|_0^2,\end{aligned}\tag{5.29}$$

where the first three terms on the right side can be bounded by

$$\begin{aligned}&\|\zeta\sqrt{\alpha_0}(2-\frac{a_5}{c^2})\alpha_0''\partial_t\partial_x u\|_0^2 + \|\zeta\sqrt{\alpha_0}\frac{\partial_x a_5}{c^2}\alpha_0'\partial_t\partial_x u\|_0^2 + \Big\|\zeta\sqrt{\alpha_0}\big[(2+\frac{a_5}{c^2})\alpha_0'\frac{\partial_t^k u}{x}\big]_x\Big\|_0^2\\ &\le M_0(\mathcal{P}_0+1)\int_0^t E(\tau)d\tau + M_0E(t)\int_0^t E(\tau)d\tau + M_0\int_0^t E^2(\tau)d\tau + M_0\mathcal{P}_0.\end{aligned}\tag{5.30}$$

By the integration by parts, it is easy to have

$$
\begin{aligned}
&\left\| \zeta \sqrt{\alpha_0}(\alpha_0 \partial_t \partial_x^3 u + (3 - \frac{a_5}{c^2})\alpha_0' \partial_t \partial_x^2 u) \right\|_0^2 \\
&= \left\| \zeta \alpha_0^{\frac{3}{2}} \partial_t \partial_x^3 u \right\|_0^2 + \int_0^1 \zeta^2 \alpha_0(x)(3 - \frac{a_5}{c^2})^2 \Big(\alpha_0'(x) \Big)^2 (\partial_t \partial_x^2 u)^2 dx \\
&+ 2 \int_0^1 \zeta^2 \alpha_0^2(x) \alpha_0'(x)(3 - \frac{a_5}{c^2}) \partial_t \partial_x^3 u \partial_t \partial_x^2 u dx.
\end{aligned} \tag{5.31}
$$

For the third term on the right side of (5.31), the integration by parts shows

$$
\begin{aligned}
&2 \int_0^1 \zeta^2 \alpha_0^2(x) \alpha_0'(x)(3 - \frac{a_5}{c^2}) \partial_t \partial_x^3 u \partial_t \partial_x^2 u dx \\
&= -2 \int_0^1 \zeta^2 \alpha_0(x)(\alpha_0'(x))^2 (3 - \frac{a_5}{c^2})(\partial_t \partial_x^2 u)^2 dx \\
&- 4 \int_0^1 \zeta \zeta_x \alpha_0^2(x) \alpha_0'(x)(3 - \frac{a_5}{c^2})(\partial_t \partial_x^2 u)^2 dx \\
&- \int_0^1 \zeta^2 \alpha_0^2(x) \alpha_0''(x)(3 - \frac{a_5}{c^2})(\partial_t \partial_x^2 u)^2 dx + \int_0^1 \zeta^2 \alpha_0^2(x) \alpha_0'(x) \frac{\partial_x a_5}{c^2} (\partial_t \partial_x^2 u)^2 dx.
\end{aligned}
$$

Then, we derive from (5.31) that

$$
\begin{aligned}
&\left\| \zeta \sqrt{\alpha_0}(\alpha_0 \partial_t \partial_x^3 u + (3 - \frac{a_5}{c^2})\alpha_0' \partial_t \partial_x^2 u) \right\|_0^2 \\
&\geq \left\| \zeta \alpha_0^{\frac{3}{2}} \partial_t \partial_x^3 u \right\|_0^2 + \int_0^1 \zeta^2 \alpha_0(x)(3 - \frac{a_5}{c^2})(1 - \frac{a_5}{c^2})(\alpha_0'(x))^2 (\partial_t \partial_x^2 u)^2 dx \\
&- M_0 \mathcal{P}_0 - M_0(\mathcal{P}_0 + 1) \int_0^t E(\tau) d\tau - M_0 E(t) \int_0^t E(\tau) d\tau - M_0 \int_0^t E^2(\tau) d\tau.
\end{aligned} \tag{5.32}
$$

Similar to the analysis of (5.21), we can have

$$
\left\| \zeta \sqrt{\alpha_0} \alpha_0 \partial_x \mathfrak{P}_1^1 \right\|_0^2 \leq M_0 E(t) \int_0^t E(\tau) d\tau.
$$

For $\partial_x \mathfrak{P}_3^1$, we have to estimate the following main difficult terms

$$
\begin{aligned}
&\left\|\zeta\sqrt{\alpha_0}\alpha_0 u_x\partial_t\partial_x^2 u\right\|_0^2\\
&=\left\|\zeta\sqrt{\alpha_0}\alpha_0 u_x\left(\partial_t\partial_x^2 u(0)+\int_0^t \partial_t^2\partial_x^2 u(\tau)d\tau\right)\right\|_0^2\\
&\le \|\sqrt{\alpha_0}\partial_t\partial_x^2 u(0)\|_0^2\|\alpha_0 u_x\|_{L^\infty}^2+M_0(\mathcal{P}(K)+1)\int_0^t \|\alpha_0\partial_t^2\partial_x^2 u(\tau)\|_0^2 d\tau,
\end{aligned}
$$

and

$$
\|\zeta\sqrt{\alpha_0}\alpha_0 r_{xx}\partial_t\partial_x u\|_0^2\le\|\sqrt{\rho_0}\partial_t\partial_x u\|_{L^\infty}^2\int_0^t\|\alpha_0 u_{xx}(\tau)\|_0^2 d\tau,
$$

$$
\|\zeta\sqrt{\alpha_0}\alpha_0 u_{xx}\partial_t\partial_x u\|_0^2\le\|\alpha_0\partial_t\partial_x u\|_{L^\infty}^2\|u_{xx}(0)\|_0^2+\|\alpha_0\partial_t\partial_x u\|_{L^\infty}^2\int_0^t\|\sqrt{\alpha_0}\partial_t\partial_x^2 u(\tau)\|_0^2 d\tau,
$$

$$
\|\zeta\sqrt{\alpha_0}\alpha_0 r_{xx}^2\partial_t\partial_x u\|_0^2\le\|\sqrt{\rho_0}\partial_t\partial_x u\|_{L^\infty}^2\int_0^t\|\alpha_0 u_{xx}(\tau)\|_{L^\infty}^2 d\tau\int_0^t\|u_{xx}(\tau)\|_0^2 d\tau.
$$

On the other hand, the Gagliardo-Nirenberg Interpolation inequality [18] shows,

$$
\|\sqrt{n_0}\partial_t\partial_x u\|_{L^\infty}^2\le C\|\sqrt{n_0}\partial_t\partial_x u\|_0^{1-\alpha}\left(\|\frac{1}{2\sqrt{\rho_0}}\|_{L^{2-\frac{a}{2}}(0,1)}\|\partial_t\partial_x u\|_{H^{\frac{1}{2}}(0,1)}+\|\sqrt{n_0}\partial_t\partial_x^2 u\|_0\right)^{\alpha},
$$

where $0<a<\frac{1}{2}$ and $0<\alpha=\frac{3-3a}{4+4a}<1$. Thus, we have that $\|\zeta\sqrt{\alpha_0}\alpha_0\partial_x\mathfrak{P}_3^1\|_0^2$ can be bounded by (5.30).

For $\partial_x\mathfrak{P}_6^1$, similar to the analysis of (5.23), we need to deal with the main difficult term $\|\zeta\sqrt{\alpha_0}\alpha_0\partial_t\partial_x^2(\frac{x^2\theta^2}{r^2r_x^3}a_3)u_x\|_0^2$. In fact, we have to handle the following estimates from (5.25)

$$
\|\zeta\sqrt{\alpha_0}\alpha_0 u_{xxx}u_x\|_0^2\le\|\zeta\alpha_0 u_{xxx}(0)\|_0^2\|\alpha_0 u_x\|_{L^\infty}^2+\|u_x\|_{L^\infty}^2\int_0^t\|\alpha_0^{\frac{3}{2}}\partial_t\partial_x^3 u(\tau)\|_{L^\infty}^2 d\tau,
$$

$$
\|\zeta\sqrt{\alpha_0}\alpha_0 r_{xx}u_{xx}u_x\|_0^2\le\|u_x\|_{L^\infty}^2\|\alpha_0 u_{xx}\|_{L^\infty}^2\int_0^t\|u_{xx}(\tau)\|_0^2 d\tau,
$$

and

$$
\|\zeta\sqrt{\alpha_0}\alpha_0 u_{xx}u_x^2\|_0^2\le\|\zeta\sqrt{\alpha_0}\alpha_0 u_{xx}(0)u_x^2(0)\|_0^2+\|\alpha_0 u_{xx}(0)\|_{L^\infty}^2\int_0^t\|\partial_t\partial_x u(\tau)\|_{L^4}^2 d\tau
$$

$$+ M_0\mathcal{P}(K)\int_0^t \|\sqrt{\alpha_0}\partial_t\partial_x^2 u(\tau)\|_0^2 d\tau.$$

Thus, the term $\|\zeta\sqrt{\alpha_0}\alpha_0\partial_t\partial_x^2(\frac{x^2\theta^2}{r^2r_x^3}a_3)u_x\|_0^2$ has a bound in (5.30). The other terms in $\sum_{i=1}^6 \|\zeta\sqrt{\alpha_0}\alpha_0\partial_x\mathfrak{P}_i^1\|_0^2$ can be similarly estimated. Therefore, we derive (5.5) from (5.29). The proof is complete. □

5.2. *Estimates for* $\partial_t^3 u$

In this subsection, the following estimate will be proved.

Lemma 5.2. *Suppose that* (r, u) *is a smooth solution to the problem* (2.21) *and* (2.23) *on* $[0, 1] \times [0, T]$ *with the property* (3.1)*. Then, there exists a small time* $0 < T_3 \le T$ *and a positive constant* $\bar{c}_2 \ge \bar{c}$ *(only depending on* $\|(\rho_0, u_0)\|_{L^\infty}$*) such that for any* $t \in (0, T_3]$ *and* $c \ge \bar{c}_3$*, it holds that*

$$\begin{aligned}&\|\xi\alpha_0\partial_t^3\partial_x^2 u\|_0^2 + \|\xi\alpha_0'\partial_t^3\partial_x u\|_0^2 + +\|\xi\alpha_0'\frac{\partial_t^3 u}{x}\|_0^2\\ &\le\ M_0\mathcal{P}_0 + M_0(\mathcal{P}(K)+1)\int_0^t (E^2(\tau)+E(\tau))d\tau + M_0(\mathcal{P}(K)+1)E(t)\int_0^t E(\tau)d\tau,\end{aligned}\tag{5.33}$$

and

$$\begin{aligned}&\|\zeta\alpha_0^{\frac{3}{2}}\partial_t^3\partial_x^2 u\|_0^2 + \|\zeta\alpha_0^{\frac{1}{2}}\alpha_0'\partial_t^3\partial_x u\|_0^2\\ &\le M_0\mathcal{P}_0 + M_0(\mathcal{P}(K)+1)\int_0^t (E^2(\tau)+E(\tau))d\tau + M_0(\mathcal{P}(K)+1)E(t)\int_0^t E(\tau)d\tau.\end{aligned}\tag{5.34}$$

Proof. We divide the proof in two steps.

Step1. Interior estimates of $\partial_t^3 u$***.***

In this step, we prove the interior estimate in (5.33). We have from (5.1) that

$$\left\|\xi\big(\alpha_0\partial_t^3\partial_x^2 u + (2-\frac{a_5}{c^2})\alpha_0'\partial_t^3\partial_x u - (2+\frac{a_5}{c^2})\alpha_0'\frac{\partial_t^3 u}{x}\big)\right\|_0^2 \le \sum_{i=1}^6 \left\|\xi\mathfrak{P}_i^3\right\|_0^2. \tag{5.35}$$

Similar to (5.16), it holds that

$$\Big\|\xi\big(\alpha_0\partial_t^3\partial_x^2u+(2-\frac{a_5}{c^2})\alpha_0'\partial_t^3\partial_xu-(2+\frac{a_5}{c^2})\frac{\partial_t^3u}{x}\big)\Big\|_0^2$$
$$\geq\|\xi\alpha_0\partial_t^3\partial_x^2u\|_0^2+\int_0^1\xi^2\Big[(2-\frac{a_5}{c^2})^2+2+3\frac{a_5}{c^2}\Big](\alpha_0'(x))^2(\partial_t^3\partial_xu)^2dx$$
$$+\int_0^1\xi^2(2+\frac{a_5}{c^2})^2(\alpha_0'(x))^2\Big(\frac{\partial_t^3u}{x}\Big)^2dx-2\int_0^1\xi^2(\alpha_0'(x))^2(2-\frac{a_5}{c^2})(2+\frac{a_5}{c^2})\partial_t^3\partial_xu\frac{\partial_t^3u}{x}dx$$
$$-M_0\mathcal{P}_0-M_0(\mathcal{P}(K)+1)\int_0^t\big(E^2(\tau)+E(\tau)\big)d\tau-M_0E(t)\int_0^tE(\tau)d\tau. \tag{5.36}$$

For $\sum_{i=1}^6\mathfrak{P}_i^3$, we only show the estimate of $\|\xi\mathfrak{P}_6^3\|_0^2$, while the other terms can be similarly estimated. In fact, we have from (5.2)

$$\|\xi\mathfrak{P}_6^3\|_0^2\leq M_0\big(\|\alpha_0\partial_t^4\partial_xu\|_0^2+\|\alpha_0\frac{\partial_t^4u}{x}\|_0^2+\|\sqrt{x\alpha_0}\partial_t^5u\|_0^2\big)$$
$$+M_0\sum_{i=1}^4\|\xi\partial_t^ia_1x\partial_t^{5-i}u\|_0^2+M_0\sum_{i=1}^6\|\xi\frac{S_i^3}{\alpha_0}\|_0^2+M_0\|\xi x\partial_t^4(\frac{x^2}{r^2\Theta^2})\Phi_0\|_0^2. \tag{5.37}$$

For the third term on the right side of (5.37), we have by Lemma 3.3

$$\sum_{i=1}^4\|\xi x\partial_t^ia_1\partial_t^{5-i}u\|_0^2\leq M_0\big(\|\xi\mathcal{H}_{t,x}^{1,0}\|_{L^\infty}^2\|\alpha_0\partial_t^4u\|_0^2+\|\mathcal{H}_{t,x}^{2,0}\|_0^2\|\alpha_0\partial_t^3u\|_{L^\infty}^2\big)$$
$$+M_0\big(\|\alpha_0\mathcal{H}_{t,x}^{3,0}\|_{L^4(0,1)}^2\|\partial_t^2u\|_{L^4(0,1)}^2+\|\alpha_0\mathcal{H}_{t,x}^{4,0}\|_0^2\|\xi\partial_tu\|_{L^\infty}^2\big).$$

For the fourth term $\sum_{i=1}^6\|\xi\frac{S_i^3}{\alpha_0}\|_0^2$ in (5.37), we only show the estimate of $\|\xi\frac{S_3^3}{\alpha_0}\|_0^2$. In fact, it follows from (4.2) that

$$\frac{S_3^3}{\alpha_0(x)}=-2\sum_{i=1}^3\mathrm{C}_3^i\alpha_0(x)\Big[\partial_t^i(\frac{x^2\Theta^2}{r^2r_x^3}a_3)\partial_t^{3-i}\partial_x^2u+\partial_t^i(\frac{x^3a_3a_4}{r^3r_x^2}\Theta^2)(\frac{\partial_t^{3-i}u}{x})_x\Big]$$
$$-4\sum_{i=1}^3\mathrm{C}_3^i\alpha_0'(x)\Big[\partial_t^i(\frac{x^2\Theta^2}{r^2r_x^3}a_3)\partial_t^{3-i}\partial_xu+\partial_t^i(\frac{x^3a_3a_4}{r^3r_x^2}\Theta^2)\frac{\partial_t^{3-i}u}{x}\Big]$$
$$-2\sum_{i=1}^3\mathrm{C}_3^i\alpha_0(x)\Big[\partial_t^i\partial_x(\frac{x^2\Theta^2}{r^2r_x^3}a_3)\partial_t^{3-i}\partial_xu+\partial_t^i\partial_x(\frac{x^3a_3a_4}{r^3r_x^2}\Theta^2)\frac{\partial_t^{3-i}u}{x}\Big]. \tag{5.38}$$

For the first term of (5.38), a simple computation shows

$$
\begin{aligned}
&-2\sum_{i=1}^{3}\mathrm{C}_3^i\alpha_0(x)\partial_t^i(\frac{x^3a_3a_4}{r^3r_x^2}\Theta^2)(\frac{\partial_t^{3-i}u}{x})_x=-2\mathrm{C}_3^1\alpha_0(x)\partial_t(\frac{x^3a_3a_4}{r^3r_x^2}\Theta^2)\big(\frac{\partial_t^2\partial_x u}{x}-\frac{\partial_t^2u}{x^2}\big)\\
&\quad-2\mathrm{C}_3^2\alpha_0(x)\partial_t^2(\frac{x^3a_3a_4}{r^3r_x^2}\Theta^2)\big(\frac{\partial_t\partial_x u}{x}-\frac{\partial_t u}{x^2}\big)-2\mathrm{C}_3^3\alpha_0(x)\partial_t^3(\frac{x^3a_3a_4}{r^3r_x^2}\Theta^2)\big(\frac{u_x}{x}-\frac{u}{x^2}\big).
\end{aligned}
\tag{5.39}
$$

Thus, we have the following estimates

$$
\begin{aligned}
&\left\|\xi\frac{\alpha_0}{x}\partial_t(\frac{x^3a_3a_4}{r^3r_x^2}\Theta^2)\frac{\partial_t^2u}{x}\right\|_0^2\le M_0\|\xi\mathcal{H}_{t,x}^{1,0}\|_{L^\infty}^2\|\frac{\partial_t^2u}{x}\|_0^2,\\
&\left\|\xi\frac{\alpha_0}{x}\partial_t^2(\frac{x^3a_3a_4}{r^3r_x^2}\Theta^2)\frac{\partial_t u}{x}\right\|_0^2\le M_0\|\frac{\partial_t u}{x}(0)\|_{L^\infty}^2\|\xi\mathcal{H}_{t,x}^{2,0}\|_0^2+M_0\|\xi\mathcal{H}_{t,x}^{2,0}\|_{L^\infty}^2\int_0^t\|\frac{\partial_t^2u}{x}(\tau)\|_0^2d\tau,
\end{aligned}
$$

and

$$
\begin{aligned}
&\left\|\xi\frac{\alpha_0}{x}\partial_t^3(\frac{x^3a_3a_4}{r^3r_x^2}\Theta^2)\frac{u}{x}\right\|_0^2\\
&\le M_0\|\xi\mathcal{H}_{t,x}^{1,0}\|_{L^\infty}^2\|\mathcal{H}_{t,x}^{2,0}\|_0^2\|\frac{u}{x}(0)\|_{L^\infty}^2+M_0\|\mathcal{H}_{t,x}^{2,0}\|_0^2\|\mathcal{H}_{t,x}^{1,0}\|_{L^\infty}^2\int_0^t\|\frac{\partial_t u}{x}(\tau)\|_{L^\infty}^2d\tau\\
&+M_0\|\frac{u}{x}\|_{L^\infty}^2\|\partial_t^2\partial_x u(0)\|_0^2+M_0\|u_x\|_{L^\infty}^2\int_0^t\|\xi\partial_t^3\partial_x u(\tau)\|_0^2d\tau\\
&+M_0\|\frac{u}{x}\|_{L^\infty}^2\|\frac{\partial_t^2u}{x}(0)\|_0^2+\|\frac{u}{x}\|_{L^\infty}^2\int_0^t\|\xi\frac{\partial_t^3u}{x}(\tau)\|_0^2d\tau,
\end{aligned}
$$

which in combination with (3.8) shows that the first term of (5.38) has a bound as in (5.30). A straightforward computation implies the third term on the right side of (5.38) as

$$
\begin{aligned}
&-2\sum_{i=1}^{3}\mathrm{C}_3^i\alpha_0(x)\Big[\partial_t^i\partial_x(\frac{x^2\Theta^2}{r^2r_x^3}a_3)\partial_t^{3-i}\partial_x u+\partial_t^i\partial_x(\frac{x^3a_3a_4}{r^3r_x^2}\Theta^2)\frac{\partial_t^{3-i}u}{x}\Big]\\
&=-2\mathrm{C}_3^1\alpha_0(x)\Big[\partial_t\partial_x(\frac{x^2\Theta^2}{r^2r_x^3}a_3)\partial_t^2\partial_x u+\partial_t\partial_x(\frac{x^3a_3a_4}{r^3r_x^2}\Theta^2)\frac{\partial_t^2u}{x}\Big]\\
&\quad-2\mathrm{C}_3^2\alpha_0(x)\Big[\partial_t^2\partial_x(\frac{x^2\Theta^2}{r^2r_x^3}a_3)\partial_t\partial_x u+\partial_t^2\partial_x(\frac{x^3a_3a_4}{r^3r_x^2}\Theta^2)\frac{\partial_t u}{x}\Big]\\
&\quad-2\mathrm{C}_3^3\alpha_0(x)\Big[\partial_t^3\partial_x(\frac{x^2\Theta^2}{r^2r_x^3}a_3)u_x+\partial_t^3\partial_x(\frac{x^3a_3a_4}{r^3r_x^2}\Theta^2)\frac{u}{x}\Big].
\end{aligned}
$$

The chain rules show

$$
\begin{aligned}
\left|\partial_t^3\partial_x(\frac{x^3a_3a_4}{r^3r_x^2}\Theta^2)\right| \le\ & M_0\big(|\partial_t^3\partial_x u|+|\partial_t^2\partial_x^2 u|+|\partial_t^3\partial_x(\frac{x}{r})|\big)\\
&+M_0\big(|u_{xtt}|+|u_{xxt}|+|\partial_t^2\partial_x(\frac{x}{r})|\big)\big(|u_t|+|u_x|+|\frac{u}{x}|\big)\\
&+M_0\big(|u_{xt}|+|u_{xx}|+|\partial_t\partial_x(\frac{x}{r})|\big)\big(|u_{tt}|+|u_{xt}|+|\frac{u_t}{x}|\big)\\
&+M_0\big(|u_{xt}|+|u_{xx}|+|\partial_t\partial_x(\frac{x}{r})|\big)\big(|u_t|^2+|u_x|^2+|\frac{u_t}{x}|^2\big)\\
&+M_0\big(|n_{0x}|+|u_{0x}|+|u_x|+|r_{xx}|+|(\frac{x}{r})_x|\big)\big(|u_t||u_{tt}|+|u_x||u_{xt}|+|\frac{u}{x}||\frac{u_t}{x}|\big)\\
&+M_0\big(|n_{0x}|+|u_{0x}|+|u_x|+|r_{xx}|+|(\frac{x}{r})_x|\big)\big(|u_t|^3+|u_x|^3+|\frac{u}{x}|^3\big). \qquad (5.40)
\end{aligned}
$$

Thus, the following estimates are needed:

$$
\|\xi\alpha_0\partial_t^3\partial_x u\frac{u}{x}\|_0^2 \le \|\frac{u}{x}\|_{L^\infty}^2\|\alpha_0\partial_t^3\partial_x u(0)\|_0^2+\|\frac{u}{x}\|_{L^\infty}^2\int_0^t\|\alpha_0\partial_t^4\partial_x u(\tau)\|_0^2d\tau,
$$

$$
\|\xi\alpha_0\partial_t^2\partial_x^2 u\frac{u}{x}\|_0^2 \le \|\frac{u}{x}\|_{L^\infty}^2\|\alpha_0\partial_t^2\partial_x^2 u(0)\|_0^2+\|\frac{u}{x}\|_{L^\infty}^2\int_0^t\|\xi\alpha_0\partial_t^3\partial_x^2 u(\tau)\|_0^2d\tau,
$$

$$
\|\xi\alpha_0\partial_t^3\partial_x(\frac{r}{x})\frac{u}{x}\|_0^2 \le M_0\left[\|u_t\|_{L^\infty}^2\|\frac{u}{x}\|_{L^\infty}^4+\|\frac{u}{x}\|_{L^\infty}^2\|\frac{\partial_t^2 u}{x}\|_0^2+\|\frac{u}{x}\|_{L^\infty}^6\right],
$$

and

$$
\begin{aligned}
\|\xi\alpha_0 u_{xx}u_{xt}\frac{u}{x}\|_0^2 \le\ & M_0\|\alpha_0 u_{xx}(0)\|_{L^\infty}^2\|\frac{u}{x}\|_{L^\infty}^2\|\xi u_{xt}\|_0^2\\
&+M_0\|\xi u_{xt}\|_{L^\infty}^2\|\frac{u}{x}\|_{L^\infty}^2\int_0^t\|\sqrt{\alpha_0}\partial_t\partial_x^2 u(\tau)\|_0^2d\tau.
\end{aligned}
$$

Thus, we can prove that the last term on the right side of (5.38) can be bounded by (5.30). Finally, we also similarly estimate the other terms of $\sum_{i=1}^6\|\xi\frac{S_i^3}{\alpha_0}\|_0^2$ and can prove (5.33).

Step 2. Boundary estimates of $\partial_t^3 u$.

In this step, we prove the boundary estimate in (5.34).

We derive from (5.1) with $k=3$

$$
\|\zeta\sqrt{\alpha_0}(\alpha_0\partial_t^3\partial_x^2 u+(2-\frac{a_5}{c^2})\alpha_0'\partial_t^3\partial_x u)\|_0^2 \le M_0\|\partial_t^3 u\|_0^2+\sum_{i=1}^6\|\zeta\sqrt{\alpha_0}\mathfrak{P}_i^3\|_0^2. \qquad (5.41)
$$

Similar to (5.36), the integration by parts shows

$$
\begin{aligned}
&\|\zeta\sqrt{\alpha_0}(\alpha_0\partial_t^3\partial_x^2 u+(2-\frac{a_5}{c^2})\alpha_0'\partial_t^3\partial_x u)\|_0^2\\
&\geq \|\zeta\alpha^{\frac{3}{2}}\partial_t^3\partial_x^2 u\|_0^2-(\varepsilon+\frac{a_5}{c^2})\int_0^1\zeta^2\alpha_0(\alpha_0'(x))^2(2-\frac{a_5}{c^2})(\partial_t^3\partial_x u)^2dx\\
&-M_0\mathcal{P}_0-M_0(\mathcal{P}(K)+1)\int_0^t(E^2(\tau)+E(\tau))d\tau-M_0E(t)\int_0^t E(\tau)d\tau. \qquad (5.42)
\end{aligned}
$$

For $\mathfrak{P}_6^3$, we have by the analogy with (5.37)

$$
\begin{aligned}
\|\zeta\sqrt{\alpha_0}\mathfrak{P}_6^3\|_0^2 &\leq M_0\left[\|\alpha_0\partial_t^4\partial_x u\|_0^2+\|\alpha_0\frac{\partial_t^4 u}{x}\|_0^2+\|\sqrt{x\alpha_0}\partial_t^5 u\|_0^2\right]\\
&+M_0\sum_{i=1}^4\|\zeta\sqrt{\alpha_0}\partial_t^3 a_1\partial_t^{3-i}u\|_0^2+M_0\sum_{i=1}^6\|\zeta\sqrt{\alpha_0}\frac{S_i^3}{\alpha_0}\|_0^2. \qquad (5.43)
\end{aligned}
$$

For the right side of (5.43), we only give the estimates for $\|\zeta\sqrt{\alpha_0}\frac{S_i^3}{\alpha_0}\|_0^2$ with the other terms of (5.43) can be handle by the similar proceeding to the estimate of $\partial_t u$ in (5.5). From (5.38), it follows that

$$
\begin{aligned}
&-2\sum_{i=1}^3 C_3^i\alpha_0(x)\partial_t^i(\frac{x^2\Theta^2}{r^2r_x^3}a_3)\partial_t^{3-i}\partial_x^2 u\\
&=-2C_3^1\alpha_0(x)\partial_t(\frac{x^2\Theta^2}{r^2r_x^3}a_3)\partial_t^2\partial_x^2 u-2C_3^2\alpha_0(x)\partial_t^2(\frac{x^2\Theta^2}{r^2r_x^3}a_3)\partial_t\partial_x^2 u\\
&-2C_3^3\alpha_0(x)\partial_t^3(\frac{x^2\Theta^2}{r^2r_x^3}a_3)\partial_x^2 u.
\end{aligned}
$$

Thus, we need the following estimates

$$
\begin{aligned}
\|\zeta\alpha_0^{\frac{3}{2}}\partial_t(\frac{x^2\Theta^2}{r^2r_x^3}a_3)\partial_t^2\partial_x^2 u\|_0^2 &\leq M_0\|\alpha_0\partial_t^2\partial_x^2 u(0)\|_0^2\|\sqrt{\alpha_0}\mathcal{H}_{t,x}^{1,0}\|_{L^\infty}^2\\
&+M_0\|\mathcal{H}_{t,x}^{1,0}\|_{L^\infty}^2\int_0^t\|\alpha_0^{\frac{3}{2}}\partial_t^3\partial_x^2 u(\tau)\|_0^2d\tau,\\
\|\zeta\alpha_0^{\frac{3}{2}}\partial_t^2(\frac{x^2\Theta^2}{r^2r_x^3}a_3)\partial_t\partial_x^2 u\|_0^2 &\leq M_0\|\sqrt{\alpha_0}\partial_t\partial_x^2 u(0)\|_0^2\|\alpha_0\mathcal{H}_{t,x}^{2,0}\|_{L^\infty}^2\\
&+M_0\|\sqrt{\alpha_0}\mathcal{H}_{t,x}^{2,0}\|_{L^\infty}^2\int_0^t\|\alpha_0\partial_t^2\partial_x^2 u(\tau)\|_0^2d\tau,
\end{aligned}
$$

and

$$\|\zeta\alpha_0^{\frac{3}{2}}\partial_t^3(\frac{x^2\Theta^2}{r^2r_x^3}a_3)u_{xx}\|_0^2$$

$$\leq M_0\|\alpha_0 u_{xx}(0)\|_{L^\infty}^2\|\zeta\sqrt{\alpha_0}\mathcal{H}_{t,x}^{3,0}\|_0^2+\|\alpha_0\mathcal{H}_{t,x}^{3,0}\|_{L^\infty}^2\int_0^t\|\sqrt{\alpha_0}\partial_t\partial_x^2u(\tau)\|_0^2d\tau. \tag{5.44}$$

On the other hand, we have form (5.40)

$$\|\zeta\alpha_0^{\frac{3}{2}}\partial_t^3\partial_x uu_x\|_0^2\leq\|\alpha_0u_2\|_{L^\infty}^2\|\sqrt{\alpha_0}\partial_t^3\partial_xu(0)\|_0^2+\|u_x\|_{L^\infty}^2\int_0^t\|\alpha_0\partial_t^4\partial_xu(\tau)\|_0^2d\tau,$$

$$\|\zeta\alpha_0^{\frac{3}{2}}\partial_t^2\partial_x^2uu_x\|_0^2\leq\|\sqrt{n_0}u_x\|_{L^\infty}^2\|\alpha_0\partial_t^2\partial_x^2u(0)\|_0^2+\|u_x\|_{L^\infty}^2\int_0^t\|\zeta\alpha_0^{\frac{3}{2}}\partial_t^3\partial_xu(\tau)\|_0^2d\tau, \tag{5.45}$$

and

$$\|\zeta\alpha_0^{\frac{3}{2}}u_{xx}u_{xt}u_x\|_0^2$$

$$\leq\|\sqrt{\rho_0}u_x\|_{L^\infty}^2\|\alpha_0u_{xx}(0)\|_{L^\infty}^2\|u_{xt}\|_0^2+\|\alpha_0u_{xt}\|_{L^\infty}^2\|u_x\|_{L^\infty}^2\int_0^t\|\sqrt{\alpha_0}\partial_t\partial_x^2u(\tau)\|_0^2d\tau.$$

Thus, we can handle the other terms of $\|\zeta\frac{S_3^3}{\alpha_0}\|_0^2$ and similarly prove (5.34). This completes the proof. □

5.3. Estimates of $E(t)$

In this subsection, we close the estimate in (2.29) as follows.

Lemma 5.3. *Suppose that (r, u) is a smooth solution to the problem (2.21) and (2.23) on $[0, 1]\times[0, T]$ with the property (3.1). Then, there exists a small time $0 < T_4 \leq T$ and a positive constant $\bar{c}_4 \geq \bar{c}$ (only depending on $\|(\rho_0, u_0)\|_{L^\infty}$) such that for any $t \in (0, T_4]$ and $c \geq \bar{c}_4$, the estimate in (2.29) holds.*

Proof. Due to (5.33) and (5.34), we can also prove the estimate for $\partial_t^2\partial_x^2u$, which is also bounded by the right side of (5.33), with the help of the fundamental theorem of calculus. Thus, we derive from (5.4), (5.5), (5.6), (5.33) and (5.34)

$$E(t) \le M_0\mathcal{P}_0 + M_0(\mathcal{P}(K)+1)\int_0^t (E^2(\tau)+E(\tau))d\tau$$
$$+ M_0(\mathcal{P}(K)+1)E(t)\int_0^t E(\tau)d\tau \tag{5.46}$$

However, we can choose $K \le \sup_{[0,t]} E(\tau)$, then we use the Gronwall inequality to obtain (2.29). □

6. Existence results

In this section, we construct the existence of smooth solutions to the problem (2.24) and (2.23) by applying the following degenerate parabolic regularization based on the priori estimate in (2.29):

$$\begin{aligned}
&x\alpha_0(x)a_1u_{tt} + \partial_t a_1 x\alpha_0(x)u_t - \partial_t\left(\frac{a_2x^2}{c^2r^2r_x^2}\alpha_0^2(x)uu_x\right)\\
&-2\partial_t\left(\frac{a_2x^2}{c^2r^2r_x}\alpha_0^2(x)u\frac{u}{x}\right) - 2\left[\alpha_0^2(x)\frac{x^2}{r^2r_x^2}a_3\left(\frac{\Theta^2}{r_x}u_x + \frac{x}{r}\Theta^2a_4\frac{u}{x} + \frac{1}{c^2}uu_t\right)\right]_x\\
&+2\frac{\alpha_0^2(x)}{x}\frac{x^3}{r^3r_x}a_3\left(\frac{a_4}{r_x}\Theta^2u_x + \frac{x}{r}(3+\frac{a_5}{c^2})\Theta^2\frac{u}{x} + \frac{1}{c^2}uu_t\right)\\
&-x^2\alpha_0(x)\partial_t\left(\frac{x^2}{r^2\Theta}\right)\Phi_0 = \mu\alpha_0^2(x)u_{xx} + \mu(2-\frac{a_5}{c^2})\alpha_0'(x)\alpha_0(x)u_x\\
&-\mu(2+\frac{a_5}{c^2})\alpha_0^2(x)\alpha_0'(x)\frac{u}{x}.
\end{aligned} \tag{6.1}$$

The existence of solutions of the regularized problem (6.1) and (2.23) can be obtained by the similar analysis in [7,18,38,39] using the fixed point theorem. Then the estimate of solutions independent of μ be similarly obtained by applying the Lemma 3.2 on the page 336 in [6] due to our estimates in (2.29), which confirms the convergence of subsequence for the solution of (6.1) to the solution of the relativistic Euler-Poisson equation (2.24) in the Langrangian form. We omit the details in here.

7. Uniqueness results

In this section, we prove the uniqueness of smooth solutions to the problem (2.21) and (2.23) given by Theorem 2.1.

Lemma 7.1. *Suppose that* (r, u) *is a smooth solution to the problem* (2.21), (2.23) *given by Theorem 2.1 satisfying*

$$\sup_{t\in[0,T]}\left\|\left(u_t, u_x, u_{tt}, n_0u_{xt}, n_0r_{xx}, \frac{u}{x}, \frac{\partial_t u}{x}\right)(t)\right\|_{L^\infty} \le K_0, \tag{7.1}$$

$$r_1 = x + \int_0^t u_1(x,\tau)d\tau,\ r_2 = x + \int_0^t u_2(x,\tau)d\tau, \tag{7.2}$$

with K_0 being a positive constant. Then, (r,u) is unique corresponding to (n_0,u_0).

Proof. We assume that (u_1,r_1) and (u_2,r_2) are two solutions of the problem (2.21) and (2.23) corresponding to (n_0,u_0) with the properties (7.1) and (7.2). In order to simplify notations, we denote by $\overline{n}_0 = \frac{1}{(1+\frac{n_0}{c^2})\Theta_0}$

$$\bar{a} := \overline{n}_0 \left(1 - \frac{1}{c^2}\overline{n}_0 n_0 \frac{\Theta}{r_x}(\frac{x}{r})^2\right)^{-1}\Theta.$$

Set

$$R := r_1 - r_2, \quad R_t = U := u_1 - u_2, \tag{7.3}$$

it follows from (2.21)

$$\begin{aligned}
&x\alpha_0\Big(a_1 u_{1t} - a_1 u_{2t}\Big) - \frac{1}{c^2}\alpha_0^2(x)\left[a_2\frac{x^2}{r_1^2 r_{1x}^2}u_1 u_{1x} - a_2\frac{x^2}{r_2^2 r_{2x}^2}u_2 u_{2x}\right]\\
&-2\frac{1}{c^2}\alpha_0^2(x)\left[a_2\frac{x^2}{r_1^2 r_{1x}}u_1\frac{u_1}{x} - a_2\frac{x^2}{r_2^2 r_{2x}}u_2\frac{u_2}{x}\right] + \left[\alpha_0^2(x)\left(\bar{a}^2\frac{x^2}{r_1^2 r_{1x}^2} - \bar{a}^2\frac{x^2}{r_2^2 r_{2x}^2}\right)\right]_x\\
&-2\frac{\alpha_0^2(x)}{x}\Big(\bar{a}^2\frac{x^3}{r_1^3 r_{1x}} - \bar{a}^2\frac{x^3}{r_2^3 r_{2x}}\Big) = 0,
\end{aligned}$$

which can be written as

$$\begin{aligned}
&x\alpha_0 a_1 U_t + \left[2\alpha_0^2(x)\left(G_2 R_x + G_3\frac{R}{x}\right)\right]_x + \frac{\alpha_0^2}{x}\Big(G_5 R_x + G_6\frac{R}{x}\Big) + P_1\Big(U, R_x, \frac{R}{x}\Big)\\
&= P_2\Big(U, R_x, \frac{R}{x}\Big)\Phi_0,
\end{aligned} \tag{7.4}$$

where we have used

$$\begin{aligned}
&\frac{x^2\bar{a}^2}{r_1^2 r_{1x}^2} - \frac{x^2\bar{a}^2}{r_2^2 r_{2x}^2} = G_1\cdot U - 2G_2\cdot R_x - 2G_3\cdot\frac{R}{x},\\
&G_1 := \int_0^1 \frac{\partial}{\partial u}\left(\frac{x^2\bar{a}^2}{r^2 r_x^2}(u_2 + \mu(u_1 - u_2), r_{1x})\right)d\mu,\\
&G_2 := \int_0^1 (\frac{x}{r_1})^2\frac{\overline{n}_0^2\Theta_1^2}{r_x^3\left(1 - \frac{1}{c^2}\overline{n}_0 n_0\frac{\Theta_1}{r_x}(\frac{x}{r_1})^2\right)^3}|_{r_x = r_{2x}+\mu(r_{1x}-r_{2x})}d\mu,
\end{aligned} \tag{7.5}$$

$$G_3 := \int_0^1 \frac{x^3\overline{n}_0^2\Theta_1^2\left(1+\frac{1}{c^2}\overline{n}_0 n_0\frac{\Theta_1}{r_{2x}}(\frac{x}{r})^2\right)}{r^3 r_{2x}^2\left(1-\frac{1}{c^2}\overline{n}_0 n_0\frac{\Theta_1}{r_{2x}}(\frac{x}{r})^2\right)^3}|_{r=r_2+\mu(r_1-r_2)}d\mu.$$

Similarly,

$$\frac{x^3\bar{a}^2}{r_1^3 r_{1x}} - \frac{x^3\bar{a}^2}{r_2^3 r_{2x}} = G_4\cdot U - G_5\cdot R_x - G_6\cdot\frac{R}{x},$$

$$G_4 := \int_0^1 \frac{\partial}{\partial u}\left(\frac{x^3\bar{a}^2}{r^3 r_x}(u_2+\mu(u_1-u_2), r_{1x})\right)d\mu,$$

$$G_5 := \int_0^1 (\frac{x}{r_1})^3\frac{\overline{n}_0^2\Theta_1^2\left(1+\frac{1}{c^2}\overline{n}_0 n_0\frac{\Theta_1}{r_x}(\frac{x}{r_1})^2\right)}{r_x^2\left(1-\frac{1}{c^2}\overline{n}_0 n_0\frac{\Theta_1}{r_x}(\frac{x}{r_1})^2\right)^3}|_{r_x=r_{2x}+\mu(r_{1x}-r_{2x})}d\mu, \tag{7.6}$$

$$G_6 := \int_0^1 \frac{x^4\overline{n}_0^2\Theta_1^2\left(3+\frac{1}{c^2}\overline{n}_0 n_0\frac{\Theta_1}{r_{2x}}(\frac{x}{r})^2\right)}{r^4 r_{2x}\left(1-\frac{1}{c^2}\overline{n}_0 n_0\frac{\Theta_1}{r_{2x}}(\frac{x}{r})^2\right)^3}|_{r=r_2+\mu(r_1-r_2)}d\mu.$$

From (7.1), there exist positive constants $C(K_0)$ such that

$$\sum_1^6 \|G_i\|_{L^\infty} \le C(K_0), \tag{7.7}$$

and $P_i\left(U, R_x, \frac{R}{x}\right)$ $(i=1,2)$ satisfy

$$\left|P_i\left(U, R_x, \frac{R}{x}\right)\right| \le C\left(|U|+|U_x|+|R_x|+|\frac{R}{x}|\right). \tag{7.8}$$

Multiplying (7.4) by $U := u_1 - u_2$ and integrating it over $(0,t)\times(0,1)$, then the integrating by parts and Cauchy-Schwarz inequality show

$$\begin{aligned}&\int_0^1 x\alpha_0(x)a_1(u_1,r_1)\frac{U^2}{2}dx + \int_0^1 \alpha_0^2(x)\left[2G_2R_x^2 + 2G_3\frac{R}{x}R_x + G_6\frac{R^2}{x^2}\right]dx\\ &\le C(K_0)\int_0^t\int_0^1 \alpha_0(x)U^2dxd\tau + C(K_0)\int_0^t\int_0^1 \alpha_0^2(x)\left(R_x^2+\frac{R^2}{x^2}+U_x^2\right)dxd\tau.\end{aligned} \tag{7.9}$$

Differentiating (7.4) with respect to t and multiplying the resulting equations by U_t, similar to (7.9), we have

$$\int_0^1 x\alpha_0(x)a_1(u_1,r_1)\frac{U_t^2}{2}dx+\int_0^1 \alpha_0^2(x)\left[G_2U_x^2+2G_3\frac{U}{x}U_x+G_6\frac{U^2}{x^2}\right]dx$$

$$\le -\int_0^t\int_0^1 \alpha_0^2(x)(G_5-G_3)\frac{U_t}{x}U_x dxd\tau + C(K_0)\int_0^1 \alpha_0^2(x)\left(R_x^2+\frac{R^2}{x^2}\right)dx$$

$$+C(K_0)\int_0^t\int_0^1 \alpha_0(x)\left(U^2+R^2\right)dxd\tau$$

$$+C(K_0)\int_0^t\int_0^1 \alpha_0^2(x)\left(R_x^2+\frac{R^2}{x^2}+U_x^2+\frac{U^2}{x^2}\right)dxd\tau. \tag{7.10}$$

A straightforward computation yields

$$|G_5-G_3|\le C(K_0)(|R|+|U|), \tag{7.11}$$

which implies

$$-\int_0^t\int_0^1 \alpha_0^2(x)(G_5-G_3)\frac{U_t}{x}U_x dxd\tau$$
$$\le C(K_0)\int_0^t\int_0^1 \alpha_0^2(x)\left(R_x^2+U^2+U_x^2+\frac{U^2}{x^2}\right)dxd\tau. \tag{7.12}$$

It is easy to obtain that there exist the positive constants T_1^* and $\varepsilon_{T_1^*}$ such that $0<T_1^*<T$ and for any $0<t\le T_1^*$,

$$\frac{1}{1+\varepsilon_{T_1^*}}\le\frac{x}{r_i}\le\frac{1}{1-\varepsilon_{T_1^*}},\qquad 1-\varepsilon_{T_1^*}\le r_{ix}\le 1+\varepsilon_{T_1^*},\qquad i=1,2, \tag{7.13}$$

where $\lim_{T_1^*\to 0}\varepsilon_{T_1^*}=0$. Thus, for any $r=r_{2x}+\mu(r_{1x}-r_{2x}),\ r=r_2+\mu(r_1-r_2)$

$$\frac{-2\varepsilon_{T_1^*}}{\left(1+\varepsilon_{T_1^*}\right)\left(1-\varepsilon_{T_1^*}\right)}+\frac{x}{r}\le\frac{x}{r_1}\le\frac{x}{r}+\frac{2\varepsilon_{T_1^*}}{\left(1+2\varepsilon_{T_1^*}\right)\left(1-\varepsilon_{T_1^*}\right)},$$
$$r_{2x}-\varepsilon_{T_1^*}\le r_x\le r_{2x}+2\varepsilon_{T_1^*}.$$

From (7.5), $G_2\ge\mathcal{F}(\mu,\varepsilon_{T_1^*})$, where

$$\mathcal{F}(\mu,\varepsilon_{T_1^*}) := \int_0^t \left(\frac{x}{r} - \frac{-2\varepsilon_{T_1^*}}{\left(1+\varepsilon_{T_1^*}\right)\left(1-\varepsilon_{T_1^*}\right)}\right)^2 \frac{\overline{n}_0^2\Theta_1^2}{\left(r_{2x}+2\varepsilon_{T_1^*}\right)^3}$$
$$\times \frac{1}{\left[1-\frac{1}{c^2}\overline{n}_0 n_0 \frac{\Theta_1}{(r_{2x}+2\varepsilon_{T_1^*})}\left(\frac{x}{r} - \frac{2\varepsilon_{T_1^*}}{\left(1+\varepsilon_{T_1^*}\right)\left(1-\varepsilon_{T_1^*}\right)}\right)^2\right]^3}\Bigg|_{r=r_2+\mu(r_1-r_2)} d\mu,$$

then we can obtain

$$\lim_{T_1^*\to 0} F(\mu,\varepsilon_{T_1^*}) = G_2^*,$$

with

$$G_2^* = \int_0^1 \left(\frac{x}{r}\right)^2 \frac{\overline{n}_0^2\Theta_1^2}{r_{2x}^3\left(1-\frac{1}{c^2}\overline{n}_0 n_0 \frac{\Theta_1}{r_{2x}}\left(\frac{x}{r}\right)^2\right)^3}\Bigg|_{r=r_2+\mu(r_1-r_2)} d\mu. \tag{7.14}$$

Then, there exists a positive constant T_2^* such that $0 < T_2^* < T_1^*$ for any $0 < t \le T_2^*$,

$$G_2 U_x^2 + 2G_3\frac{U}{x}U_x + G_6\frac{U^2}{x^2} \ge G_2^* U_x^2 + 2G_3 \cdot \frac{U}{x}U_x + G_6\frac{U^2}{x^2}, \tag{7.15}$$

which together with (7.5) and (7.6) shows there exist the positive constants $c_1^* \ge \overline{c}$ and M such that

$$G_2^* U_x^2 + 2G_3 \cdot \frac{U}{x}U_x + G_6\frac{U^2}{x^2} \ge M\left(U_x^2 + \frac{U^2}{x^2}\right),$$

with the help of Young's inequality.

Finally, we have for any $0 < t < \min\{T_1^*, T_2^*\}$

$$G_2 U_x^2 + 2G_3\frac{U}{x}U_x + G_6\frac{U^2}{x^2} \ge M\left(U_x^2 + \frac{U^2}{x^2}\right). \tag{7.16}$$

Due to (7.7)-(7.10) and (7.16), we can obtain $U = R = 0$ by using Gronwall's inequality. The proof of Lemma 7.1 is completed. □

8. Non-relativistic limits

This section is to devoted to investigating the non-relativistic limit of smooth solutions to the problem (2.21) and (2.23) obtained in Theorem 2.1.

Lemma 8.1. *Suppose that (r^c, u^c) is a smooth solution to the problem (2.21) and (2.23) satisfying Theorem 2.1 for any $(x,t) \in [0,1]\times[0,T_{c_0}]$ and (r,u) is a smooth solution of the problem (2.24) and (2.23) satisfying (2.29) for any $(x,t) \in [0,t]\times[0,T_0]$. Then, there exists a positive constant*

T_0^* *such that* $0 < T_0^* \le \min\{T_{c_0}, T_0\}$, *and for any* $0 < t < T_0^*$ *and* $c \ge c_0$ *the estimate in* (2.30) *holds.*

Proof. Based on the uniform estimate in (2.29) with respect to c, the standard Sobolev embedding shows that there exists a subsequence (r^c, u^c), which converges to (r, u).

We only establish the convergence rate estimate in (2.30). Set $R_t^c = U^c = u^c - u$, $R^c = r^c - r$. Subtracting (2.21) from (2.24) and using Taylor's expansion, it concludes that

$$\begin{aligned}
&x\alpha_0(x)U_t^c - \left\{\alpha_0^2(x)\left[(\frac{x}{r})^2\frac{1}{(r_x^c)^2}\frac{(r^c+r)}{r^c}\frac{x}{r^c}\frac{R^c}{x} + (\frac{x}{r})^2\frac{(r_x^c+r_x)}{(r_x^c r_x)^2}R_x^c + O(\frac{1}{c^2})\right]\right\}_x \\
&+ \frac{\alpha_0^2(x)}{x}(\frac{x}{r})^3\frac{1}{r_x^c}\frac{(r^2+r^c r+(r^c)^2)}{(r^c)^2}\frac{x}{r^c}\frac{R^c}{x} + \frac{\alpha_0^2(x)}{x}(\frac{x}{r})^3\frac{1}{r_x^c r_x}R_x^c + \left(\alpha_0^2(x) + \frac{\alpha_0^2(x)}{x}\right)O(\frac{1}{c^2}) \\
&= x^2\alpha_0^2(x)\frac{x^2}{rr^c}(r^c+r)R^c\frac{1}{x^3}\int_0^x \rho_0(y)y^2dy + O(\frac{1}{c^2}).
\end{aligned} \tag{8.1}$$

Multiplying (8.1) by U^c and integrating it over (0, 1), it holds that

$$\begin{aligned}
&\frac{d}{dt}\int_0^1 x\alpha_0(x)\frac{(U^c)^2}{2}dx + \frac{d}{dt}\int_0^1 \alpha_0^2(x)(\frac{x}{r})^2\frac{(r_x^c+r_x)}{(r_x^c)^2 r_x^2}R_x^c dx \\
&+ \frac{d}{dt}\int_0^1 \alpha_0^2(x)(\frac{x}{r})^2\frac{1}{(r_x^c)^2}\frac{(r^c+r)}{r^c}\frac{x}{r^c}\frac{R^c}{x}R_x^c dx \\
&+ \frac{d}{dt}\int_0^1 \frac{\alpha_0^2(x)}{x}(\frac{x}{r})^3\frac{1}{r_x^c}\frac{(r^2+r^c r+(r^c)^2)}{(r^c)^2}\frac{x}{r^c}\frac{(R^c)^2}{x^2}dx \\
&\le C(K_0)\int_0^1 \alpha_0^2(x)\left((U^c)^2 + (R^c)^2\right)dx \\
&+ C(K_0)\int_0^1 \alpha_0^2(x)\left((R_x^c)^2 + (\frac{R^c}{x})^2\right)dx + O(\frac{1}{c^4}).
\end{aligned} \tag{8.2}$$

By the analogy with (7.13), there exists a positive constant T_0^* such that $0 < T_0^* \le \min\{T_{c_0}, T_0^*\}$, and for any $0 < t < T_0^*$,

$$\frac{1}{1+\varepsilon_{T_0^*}} \le \frac{x}{r},\ \frac{x}{r^c} \le \frac{1}{1-\varepsilon_{T_0^*}}, \qquad 1-\varepsilon_{T_0^*} \le r_x,\ r_x^c \le 1+\varepsilon_{T_0^*}. \tag{8.3}$$

For small T_0^* and large enough c, there exists a positive constant $C(K_0)$ such that

$$
\begin{aligned}
&(\frac{x}{r})^2\frac{(r_x^c+r_x)}{(r_x^c)^2r_x^2}(R_x^c)^2+(\frac{x}{r})^2\frac{1}{(r_x^c)^2}\frac{(r^c+r)}{r^c}\frac{x}{r^c}\frac{R^c}{x}R_x^c\\
&+(\frac{x}{r})^3\frac{1}{r_x^c}\frac{(r^2+r^cr+(r^c)^2)}{(r^c)^2}\frac{x}{r^c}\frac{(R^c)^2}{x^2}\ge C(K_0)\left((R_x^c)^2+(\frac{R^c}{x})^2\right). \qquad (8.4)
\end{aligned}
$$

Then, we can obtain

$$
\int_0^1 x\alpha_0(x)(U^c)^2dx+\int_0^1\alpha_0^2\left((R_x^c)^2+(\frac{R^c}{x})^2\right)dx\le O(\frac{1}{c^4}).
$$

Finally, we can prove (2.30) with the help of the weighted embedding estimate in (1.7)-(1.8) and $H^1(0,1)\hookrightarrow C^0[0,1]$. This completes the proof. $\square$

Acknowledgments

The research of L. Mai was partially supported by the National Natural Science Foundation of China (No. 11601246, and No. 11971014), the Program of Higher Level Talents of Inner Mongolia University (No. 21100-5165105), and the Young Science and Technology Talents Cultivation Project of Inner Mongolia University (No. 21221505). The research of M. Mei was partially supported by NSERC grant RGPIN 354724-2016 and FRQNT grant 2019-CO-256440.

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