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Global stability of traveling waves for nonlocal time-delayed degenerate diffusion equation

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Abstract

This paper is concerned with a class of nonlocal reaction-diffusion equations with time-delay and degenerate diffusion. Affected by the degeneracy of diffusion, it is proved that, the Cauchy problem of the equation possesses the Hölder-continuous solution. Furthermore, the non-critical traveling waves are proved to be globally L^1-stable, which is the first frame work on L^1-wavefront-stability for the degenerate diffusion equations. The time-exponential convergence rate is also derived. The adopted approach for the proof is the technical L^1-weighted energy estimates combining the compactness analysis, but with some new development.

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1. Introduction

In this paper, we consider the time-delayed degenerate diffusion equation with nonlocality:

$$\frac{\partial u}{\partial t} - D(u^m)_{xx} + d(u) = \int_{\mathbb{R}} f_\alpha(y) b(u(t-r, x-y))dy, \ t \geq 0, \ x \in \mathbb{R}, \tag{1.1}$$

subject to initial datum

$$u|_{t=s} = u_0(s, x), \quad s \in [-r, 0], \ x \in \mathbb{R}. \tag{1.2}$$

Here, u represents the total population of the mature at location x and time $t > 0$, $D > 0$ is the diffusion rate of mature population. $d(u)$ and $b(u)$ are the death rate function and the birth rate function of the mature species, respectively. $m > 1$ is the index of degeneracy for diffusion. $r > 0$ denotes the age for maturation population, the so-called time-delay. The constant α satisfying $0 < \alpha \leq rD$ expresses the total amount of diffusion for the immature population. $f_\alpha(y)$ is the heat kernel and satisfies

$$f_\alpha(y) = \frac{1}{\sqrt{4\pi\alpha}} e^{-\frac{y^2}{4\alpha}} \quad \text{with} \quad \int_{-\infty}^{+\infty} f_\alpha(y)dy = 1. \tag{1.3}$$

In particular, when the immature population is almost immobile, *i.e.*, $\alpha \to 0^+$, then

$$\lim_{\alpha \to 0^+} \int_{\mathbb{R}} f_\alpha(y) b(u(t-r, x-y))dy = b(u(t-r, x)),$$

and the nonlocal equation (1.1) is reduced to the local equation with degenerate diffusion:

$$\frac{\partial u}{\partial t} - D(u^m)_{xx} + d(u) = b(u(t-r, x)), \ t \geq 0, \ x \in \mathbb{R}. \tag{1.4}$$

Throughout this paper, we assume the birth rate function $b(u)$ and the death rate function $d(u)$ satisfying

- (H_1) Two constant equilibria: $u_- = 0$ and $u_+ > 0$, such that $d(0) = b(0) = 0$, $d(u_+) = b(u_+)$, and $d(u), b(u) \in C^2([0, u_+])$;
- (H_2) Mono-stable type: $0 < d'(0) < b'(0)$, and $d'(u_+) > b'(u_+) > 0$. Namely, $u = 0$ is an unstable node, and $u = u_+$ is a stable node;
- (H_3) Monotonicity and concavity: $d'(u) \geq 0$, $b'(u) \geq 0$, $d''(u) \geq 0$, $b''(u) \leq 0$, but either $d''(u) > 0$ or $b''(u) < 0$, for $u \in [0, u_+]$.

The typical examples include three well-known dynamic models of population:

- the dynamic population model with age structure [2,16,30] for $d(u) = \beta u^2$ and $b(u) = \alpha e^{\gamma r} u(t-r, x)$, where $\beta > 0$, $\alpha > 0$, $\gamma > 0$, and $r > 0$, which is reduced to Fisher-KPP equation by taking $\beta = 1$ and $\alpha e^{\gamma r} = 1$:

$$\frac{\partial u}{\partial t} - D(u^m)_{xx} + u^2 = \int_{\mathbb{R}} f_\alpha(y) u(t-r, x-y) dy, \tag{1.5}$$

with $u_- = 0$ and $u_+ = 1$;

- Nicholson's blowflies equation [3,9,15,18,26,33–35,46] with $d(u) = \delta u$ and $b(u) = pue^{-au}$:

$$\frac{\partial u}{\partial t} - D(u^m)_{xx} + \delta u = p \int_{\mathbb{R}} f_\alpha(y) u(t-r, x-y) e^{-au(t-r,x-y)} dy, \tag{1.6}$$

where $\delta > 0$, $p > 0$, $a > 0$, $1 < \frac{p}{\delta} \le e$, $u_- = 0$ and $u_+ = \frac{1}{a} \ln \frac{p}{\delta}$;

- Mackey-Glass equation [22,24,25]with $d(u) = \delta u$ and the Beverton-Holt type function $b(u) = \frac{pu}{1+au}$:

$$\frac{\partial u}{\partial t} - D(u^m)_{xx} + \delta u = p \int_{\mathbb{R}} f_\alpha(y) \frac{u(t-r, x-y)}{1 + au(t-r, x-y)} dy, \tag{1.7}$$

where $0 < \delta < p$, $a > 0$, $u_- = 0$ and $u_+ = \frac{1}{a}(\frac{p}{\delta} - 1)$.

A traveling wave to (1.1) is a special solution in the form of $u = \phi(x + ct)$ with speed $c > 0$ satisfying

$$\begin{cases} c\phi'(\xi) - D(\phi^m(\xi))'' + d(\phi(\xi)) = \int_{\mathbb{R}} f_\alpha(y) b(\phi(\xi - y - cr)) dy, \\ \phi(\pm\infty) = u_\pm, \end{cases} \tag{1.8}$$

where $\xi = x + ct$ and $' = \partial/\partial\xi$. The main purpose in this paper is to study the global stability of traveling waves $u = \phi(x + ct)$ to (1.8).

Background. When $m = 1$, (1.1) and (1.4) are the regular diffusion equations with time-delay, and have been extensively studied, for example, [20,23,26,29,33] and the references therein. For the local equation with regular diffusion, namely $\alpha \to 0^+$ and $m = 1$, Schaaf [32] in 1984 first investigated the existence of the monotone traveling waves and the linear stability for the non-critical traveling waves by the spectral analysis method. Later then, the existence of different types of traveling waves (monotone and oscillatory waves) were studied in different cases based on the monotonicity / non-monotonicity of the equations as well as the size of time-delay, see [5, 7,8,10,11,21,33,34,38,39] and the references therein. The asymptotic stability of traveling waves (critical/non-critical waves, oscillating waves) were further proved in different cases in [3,13, 14,16–18,24–30,32,36,40], where the adopted approaches are various, including the monotonic technique, the weighted-energy method, the Green function method with Fourier transform, and the spectral method, and so on.

However, from the ecological sense, the diffusion coefficient usually is not a constant, but depends on the population density u. Consequently, it is more practical and interesting to consider the reaction-diffusion equations with degeneracy of diffusion like $(u^{m-1}u_x)_x$ with $m > 1$, cf. [12,42–44]. The relevant study in this case is quite limited.

Regarding the degenerate diffusion equations without time-delay, namely $m > 1$ and $r = 0$, Aronson [1] first studied Fisher-KPP equation with degenerate diffusion

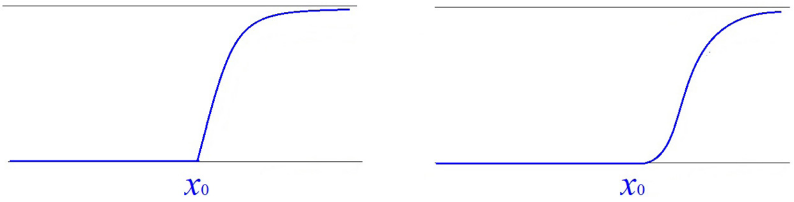

Fig. 1. Sharp traveling waves

$$u_t - (u^m)_{xx} = u(1-u), \quad m > 1$$

and proved the existence of the traveling wave $\phi(x+ct)$ for $c \geq c_* > 0$, where c_* is the critical wave speed. In particular, such a critical traveling wave was recognized as the sharp-type in the first time. Namely, there exists a number ξ_0 such that the critical wave $\phi(\xi) = 0$ for $\xi \in (-\infty, \xi_0]$ and $\phi(\xi) > 0$ for $\xi \in (\xi_0, \infty)$, see Fig. 1. Then, in the special case of $m = 2$, Gilding and Kersner [6] constructed the exact critical traveling wave $\phi(x + c_* t)$. Furthermore, De Pablo and Vázquez [4] investigated the existence of traveling waves for the degenerate Fisher-KPP equation in the more general form

$$u_t - (u^m)_{xx} = u^n(1-u), \quad m > 1.$$

For time-delayed reaction-diffusion equations with degenerate diffusion, namely, $m > 1$ and $r > 0$, Huang-Jin-Mei-Yin [12] first considered the local equation (1.4). By using the perturbation method, they proved the existence of traveling waves with large wave speed c, when the time-delay is small enough $r \ll 1$, and showed the stability of the traveling waves by the weighted L^1_ω energy method associated with a certain weight function $\omega(x) > 0$. Later then, by means of the viscosity vanishing method and the monotonic technique, Xu-Ji-Mei-Yin [42] proved that, for the nonlocal degenerate equation (1.1) with monotonicity, the traveling waves $\phi(x+ct)$ with $c \geq c_*$ (including the critical traveling waves $\phi(x + c_* t)$) all exist. By observing that the nonlocality involving the heat kernel $f_\alpha(y)$ plays a mollification role in some sense for the dynamic system, even if the equation (1.1) is with degenerate diffusion $-(u^m)_{xx}$, all traveling waves with $c \geq c_* > 0$ are proved to be smooth. In [42], they further showed that, for the local equation (1.4) with monotonicity, the non-critical traveling waves with $c > c_*$ exist, and these wavefronts are also smooth. But the existence of the critical traveling waves $\phi(x + c_* t)$ for the local equation (1.4) kept open at that moment. Subsequently, by proposing a new variation scheme, Xu-Ji-Mei-Yin [43] succeeded in obtaining the existence of the critical traveling waves $\phi(x + c_* t)$ for the local equation (1.4) with monotonicity. Such critical waves are of sharp-type, and C^1-piecewise smooth for $m \geq 2$, and C^1 smooth for $1 < m < 2$. All the non-critical traveling waves with $c > c_*$ are C^2-smooth. Furthermore, when the local equation (1.4) loses its monotonicity, Xu-Ji-Mei-Yin in [44] recently proved the existence of traveling waves and semi-traveling-waves, and recognized that the traveling waves may be oscillating once the time-delay is a bit large, and showed the criteria to classify all types of waves, including sharp waves, oscillating waves, smooth waves, and semi-waves, and so on. For the forced waves of reaction-diffusion model with density-dependent degenerate diffusion, Liu-Xu-Yin [19] further proved the stability of forced wave.

Main purpose. In this paper, we focus on the nonlocal equation (1.1). First of all, we will establish the existence and uniqueness of the solutions to the initial value problem (1.1) and

(1.2), as well as the regularity of the solutions. Once the initial data is in L^∞, the solution $u(t,x)$ of (1.1) and (1.2) is proved to be Hölder continuous with respect to t in the form of $C^{\frac{1}{4m}}$, and with respect to x in the form of $C^{\frac{1}{2m}}$. The approach adopted is the compactness method by constructing Holmgren's sequence. Secondly, we will prove the global stability of the non-critical traveling waves with $c > c_*$. The method for the proof is the weighted L^1-energy method with new development on compactness analysis. Here the initial perturbations around the traveling waves can be allowed arbitrarily large in L^1-sense.

Technical issues. Although the weighted energy method for proving wave stability for time-delayed reaction-diffusion equations was recently developed by Mei and his collaborators in [3,13,16,18,24–31,45], however, to treat the case with nonlocality and the degenerate diffusion, it is still a new and challenging attempt, because the degeneracy and the nonlocality cause some difficulty in the energy estimates. Here we will develop it with some new techniques, in particular, we obtain the stability result in the regular L^1-space, which is the first result for the degenerate diffusion case. More precisely, to obtain the L^1-stability, we note that the chosen weight function is somewhat like $\omega(x) = e^{-\lambda x}$ with $\lambda > 0$, which is asymptotically vanishing as $x \to \infty$, such that the weighted L^1_ω-stability never guarantees the L^1-stability, so we need to develop some new idea to treat it. We observe that mono-stability (H_2) of the equation (1.1), implies $d'(u) > b'(u)$ for $u \to u_+$, namely $x \to \infty$, so then we technically introduce a new cutting-function such that the L^1-stability on the far field of $[x_0, \infty)$ for $x_0 \gg 1$ can be derived by the L^1-compactness approximation. For details, see Step 4 in Section 4.

Notations. Let $\Omega \subset \mathbb{R}$ be a nonempty open set in 1-dimensional real Euclidean space $\mathbb{R}$, $\partial\Omega$ is its boundary, $\overline{\Omega} = \Omega \cup \partial\Omega$. $\|\cdot\|_X$ represents the norm of space X. If the norm of a function $f(x)$ is bounded, that is, $\|f(x)\|_X < \infty$, we say that $f(x)$ belongs to X, writing $f(x) \in X$. Here and hereafter, the associated subset Ω is either $\Omega \subset \mathbb{R}$ or $\Omega = \mathbb{R}$. $L^p(\Omega)$ $(p > 0)$ consists of all measurable functions $f(x)$ defined on $\mathbb{R}$ for which

$$\|f(x)\|_{L^p(\Omega)} = \left(\int_\Omega |f(x)|^p dx \right)^{\frac{1}{p}} < \infty.$$

If $p = \infty$, the norm of $f(x) \in L^\infty(\Omega)$ is given by $\|f(x)\|_{L^\infty(\Omega)} = \sup\limits_{x\in\Omega} f(x)$. The Sobolev space $H^k(\Omega)$ is defined as

$$H^k(\Omega) := \{f \mid \partial^j f \in L^2(\Omega) \ \text{ for } \ 0 \le j \le k\},$$

where ∂^j is a differential operator of order j and $\partial^0 f = f$. $C(\Omega)$ is the space of all continuous functions defined on Ω. $C^k(\Omega)$ is the space consisting of all functions $f(x)$ which, together with all their partial derivatives $\partial^j f$ of orders $0 \le j \le k$, are continuous on Ω.

For a given weight function $\omega(x) > 0$, $x \in \Omega$, we denote the weighted Sobolev spaces $L^p_\omega(\Omega)$ and $H^1_\omega(\Omega)$ by

$$L^p_\omega(\Omega) = \{f \mid \omega^{\frac{1}{p}} f \in L^p(\Omega)\},$$

and

$$H^1_\omega(\Omega) = \{f \mid \omega^{\frac{1}{2}} f,\ \omega^{\frac{1}{2}} f_x \in L^2(\Omega)\},$$

where f_x is the partial derivative of f with respect to x. The Hölder continuous space $C^\alpha(\Omega)$ $(0 < \alpha < 1)$ is equipped with

$$\|f\|_{C^\alpha(\Omega)} = \sup_{x\in\Omega} |f(x)| + \sup_{x,y\in\Omega; x\neq y} \frac{|f(x) - f(y)|}{|x-y|^\alpha}.$$

After introducing the time variable t, let $T > 0$ be a number and $\mathcal{B}$ be a Banach space, we denote $L^\infty([0,T];\mathcal{B})$ is the space of the $\mathcal{B}$-valued L^∞-functions on $[0,T]$. In particular,

$$\|f\|_{L^\infty(Q_T)} = \sup_{(t,x)\in Q_T} |f(t,x)|,$$

$$\|f\|_{L^p(Q_T)} = \left(\int_0^T \int_\Omega |f(t,x)|^p dxdt \right)^{\frac{1}{p}}.$$

A norm on the space $C^{\alpha,\beta}(Q_T)$ with $Q_T := [0,T] \times \Omega)$ can be defined by

$$\|f\|_{C^{\alpha,\beta}(Q_T)} = \sup_{(t,x)\in Q_T} |f(t,x)| + \sup_{P,Q\in Q_T; P\neq Q} \frac{|f(t_1,x) - f(t_2,y)|}{|t_1-t_2|^\alpha + |x-y|^\beta},$$

where P and Q are points of (t_1,x) and (t_2,y), respectively.

Throughout this paper, C denotes a generic positive constant, and $C_i > 0$ $(i = 0,1,2,\ldots)$ represent some special constants.

In what follows, we will state our main theorems on the existence, uniqueness and Hölder-continuity of the solutions to (1.1) and (1.2), and the stability of the traveling waves in Section 2. Then we will give the proof of the existence, uniqueness, and regularity of the solutions to (1.1) and (1.2) in Section 3. Finally, we will show the global stability of the non-critical traveling waves in Section 4.

2. Main results

In this section, we are going to state our main results. Since (1.1) is degenerate for $u = 0$, the solution is usually non-smooth, and we need to employ the following definition of weak solutions, or say the generalized solutions.

Definition 2.1. A function $u \in L^2_{\text{loc}}((0,+\infty)\times\mathbb{R})$ is called a weak solution of (1.1) if $0 \le u \le u_+$, $\nabla u^m \in L^2_{\text{loc}}((0,+\infty)\times\mathbb{R})$, and for any $T > 0$ and $\psi \in C_0^\infty((-r,T)\times\mathbb{R})$

$$-\int_0^T \int_{\mathbb{R}} u(t,x) \frac{\partial\psi}{\partial t} dxdt + D \int_0^T \int_{\mathbb{R}} \nabla u^m \cdot \nabla\psi dxdt + \int_0^T \int_{\mathbb{R}} d(u(t,x))\psi dxdt$$

$$= \int_{\mathbb{R}} u_0(0,x)\psi(0,x)dx + \int_r^{\max\{T,r\}} \int_{\mathbb{R}} \int_{-\infty}^{+\infty} b(u(t-r,y)) f_\alpha(x-y)\psi(x,t) dydxdt$$

$$+\int_0^{\min\{T,r\}}\int_{\mathbb{R}}\int_{-\infty}^{+\infty} b(u_0(t-r,y))f_\alpha(x-y)\psi(x,t)dydxdt.$$

Our first theorem is on the existence, uniqueness and regularity of the solution to (1.1) and (1.2).

Theorem 2.1 *(Existence, uniqueness and regularity). Let the initial data be*

$$0<u_0(s,x)<u_+ \ \ \textit{for } (s,x)\in[-r,0]\times\mathbb{R},$$

and $u_0\in L^\infty([-r,0]\times\mathbb{R})$*, and*

$$\lim_{x\to-\infty}u_0(s,x)=0,\ \lim_{x\to+\infty}u_0(s,x)=u_+,\ \textit{uniformly in } s\in[-r,0]. \tag{2.1}$$

Then (1.1) *admits a uniquely global solution* $u\in\mathcal{D}$*, where*

$$\mathcal{D}=\{u\mid u\in L^\infty(\mathbb{R}_+\times\mathbb{R})\cap C^{\frac{1}{4m},\frac{1}{2m}}(\mathbb{R}_+\times\mathbb{R});\ \ (u^m)_x\in L^\infty(\mathbb{R}_+;L^2_{loc}(\mathbb{R})),$$
$$(u^m)_t\in L^2_{loc}(\mathbb{R}_+\times\mathbb{R})\},$$

and

$$0<u(t,x)<u_+\ \textit{for } (t,x)\in\mathbb{R}_+\times\mathbb{R}, \tag{2.2}$$

and

$$\lim_{x\to-\infty}u(t,x)=0,\ \lim_{x\to+\infty}u(t,x)=u_+,\ \textit{uniformly in } t\in\mathbb{R}_+. \tag{2.3}$$

In order to state our second theorem on the stability of traveling waves, let us recall the property of traveling waves. Regarding the traveling waves $\phi(x+ct)$ to (1.8), it was showed in [42] that there exists the critical wave speed $c_*>0$ such that, for all $c\geq c_*$, the traveling waves $\phi(x+ct)$ exist. These waves are smooth and monotonic increasing from 0 to u_+. See Fig. 2 for numerical results as showed in [42].

Clearly, the characteristic equation associated with the wave speed $c>0$ and the corresponding eigenvalue $\lambda>0$ is given by

$$\Delta_{c,\lambda}:=d'(0)+c\lambda-b'(0)e^{-\lambda cr+\alpha\lambda^2}=0. \tag{2.4}$$

Define

$$G_c(\lambda):=d'(0)+c\lambda,\ \ \text{and}\ \ H_c(\lambda):=b'(0)e^{-\lambda cr+\alpha\lambda^2}.$$

It is known that, from the graphs of $G_c(\lambda)$ and $H_c(\lambda)$ (see Fig. 3), there exists a unique tangent point (c_*,λ_*) with $c_*>0$ and $\lambda_*>0$ for these two curves $G_c(\lambda)$ and $H_c(\lambda)$ such that:

- when $c<c_*$, the characteristic equation $\Delta_{c,\lambda}=0$ has no solution;

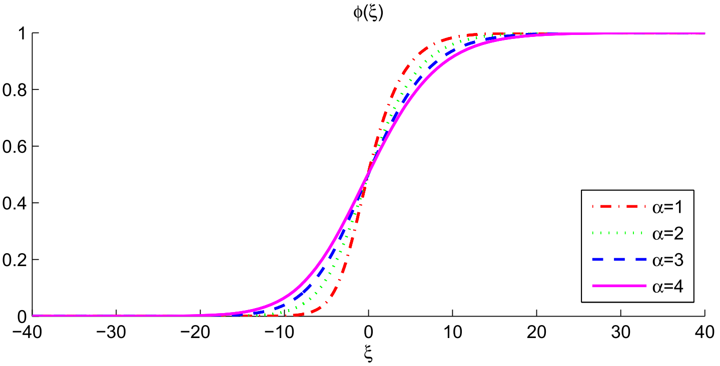

Fig. 2. Smooth traveling waves

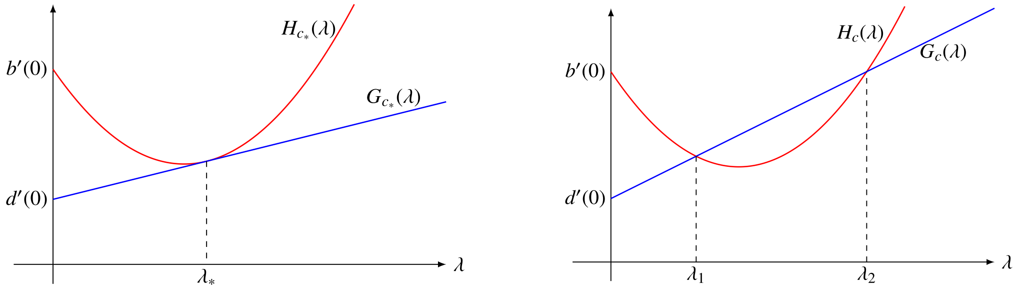

Fig. 3. The graphs of $G_c(\lambda)$ and $H_c(\lambda)$ for $c = c_*$ and $c > c_*$, respectively.

- when $c = c_*$, the characteristic equation $\Delta_{c,\lambda} = 0$ has a unique pair of solution (c_*, λ_*);
- when $c > c_*$, the characteristic equation $\Delta_{c,\lambda} = 0$ has two pairs of solutions (c, λ_1) and (c, λ_2), and $\Delta_{c,\lambda} > 0$ for $\lambda \in (\lambda_1, \lambda_2)$, and $\Delta_{c,\lambda} < 0$ for $\lambda \in (0, \lambda_1) \cup (\lambda_2, \infty)$.

For details, see Fig. 3.

Furthermore, let us define the other two functions:

$$\overline{G}_c(\lambda) := d'(0) + c\lambda - D\lambda^2 m u_+^{m-1}, \quad \text{and} \quad \overline{H}_c(\lambda) := b'(0)e^{-\lambda c r + \alpha\lambda^2}.$$

From the graphes of $\overline{G}_c(\lambda)$ and $\overline{H}_c(\lambda)$ (see Fig. 4), there exists another tangent point $(\bar{c}_*, \bar{\lambda}_*)$ for the curves of $\overline{G}_c(\lambda)$ and $\overline{H}_c(\lambda)$, with $\bar{c}_* > c_* > 0$ and $\bar{\lambda}_* > 0$, such that:

$$\begin{aligned} \overline{\Delta}_{c,\lambda}\Big|_{(c,\lambda)=(\bar{c}_*,\bar{\lambda}_*)} &:= \Big(\overline{G}_c(\lambda) - \overline{H}_c(\lambda)\Big)\Big|_{(c,\lambda)=(\bar{c}_*,\bar{\lambda}_*)} \\ &= d'(0) + \bar{c}_*\bar{\lambda}_* - D\bar{\lambda}_*^2 m u_+^{m-1} - b'(0)e^{-\bar{\lambda}_*\bar{c}_* r + \alpha\bar{\lambda}_*^2} = 0, \end{aligned} \tag{2.5}$$

$$\begin{aligned} \partial_\lambda \overline{\Delta}_{c,\lambda}|_{(c,\lambda)=(\bar{c}_*,\bar{\lambda}_*)} &:= \Big(\overline{G}_c'(\lambda) - \overline{H}_c'(\lambda)\Big)\Big|_{(c,\lambda)=(\bar{c}_*,\bar{\lambda}_*)} \\ &= \bar{c}_* - 2D\bar{\lambda}_* m u_+^{m-1} - b'(0)e^{-\bar{\lambda}_*\bar{c}_* r + \alpha\bar{\lambda}_*^2}(-\bar{c}_* r + 2\alpha\bar{\lambda}_*) = 0. \end{aligned} \tag{2.6}$$

When $c > \bar{c}_*$, the equation

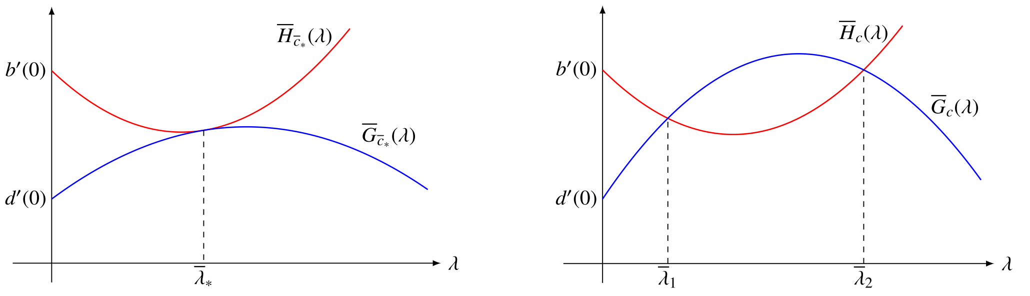

Fig. 4. The graphs of $\overline{G}_c(\lambda)$ and $\overline{H}_c(\lambda)$ for $c = \bar{c}_* > c_*$ and $c > \bar{c}_*$, respectively.

$$\overline{\Delta}_{c,\lambda} := d'(0) + c\lambda - D\lambda^2 m u_+^{m-1} - b'(0)e^{-\lambda c r + \alpha\lambda^2} = 0$$

possesses two solutions

$$\lambda = \bar{\lambda}_1(c) > 0 \quad \text{and} \quad \lambda = \bar{\lambda}_2(c) > 0, \tag{2.7}$$

such that

$$\overline{\Delta}_{c,\lambda} := d'(0) + c\lambda - D\lambda^2 m u_+^{m-1} - b'(0)e^{-\lambda c r + \alpha\lambda^2} > 0 \quad \text{for} \quad \lambda \in (\bar{\lambda}_1, \bar{\lambda}_2). \tag{2.8}$$

See Fig. 4 for details.

Now we are ready to state the global stability of traveling waves.

Theorem 2.2 *(Global stability of traveling waves). Let $b(u)$ and $d(u)$ satisfy $(H_1)-(H_3)$, and $\phi(x+ct)$ be a smooth non-critical traveling wave with $c > \bar{c}_*$, and $\omega(\xi) = e^{-\lambda\xi}$ with $\xi = x + ct$ be the weight function with $\lambda \in (\bar{\lambda}_1, \bar{\lambda}_2)$, where $\bar{c}_*$ is determined by (2.5) and (2.6), and $\bar{\lambda}_1$ and $\bar{\lambda}_2$ are given in (2.7). When the initial perturbation around the smooth traveling wave satisfies*

$$v_0(s,\xi) := u_0(s,x) - \phi(x+cs) \in C([-r,0]; L^1_\omega(\mathbb{R}) \cap L^1(\mathbb{R})),$$

then

$$v(t,\xi) := u(t,x) - \phi(\xi) \in C(R_+; L^1_\omega(\mathbb{R}) \cap L^1(\mathbb{R})),$$

and

$$\|(u-\phi)(t)\|_{L^1(\mathbb{R})} \le C e^{-\mu t} \left(\|v_0\|_{L^1(-r,0;L^1_\omega(\mathbb{R}))} + \|v_0(0)\|_{L^1_\omega(\mathbb{R})\cap L^1(\mathbb{R})} \right),$$

provided with some constant $\mu > 0$.

Remark 2.1.

1. In Theorem 2.2, we prove the stability for the fast traveling waves with a large wave speed $c > \bar{c}_*$. When the wave speed c is close to the critical wave speed c_*, the stability of traveling waves is unknown. In particular, when $c = c_*$, the asymptotic stability of the critical traveling

waves $\phi(x + c_*t)$ is completely open. We realize that, the approach adopted in this paper cannot be applied to these cases, and a new technique is expected. These will be our targets in future.

2. In this paper, the birth rate function $b(u)$ and the death rate function $d(u)$ both request to be monotone (see (H_3)). If both of them are non-monotone, the traveling waves are usually oscillatory when the time-delay r is big. In this case, the adopted method in this paper is failed, and the study of the stability of the oscillating waves is more challenging as we know. Usually these waves could not be expected to be globally stable. This topic will be also interesting for future study.

3. Existence, uniqueness and regularity of solutions

In this section, we are going to prove the existence, uniqueness and regularity of the solution $u(t,x)$ to (1.1) and (1.2). Adopted approach is the Holmgren's approximation scheme and compactness analysis.

Let us define a sequence $u_0^{(l)}(s,x)$ with $l \gg 1$ (of course, $l \gg u_+$), which is sufficiently smooth and satisfies

$$0 < u_0^{(l)}(s,x) < u_+, \quad \text{for } (s,x) \in [-r,0] \times [-l,l],$$

and

$$\lim_{l\to\infty} u_0^{(l)}(s,x) = u_0(s,x), \text{ uniformly in } x \text{ and } s.$$

In order to construct a proper series of approximations by localizing the original problem (1.1), let us consider the following initial-boundary-value problem

$$\begin{cases} \dfrac{\partial u}{\partial t} - D(u^m)_{xx} + d(u) = \displaystyle\int_{\mathbb{R}} f_\alpha(y) b(u(t-r,x-y))dy, & x \in (-l,l),\ t>0, \\ u(t,x) = \frac{1}{l}, & x \in (-\infty,-l],\ t>0, \\ u(t,x) = u_+ - \frac{1}{l}, & x \in [l,\infty),\ t>0, \\ u|_{t=s} = u_0^{(l)}(s,x), & (s,x) \in [-r,0]\times[-l,l]. \end{cases} \tag{3.1}$$

We define the upper and lower solutions for (3.1) as follows.

Definition 3.1. A function $\underline{u} \in C(\mathbb{R}_+ \times [-l,l])$ with $\underline{u}^m \in W^{1,2}_{loc}(\mathbb{R}_+ \times [-l,l])$ is called a lower solution of (3.1), if $\underline{u}(t,x)$ satisfies

$$\begin{cases} \dfrac{\partial \underline{u}}{\partial t} - D(\underline{u}^m)_{xx} + d(\underline{u}) \le \displaystyle\int_{\mathbb{R}} f_\alpha(y) b(u(t-r,x-y))dy, & t>0,\ x\in[-l,l], \\ \underline{u}|_{x\le -l} \le \frac{1}{l},\ \ \underline{u}|_{x\ge l} \le u_+ - \frac{1}{l}, & t>0, \\ \underline{u}|_{t=s} \le u_0^{(l)}(s,x), & (s,x)\in[-r,0]\times[-l,l]. \end{cases} \tag{3.2}$$

A function $\bar{u} \in C(\mathbb{R}_+ \times [-l,l])$ with $\bar{u}^m \in W^{1,2}_{loc}(\mathbb{R}_+ \times [-l,l])$ is called an upper solution of (3.1), if $\bar{u}(t,x)$ satisfies the opposite inequalities of (3.2).

Let $T > 0$ be a given number, $Q_T := [0, T] \times [-l, l]$, and define

$$C_+([0,T]; H^1(-l,l)) = \{v(t,x) \mid v(t,x) > 0 \text{ for } (t,x) \in Q_T, \\ \text{and } v \in C([0,T]; H^1(-l,l))\},$$

and let $v(t,x)$ be in $C_+([0,T]; H^1(-l,l)]$ with the given extensions out of $[-l,l]$

$$v|_{x\le -l} = \frac{1}{l}, \quad v|_{x\ge l} = u_+ - \frac{1}{l}, \quad \text{for } t \in [0,T],$$

and let us define

$$F(v) := d'(u_+)v(t,x) - d(v(t,x)) + \int_{\mathbb{R}} f_\alpha(y) b(v(t-r, x-y))dy. \tag{3.3}$$

We first consider the following IBVP of porous media equation:

$$\begin{cases} \dfrac{\partial u}{\partial t} - D(u^m)_{xx} + d'(u_+)u = F(v), & (t,x) \in Q_T, \\ u(t,-l) = \frac{1}{l}, \ \ u(t,l) = u_+ - \frac{1}{l}, & t \in [0,T], \\ u|_{t=s} = u_0^{(l)}(s,x), & (s,x) \in [-r,0] \times [-l,l], \end{cases} \tag{3.4}$$

and define the values of $u(t,x)$ out of the above mentioned region $[0,T] \times [-l,l]$ as follows

$$u|_{x\le -l} = \frac{1}{l}, \quad \text{and} \quad u|_{x\ge l} = u_+ - \frac{1}{l}, \quad \text{for } t \in [0,T]. \tag{3.5}$$

Lemma 3.1 *(Positivity and boundedness). Let $u(t,x)$ be the solution of the IBVP (3.4) with $0 < v(t,x) < u_+$ for $(t,x) \in Q_T$ and*

$$v|_{x\le -l} = \frac{1}{l}, \quad \textit{and} \quad v|_{x\ge l} = u_+ - \frac{1}{l}, \quad \textit{for } t \in [0,T],$$

and let $0 < u_0^{(l)}(s,x) < u_+$ for $(s,x) \in [-r,0] \times [-l,l]$. Then

$$0 < u(t,x) < u_+, \quad (t,x) \in Q_T. \tag{3.6}$$

Proof. From (H_1)-(H_3), we have

$$d'(u_+)v - d(v) = d'(u_+)v - [d(v) - d(0)] = [d'(u_+) - d'(\theta_0)]v \ge 0,$$

where $\theta_0 = \theta_0(t,x)$ is some point in $(0, v(t,x)) \subset (0, u_+)$ due to Taylor's formula, and $d'(u_+) \ge d'(\theta_0)$ due to the condition $d''(s) > 0$ given in (H_3). On the other hand, since $b(v) \ge 0$, we have

$$\int_{\mathbb{R}} f_\alpha(y) b(v(t-r, x-y))dy \ge 0.$$

Thus, we immediately get

$$F(v) \geq 0 \ \text{ for } (t,x) \in Q_T,$$

which reduces (3.4) to

$$\begin{cases} \dfrac{\partial u}{\partial t} - D(u^m)_{xx} + d'(u_+)u \geq 0, & (t,x) \in Q_T, \\ u(t,-l) = \frac{1}{l} > 0, \ \ u(t,l) = u_+ - \frac{1}{l} > 0, & t \in [0,T], \\ u|_{t=0} = u_0^{(l)}(0,x) > 0, & x \in [-l,l]. \end{cases} \tag{3.7}$$

In the standard theory of quasi-linear parabolic equations (see the textbook [37,41]), we immediately obtain

$$u(t,x) > 0 \quad \text{for } (t,x) \in Q_T.$$

On the other hand, let

$$w := u_+ - u, \ \text{ and } \ A(w) := \frac{u_+^m - u^m}{u_+ - u} = \frac{u_+^m - (u_+ - w)^m}{w} > 0,$$

then $w(t,x)$ satisfies

$$\begin{cases} \dfrac{\partial w}{\partial t} - D(A(w)w)_{xx} + d'(u_+)w \geq 0, & (t,x) \in Q_T, \\ w(t,-l) = u_+ - \frac{1}{l} > 0, \ \ w(t,l) = \frac{1}{l} > 0, & t \in [0,T], \\ w|_{t=0} = u_+ - u_0^{(l)}(0,x) > 0, & x \in [-l,l], \end{cases} \tag{3.8}$$

which, again from [37,41], implies

$$w(t,x) > 0, \quad \text{for } (t,x) \in Q_T.$$

Therefore, we prove

$$0 < u(t,x) < u_+, \quad \text{for } (t,x) \in Q_T.$$

The proof is complete. □

Lemma 3.2 *(Comparison Principle). Let*

$$0 < v_1(t,x) \leq v_2(t,x) < u_+, \ \ (t,x) \in Q_T,$$

with the given extensions

$$v_1|_{x \leq -l} = v_2|_{x \leq -l} = \frac{1}{l} \ \ \textit{and} \ \ v_1|_{x \geq l} = v_2|_{x \geq l} = u_+ - \frac{1}{l}, \ \ \textit{for } t \in [0,T],$$

and let $u_1(t,x)$ and $u_2(t,x)$ be the solutions to the IBVP (3.4) with respect to v_1 and v_2, respectively. Then

$$0 < u_1(t,x) \le u_2(t,x) < u_+, \quad (t,x) \in Q_T. \tag{3.9}$$

Proof. Let $U := u_1 - u_2$. From (3.4), U satisfies

$$\begin{cases} \dfrac{\partial U}{\partial t} - D(u_1^m - u_2^m)_{xx} + d'(u_+)U = F(v_1) - F(v_2), & (t,x) \in Q_T, \\ U(t,-l) = 0, \quad U(t,l) = 0, & t \in [0,T], \\ U|_{t=s} = 0, & (s,x) \in [-r,0] \times [-l,l]. \end{cases} \tag{3.10}$$

Since $F(v)$ is increasing with respect to $v > 0$ due to the condition (H_2)-(H_3), we have

$$F(v_1) \le F(v_2), \quad \text{for } 0 < v_1 \le v_2.$$

Then, (3.10) is reduced to

$$\begin{cases} \dfrac{\partial U}{\partial t} - D(u_1^m - u_2^m)_{xx} + d'(u_+)U \le 0, & (t,x) \in Q_T, \\ U(t,-l) = 0, \quad U(t,l) = 0, & t \in [0,T], \\ U|_{t=0} = 0, & x \in [-l,l], \end{cases}$$

which, by the comparison principle of nonlinear diffusion equations [41], immediately implies

$$U(t,x) \le 0 \quad \text{for } (t,x) \in Q_T,$$

namely,

$$0 < u_1(t,x) \le u_2(t,x) < u_+, \quad \text{for } (t,x) \in Q_T.$$

The proof is complete. □

Lemma 3.3 *(Existence and uniqueness of approximate solutions). Let $u_0^{(l)} \in C_+([-r,0] \times [-l,l])$ satisfy $0 < u_0^{(l)}(s,x) < u_+$. Then the IBVP (3.4) exists a unique smooth solution $u(t,x)$ satisfying*

$$u \in C_+([0,T]; H^1(-l,l)) \quad \textit{for any } T > 0, \tag{3.11}$$

and

$$0 < u(t,x) < u_+, \quad (t,x) \in Q_T. \tag{3.12}$$

Proof. Define

$$\underline{u} := 0, \quad \text{and} \quad \bar{u} := u_+. \tag{3.13}$$

Obviously, $\underline{u} := 0$ and $\bar{u} = u_+$ are a pair of lower and upper solutions of (3.1).

For given $v \in C_+([0,T]; H^1(-l,l))$ with the extensions out of $[-l,l]$

$$v|_{x\le -l} = \frac{1}{l}, \ \text{ and } \ v|_{x\ge l} = u_+ - \frac{1}{l}, \quad \text{for } t \in [0,T],$$

the IBVP (3.4) defines a differential operator

$$u = Pv, \quad \text{in } Q_T,$$

with the given extensions

$$u|_{x\le -l} = \frac{1}{l}, \ \text{ and } \ u|_{x\ge l} = u_+ - \frac{1}{l}, \quad \text{for } t \in [0,T].$$

Let us make the iterations of $u = Pv$ from (3.4) by choosing the initial functions $\underline{u}^{(0)} = \underline{u}$ and $\bar{u}^{(0)} = \bar{u}$ defined in (3.13), respectively, and denote the corresponding sequences by

$$\underline{u}^{(n)} = P\underline{u}^{(n-1)}, \ \text{ and } \ \bar{u}^{(n)} = P\bar{u}^{(n-1)}, \ \ n = 1, 2, \cdots.$$

Namely, $\underline{u}^{(n)}$ satisfies

$$\begin{cases} \dfrac{\partial \underline{u}^{(n)}}{\partial t} - D((\underline{u}^{(n)})^m)_{xx} + d'(u_+)\underline{u}^{(n)} = F(\underline{u}^{(n-1)}), & (t,x) \in Q_T, \\ \underline{u}^{(n)}(t,-l) = \frac{1}{l}, \ \ \underline{u}^{(n)}(t,l) = u_+ - \frac{1}{l}, & t \in [0,T], \\ \underline{u}^{(n)}|_{t=s} = u_0^{(l)}(s,x), & (s,x) \in [-r,0] \times [-l,l], \end{cases} \tag{3.14}$$

with the extensions

$$\underline{u}^{(n)}|_{x\le -l} = \frac{1}{l}, \ \text{ and } \ \underline{u}^{(n)}|_{x\ge 1} = u_+ - \frac{1}{l}, \quad \text{for } t \in [0,T],$$

and $\bar{u}^{(n)}$ satisfies

$$\begin{cases} \dfrac{\partial \bar{u}^{(n)}}{\partial t} - D((\bar{u}^{(n)})^m)_{xx} + d'(u_+)\bar{u}^{(n)} = F(\bar{u}^{(n-1)}), & (t,x) \in Q_T, \\ \bar{u}^{(n)}(t,-l) = \frac{1}{l}, \ \ \bar{u}^{(n)}(t,l) = u_+ - \frac{1}{l}, & t \in [0,T], \\ \bar{u}^{(n)}|_{t=s} = u_0^{(l)}(s,x), & (s,x) \in [-r,0] \times [-l,l], \end{cases} \tag{3.15}$$

with the extensions

$$\bar{u}^{(n)}|_{x\le -l} = \frac{1}{l}, \ \text{ and } \ \bar{u}^{(n)}|_{x\ge 1} = u_+ - \frac{1}{l}, \quad \text{for } t \in [0,T].$$

We claim that the sequences $\{\underline{u}^{(n)}\}$ and $\{\bar{u}^{(n)}\}$ both belong to $C_+([0,T]; H^1(-l,l))$, and satisfy

$$0 = \underline{u}^{(0)} < \underline{u}^{(1)} \le \cdots \le \underline{u}^{(n)} \le \cdots \le \bar{u}^{(n)} \le \cdots \le \bar{u}^{(1)} < \bar{u}^{(0)} = u_+, \quad \text{in } Q_T. \tag{3.16}$$

Clearly, the monotonicity of (3.16) can be directly obtained from the boundedness of Lemma 3.1 and the comparison principle of Lemma 3.2. So, there exist two limit functions such that

$$\hat{u}(t,x) := \lim_{n\to\infty} \underline{u}^{(n)}(t,x), \quad \check{u}(t,x) := \lim_{n\to\infty} \bar{u}^{(n)}(t,x), \quad \text{for } (t,x) \in Q_T, \tag{3.17}$$

satisfying

$$0 < \hat{u}(t,x) \le \check{u}(t,x) < u_+, \quad \text{for } (t,x) \in Q_T, \tag{3.18}$$

with the extensions

$$\hat{u}|_{x\le -l} = \frac{1}{l}, \quad \text{and} \quad \hat{u}|_{x\ge 1} = u_+ - \frac{1}{l}, \quad \text{for } t \in [0,T],$$

and

$$\check{u}|_{x\le -l} = \frac{1}{l}, \quad \text{and} \quad \check{u}|_{x\ge 1} = u_+ - \frac{1}{l}, \quad \text{for } t \in [0,T].$$

Now we prove that $\hat{u}(t,x) = \check{u}(t,x)$ in Q_T is the unique solution of the IBVP (3.1).

Let $w := \check{u} - \hat{u}$, and

$$A(t,x) := \begin{cases} \dfrac{\check{u}^m - \hat{u}^m}{\check{u} - \hat{u}}, & \text{for } \check{u} \ne \hat{u}, \\ m\check{u}^{m-1}, & \text{for } \check{u} = \hat{u}, \end{cases}$$

and

$$B(t,x) := \begin{cases} \dfrac{d(\check{u}) - d(\hat{u})}{\check{u} - \hat{u}}, & \text{for } \check{u} \ne \hat{u}, \\ d'(\check{u}), & \text{for } \check{u} = \hat{u}. \end{cases}$$

Clearly, it holds

$$A(t,x) > 0 \text{ for } (t,x) \in Q_T, \quad \text{and } A(\cdot,\cdot) \in C(Q_T),$$

and

$$B(t,x) > 0 \text{ for } (t,x) \in Q_T, \quad \text{and } B(\cdot,\cdot) \in C(Q_T),$$

due to $\check{u} \ge \hat{u} > 0$, and $d(u)$ is increasing. Then w satisfies the following IBVP

$$\begin{cases} \partial_t w - D\partial_x^2(A(t,x)w)) + B(t,x)w \\ \qquad = \int_{\mathbb{R}} f_\alpha(y)[b(\check{u}(t-r,x-y)) - b(\hat{u}(t-r,x-y))]dy, & (t,x) \in Q_T, \\ w(t,-l) = 0, \quad w(t,l) = 0, & t \in [0,T], \\ w|_{t=s} = 0, & (s,x) \in [-r,0] \times [-l,l]. \end{cases} \tag{3.19}$$

When $t \in [0,r]$, namely, $t-r \in [-r,0]$, then

$$\check{u}(t-r,x) = \hat{u}(t-r,x) = u_0^{(l)}(t-r,x), \quad \text{for } t \in [0,r], x \in [-l,l],$$

with the extensions

$$\check{u}|_{x\le -l} = \frac{1}{l}, \ \text{ and } \ \hat{u}|_{x\ge 1} = u_+ - \frac{1}{l}, \quad \text{for } t \in [0,r],$$

which implies

$$b(\check{u}(t-r,x-y)) - b(\hat{u}(t-r,x-y)) = 0, \quad \text{for } t \in [0,r], \ \text{and} \ \ x-y \in \mathbb{R}.$$

Thus, (3.19) can be reduced to

$$\begin{cases} \partial_t w - D\partial_x^2(A(t,x)w)) + B(t,x)w = 0, & (t,x) \in [0,r] \times [-l,l], \\ w(t,-l) = 0, \quad w(t,l) = 0, & t \in [0,r], \\ w|_{t=0} = 0, & x \in [-l,l]. \end{cases} \tag{3.20}$$

By using the standard theory of linear parabolic equations [37], we obtain

$$w(t,x) = 0, \ i.e., \ \check{u}(t,x) = \hat{u}(t,x), \quad \text{for } (t,x) \in [0,r] \times [-l,l]. \tag{3.21}$$

When $t \in [r,2r]$, by the same fashion, we can prove

$$w(t,x) = 0, \ i.e., \ \check{u}(t,x) = \hat{u}(t,x), \quad \text{for } (t,x) \in [r,2r] \times [-l,l], \tag{3.22}$$

with the same extensions

$$\hat{u}|_{x\le -l} = \frac{1}{l}, \ \text{ and } \ \hat{u}|_{x\ge 1} = u_+ - \frac{1}{l}, \quad \text{for } t \in [r,2r],$$

and

$$\check{u}|_{x\le -l} = \frac{1}{l}, \ \text{ and } \ \check{u}|_{x\ge 1} = u_+ - \frac{1}{l}, \quad \text{for } t \in [r,2r],$$

Repeating this procedure, when $t \in [nr,(n+1)r]$ for any integer $n > 1$, we can show

$$w(t,x) = 0, \ i.e., \ \check{u}(t,x) = \hat{u}(t,x), \quad \text{for } (t,x) \in [nr,(n+1)r] \times [-l,l], \tag{3.23}$$

with the same extensions

$$\hat{u}|_{x\le -l} = \frac{1}{l}, \text{ and } \hat{u}|_{x\ge 1} = u_+ - \frac{1}{l}, \quad \text{for } t \in [nr, (n+1)r],$$

and

$$\check{u}|_{x\le -l} = \frac{1}{l}, \text{ and } \check{u}|_{x\ge 1} = u_+ - \frac{1}{l}, \quad \text{for } t \in [nr, (n+1)r].$$

Combining (3.21)-(3.23), we have proved

$$w(t,x) = 0, \ i.e., \ \check{u}(t,x) = \hat{u}(t,x), \quad \text{for } (t,x) \in Q_T. \tag{3.24}$$

The proof is complete. □

Proposition 3.1 *(Existence and regularity of original solution). Under the assumptions in Theorem 2.1, then (1.1) and (1.2) exists a Hölder continuous $C^{\frac{1}{4m},\frac{1}{2m}}$-solution $u \in \mathcal{D}$, where $\mathcal{D}$ is defined in Theorem 2.1.*

Proof. For any $a \in (-l+2, l-2)$, then $(a-2, a+2)$ is any given subset of $(-l, l)$ with an equivalent size of 4. Let $\eta(x)$ be a function such that $\eta(x) \in C_0^\infty(a-2, a+2)$, $0 \le \eta(x) \le 1$, $|\eta'(x)| \le 1$ and $\eta(x) = 1$ for $x \in (a-1, a+1)$.

Multiplying the first equation of (3.1) by $u^k\eta^2$ with a number $k > 0$, and integrating it with respect to x over $[a-2, a+2] \subset [-l, l]$, we get

$$\begin{aligned}
&\frac{1}{k+1}\frac{d}{dt}\int_{a-2}^{a+2} u^{k+1}\eta^2 dx + Dmk\int_{a-2}^{a+2} u^{m+k-2}|u_x|^2\eta^2 dx \\
&+ 2Dm\int_{a-2}^{a+2} u^{m+k-1}u_x\eta\eta_x dx + \int_{a-2}^{a+2} d(u)u^k\eta^2 dx \\
&= \int_{a-2}^{a+2}\int_{\mathbb{R}} f_\alpha(y)b(u(t-r, x-y))u^k\eta^2 dy dx.
\end{aligned} \tag{3.25}$$

By applying Young's inequality and (3.12), then (3.25) yields

$$\begin{aligned}
&\frac{1}{k+1}\frac{d}{dt}\int_{a-2}^{a+2} u^{k+1}\eta^2 dx + Dmk\int_{a-2}^{a+2} u^{m+k-2}|u_x|^2\eta^2 dx + \int_{a-2}^{a+2} d(u)u^k\eta^2 dx \\
&= -2Dm\int_{a-2}^{a+2} u^{m+k-1}u_x\eta\eta_x dx + \int_{a-2}^{a+2}\int_{\mathbb{R}} f_\alpha(y)b(u(t-r, x-y))u^k\eta^2 dy dx \\
&\le \frac{Dmk}{2}\int_{a-2}^{a+2} u^{m+k-2}|u_x|^2\eta^2 dx + \frac{2Dm}{k}\int_{a-2}^{a+2} u^{m+k}\eta_x^2 dx + C\int_{a-2}^{a+2} u^k\eta^2 dx
\end{aligned}$$

$$\leq \frac{Dmk}{2} \int_{a-2}^{a+2} u^{m+k-2}|u_x|^2\eta^2 dx + C. \tag{3.26}$$

Thus, for any given $\sigma > 0$, and any $t > 0$, we see that

$$\int_t^{t+\sigma} \int_{a-2}^{a+2} u^{m+k-2}|u_x|^2\eta^2 dxds \leq C, \tag{3.27}$$

where C only depends on r and σ. Thanks to the mean value theorem of integrals, there exists $t_0 \in (t, t+\sigma)$ such that

$$\int_{a-2}^{a+2} u^{m+k-2}(t_0)|u_x(t_0,x)|^2\eta^2(x)dx \leq C. \tag{3.28}$$

Multiplying (3.1) by $(u^m)_t\eta^4$ and integrating the resultant equation with respect to x over $[a-2, a+2]$, using integration by parts and the properties of $b(u)$ and $d(u)$ in (H_1)-(H_3), and noting

$$\int_{a-2}^{a+2} d(u)(u^m)_t\eta^4 dx = m\int_{a-2}^{a+2} (d(u)-d(0))u^{m-1}u_t\eta^4 dx = m\int_{a-2}^{a+2} d'(\theta_1)u^m u_t\eta^4 dx,$$

with some number $\theta_1 \in (0, u_+)$, then we have

$$\begin{aligned}
0 &= \int_{a-2}^{a+2} (u^m)_t u_t\eta^4 dx - D\int_{a-2}^{a+2} (u^m)_{xx}(u^m)_t\eta^4 dx + \int_{a-2}^{a+2} d(u)(u^m)_t\eta^4 dx \\
&\quad - \int_{a-2}^{a+2}\int_{\mathbb{R}} f_\alpha(y)b(u(t-r,x-y))dy(u^m)_t\eta^4 dx \\
&= m\int_{a-2}^{a+2} u^{m-1}(u_t)^2\eta^4 dx + D\int_{a-2}^{a+2} (u^m)_{xt}(u^m)_x\eta^4 dx \\
&\quad + 4D\int_{a-2}^{a+2} (u^m)_t(u^m)_x\eta^3\eta_x dx + m\int_{a-2}^{a+2} d'(\theta_1)u^m u_t\eta^4 dx \\
&\quad - \int_{a-2}^{a+2}\int_{\mathbb{R}} f_\alpha(y)b(u(t-r,x-y))dy(u^m)_t\eta^4 dx
\end{aligned}$$

$$
\begin{aligned}
&= m\int_{a-2}^{a+2} u^{m-1}u_t^2\eta^4 dx + \frac{D}{2}\frac{d}{dt}\int_{a-2}^{a+2} |(u^m)_x|^2\eta^4 dx \\
&\quad + \frac{m}{m+1}\frac{d}{dt}\int_{a-2}^{a+2} d'(\theta_1)u^{m+1}\eta^4 dx + 4Dm^2\int_{a-2}^{a+2} u^{2m-2}u_x u_t\eta^3\eta_x dx \\
&\quad - \int_{a-2}^{a+2}\int_{\mathbb{R}} f_\alpha(y)b(u(t-r,x-y))dy(u^m)_t\eta^4 dx \\
&\geq m\int_{a-2}^{a+2} u^{m-1}u_t^2\eta^4 dx + \frac{D}{2}\frac{d}{dt}\int_{a-2}^{a+2} |(u^m)_x|^2\eta^4 dx \\
&\quad + d'(0)\frac{m}{m+1}\frac{d}{dt}\int_{a-2}^{a+2} u^{m+1}\eta^4 dx + 4Dm^2\int_{a-2}^{a+2} u^{2m-2}u_x u_t\eta^3\eta_x dx \\
&\quad - \int_{a-2}^{a+2}\int_{\mathbb{R}} f_\alpha(y)b(u(t-r,x-y))dy(u^m)_t\eta^4 dx. \qquad (3.29)
\end{aligned}
$$

On the other hand, by using Cauchy-Schwarz inequality ($ab \leq \varepsilon a^2 + \frac{1}{4\varepsilon}b^2$ for any $\varepsilon > 0$), and noting (3.26), we get

$$
\begin{aligned}
&4Dm^2\int_{a-2}^{a+2} u^{2m-2}u_x u_t\eta^3\eta_x dx \\
&\leq \frac{m}{4}\int_{a-2}^{a+2} u^{m-1}u_t^2\eta^4 dx + 16D^2m^3\int_{a-2}^{a+2} u^{3(m-1)}u_x^2\eta^2\eta_x^2 dx \\
&\leq \frac{m}{4}\int_{a-2}^{a+2} u^{m-1}u_t^2\eta^4 dx + C\int_{a-2}^{a+2} u^{3(m-1)}u_x^2\eta^2 dx \\
&\leq \frac{m}{4}\int_{a-2}^{a+2} u^{m-1}u_t^2\eta^4 dx + C, \qquad (3.30)
\end{aligned}
$$

and

$$
\int_{a-2}^{a+2}\Big(\int_{\mathbb{R}} f_\alpha(y)b(u(t-r,x-y))dy\Big)(u^m)_t\eta^4 dx
$$

$$= m \int_{a-2}^{a+2} \Big(\int_{\mathbb{R}} f_\alpha(y) b(u(t-r, x-y)) dy \Big) u^{m-1} u_t \eta^4 dx$$

$$\leq \int_{a-2}^{a+2} \int_{\mathbb{R}} f_\alpha(y) \Big[m b(u(t-r, x-y))^2 + \frac{m}{4} u_t^2 \Big] u^{m-1} \eta^4 dy dx$$

$$\leq \frac{m}{4} \int_{a-2}^{a+2} u^{m-1} u_t^2 \eta^4 dx + m \int_{a-2}^{a+2} u^{m-1} \eta^4 \Big(\int_{\mathbb{R}} f_\alpha(y) b(u(t-r, x-y))^2 dy \Big) dx$$

$$\leq \frac{m}{4} \int_{a-2}^{a+2} u^{m-1} u_t^2 \eta^4 dx + C. \tag{3.31}$$

Plugging (3.30) and (3.31) into (3.29), we get

$$\int_t^{t+\sigma} \int_{a-2}^{a+2} u^{m-1} |u_s|^2 \eta^4 dx ds + \sup_t \int_{a-2}^{a+2} |(u^m)_x|^2 \eta^4 dx + \sup_t \int_{a-2}^{a+2} u^{m+1} \eta^4 dx \leq C, \tag{3.32}$$

where C is independent of l. We denote the weak limit of $u^{(l)}$ by u as $l \to \infty$. Letting $l \to \infty$, we prove the existence of generalized solutions for the problem (1.1) and (1.2) equipping with (3.27), (3.28) and

$$0 \leq u \leq u_+.$$

We further have

$$\int_t^{t+\sigma} \int_{a-2}^{a+2} |u_s|^2 \eta^4 dx ds + \sup_t \int_{a-2}^{a+2} |(u^m)_x|^2 \eta^4 dx \leq C. \tag{3.33}$$

In view of the above results, $u^m \in L^\infty(\mathbb{R}_+; H^1_{loc}(\mathbb{R}))$, due to the Sobolev's inequality, we have $u^m \in L^\infty(\mathbb{R}_+; C^{\frac{1}{2}}(\mathbb{R}))$, namely

$$u(t, x) \in C^{\frac{1}{2m}}(\mathbb{R}), \quad \text{for } t \in \mathbb{R}_+.$$

Since $(u^m)_t \in L^2((t, t+\sigma) \times (a-1, a+1))$, for any $t_1 \geq t_2 \geq 0$, $x \in \mathbb{R}$, denoting B_r as a ball with a radius $r = |t_1 - t_2|^{\frac{1}{2}}$ centered at x. From (3.33), we obtain

$$\int_{B_r} |u^m(t_1, x) - u^m(t_2, x)| dx = \int_{B_r} \Big| \int_{t_2}^{t_1} \frac{\partial}{\partial s} u^m(s, x) ds \Big| dx$$

$$
\begin{aligned}
&\le C\int_{B_r}\left(\int_{t_2}^{t_1}\left|\frac{\partial}{\partial s}u^m(s,x)\right|^2 ds\right)^{\frac{1}{2}}|t_1-t_2|^{\frac{1}{2}}dx\\
&\le C\left(\int_{B_r}\int_{t_2}^{t_1}\left|\frac{\partial}{\partial s}u^m(s,x)\right|^2 dsdx\right)^{\frac{1}{2}}|t_1-t_2|^{\frac{1}{2}}|r|^{\frac{1}{2}}\\
&\le C|t_1-t_2|^{\frac{1}{2}}|r|^{\frac{1}{2}}.
\end{aligned}
\tag{3.34}
$$

Meanwhile, using the mean value theorem and $r=|t_1-t_2|^{\frac{1}{2}}$, there exists $x^*\in B_r$ such that

$$
\int_{B_r}|u^m(t_1,x)-u^m(t_2,x)|dx=|u^m(t_1,x^*)-u^m(t_2,x^*)||r|\le C|t_1-t_2|^{\frac{1}{2}}|r|^{\frac{1}{2}},
$$

namely,

$$
|u^m(t_1,x^*)-u^m(t_2,x^*)|\le C|t_1-t_2|^{\frac{1}{4}}.
$$

Therefore, we have

$$
\begin{aligned}
&|u^m(t_1,x)-u^m(t_2,x)|\\
&\le |u^m(t_1,x)-u^m(t_1,x^*)|+|u^m(t_1,x^*)-u^m(t_2,x^*)|\\
&\quad +|u^m(t_2,x)-u^m(t_2,x^*)|\\
&\le C(|x-x^*|^{\frac{1}{2}}+|t_1-t_2|^{\frac{1}{4}})\\
&\le C|t_1-t_2|^{\frac{1}{4}}.
\end{aligned}
\tag{3.35}
$$

This implies $u^m\in C^{\frac{1}{4},\frac{1}{2}}(\mathbb{R}_+\times\mathbb{R})$, equivalently, $u\in C^{\frac{1}{4m},\frac{1}{2m}}(\mathbb{R}_+\times\mathbb{R})$. □

Proposition 3.2 *(Uniqueness of original solution). Under the assumptions in Theorem 2.1, then the solution of* (1.1) *and* (1.2) *is unique.*

Proof. Let $u_1, u_2\in\mathcal{D}$ be two solutions of (1.1) and (1.2), and denote $u=u_1-u_2$. When $t\in(0,r)$, $b(u_1(t-r))=b(u_2(t-r))$, then $u(t,x)$ satisfies

$$
\begin{cases}
\dfrac{\partial u}{\partial t}-D(u_1^m-u_2^m)_{xx}+d(u_1)-d(u_2)=0, & (t,x)\in(0,r)\times\mathbb{R},\\
\lim_{x\to\pm\infty}u(t,x)=0, & t\in(0,r),\\
u(s,x)=0, & s\in[-r,0],\ x\in\mathbb{R}.
\end{cases}
\tag{3.36}
$$

From the definition of weak solution, the weak form for (3.36) is reduced to

$$-\int_0^r\int_{\mathbb{R}} u(t,x)\frac{\partial\psi}{\partial t}dxdt + D\int_0^r\int_{\mathbb{R}} \nabla(u_1^m-u_2^m)\cdot\nabla\psi dxdt$$
$$+\int_0^r\int_{\mathbb{R}}[d(u_1)-d(u_2)]\psi dxdt = 0, \tag{3.37}$$

with any test function $\psi \in C_0^\infty([-r,T]\times\mathbb{R})$. Namely, we have

$$\int_0^r\int_{\mathbb{R}} u(t,x)\Big[\frac{\partial\psi}{\partial t} + DA(u_1,u_2)\frac{\partial^2\psi}{\partial x^2} - B(u_1,u_2)\psi\Big]dxdt = 0, \tag{3.38}$$

where

$$A(u_1,u_2) = \begin{cases} \dfrac{u_1^m-u_2^m}{u_1-u_2}, & \text{for } u_1 \neq u_2,\\ mu_1^{m-1}, & \text{for } u_1 = u_2, \end{cases} \tag{3.39}$$

and

$$B(u_1,u_2) = \begin{cases} \dfrac{d(u_1)-d(u_2)}{u_1-u_2}, & \text{for } u_1 \neq u_2,\\ d'(u_1), & \text{for } u_1 = u_2. \end{cases} \tag{3.40}$$

For any smooth functions $g(x)\in C_0^\infty(R)$ and $h(t)\in C_0^\infty(0,r)$, let B_{R_0} be the ball with radius $R_0>0$ such that supp $g\subset B_{R_0}$. For any $R>R_0+1$, $\eta>0$, and $\varepsilon>0$, we consider the following adjoint problem in $C_0^\infty([0,r]\times\mathbb{R})$:

$$\begin{cases} \dfrac{\partial\varphi}{\partial t} + (DA_\varepsilon(u_1,u_2)+\eta)\dfrac{\partial^2\varphi}{\partial x^2} - B_\varepsilon(u_1,u_2)\varphi = -g(x)h(t), & |x|<R,\ t\in(0,r],\\ \frac{\partial\varphi}{\partial \nu} = 0, & |x|=R,\ t\in(0,r],\\ \varphi(x,r)=0, & |x|<R, \end{cases} \tag{3.41}$$

where $A_\varepsilon(u_1,u_2)$ and $B_\varepsilon(u_1,u_2)$ are the smooth approximations of $A(u_1,u_2)$ and $B(u_1,u_2)$ respectively, with

$$A(u_1,u_2)\le A_\varepsilon(u_1,u_2)\le A(u_1,u_2)+\varepsilon \quad \text{and} \quad B(u_1,u_2)\le B_\varepsilon(u_1,u_2)\le B(u_1,u_2)+\varepsilon.$$

Clearly, the IBVP (3.41) is the linear and regular heat equation with the given "initial data" imposed at $t=r$, which possesses a classic solution $\varphi(t,x)$. Let $\varphi(t,x)=\varphi^R_{\varepsilon,\eta}(t,x)$ be the solution of (3.41) with zero extension to the whole space of $\mathbb{R}$, and let $\xi_R(x)\in C_0^\infty(\mathbb{R})$ be a cut-off function such that $0\le\xi_R(\xi)\le 1$, $\xi_R(x)=1$ for $|x|\le R-1$, and $\xi_R(x)=0$ for $|x|>R-\frac{1}{2}$.

Particularly, by taking $\psi(t,x)=\xi_R(x)\varphi^R_{\varepsilon,\eta}(t,x)$ as the test function in (3.38), we have

$$\begin{aligned}
&\frac{\partial\psi}{\partial t}+DA(u_1,u_2)\frac{\partial^2\psi}{\partial x^2}-B(u_1,u_2)\psi\\
&=\xi_R(x)\Big(\frac{\partial\varphi}{\partial t}+DA(u_1,u_2)\frac{\partial^2\varphi}{\partial x^2}-B(u_1,u_2)\varphi\Big)\\
&\quad+DA(u_1,u_2)(\xi_{Rxx}\varphi+2\xi_{Rx}\varphi_x)\\
&=\xi_R(x)\Big(\frac{\partial\varphi}{\partial t}+(DA_\varepsilon(u_1,u_2)+\eta)\frac{\partial^2\varphi}{\partial x^2}-B_\varepsilon(u_1,u_2)\varphi\Big)\\
&\quad-[D(A_\varepsilon(u_1,u_2)-A(u_1,u_2))+\eta]\xi_R\varphi_{xx}\\
&\quad+[B_\varepsilon(u_1,u_2)-B(u_1,u_2)]\varphi\\
&\quad+DA(u_1,u_2)(\xi_{Rxx}\varphi+2\xi_{Rx}\varphi_x)\\
&=-g(x)h(t)\xi_R-[D(A_\varepsilon(u_1,u_2)-A(u_1,u_2))+\eta]\xi_R\varphi_{xx}\\
&\quad+[B_\varepsilon(u_1,u_2)-B(u_1,u_2)]\varphi\\
&\quad+DA(u_1,u_2)(\xi_{Rxx}\varphi+2\xi_{Rx}\varphi_x).
\end{aligned}\tag{3.42}$$

Substituting (3.42) into (3.38), we obtain

$$\begin{aligned}
&\int_0^r\int_{\mathbb{R}} u(t,x)g(x)h(t)\xi_R(x)dxdt\\
&=-\int_0^r\int_{\mathbb{R}} u(t,x)[D(A_\varepsilon(u_1,u_2)-A(u_1,u_2))+\eta]\xi_R\varphi_{xx}dxdt\\
&\quad+\int_0^r\int_{\mathbb{R}} u(t,x)[B_\varepsilon(u_1,u_2)-B(u_1,u_2)]\varphi dxdt\\
&\quad+\int_0^r\int_{\mathbb{R}} u(t,x)DA(u_1,u_2)(\xi_{Rxx}\varphi+2\xi_{Rx}\varphi_x)dxdt\\
&=: I_1+I_2+I_3.
\end{aligned}\tag{3.43}$$

As showed in the estimates in Lemma 3.1 and Lemma 3.2 in [41], we can similarly prove

$$I_1\to 0,\quad \text{as } \varepsilon\to 0,\ \eta\to 0,\tag{3.44}$$

and

$$I_2\to 0,\quad \text{as } \varepsilon\to 0.\tag{3.45}$$

On the other hand, we know that

$$\int_0^r \int_{B_R} \varphi^2 dxdt \le 2Rr \|\varphi\|^2_{L^\infty((0,r)\times B_R)},$$

and $|\varphi(t,x)| \le C(\|g\|_{L^\infty(B_R)}, \|h\|_{L^\infty(0,r)}, r)$ independent of ε, η, and R, based on the maximum principle. Note that

$$\begin{aligned} &D\int_0^r \int_{\mathbb{R}} |u(t,x)A(u_1,u_2)\xi_{Rxx}\varphi|dxdt \\ &\le C\int_0^r \int_{B_R\setminus B_{R-1}} |u_1^m(t,x)-u_2^m(t,x)|dxdt \\ &\to 0, \quad \text{as } R\to\infty, \end{aligned}$$

because $u_1(t,x)$ and $u_2(t,x)$ converge to the same value uniformly for $x\to\pm\infty$, and

$$\begin{aligned} &2D\int_0^r \int_{\mathbb{R}} |u(t,x)\xi_{Rx}\varphi_x|dxdt \\ &\le C\int_0^r \int_{B_R\setminus B_{R-1}} |u_1^m(t,x)-u_2^m(t,x)||\varphi_x|dxdt \\ &\to 0, \quad \text{as } R\to\infty, \end{aligned}$$

due to the fact that $\|\varphi_x\|_{L^1((0,r)\times B_R)}$ is uniformly bounded according to Lemma 3.2 in [41]. Therefore, we prove

$$I_3 \to 0 \text{ as } R\to\infty. \tag{3.46}$$

Thus, taking $\varepsilon\to 0$, $\eta\to 0$ and $R\to\infty$, then (3.43) with the help of (3.44)-(3.46) implies

$$\int_0^r \int_{\mathbb{R}} u(t,x)g(x)h(t)dxdt = 0.$$

Since $g(x)$ and $h(t)$ are arbitrarily given, we then obtain

$$u(t,x)=0, \ \ i.e., \ \ u_1(t,x)=u_2(t,x), \quad \text{for } (t,x)\in[0,r]\times\mathbb{R}.$$

When $t\in[r,2r]$, we can similarly prove

$$u_1(t,x)=u_2(t,x), \quad \text{for } (t,x)\in[r,2r]\times\mathbb{R}.$$

Repeating the same procedure, we can prove

$$u_1(t,x) = u_2(t,x), \quad \text{for } (t,x) \in [nr, (n+1)r] \times \mathbb{R},$$

for any integer n. Therefore, we finally prove the uniqueness

$$u_1(t,x) = u_2(t,x), \quad \text{for } (t,x) \in \mathbb{R}_+ \times \mathbb{R}.$$

The proof is complete. □

Proof of Theorem 2.1. Based on Proposition 3.1 and Proposition 3.2, we immediately obtain Theorem 2.1. □

4. Global stability of traveling waves

Let $\phi(x+ct)$ be a given traveling wave connecting 0 and u_+ with $c \geq c^*$, and $u(t,x)$ be the solution of (1.1). Denote

$$v(t,x) := u(t,x) - \phi(x+ct),$$

and the initial datum

$$v_0(s,x) := u(s,x) - \phi(x+cs), \quad s \in [-r,0].$$

Then $v(t,x)$ satisfies

$$\begin{cases} \dfrac{\partial v}{\partial t} - D\dfrac{\partial^2(u^m - \phi^m)}{\partial x^2} + d(v+\phi) - d(\phi) \\ = \displaystyle\int_{\mathbb{R}} f_\alpha(y)[b(v(t-r,x-y) + \phi(x+ct-cr-y)) - b(\phi(x+ct-cr-y))]dy \\ v(t,x)|_{t=s} = v_0(s,x) = u_0(s,x) - \phi(x+cs), \; s \in [-r,0]. \end{cases} \tag{4.1}$$

Now we are going to prove the global stability of traveling waves in L^1-sense. The proof will be divided in four steps. In Step 1 and Step 2, we will prove the regularity of v in the weighted $L^1_\omega(\mathbb{R})$-space and the regular $L^1(\mathbb{R})$-space, respectively. Then in Step 3 we show the exponential stability of traveling wave in the weighted $L^1_\omega(\mathbb{R})$-space. In Step 4, based on such an exponential convergence, we finally prove the exponential stability of traveling wave in the regular $L^1(\mathbb{R})$-space. The adopted approach is the energy estimates with the compactness analysis together.

Step 1. L^1_ω-weighted regularity of $v \in L^\infty(\mathbb{R}_+, L^1_\omega(\mathbb{R}))$ for the perturbation equation.

Let $\mathbf{sgn}(s)$ be the sign function, and $J_\varepsilon(s)$ be its smooth approximation. Namely, $J_\varepsilon(s) \in C^1(\mathbb{R})$ and $J_\varepsilon(s) \to \mathbf{sgn}(s)$, $J'_\varepsilon(s) \to \delta(s)$, as $\varepsilon \to 0$, where $\delta(s)$ is the Delta function. Define

$$\hat{v} := u^m - \phi^m,$$

and a smooth concave function $\alpha_n(x) \in C_0^\infty(\mathbb{R})$ satisfying

$$0 \leq \alpha_n(x) \leq 1, \; |\alpha'_n(x)| \leq 1, \; \text{and } \alpha_n(x) = \begin{cases} 1, & |x| \leq n, \\ 0, & |x| \geq n+1, \end{cases}$$

where $n > 0$ is a given number.

Multiplying (4.1) by $J_\varepsilon(\hat{v})\omega(x)\alpha_n(x)$ with the weight function $\omega(x) = e^{-\lambda x}$, and integrating the resultant equation with respect to x over $\mathbb{R}$, we get

$$\begin{aligned}
&\int_{\mathbb{R}} \partial_t(v\omega\alpha_n)J_\varepsilon(\hat{v})dx - D\int_{\mathbb{R}} \hat{v}_{xx}J_\varepsilon(\hat{v})\omega\alpha_n dx \\
&\quad + \int_{\mathbb{R}} [d(v+\phi) - d(\phi)]J_\varepsilon(\hat{v})\omega\alpha_n dx \\
&= \int_{\mathbb{R}} \Big(\int_{\mathbb{R}} f_\alpha(y)[b(v_{cr}+\phi_{cr}) - b(\phi_{cr})]dy\Big) J_\varepsilon(\hat{v})\omega\alpha_n dx,
\end{aligned} \tag{4.2}$$

where

$$v_{cr} := v(t-r, x-y), \quad \text{and} \quad \phi_{cr} := \phi(x-y+ct-cr)).$$

Let $h_\varepsilon(\hat{v}) = \int_0^{\hat{v}} J_\varepsilon(s)ds$, then we know that $\partial_x h_\varepsilon(\hat{v}) = J_\varepsilon(\hat{v})\hat{v}_x$, and $h_\varepsilon(\hat{v}) \to |\hat{v}|$ as $\varepsilon \to 0$. Note that

$$\begin{aligned}
&D\int_{\mathbb{R}} \hat{v}_{xx}J_\varepsilon(\hat{v})\omega\alpha_n dx \\
&= -D\int_{\mathbb{R}} |\hat{v}_x|^2 J_\varepsilon'(\hat{v})\omega\alpha_n dx - D\int_{\mathbb{R}} \hat{v}_x J_\varepsilon(\hat{v})\omega'\alpha_n dx - D\int_{\mathbb{R}} \hat{v}_x J_\varepsilon(\hat{v})\omega\alpha_n' dx \\
&= -D\int_{\mathbb{R}} |\hat{v}_x|^2 J_\varepsilon'(\hat{v})\omega\alpha_n dx - D\int_{\mathbb{R}} [\omega'\alpha_n + \omega\alpha_n']\partial_x h_\varepsilon(\hat{v})dx \\
&= -D\int_{\mathbb{R}} |\hat{v}_x|^2 J_\varepsilon'(\hat{v})\omega\alpha_n dx + D\int_{\mathbb{R}} h_\varepsilon(\hat{v})(\omega''\alpha_n + 2\omega'\alpha_n' + \omega\alpha_n'')dx,
\end{aligned}$$

then (4.2) is reduced to

$$\begin{aligned}
&\int_{\mathbb{R}} \partial_t(v\omega\alpha_n)J_\varepsilon(\hat{v})dx + D\int_{\mathbb{R}} |\hat{v}_x|^2 J_\varepsilon'(\hat{v})\omega\alpha_n dx \\
&\quad - D\int_{\mathbb{R}} h_\varepsilon(\hat{v})(\omega''\alpha_n + 2\omega'\alpha_n' + \omega\alpha_n'')dx \\
&\quad + \int_{\mathbb{R}} [d(v+\phi) - d(\phi)]J_\varepsilon(\hat{v})\omega\alpha_n dx \\
&= \int_{\mathbb{R}} \Big(\int_{\mathbb{R}} f_\alpha(y)[b(v_{cr}+\phi_{cr}) - b(\phi_{cr})]dy\Big) J_\varepsilon(\hat{v})\omega\alpha_n dx.
\end{aligned} \tag{4.3}$$

From (3.33), we know that $v, \hat{v}_x \in L^2_{loc}(\mathbb{R})$. Thus, letting $\varepsilon \to 0$, we have

$$\int_{\mathbb{R}} J_\varepsilon'(\hat{v})\hat{v}_x v\omega\alpha_n dx \to \int_{\mathbb{R}} \delta(\hat{v})\hat{v}_x v\omega\alpha_n dx = 0, \quad \text{as } \varepsilon \to 0,$$

and further from (4.3) we reach, by taking $\varepsilon \to 0$,

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}} v\,\mathbf{sgn}(\hat{v})\omega\alpha_n dx + D\int_{\mathbb{R}} |\hat{v}_x|^2\delta(\hat{v})\omega\alpha_n dx \\
&\quad + \int_{\mathbb{R}} [d(v+\phi) - d(\phi)]\,\mathbf{sgn}(\hat{v})\omega\alpha_n dx \\
&\quad - D\int_{\mathbb{R}} |\hat{v}|(\omega''\alpha_n + 2\omega'\alpha_n' + \omega\alpha_n'')dx \\
&= \int_{\mathbb{R}} \Big(\int_{\mathbb{R}} f_\alpha(y)[b(v_{cr}+\phi_{cr}) - b(\phi_{cr})]dy\Big)\,\mathbf{sgn}(\hat{v})\omega\alpha_n dx. \qquad (4.4)
\end{aligned}$$

Note that

$$\mathbf{sgn}(\hat{v}) = \mathbf{sgn}(u^m - \phi^m) = \mathbf{sgn}(u-\phi) = \mathbf{sgn}(v),$$

and $d(v)$ is increasing, then

$$\mathbf{sgn}(\hat{v}) = \mathbf{sgn}(u^m - \phi^m) = \mathbf{sgn}(u-\phi) = \mathbf{sgn}(d(u) - d(\phi)),$$

we obtain from (4.4) that

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}} |v|\omega\alpha_n dx + D\int_{\mathbb{R}} |\hat{v}_x|^2\delta(\hat{v})\omega\alpha_n dx + \int_{\mathbb{R}} |d(v+\phi) - d(\phi)|\omega\alpha_n dx \\
&\quad - D\int_{\mathbb{R}} |\hat{v}|(\omega''\alpha_n + 2\omega'\alpha_n' + \omega\alpha_n'')dx \\
&= \int_{\mathbb{R}} \Big(\int_{\mathbb{R}} f_\alpha(y)[b(v_{cr}+\phi_{cr}) - b(\phi_{cr})]dy\Big)\,\mathbf{sgn}(\hat{v})\omega\alpha_n dx,
\end{aligned}$$

namely,

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}} |v|\omega\alpha_n dx + D\int_{\mathbb{R}} |\hat{v}_x|^2\delta(\hat{v})\omega\alpha_n dx \\
&\quad + \int_{\mathbb{R}} |d(v+\phi) - d(\phi)|\omega\alpha_n dx - D\int_{\mathbb{R}} |\hat{v}|\omega\alpha_n'' dx
\end{aligned}$$

$$= D\int_{\mathbb{R}} |\hat{v}|(\lambda^2\omega\alpha_n - 2\lambda\omega\alpha_n')dx$$
$$+\int_{\mathbb{R}} \Big(\int_{\mathbb{R}} f_\alpha(y)[b(v_{cr}+\phi_{cr}) - b(\phi_{cr})]dy\Big)\,\mathbf{sgn}(\hat{v})\omega\alpha_n dx. \tag{4.5}$$

Since $\alpha_n''(x) < 0$ for $x \in (-n-1,-n)\cup(n,n+1)$, and $\alpha_n'(x) > 0$ for $x \in (-n-1,-n)$, $\alpha_n'(x) < 0$ for $x \in (n,n+1)$, $0 < \alpha_n(x) < 1$ and $-1 < \alpha_n'(x) < 1$ for $x \in (-n-1,-n)\cup(n,n+1)$, and noting

$$|d(v+\phi) - d(\phi)| \geq d'(0)|v|,$$

due to $d'(s) > 0$ being increasing (see (H_3)), then from (4.5) we have

$$\frac{d}{dt}\int_{\mathbb{R}} |v|\omega\alpha_n dx + D\int_{\mathbb{R}} |\hat{v}_x|^2\delta(\hat{v})\omega\alpha_n dx$$
$$+ d'(0)\int_{\mathbb{R}} |v|\omega\alpha_n dx + D\int_{\mathbb{R}} \omega|\hat{v}\alpha_n''|dx$$
$$\leq D\int_{\mathbb{R}} |\hat{v}|(\lambda^2\omega\alpha_n - 2\lambda\omega\alpha_n')dx$$
$$+\int_{\mathbb{R}}\Big(\int_{\mathbb{R}} f_\alpha(y)|b(v_{cr}+\phi_{cr}) - b(\phi_{cr})|dy\Big)\omega\alpha_n dx$$
$$\leq C_0\int_{\mathbb{R}} |v|\omega\alpha_n dx + 2\lambda\int_n^{n+1} e^{-\lambda x}|\hat{v}|dx$$
$$+ C\int_{\mathbb{R}}\Big(\int_{\mathbb{R}} f_\alpha(y)|v_{cr}|dy\Big)\omega\alpha_n dx$$
$$\leq C_0\int_{\mathbb{R}} |v|\omega\alpha_n dx + C\int_n^{n+1} e^{-\lambda x}dx$$
$$+ C\int_{\mathbb{R}}\Big(\int_{\mathbb{R}} f_\alpha(y)\omega(x-y)|v_{cr}|dy\Big)\frac{\omega(x)}{\omega(x-y)}\alpha_n(x)dx$$
$$\leq C_0\int_{\mathbb{R}} |v|\omega\alpha_n dx + C + C\|v(t-r)\|_{L^1_\omega(\mathbb{R})}, \tag{4.6}$$

for some constant $C_0 > 0$. Namely, it is

$$\frac{d}{dt}\int_{\mathbb{R}} |v|\omega\alpha_n dx + [d'(0) - C_0]\int_{\mathbb{R}} |v|\omega\alpha_n dx \le C + C\|v(t-r)\|_{L^1_\omega(\mathbb{R})}. \tag{4.7}$$

When $t \in [0, r]$, we have

$$\|v(t-r)\|_{L^1_\omega(\mathbb{R})} = \|v_0(t-r)\|_{L^1_\omega(\mathbb{R})} \le C,$$

which together with (4.7) gives

$$\frac{d}{dt}\Big(e^{[d'(0)-C_0]t}\int_{\mathbb{R}} |v|\omega\alpha_n dx\Big) \le Ce^{[d'(0)-C_0]t}, \quad t \in [0, r],$$

and then

$$\begin{aligned}\int_{\mathbb{R}} |v|\omega\alpha_n dx &\le e^{-[d'(0)-C_0]t}\int_{\mathbb{R}} |v_0|\omega(x)\alpha_n(x)dx + C\int_0^t e^{-[d'(0)-C_0](t-s)}ds \\ &\le C\int_{\mathbb{R}} |v_0|\omega(x)\alpha_n(x)dx + C \\ &\le C, \quad \text{for } t \in [0, r].\end{aligned}$$

Taking $n \to \infty$, we have

$$\sup_{t\in[0,r]}\int_{\mathbb{R}} \omega(x)|v(t,x)|dx \le C.$$

Similarly, repeating the same procedure, for $t \in [nr, (n+1)r]$, we can prove

$$\sup_{t\in[nr,(n+1)r]}\int_{\mathbb{R}} \omega(x)|v(t,x)|dx \le C, \quad n = 1, 2, \cdots.$$

Thus, we have

$$\sup_{t\in[0,\infty)}\int_{\mathbb{R}} \omega(x)|v(t,x)|dx \le C.$$

This proves $v \in L^\infty(\mathbb{R}_+, L^1_\omega(\mathbb{R}))$.

Step 2. L^1-regularity: $v \in L^\infty(\mathbb{R}_+, L^1(\mathbb{R}))$.

Multiplying (4.1) by $J_\varepsilon(\hat{v})\alpha_n(x)$, and integrating the resultant equation with respect to x over $\mathbb{R}$, we get

$$
\begin{aligned}
&\int_{\mathbb{R}} \partial_t(v\alpha_n) J_\varepsilon(\hat{v}) dx - D \int_{\mathbb{R}} \hat{v}_{xx} J_\varepsilon(\hat{v}) \alpha_n dx \\
&\quad + \int_{\mathbb{R}} [d(v+\phi) - d(\phi)] J_\varepsilon(\hat{v}) \alpha_n dx \\
&= \int_{\mathbb{R}} \Big(\int_{\mathbb{R}} f_\alpha(y)[b(v_{cr}+\phi_{cr}) - b(\phi_{cr})] dy \Big) J_\varepsilon(\hat{v}) \alpha_n dx. \qquad (4.8)
\end{aligned}
$$

Taking $\varepsilon \to 0$, as showed in (4.5), we can similarly prove

$$
\begin{aligned}
&\frac{d}{dt} \int_{\mathbb{R}} |v| \alpha_n dx + D \int_{\mathbb{R}} |\hat{v}_x|^2 \delta(\hat{v}) \alpha_n dx \\
&\quad + \int_{\mathbb{R}} |d(v+\phi) - d(\phi)| \alpha_n dx - D \int_{\mathbb{R}} |\hat{v}| \alpha_n'' dx \\
&= \int_{\mathbb{R}} \Big(\int_{\mathbb{R}} f_\alpha(y)[b(v_{cr}+\phi_{cr}) - b(\phi_{cr})] dy \Big) \, \mathbf{sgn}(\hat{v}) \alpha_n dx \\
&\le C \int_{\mathbb{R}} \Big(\int_{\mathbb{R}} f_\alpha(y) |v(t-r, x-y-cr)| dy \Big) \alpha_n dx. \qquad (4.9)
\end{aligned}
$$

Taking $n \to \infty$, we further have

$$
\begin{aligned}
&\frac{d}{dt} \int_{\mathbb{R}} |v| dx + D \int_{\mathbb{R}} |\hat{v}_x|^2 \delta(\hat{v}) dx \\
&\quad + \int_{\mathbb{R}} |d(v+\phi) - d(\phi)| dx \\
&\le C \int_{\mathbb{R}} \Big(\int_{\mathbb{R}} f_\alpha(y) |v(t-r, x-y-cr)| dy \Big) dx \\
&\le C \|v(t-r)\|_{L^1(\mathbb{R})}. \qquad (4.10)
\end{aligned}
$$

When $t \in [0, r]$, (4.10) implies

$$
\frac{d}{dt} \int_{\mathbb{R}} |v| dx \le C \|v(t-r)\|_{L^1(\mathbb{R})} = \|v_0(t-r)\|_{L^1(\mathbb{R})},
$$

which implies

$$
\int_{\mathbb{R}} |v(t,x)| dx \le C, \quad \text{for } t \in [0, r].
$$

Similarly, we can prove

$$\int_{\mathbb{R}} |v(t,x)|dx \le C, \quad \text{for } t \in [nr, (n+1)r],$$

and further

$$\int_{\mathbb{R}} |v(t,x)|dx \le C, \quad \text{for } t \in [0,\infty).$$

Namely, we prove $v \in L^\infty(\mathbb{R}_+; L^1(\mathbb{R}))$.

Step 3. Exponential stability of $\phi(x+ct)$ in the weighted $L^1_\omega(\mathbb{R})$ space.

Now we consider $v = u(t,x) - \phi(x+ct)$ in the moving coordinate $\xi = x + ct$. Namely $v = v(t,\xi)$ satisfies the following equation

$$\begin{cases} \dfrac{\partial v}{\partial t} + c\dfrac{\partial v}{\partial \xi} - D\dfrac{\partial^2(u^m - \phi^m)}{\partial \xi^2} + d(v+\phi) - d(\phi) \\ \qquad = \displaystyle\int_{\mathbb{R}} f_\alpha(y)[b(v_{cr}+\phi_{cr}) - b(\phi_{cr})]dy, \\ v(s,\xi) = v_0(s,\xi) = u_0(s,\xi - cs) - \phi(\xi),\ s \in [-r,0], \end{cases} \tag{4.11}$$

with $v_{cr} := v(t-r, \xi - cr - y)$ and $\phi_{cr} := \phi(\xi - cr - y)$.

As we showed in (4.5), let us multiply (4.11) by $J_\varepsilon(\hat{v})\omega(\xi)\alpha_n(\xi)$ with the weight function $\omega(\xi) = e^{-\lambda\xi}$, and integrating the resultant equation with respect to ξ over $\mathbb{R}$, we get

$$\begin{aligned} &\frac{d}{dt}\int_{\mathbb{R}} |v|\omega\alpha_n d\xi + D\int_{\mathbb{R}} |\hat{v}_\xi|^2\delta(\hat{v})\omega\alpha_n d\xi \\ &\quad - D\int_{\mathbb{R}} |\hat{v}|\Big(\lambda^2\omega\alpha_n - 2\lambda\omega\alpha_n' + \omega\alpha_n''\Big)d\xi \\ &\quad + \int_{\mathbb{R}} |d(v+\phi) - d(\phi)|\omega\alpha_n d\xi \\ &\quad - c\int_{\mathbb{R}} \Big(|v|\alpha_n'(\xi)\omega(\xi) + |v|\alpha_n(\xi)\omega'(\xi)\Big)d\xi \\ &= \int_{\mathbb{R}} \Big(\int_{\mathbb{R}} f_\alpha(y)[b(v_{cr}+\phi_{cr}) - b(\phi_{cr})]dy\Big)\,\mathbf{sgn}(\hat{v})\omega\alpha_n d\xi. \end{aligned} \tag{4.12}$$

Since $v \in L^1_\omega(\mathbb{R})$ (see Step 1), from the definition of $\alpha_n(\xi)$, by taking $n \to \infty$ to (4.12), we obtain

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}} |v|\omega(\xi)d\xi + D\int_{\mathbb{R}} |\hat{v}_\xi|^2\delta(\hat{v})\omega(\xi)d\xi \\
&\quad + \int_{\mathbb{R}} |d(v+\phi)-d(\phi)|\omega(\xi)d\xi - c\int_{\mathbb{R}} |v|\omega'(\xi)d\xi \\
&\quad - D\int_{\mathbb{R}} |\hat{v}|(\lambda^2\omega(\xi))d\xi \\
&\le \int_{\mathbb{R}}\Big(\int_{\mathbb{R}} f_\alpha(y)|b(v_{cr}+\phi_{cr})-b(\phi_{cr})|dy\Big)\omega(\xi)d\xi.
\end{aligned} \tag{4.13}$$

Note that $d''(u) > 0$ and $b''(u) < 0$ for $u \in [0, u_+]$ (see (H_3)), there exist some points $\tilde{v}$ between ϕ and $v + \phi$, and $\tilde{v}_{cr}$ between ϕ_{cr} and $v_{cr} + \phi_{cr}$, such that

$$\begin{aligned}
&|d(v+\phi)-d(\phi)| = |d'(\tilde{v})||v| \ge d'(0)|v| \ge 0, \\
&|b(v_{cr}+\phi_{cr})-b(\phi_{cr})| = |b'(\tilde{v}_{cr})||v_{cr}| \le b'(0)|v_{cr}|.
\end{aligned}$$

These can guarantee that

$$\int_{\mathbb{R}} |d(v+\phi)-d(\phi)|\omega d\xi \ge d'(0)\int_{\mathbb{R}} |v|\omega d\xi, \tag{4.14}$$

and

$$\begin{aligned}
&\int_{\mathbb{R}}\Big(\int_{\mathbb{R}} f_\alpha(y)|b(v_{cr}+\phi_{cr})-b(\phi_{cr})|dy\Big)\omega(\xi)d\xi \\
&\le b'(0)\int_{\mathbb{R}}\Big(\int_{\mathbb{R}} f_\alpha(y)|v(t-r,\xi-y-cr)|dy\Big)\omega(\xi)d\xi \\
&= b'(0)\int_{\mathbb{R}}\Big(\int_{\mathbb{R}} f_\alpha(y)|v(t-r,\xi-y-cr)|\omega(\xi-y-cr)dy\Big)\frac{\omega(\xi)}{\omega(\xi-y-cr)}d\xi \\
&= b'(0)e^{-\lambda cr}\int_{\mathbb{R}}\Big(\int_{\mathbb{R}} f_\alpha(y)e^{-\lambda y}|v(t-r,\xi-y-cr)|\omega(\xi-y-cr)dy\Big)d\xi \\
&= b'(0)e^{-\lambda cr}\int_{\mathbb{R}} f_\alpha(y)e^{-\lambda y}\Big(\int_{\mathbb{R}} |v(t-r,\xi-y-cr)|\omega(\xi-y-cr)d\xi\Big)dy \\
&= b'(0)e^{-\lambda cr}\Big(\int_{\mathbb{R}} f_\alpha(y)e^{-\lambda y}dy\Big)\|v(t-r)\|_{L^1_\omega(\mathbb{R})} \\
&= b'(0)e^{-\lambda cr+\alpha\lambda^2}\|v(t-r)\|_{L^1_\omega(\mathbb{R})}.
\end{aligned} \tag{4.15}$$

Substituting (4.14) and (4.15) into (4.13), we get

$$\begin{aligned}&\frac{d}{dt}\int_{\mathbb{R}}|v|\omega(\xi)d\xi+D\int_{\mathbb{R}}|\hat{v}_\xi|^2\delta(\hat{v})\omega(\xi)d\xi\\&+\int_{\mathbb{R}}\Big(d'(0)+c\lambda-D\lambda^2\frac{|\hat{v}|}{|v|}\Big)|v(t,\xi)|\omega(\xi)d\xi\\&\le b'(0)e^{-\lambda cr+\alpha\lambda^2}\|v(t-r)\|_{L^1_\omega(\mathbb{R})}.\end{aligned}\tag{4.16}$$

Multiplying (4.16) by $e^{\mu_1 t}$ and integrating it with respect to t over $[0,t]$, where $\mu_1>0$ is a number which will be determined later, then we have

$$\begin{aligned}&e^{\mu_1 t}\int_{\mathbb{R}}|v(t,\xi)|\omega(\xi)d\xi+D\int_0^t e^{\mu_1\tau}\int_{\mathbb{R}}|\hat{v}_\xi(\tau,\xi)|^2\delta(\hat{v})\omega(\xi)d\xi d\tau\\&+\int_0^t e^{\mu_1\tau}\int_{\mathbb{R}}\Big(d'(0)+c\lambda-D\lambda^2\frac{|\hat{v}|}{|v|}-\mu_1\Big)|v(t,\xi)|\omega(\xi)d\xi d\tau\\&\le\int_{\mathbb{R}}|v_0(0,\xi)|\omega(\xi)d\xi+b'(0)e^{-\lambda cr+\alpha\lambda^2}\int_0^t e^{\mu_1\tau}\|v(\tau-r)\|_{L^1_\omega(\mathbb{R})}d\tau.\end{aligned}\tag{4.17}$$

Note that, by change of variables $\tau-r\to\tau$, we have

$$\begin{aligned}&b'(0)e^{-\lambda cr+\alpha\lambda^2}\int_0^t e^{\mu_1\tau}\|v(\tau-r)\|_{L^1_\omega(\mathbb{R})}d\tau\\&=b'(0)e^{-\lambda cr+\alpha\lambda^2}\int_{-r}^{t-r}e^{\mu_1(\tau+r)}\|v(\tau)\|_{L^1_\omega(\mathbb{R})}d\tau\\&\le b'(0)e^{-\lambda cr+\alpha\lambda^2}\int_0^t e^{\mu_1(\tau+r)}\|v(\tau)\|_{L^1_\omega(\mathbb{R})}d\tau\\&\quad+b'(0)e^{-\lambda cr+\alpha\lambda^2}\int_{-r}^0 e^{\mu_1(\tau+r)}\|v_0(\tau)\|_{L^1_\omega(\mathbb{R})}d\tau.\end{aligned}\tag{4.18}$$

Substituting (4.18) into (4.17), we get

$$e^{\mu_1 t}\int_{\mathbb{R}}|v(t,\xi)|\omega(\xi)d\xi+D\int_0^t e^{\mu_1\tau}\int_{\mathbb{R}}|\hat{v}_\xi(\tau,\xi)|^2\delta(\hat{v})\omega(\xi)d\xi d\tau$$

$$+\int_0^t e^{\mu_1\tau}\int_{\mathbb{R}} \mathcal{A}_{c,\mu_1}(\tau,\xi)|v(t,\xi)|\omega(\xi)d\xi d\tau$$

$$\le \int_{\mathbb{R}} |v_0(0,\xi)|\omega(\xi)d\xi + b'(0)e^{-\lambda cr+\alpha\lambda^2}\int_{-r}^{0} e^{\mu_1(\tau+r)}\|v_0(\tau)\|_{L^1_\omega(\mathbb{R})}d\tau, \tag{4.19}$$

where

$$\mathcal{A}_{c,\mu_1}(t,\xi) := d'(0) + c\lambda - D\lambda^2\frac{|\hat{v}(t,\xi)|}{|v(t,\xi)|} - \mu_1 - b'(0)e^{-\lambda cr+\alpha\lambda^2}e^{\mu_1 r}. \tag{4.20}$$

Since

$$\begin{aligned}\frac{|\hat{v}(t,\xi)|}{|v(t,\xi)|} &= \frac{|u^m(t,\xi)-\phi^m(\xi)|}{|u(t,\xi)-\phi(\xi)|}\\ &= \begin{cases} m\tilde{\phi}^{m-1}, & \text{as } u\ne\phi,\ \text{for some } \tilde{\phi}\in[0,u_+],\\ m\phi^{m-1}, & \text{as } u=\phi,\end{cases}\\ &\le mu_+^{m-1},\end{aligned} \tag{4.21}$$

there exists a large wave speed $c > \bar{c}_*$, by taking $\lambda \in (\bar{\lambda}_1, \bar{\lambda}_2)$, such that

$$\begin{aligned}\mathcal{B}_c(t,\xi) :&= d'(0) + c\lambda - D\lambda^2\frac{|\hat{v}(t,\xi)|}{|v(t,\xi)|} - b'(0)e^{-\lambda cr+\alpha\lambda^2}\\ &\ge d'(0) + c\lambda - D\lambda^2 mu_+^{m-1} - b'(0)e^{-\lambda cr+\alpha\lambda^2}\\ &=: C_2 > 0 \quad (\text{see } (2.8)).\end{aligned} \tag{4.22}$$

Thus, there exists a small $0<\mu_1\ll 1$ such that

$$\begin{aligned}\mathcal{A}_{c,\mu_1}(t,\xi) :&= d'(0) + c\lambda - D\lambda^2\frac{|\hat{v}(t,\xi)|}{|v(t,\xi)|} - \mu_1 - b'(0)e^{-\lambda cr+\alpha\lambda^2}e^{\mu_1 r}\\ &\ge d'(0) + c\lambda - D\lambda^2 mu_+^{m-1} - b'(0)e^{-\lambda cr+\alpha\lambda^2}\\ &\quad - \mu_1 - b'(0)e^{-\lambda cr+\alpha\lambda^2}(e^{\mu_1 r}-1)\\ &\ge C_2 - \mu_1 - b'(0)e^{-\lambda cr+\alpha\lambda^2}(e^{\mu_1 r}-1)\\ &> 0.\end{aligned} \tag{4.23}$$

Combining (4.19) and (4.23), we prove

$$\|v(t)\|_{L^1_\omega(\mathbb{R})} \le e^{-\mu_1 t}\Big(\|v_0(0)\|_{L^1_\omega(\mathbb{R})} + C\|v_0\|_{L^1([-r,0];L^1_\omega(\mathbb{R}))}\Big). \tag{4.24}$$

Step 4. Exponential stability of traveling wave in $L^1(\mathbb{R})$.

Since $d'(u_+) > b'(u_+)$, and $\phi(\xi) \in C^1(\mathbb{R})$ is increasing in $[0, u_+]$, there exists a sufficiently large number $\xi_0 \gg 1$ such that

$$d'(\phi(\xi)) > b'(\phi(\xi)), \quad \text{for } \xi \in [\xi_0, \infty).$$

Let us introduce a new smooth function $\bar{\alpha}_n(\xi) \in C_0^\infty(\mathbb{R})$ such that

$$\bar{\alpha}_n(\xi) = \begin{cases} 0, & \xi \le \xi_0 - 1, \\ \tilde{\alpha}(\xi), & \xi_0 - 1 < \xi < \xi_0, \\ 1, & \xi_0 < \xi \le n, \\ 0, & \xi \ge n+1, \end{cases}$$

and

$$0 \le \bar{\alpha}_n(\xi) \le 1, \ |\bar{\alpha}_n'(\xi)| \le 1,$$

where n satisfying $n > \xi_0$ is a given number, and $\tilde{\alpha}(\xi) \in C^\infty(\xi_0 - 1, \xi_0)$ satisfies

$$\tilde{\alpha}((\xi_0 - 1)^+) = 0, \quad \tilde{\alpha}((\xi_0)^-) = 1,$$

and

$$\partial_\xi^j \tilde{\alpha}|_{\xi=(\xi_0-1)^+} = 0, \quad \partial_\xi^j \tilde{\alpha}|_{\xi=(\xi_0)^-} = 0, \ j = 1, 2 \cdots .$$

Multiplying (4.11) by $J_\varepsilon(\hat{v})\bar{\alpha}_n(\xi)$, and integrating the resultant equation with respect to ξ over $\mathbb{R}$, as showed in (4.12), where the nonlocal term is switched to, by taking $\xi - y - cr \to y$,

$$\int_{\mathbb{R}} f_\alpha(y)[b(v_{cr} + \phi_{cr}) - b(\phi_{cr})]dy = \int_{\mathbb{R}} f_\alpha(\xi - y - cr)[b(v(t-r, y) + \phi(y)) - b(\phi(y))]dy,$$

then we can similarly have

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}} |v|\bar{\alpha}_n(\xi)d\xi + D\int_{\mathbb{R}} |\hat{v}_\xi|^2 \delta(\hat{v})\bar{\alpha}_n(\xi)d\xi \\
&\quad - D\int_{\mathbb{R}} |v|\bar{\alpha}_n''(\xi)d\xi - c\int_{\mathbb{R}} |v|\bar{\alpha}_n'(\xi)d\xi \\
&\quad + \int_{\mathbb{R}} |d(v+\phi) - d(\phi)|\bar{\alpha}_n(\xi)d\xi \\
&= \int_{\mathbb{R}} \Big(\int_{\mathbb{R}} f_\alpha(y - \xi - cr)[b(v(t-r, y) + \phi(y)) - b(\phi(y))]dy \Big) \mathbf{sgn}(\hat{v})\bar{\alpha}_n(\xi)d\xi
\end{aligned}$$

$$= \int_{\mathbb{R}} [b(v(t-r,y)+\phi(y)) - b(\phi(y))]\Big(\int_{\mathbb{R}} f_\alpha(y-\xi-cr)\mathbf{sgn}(\hat{v})\bar{\alpha}_n(\xi)d\xi\Big)dy. \quad (4.25)$$

Taking $n \to \infty$ to (4.25), and noting the properties of $\bar{\alpha}_n(\xi)$, we get

$$\begin{aligned}
&\frac{d}{dt}\int_{\xi_0-1}^{\xi_0} |v|\tilde{\alpha}(\xi)d\xi + \frac{d}{dt}\int_{\xi_0}^{\infty}|v|d\xi \\
&\quad + D\int_{\xi_0-1}^{\xi_0} |\hat{v}_\xi|^2\delta(\hat{v})\tilde{\alpha}(\xi)d\xi + D\int_{\xi_0}^{\infty}|\hat{v}_\xi|^2\delta(\hat{v})d\xi \\
&\quad - D\int_{\xi_0-1}^{\xi_0} |v|\tilde{\alpha}''(\xi)d\xi - c\int_{\xi_0-1}^{\xi_0} |v|\tilde{\alpha}'(\xi)d\xi \\
&\quad + \int_{\xi_0-1}^{\xi_0} |d(v+\phi)-d(\phi)|\tilde{\alpha}(\xi)d\xi + \int_{\xi_0}^{\infty}|d(v+\phi)-d(\phi)|d\xi \\
&\le \int_{\mathbb{R}} |b(v(t-r,y)+\phi(y)) - b(\phi(y))|\Big(\int_{\mathbb{R}} f_\alpha(y-\xi-cr)d\xi\Big)dy \\
&\le \int_{\mathbb{R}} |b(v(t-r,y)+\phi(y)) - b(\phi(y))|dy,
\end{aligned}$$

namely,

$$\begin{aligned}
&\frac{d}{dt}\int_{\xi_0-1}^{\xi_0} |v|\tilde{\alpha}(\xi)d\xi + \frac{d}{dt}\int_{\xi_0}^{\infty}|v|d\xi \\
&\quad + D\int_{\xi_0}^{\infty}|\hat{v}_\xi|^2\delta(\hat{v})d\xi + D\int_{\xi_0-1}^{\xi_0} |\hat{v}_\xi|^2\delta(\hat{v})\tilde{\alpha}(\xi)d\xi \\
&\quad + \int_{\xi_0}^{\infty}|d(v+\phi)-d(\phi)|d\xi \\
&\le D\int_{\xi_0-1}^{\xi_0} |v|\tilde{\alpha}''(\xi)d\xi + c\int_{\xi_0-1}^{\xi_0} |v|\tilde{\alpha}'(\xi)d\xi
\end{aligned}$$

$$
\begin{aligned}
&- \int_{\xi_0-1}^{\xi_0} |d(v+\phi) - d(\phi)|\tilde{\alpha}(\xi)d\xi \\
&+ \int_{-\infty}^{\xi_0} |b(v(t-r,y)+\phi(y)) - b(\phi(y))|dy \\
&+ \int_{\xi_0}^{\infty} |b(v(t-r,y)+\phi(y)) - b(\phi(y))|dy. \qquad (4.26)
\end{aligned}
$$

Since

$$
D\int_{\xi_0-1}^{\xi_0} |\hat{v}_\xi|^2\delta(\hat{v})\tilde{\alpha}(\xi)d\xi \geq 0,
$$

and from (H_3) and the mean value theorem that there exists $\tilde{\phi} \in (\phi(\xi_0), u_+)$ such that

$$
|d(v+\phi) - d(\phi)| = |d'(\tilde{\phi})||v| \geq d'(\phi(\xi_0))|v|, \quad \text{for } \xi \in (\xi_0, \infty),
$$

and

$$
|b(v_{cr}+\phi_{cr}) - b(\phi_{cr})| = |b'(\tilde{\tilde{\phi}})v_{cr}| \leq b'(\phi(\xi_0))|v_{cr}|, \quad \text{for } \xi \in (\xi_0, \infty),
$$

where $\tilde{\tilde{\phi}}$ is a mean value between $\phi(\xi_0)$ and u_+, and we also note that the exponential decays showed in (4.18) and (4.24), which imply

$$
\int_{\xi_0-1}^{\xi_0} |v(t,\xi)|d\xi \leq Ce^{-\mu_1 t},
$$

and

$$
\int_{-\infty}^{\xi_0} |v(t,\xi)|d\xi \leq C\int_{-\infty}^{\xi_0} \omega(\xi)|v(t,\xi)|d\xi \leq Ce^{-\mu_1 t}, \qquad (4.27)
$$

then (4.26) is reduced to

$$
\frac{d}{dt}\int_{\xi_0-1}^{\xi_0} |v|\tilde{\alpha}(\xi)d\xi + \frac{d}{dt}\int_{\xi_0}^{\infty} |v|d\xi
$$

$$
\begin{aligned}
&+ D\int_{\xi_0}^{\infty} |\hat{v}_\xi|^2\delta(\hat{v})d\xi + d'(\phi(\xi_0))\int_{\xi_0}^{\infty} |v(t,\xi)|d\xi \\
&\le C\int_{\xi_0-1}^{\xi_0} |v(t,\xi)|d\xi + C\int_{\xi_0-1}^{\xi_0} |v(t-r,\xi)|d\xi \\
&+ \int_{-\infty}^{\xi_0} |b(v(t-r,y)+\phi(y)) - b(\phi(y))|dy \\
&+ \int_{\xi_0}^{\infty} |b(v(t-r,y)+\phi(y)) - b(\phi(y))|dy \\
&\le C\int_{\xi_0-1}^{\xi_0} |v(t,\xi)|d\xi + C\int_{\xi_0-1}^{\xi_0} |v(t-r,\xi)|d\xi \\
&+ C\int_{-\infty}^{\xi_0} |v(t-r,y)|dy \\
&+ b'(\phi(\xi_0))\int_{\xi_0}^{\infty} |v(t-r,y)|dy \\
&\le Ce^{-\mu_1 t} + b'(\phi(\xi_0))\int_{\xi_0}^{\infty} |v(t-r,y)|dy. \qquad (4.28)
\end{aligned}
$$

Multiplying (4.28) by $e^{\mu_2 t}$ and integrating it with respect to t, where $0 < \mu_2 < \mu_1$ will be determined later, then we have

$$
\begin{aligned}
&e^{\mu_2 t}\int_{\xi_0-1}^{\xi_0} |v|\tilde{\alpha}(\xi)d\xi + e^{\mu_2 t}\int_{\xi_0}^{\infty} |v|d\xi \\
&+ D\int_0^t e^{\mu_2\tau}\int_{\xi_0}^{\infty} |\hat{v}_\xi|^2\delta(\hat{v})d\xi d\tau + d'(\phi(\xi_0))\int_0^t e^{\mu_2\tau}\int_{\xi_0}^{\infty} |v(\tau,\xi)|d\xi d\tau \\
&\le C\int_{\xi_0-1}^{\xi_0} |v_0(0,\xi)|d\xi + C\int_{\xi_0}^{\infty} |v_0(0,\xi)|d\xi
\end{aligned}
$$

$$+\mu_2\int_0^t e^{\mu_2\tau}\int_{\xi_0-1}^{\xi_0}|v|\tilde{\alpha}(\xi)d\xi d\tau+\mu_2\int_0^t e^{\mu_2\tau}\int_{\xi_0}^{\infty}|v(\tau,\xi)|d\xi d\tau$$

$$+C\int_0^t e^{-(\mu_1-\mu_2)\tau}d\tau+b'(\phi(\xi_0))\int_0^t e^{\mu_2\tau}\int_{\xi_0}^{\infty}|v(\tau-r,y)|dyd\tau$$

$$\le C\|v_0(0)\|_{L^1(\mathbb{R})}+\mu_2\int_0^t e^{\mu_2\tau}\int_{\xi_0}^{\infty}|v(\tau,\xi)|d\xi d\tau$$

$$+b'(\phi(\xi_0))\int_0^{t-r} e^{\mu_2\tau}\int_{\xi_0}^{\infty}|v(\tau,y)|dyd\tau$$

$$+b'(\phi(\xi_0))\int_{-r}^{0} e^{\mu_2\tau}\int_{\xi_0}^{\infty}|v_0(\tau,y)|dyd\tau. \tag{4.29}$$

Thus, (4.29) can be estimated as

$$e^{\mu_2 t}\int_{\xi_0}^{\infty}|v(t,\xi)|d\xi+[d'(\phi(\xi_0))-b'(\phi(\xi_0))-\mu_2]\int_0^t e^{\mu_2\tau}\int_{\xi_0}^{\infty}|v(\tau,\xi)|d\xi d\tau$$

$$\le C\|v_0(0)\|_{L^1(\mathbb{R})}+b'(\phi(\xi_0))\int_{-r}^{0} e^{\mu_2\tau}\int_{\xi_0}^{\infty}|v_0(\tau,y)|dyd\tau. \tag{4.30}$$

Let μ_2 be small enough such that

$$0<\mu_2<\min\{\mu_1, d'(\phi(\xi_0))-b'(\phi(\xi_0))\}.$$

Thus, from (4.30), we get

$$\int_{\xi_0}^{\infty}|v(t,\xi)|d\xi\le Ce^{-\mu_2 t}. \tag{4.31}$$

Combining (4.27) and (4.31), and taking

$$\mu=\min\{\mu_1,\mu_2\}=\mu_2,$$

we finally prove the exponential L^1-stability for the traveling wave:

$$\|v(t)\|_{L^1(\mathbb{R})}\le Ce^{-\mu t}.$$

The proof is complete.

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