McGill University Nov. 3, 2016

Midterm examination (Solutions)

Advanced Calculus for Engineers

Math 264

Nov. 3, 2016

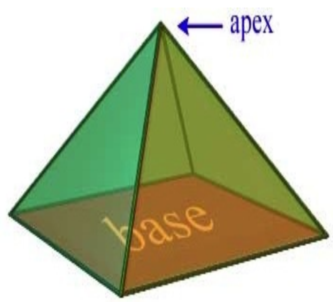
Time: 14:35 - 16:00

Prof. R. Choksi

Student name (last, first)	**Student number (McGill ID)**

INSTRUCTIONS

1. You are allowed one double-sided sheet of information on 8.5 x 11 inches paper.
2. Calculators are NOT permitted.
3. The exam is 85 minutes.
4. Make sure you READ CAREFULLY the question before embarking on the solution.
5. Show your work.
6. This exam comprises 6 pages (including the cover page). Please provide all your answers on this exam.



Problem	1	2	3	4	5	Total
Mark						
Out of	10	10	10	10	10	50

Question 1a (5 pts) Consider a 2D force field defined by

$$\mathbf{F}(x, y) \;=\; \langle\, 2x \,,\, 2y \,\rangle\,.$$

Find the work done by the force field in moving a particle along the curve

$$\mathbf{r}(t) \;=\; \left\langle 2t, \sqrt{t^2+1}\right\rangle, \qquad \text{from } t = 0 \text{ to } t = 1.$$

Solution: This field is conservative. The potential is $\phi(x, y) = x^2 + y^2$. The curve goes from the point $(0, 1)$ to $(2, \sqrt{2})$. Hence the work done is

$$\phi(2, \sqrt{2}) - \phi(0, 1) \;=\; 4 + 2 - 1 \;=\; 5.$$

Question 1b (5 pts) Let $\mathbf{F} = \langle\, xy \,,\, yz \,,\, xz\rangle$ Let $\mathcal{S}$ be the part of the cylinder $x^2 + z^2 = 1$ which lies above the xy plane and between the planes $y = -1$ and $y = 1$. Orient the surface with the outer normal. Write down a double iterated integral with respect to x and y which gives

$$\iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S}.$$

Solution: Best to parametrize the surface with x and y, and view this as the graph of the level set of $G(x, y, z) = x^2 + z^2 - 1 = 0$ which lies above the square in the xy plane: $-1 \le x \le 1$ and $-1 \le y \le 1$. Note that on the surface we can solve for z in terms of x and y, indeed, $z(x, y) = \sqrt{1 - x^2}$. Hence

$$\begin{aligned}
\iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S} \;&=\; \int_{-1}^{1}\int_{-1}^{1} \mathbf{F}(x, y, z(x, y)) \cdot \frac{\nabla G}{G_3}\, dx\, dy. \\
&=\; \int_{-1}^{1}\int_{-1}^{1} \left\langle\, xy \,,\, y\sqrt{1-x^2} \,,\, x\sqrt{1-x^2}\right\rangle \cdot \frac{\left\langle\, 2x \,,\, 0 \,,\, 2\sqrt{1-x^2}\right\rangle}{2\sqrt{1-x^2}}\, dx\, dy \\
&=\; \int_{-1}^{1}\int_{-1}^{1} \frac{x^2 y \,+\, x(1-x^2)}{\sqrt{1-x^2}}\, dx\, dy.
\end{aligned}$$

Note here we chose the positive sign for $\frac{\nabla G}{G_3}$ as this points upwards.

Question 2 Suppose a bug lives in the xy plane. Suppose that at all times, the temperature at any point (x, y) in the xy plane is given by

$$T(x, y) = x + \frac{y^2}{2}.$$

The bug always travels at a constant speed and starts at the point $(1, 1)$.
a) (5 pts) Suppose the bug moves in such a way as to **get warmest the fastest**. On what curve does the bug move? Describe the curve via a precise equation involving x and y.

Solution: The bug will move in the direction of the gradient

$$\nabla T(x, y) = \langle 1 \, , \, y \rangle \, .$$

Recall from class, to find curves whose tangent is parallel to this vector field we solve

$$\frac{dy}{dx} = \frac{y}{1} = y.$$

These curves have the form

$$y = Ce^x,$$

for some constant C. The curve which contains the point $(1, 1)$ has constant $C = \frac{1}{e}$. Thus we move along the curve

$$y = \frac{1}{e}e^x = e^{x-1},$$

in the direction of increasing x.

b) (5 pts) Suppose the bug moves in such a way that its temperature **does not change**. On what curve does the bug move? Describe the curve via a precise equation involving x and y.

Solution: $T(1, 1) = \frac{3}{2}$. We move on the curve defined by the equation

$$T(x, y) = \frac{3}{2}.$$

This is the curve

$$x + \frac{y^2}{2} = \frac{3}{2}.$$

Question 3 (10 points) a) (5 pts) Suppose the velocity of the fluid particle at any point (x, y, z) is given by

$$\mathbf{v}(x, y, z) = \langle 2, 5, 7 \rangle.$$

The unit of length for x,y, and z is meters, and the units of each component of $\mathbf{v}$ are in meters per second. Suppose the fluid has a constant density of 100 kg per meter^3. Let $\mathcal{S}$ be the inside of the triangle in the yz plane with vertices at $(0,0,0), (0,1,0), (0,0,1)$. What is the mass of fluid which flows through $\mathcal{S}$ in the $\mathbf{i}$ direction in a 10 second interval?

Solution: We first find the fluid flow rate

$$\iint_{\mathcal{S}} \mathbf{v} \cdot d\mathbf{S}$$

which gives the volume per unit time of fluid flowing through $\mathcal{S}$ in the $\mathbf{i}$ direction at any time. This is easy, it is simply 2 times the area of the triangle which is 1/2. Hence it is 1. Then we multiply by 100 kg/meter^3 to the mass per unit time, and finally multiply by 10 seconds. Hence the answer is 1000 kg.

b) (5 pts) Now suppose the velocity of the fluid particle at any point (x, y, z) depends on time as well and is given by

$$\mathbf{v}(x, y, z, t) = \langle 2t, 5t, 7t \rangle.$$

Suppose again the units of each component of $\mathbf{v}$ are in meters per second (this means the constants have units as well). What is the mass of fluid which flows through $\mathcal{S}$ (given in part a) in the $\mathbf{i}$ direction from $t = 1$ to $t = 10$ seconds.

Solution: Now we find that the fluid flow rate

$$\iint_{\mathcal{S}} \mathbf{v} \cdot d\mathbf{S}$$

depends on time. It is t meters^3 per second. We again multiply by the density 100 kg per meter^3 to find that the rate of mass per unit time is $100t$ kg/s. Finally we must integrate this from $t = 0$ to $t = 10$. That is, the mass of fluid which flows through $\mathcal{S}$ (given in part a) in the $\mathbf{i}$ direction from $t = 1$ to $t = 10$ seconds is

$$100 \int_1^{10} t\, dt = 100 \left(\frac{100}{2} - \frac{1}{2} \right) = 5000 - 50 = 4950 \text{ kg}.$$

Question 4 (10 pts) Let

$$\mathbf{F} = \langle z\,,\ \sqrt{x^2+y^4+z^2}\,,\ xy\rangle.$$

Consider the **boundary of the pyramid** with **base vertices** $(1,0,1),(1,0,-1),(-1,0,1),(-1,0,-1)$ lying in the xz plane and its **apex** point at $(0,10,0)$. I included a picture of a pyramid on the cover but there the base appears to be in the xy plane: Now the base lies in the xz plane. Let $\mathcal{S}$ be the **part** of this boundary which **does not include the base** (i.e. the square in the xz plane). Orient $\mathcal{S}$ with the outer normal. Use Stokes' Theorem to evaluate

$$\iint_{\mathcal{S}} \operatorname{curl}\mathbf{F}\cdot d\mathbf{S}.$$

Solution: By Stokes' Theorem this flux is

$$\int_{\mathcal{C}} \mathbf{F}\cdot d\mathbf{r},$$

where $\mathcal{C}$ is the base square in the xz plane oriented couterclockwise from the perspective of the apex point. This square has four sides. On the sides where x is fixed (either ± 1) but z changes form -1 to 1, the integrals are zero since in both cases $d\mathbf{r}$ (which points in the z direction) and on these line segments, the third component of $\mathbf{F}$ is 0.
On **each** side where z is fixed (either ± 1) but x changes from -1 to 1, we find that the line integral is simply 2. Hence

$$\int_{\mathcal{C}} \mathbf{F}\cdot d\mathbf{r} = 4.$$

Though since I also got confused with the orientation when I first did the problem, I accepted for full credit -4 as well!!!

Question 5 (10 pts) Suppose $\mathbf{F}$ is a 3D vector field for which the following two pieces of information are given:

1. $\mathbf{F}(x, y, z) = \langle x, y, z\rangle \qquad$ if $2 \leq \sqrt{x^2 + y^2 + z^2} \leq 7$.
2. The flux of $\mathbf{F}$ **out** of the sphere radius 3 centred at the origin **is** 8π.

From this information alone **find the flux out the sphere of radius** 5 **centred at the origin**. That is, evaluate

$$\iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S},$$

where $\mathcal{S}$ is the sphere of radius 5 centred at the origin oriented with the outer normal. Hint: use the Divergence Theorem.

Solution: Here are two ways to solve the problem. They give different answers!!! I accepted both answers for full credit and gave extra credit if there was any discussion of what happened. I will give you both answers and then explain why the problem yielded two answers – in other words, why there was "a problem with the problem!"

Answer 1: Here I could ignore the hint and only use the first piece of information since I give you the flux on and around the sphere of radius 5. This is one of the easy integrals we talked about: since $\mathbf{F} \cdot \mathbf{N} = 5$ on the sphere the flus is simply 3 times the area of the sphere:

$$5 \times 4\pi 25 \; = \; 500\pi.$$

However: note that you can **NOT** use the Divergence Theorem with only first piece of information as you only know the divergence in the annular region $2 \leq \sqrt{x^2 + y^2 + z^2} \leq 7$.

Answer 2: Let $\mathcal{S}$ be the sphere of radius 5 centered at the origin oriented with the outer normal. Note that

$$\iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S} \; = \; \iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S} + (\text{flux } \textbf{into} \text{ the sphere of radius 3}) + (\text{flux } \textbf{out} \text{ of the sphere of radius 3}).$$

From information 2, the flux out of the sphere of radius 3 is 8π. On the other hand, by the Divergence Theorem

$$\iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S} \; + \; (\text{flux } \textbf{into} \text{ the sphere of radius 3}) \; = \; \iiint_{\text{annular region}} \operatorname{div} \mathbf{F} \, dxdydz.$$

In the annular region, the divergence is the constant 3. Hence the integral over the annular region is 3 times its volume,

$$3\left(\frac{4\pi}{3}5^3 \; - \; \frac{4\pi}{3}3^3\right) \; = \; 4\pi(125 - 27) \; = \; (4 \times 98)\pi.$$

Thus the answer is $8\pi + (4 \times 98)\pi = 400\pi$

So what happened?: First off, the first piece of information should have been given from $3 < \sqrt{x^2 + y^2 + z^2} \leq 7$ so as to not contradict the second piece of information. But even then we would have gotten two different answers. This was because by joining two pieces of information, we invariably introduced discontinuities in the vector fields. These discontinuities mean that we can not piece the two together the way we did in the first equation of Answer 2. For that to work, we need $\mathbf{F}$ to be continuous across the sphere of radius 3. So to conclude, Answer 1 is correct BUT given the hint, I accepted both even if you did not see the issue with Answer 2.