McGill University April 16, 2014
Faculty of Science Final examination

Advanced Calculus for Engineers

Math 264

April 16, 2014

Time: 6PM-9PM

Examiner: Prof. R. Choksi Associate Examiner: Prof. A. Hundemer

Student name (last, first)	**Student number (McGill ID)**

Here I include solutions to the first 7 problems – the last three were discussed last class.

A paddle wheel illustration for Question 7:

Problem	1	2	3	4	5	6	7	8	9	10	Total
Mark											
Out of	10	10	10	12	10	10	8	10	10	10	100

Question 1a (5 pts):
Find the flux of the vector field $\mathbf{F} = \langle x, y, z\rangle$ through the surface which is **the part of the paraboloid** $z = x^2 + y^2$ which lies **below** the plane $z = 4$ (oriented with the upward normal). **Leave** your answer as an iterated double integral either with respect to x and y **or** polar coordinates r and θ.

Solution: Use x and y to parametrize the surface which is simply the graph of $z = g(x, y) = x^2+y^2$ which lies above the disc D in the xy plane centred at the origin of radius 2: $x^2 + y^2 \leq 4$. Hence

$$\iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S} \;=\; \iint_D \langle x, y, x^2 + y^2\rangle \cdot \left\langle -\frac{\partial g}{\partial x}, -\frac{\partial g}{\partial y}, 1\right\rangle \, dx\, dy,$$

noting here that I have chosen the sign in the normal vector to point upwards. Taking into account the form of g gives

$$\iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S} \;=\; \iint_D \langle x, y, x^2 + y^2\rangle \cdot \langle -2x, -2y, 1\rangle \; dx\, dy \;=\; \iint_D -x^2 - y^2 \, dx\, dy.$$

Since the disc is easier to write in polar coordinates, we use polar and find

$$\iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S} \;=\; \int_0^{2\pi} \int_0^2 -r^2 \, r dr \; d\theta \;=\; \int_0^{2\pi} \int_0^2 -r^3 \, dr \; d\theta$$

Question 1b (5 pts): Find the work done by the force field $\mathbf{F} = \langle y, z, x\rangle$ on moving a particle along the curve

$$\mathbf{r}(t) \;=\; \langle t, t^2, t^3\rangle, \qquad 0 \leq t \leq 1.$$

Solution: We want

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} \;=\; \int_0^1 \langle t^2, t^3, t\rangle \cdot \langle 1, 2t, 3t^2\rangle \, dt \;=\; \int_0^1 t^2 + 2t^4 + 3t^3 \, dt.$$

Question 2a (5 pts). Write down a double iterated integral with respect to x and y which gives the surface area of the top half of the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

Do not evaluate the integral.

Solution: The surface $\mathcal{S}$ can be described as the graph of the part of the level set

$$G(x,y,z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 = 0,$$

which lies above the elliptical region E of the xy plane which lies inside the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Hence we can use x and y to parameterize the surface and use the fact that surface area is

$$\iint_{\mathcal{S}} 1\, dS = \iint_E \frac{|\nabla G|}{|G_3|}\, dy\, dx = \int_{-a}^{a} \int_{-\frac{b}{a}\sqrt{a^2-x^2}}^{\frac{b}{a}\sqrt{a^2-x^2}} \frac{|\nabla G|}{|G_3|}\, dy\, dx.$$

Now we compute

$$|\nabla G| = \sqrt{\frac{4x^2}{a^4} + \frac{4y^2}{b^4} + \frac{4z^2}{c^4}} \qquad |G_3| = \frac{2z}{c^2},$$

and note that on the surface

$$z = c\sqrt{1 - \frac{x^2}{a^2} + \frac{y^2}{b^2}}.$$

Thus in terms of x and y the integrand is

$$\frac{|\nabla G|}{|G_3|} = c^2 \frac{\sqrt{\frac{4x^2}{a^4} + \frac{4y^2}{b^4} + \frac{4c^2(1-\frac{x^2}{a^2}+\frac{y^2}{b^2})}{c^4}}}{2c\sqrt{1 - \frac{x^2}{a^2} + \frac{y^2}{b^2}}} = c\sqrt{\frac{1}{1 - \frac{x^2}{a^2} + \frac{y^2}{b^2}} \left(\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{1 - \frac{x^2}{a^2} + \frac{y^2}{b^2})}{c^2} \right)}.$$

Yikes!!! Poor students in 2014!

Question 2b (5 pts). Write down an example of a **conservative vector field** for which the previous ellipsoid is an **equipotential surface**.

Solution: Let

$$\phi(x,y,z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1$$

and take

$$\mathbf{F} = \nabla\phi = \left\langle \frac{2x}{a^2}, \frac{2y}{b^2}, \frac{2z}{c^2} \right\rangle.$$

Question 3a (5 pts). Let $\mathcal{S}$ be the **boundary surface**, oriented with the outer normal, **of the region** V which lies above the xy plane (i.e. $z \geq 0$) and is **bounded by the two planes and cylinder**:

$$y = 0, \qquad y = 2, \qquad x^2 + z^2 = 1.$$

Let

$$\mathbf{F} = \langle x + \cos y, y + \sin z, z + e^x \rangle.$$

Evaluate the flux out of V, i.e. the flux integral

$$\iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S}.$$

Solution: Draw a picture! This is a prime example where we want to use the divergence theorem which states that

$$\iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S} \;=\; \iiint_V \text{div } \mathbf{F} \, dx\, dy\, dz.$$

Since div $\mathbf{F} = 3$ (a constant) and the volume of V is just one half the volume inside the cylinder piece (which is $\pi\, 1^2\, 2 = 2\pi$), the answer is simply 3π.

Question 3b (5 pts) Suppose a smooth vector field $\mathbf{F}$ has divergence equal to 3 at the origin, i.e. div $\mathbf{F}(0,0,0) = 3$. Use only this information to approximate the flux of $\mathbf{F}$ out of the sphere centred at the origin of radius 2.

Solution: The divergence gives a measure of the instantaneous flux out per unit volume. Hence an approximation to the total flux out of the sphere would be 3 times the volume of the sphere, i.e.

$$3\left(\frac{4}{3}\,\pi\, 2^3\right) \;=\; 32\pi.$$

Alternatively and equivalently we could use the divergence theorem to argue that

$$\iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S} \;=\; \iiint_V \text{div } \mathbf{F} \, dx\, dy\, dz \;\approx\; \iiint_V \text{div } \mathbf{F}(0,0,0) \, dx\, dy\, dz \;=\; \iiint_V 3 \, dx\, dy\, dz \;=\; 32\pi.$$

Question 4a (8 pts). Use the change of variables

$$u = x + y \qquad v = x - 2y$$

to **write the integral**

$$\iint_R (2x - y)\, dx\, dy,$$

where R is the trapezoidal region in the xy plane with boundary points $(0, -1), (0, -2), (2, 0), (4, 0)$, **as an iterated integral** with respect to u and v. Make sure you clearly state the limits of integration.

Solution: First draw a picture!!!!

- Integrand in terms of u, v is $u + v$.
- Region of integration transforms to $2 < v < 4, -\frac{v}{2} < u < v$.
- Jacobian is $\frac{1}{3}$.

Note that to find the transformed region of integration: we looked at how the boundary curves of R transform under the change of variables. This is what we did in class. In this problem, you could alternatively look at the images of the four boundary points of R, and then connect the lines. However, this would work because the change of variables was **linear**. If it was not linear, you would have to look at how the boundary curves transform (as they would not be line segments).

Thus

$$\iint_R (2x - y)\, dx\, dy \;=\; \int_2^4 \int_{-\frac{v}{2}}^{v} (u + v)\; \frac{1}{3}\, du\, dv.$$

Question 4b (4 pts). Verify the **Divergence Theorem** for

$$\mathbf{F} = \langle x, y, z \rangle,$$

with D the solid ball centred at the origin of radius 1 and $\mathcal{S}$ its boundary surface (the unit sphere).

Solution:

$$\iiint_D \text{div } \mathbf{F}\, dx\, dy\, dz \;=\; \iiint_D 3\, dx\, dy\, dz \;=\; 3\, Vol(D) \;=\; 4\pi.$$

On the other hand, the flux integral is easy to do (we talked about this in class many times). Why? Because at any point of the sphere, $\mathbf{F}$ is normal to the sphere and hence its normal component is just its length which in this case in the constant 1. Thus

$$\iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S} \;=\; \iint_{\mathcal{S}} \mathbf{F} \cdot \mathbf{N}\, dS \;=\; 1 \times Area(\mathcal{S}) = 4\pi.$$

Question 5b (7 pts). Use **Green's Theorem to find the area** of the region which lies inside both circles

$$x^2 + y^2 = 4 \qquad \text{and} \qquad (x-2)^2 + y^2 = 4.$$

Leave your answer as a single definite integral (or sum of two integrals) with respect to parameter t.

Hint: first **draw a picture**. For the intersection points of the circles, consider the triangle formed by one of these intersection points and the centres of the circles. What type of triangle is it?

Solution: We apply Green's Theorem with vector field $\langle P, Q\rangle = \langle 0, x\rangle$. The region has a boundary consisting of two pieces of circles – DRAW A PICTURE! Call them $\mathcal{C}_1$ (piece of circle centred at the origin) and $\mathcal{C}_2$ (piece of the other circle) and note that they are oriented counterclockwise. This means

$$Area = \iint_{region} 1\,dx\,dy \;=\; \int_{\mathcal{C}_1} x\,dy \;+\; \int_{\mathcal{C}_2} x\,dy.$$

We can parametrize $\mathcal{C}_1$ with

$$\langle 2\cos t, 2\sin t\rangle \qquad -\frac{\pi}{3} \le t \le \frac{\pi}{3}.$$

We can $\mathcal{C}_2$ with

$$\langle 2 + 2\cos t, 2\sin t\rangle \qquad \frac{2\pi}{3} \le t \le \frac{4\pi}{3}.$$

Hence the area of the region is

$$\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} 4\cos^2 t\,dt \;+\; \int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} 4\cos t\,(1+\cos t)\,dt.$$

Question 6a (7 pts). Consider the unit cube with vertices (corner points)

$$(0,0,0),\ (0,1,0),\ (1,0,0),\ (1,1,0),\ (0,0,1),\ (0,1,1),\ (1,0,1),\ (1,1,1).$$

Let $\mathcal{S}$ be the **boundary of the cube minus (i.e. not including) the bottom square** (the side which lies in the xy plane). Orient $\mathcal{S}$ with the normal which points out of the cube. Let

$$\mathbf{F} = \big\langle -y\,,\, x\,,\, y^2e^x \big\rangle.$$

Evaluate

$$\iint_{\mathcal{S}} (\text{curl } \mathbf{F}) \cdot d\mathbf{S}.$$

Make sure you carefully compute the curl. **Hint:** Use Stokes' Theorem twice to write the flux integral as an equal flux integral but over a different surface.

Solution: First we compute the curl

$$\text{curl } \mathbf{F} = \langle 2ye^x, -y^2e^x, 2\rangle.$$

Next we note that using Stokes Theorem twice, we have

$$\iint_{\mathcal{S}} (\text{curl } \mathbf{F}) \cdot d\mathbf{S} = \iint_{D} (\text{curl } \mathbf{F}) \cdot d\mathbf{S},$$

where D is the inside of the square in the xy plane which is the boundary of $\mathcal{S}$. But note that here D is oriented with upward normal. Hence

$$\iint_{D} (\text{curl } \mathbf{F}) \cdot d\mathbf{S} = \iint_{D} (\text{curl } \mathbf{F}) \cdot \langle 0,0,1\rangle \cdot dS == \iint_{D} 2 \cdot dS = 2 \times Area(D) = 2.$$

Question 6b (3 pts). Let $f(x,y,z) = x^2 + y^3 + z^2$ and let $\mathcal{C}$ be the helix with parametrization

$$\mathbf{r}(t) \;=\; \langle 2\cos t, 2\sin t, t^2\rangle, \qquad t \in [0,\pi].$$

Find

$$\int_{\mathcal{C}} (\nabla f)\cdot d\mathbf{r}.$$

Solution: Easy!!!!!

$$\int_{\mathcal{C}} (\nabla f)\cdot d\mathbf{r} \;=\; f(\mathbf{r}(\pi)) \;-\; f(\mathbf{r}(0)) \;=\; f(-2,0,\pi^2) \;-\; f(2,0,0) \;=\; \pi^4.$$

Question 7 (8 pts total). Consider the following experiment associated with a fluid moving in space. Two paddle wheels of radius 1 lying in the xy plane are placed with centres at two different points. In all cases, the centre is fixed but the wheel is free to turn (depending on what the fluid is doing). For each spatial velocity vector field $\mathbf{v}$ of the fluid and choices of centres for the paddle wheels, decide whether or not each paddle wheel will turn, and if so which way (clockwise or counter clockwise in the xy plane).

a) $\mathbf{v} = \langle x, y, z\rangle$ and centres at $(x, y) = (0, 0)$ and at $(x, y) = (4, 4)$.

Solution: curl $\mathbf{v} = \mathbf{0}$ at all points. Hence neither of the paddles will turn.

b) $\mathbf{v} = \langle y, -x, 0\rangle$ and centres at $(x, y) = (0, 0)$ and at $(x, y) = (4, 4)$.

Solution: curl $\mathbf{v} = \langle 0, 0, -2\rangle = -2\mathbf{k}$ at all points. Hence both paddles turn clockwise.

c)

$$\mathbf{v} = \left\langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}, 0 \right\rangle \qquad \text{and centres at } (x, y) = (0, 0) \text{ and at } (x, y) = (4, 4).$$

Solution: curl $\mathbf{v} = \mathbf{0}$ at all points except the origin (where it is all concentrated). Hence the paddle centred at $(4, 4)$ will not turn. On the other hand the paddle centred at $(0, 0)$ spins counterclockwise (saw this in class a while back).

Question 8 (10 pts total). Consider the function $\phi(x) = x^2 + 1$.
a) What are the Fourier sine series and the Fourier cosine series of $\phi(x)$ on the interval $x \in (0, 1)$? Write down these series but leave your coefficients as integrals.

b) At the point $x = 0$, what number does the Fourier sine series converge to? What number does the Fourier cosine series converge to? In other words, if we take more and more terms in the sum and evaluate at $x = 0$, what number will we get close to?

c) For each series, plot the function that the Fourier series will get closer to (converge to) on the interval $x \in (-2, 2)$.

Question 9a (5 pts). What are all the positive eigenvalues and corresponding eigenfunctions of the BVP for $y(x), x \in [0,1]$:

$$y'' + \lambda y = 0 \qquad y'(0) = 0 \quad y(1) = 0.$$

Read the above line carefully, i.e. note the prime (derivative) in one of the boundary conditions.

Question 9b (5 pts). Consider the BVP for the heat equation:

$$u_t = u_{xx} \qquad \text{for } x \in (0,1),\ t \geq 0$$
$$u_x(0,t) = 0 \qquad u(1,t) = 10 \qquad \text{for } t > 0,$$
$$u(x,0) = x^2 \qquad \text{for } x \in [0,1].$$

What is a physical model for which this BVP applies, i.e. **a physical interpretation** of u, the domain $x \in [0,1]$, the condition $u(x,0) = x^2$, and the boundary conditions.

Question 10 (10 pts). Consider the BVP from the previous question:

$$
\begin{gathered}
u_t = u_{xx} \qquad \text{for } x \in (0,1),\ t \geq 0 \\
u_x(0,t) = 0 \qquad u(1,t) = 10 \qquad \text{for } t > 0, \\
u(x,0) = x^2 \qquad \text{for } x \in [0,1].
\end{gathered}
$$

Find the steady state solution **and** then use the method of separation of variables **to solve** the BVP – **leave** all the coefficients as integrals.