



STATIONARY SOLUTION TO STOCHASTICALLY FORCED EULER-POISSON EQUATIONS IN BOUNDED DOMAIN: PART 1. 3-D INSULATING BOUNDARY

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ABSTRACT. This paper is concerned with 3-D stochastic Euler-Poisson equations with insulating boundary conditions forced by the Wiener process. We first establish the global existence and uniqueness of the solution to the system, then we prove that the solution converges to its steady-state time-asymptotically. To obtain the converging rate, although the weighted energy estimates are unnecessary for the deterministic case, we must develop the weighted energy estimates for stochastic system. Moreover, we observe that the invariant measure is just the Dirac measure generated by the steady-state, in which the time-exponential convergence rate to the steady-state plays an essential role.

1. Introduction. The Euler-Poisson equations are important in the analysis and design of semiconductor devices, offering a more precise description of physical phenomena [21] compared to the conventional drift-diffusion model. Furthermore, in the extreme ultraviolet (EUV) lithography, stochastic effects sometimes cause unwanted defects and pattern roughness in chips [3] that may impact the performance of a chip or cause a device to fail. Hence, there is a pressing need to investigate the dynamical model of semiconductors perturbed by stochastic forces within mathematical frameworks. The stochastically forced Euler-Poisson equations (SEP for

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short) in a bounded smooth domain $U \subset \mathbb{R}^3$ reads as

$$\begin{cases} \rho_t + \nabla \cdot (\rho \mathbf{u}) = 0, \\ d(\rho \mathbf{u}) + (\nabla \cdot (\rho \mathbf{u} \otimes \mathbf{u}) + \nabla P(\rho) - \rho \nabla \Phi) dt = -\frac{\rho \mathbf{u}}{\tau} dt + \mathbb{F}(\rho, \mathbf{u}) dW, \\ \Delta \Phi = \rho - b, \end{cases} \quad (1)$$

where “d” in (1) is the differential notation with respect to time t , in comparison to gradient ∇ and Laplacian Δ for spatial derivatives, ρ is the electron density of semiconductors, $\mathbf{u}$ denotes the particle velocity, $P(\rho)$ is the pressure, Φ is the electrostatic potential, τ is the velocity relaxation time, and $b(x)$ is called the doping profile, which is positive and immobile. The above mentioned unknowns $\rho = \rho(\omega, t, x)$, $\mathbf{u} = \mathbf{u}(\omega, t, x)$, $\Phi = \Phi(\omega, t, x)$, and $P(\rho) = P(\rho(\omega, t, x))$ are stochastic processes as functions with respect to ω , t , and x , where ω is a sample in the complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. For convenience, we use the simplified notation ρ , $\mathbf{u}$, Φ , and $P(\rho)$ here and hereafter. W is an $\mathcal{H}$ -valued cylindrical Brownian motion defined on the filtrated probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where $\mathcal{H}$ is an auxiliary separable Hilbert space, $\mathcal{F}$ is the filtration, see the definitions of filtration and Wiener process in Appendix A.

Let $\{e_k\}_{k=1}^{+\infty}$ be an orthonormal basis in $\mathcal{H}$. Then, the Brownian motion W can be written in the form of $W = \sum_{k=1}^{+\infty} e_k \beta_k$, where $\{\beta_k(t); k \in \mathbb{N}, t \geq 0\}$ is a sequence of independent, real-valued standard Brownian motions. Let H be a Bochner space. $\mathbb{F}(\rho, \mathbf{u})$ is an H -valued operator from $\mathcal{H}$ to $\mathcal{H}$. Denoting the inner product in $\mathcal{H}$ as $\langle \cdot, \cdot \rangle$, the inner product

$$\langle \mathbb{F}(\rho, \mathbf{u}), e_k \rangle = \mathbf{F}_k(\rho, \mathbf{u}) \quad (2)$$

is an H -valued vector function, which shows the strength of the external stochastic forces by

$$\mathbb{F}(\rho, \mathbf{u}) dW = \sum_{k=1}^{+\infty} \mathbf{F}_k(\rho, \mathbf{u}) d\beta_k. \quad (3)$$

Throughout the paper, we assume that

$$\mathbf{F}_k(\rho, \mathbf{u}) = a_k \rho \mathbf{u} Y(\rho, \mathbf{u}), \quad (4)$$

where a_k are positive constants, and $Y(\rho, \mathbf{u})$ is a smooth function of ρ and $\mathbf{u}$, where $Y(\rho, \mathbf{u})$ can be bounded by the homogeneous polynomials.

Subjected to the stochastic Euler-Poisson equations (1), the proposed boundary is the insulating boundary:

$$\mathbf{u} \cdot \nu = 0, \quad \nabla \Phi \cdot \nu = 0, \quad (5)$$

where ν is the outer normal vector of U . The initial data is

$$(\rho, \mathbf{u}_0, \Phi)|_{t=0} = (\rho_0(\omega, x), \mathbf{u}_0(\omega, x), \Phi_0(\omega, x)), \quad (6)$$

which is given in the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, $\rho_0(\omega, x) > 0$. Here and hereafter, we simply denote the initial data by $(\rho_0, \mathbf{u}_0, \Phi_0)$ without confusion.

The hydrodynamic model of semiconductors was first introduced by Blotekjaer [4], which is the deterministically dynamical model presented by the Euler-Poisson equations mathematically. For the 1-D case, the initial boundary value problems of the Euler-Poisson equations with the insulating boundary and the Ohmic contact boundary were studied by Hsiao-Yang [29] and Li-Markowich-Mei [39], respectively, where the solutions are shown to converge to the corresponding subsonic steady-states time-asymptotically, where the doping profile is required to

be flat: $|b'(x)| \ll 1$. Such a restriction was then released by Nishibata-Suzuki [44] and Guo-Strauss [21] independently. For the N -D case, Guo-Strauss [21] first considered the deterministic 3-D Euler-Poisson equations in a bounded domain with insulating boundary, and showed the convergence of solutions to the 3-D subsonic steady-states. Subsequently, Mei-Wu-Zhang [42] investigated the convergence to the steady-states for the N -D radial Euler-Poisson equations with the Ohmic contact boundary. For the whole space without boundary effects, the Cauchy problems to deterministic Euler-Poisson equations were extensively studied in [10, 30, 31, 32, 33, 36]. For the case of free boundary with vacuum, we refer to [40, 43, 51] and the references therein. For the formulation of singularities in compressible Euler-Poisson equations and the large time behavior of Euler equations with damping, one can refer to [49] and [50], respectively.

When the hydrodynamic model of semiconductors is counted into the stochastic affections, it then becomes the stochastic Euler-Poisson equations with uncertain extra disturbances, see (1) with the Wiener process $\mathbb{F}(\rho, \mathbf{u}) dW$. This is a new model for semiconductor devices and has not been touched yet. The main issue of the paper is to investigate this 3-D SEP in a bounded domain with insulating boundary and prove the convergence of solutions to the stochastic steady states. The coefficient function of the Wiener process $\mathbf{F}_k(\rho, \mathbf{u})$, depending on the solutions ρ and $\mathbf{u}$, is called the multiplicative noise. In most cases, the multiplicative noise magnifies the perturbation, thereby complicating the well-posedness of solutions for evolution systems. The stochastic forces are at most Hölder- $\frac{1}{2}$ -continuous in time t , resulting in reduced regularity of velocity with respect to time. So, from a mathematical standpoint, the study of the stochastic problem helps us to study how the solutions to the stochastic Euler-Poisson equations behave in the absence of strong regularity in time. Further, this encourages exploring whether the desirable property remains under the influence of particular types of noise. This is the first attempt to study the asymptotic behavior of solutions to stochastic 3-D Euler-Poisson equations.

For stochastic evolution systems, the solution is called the stationary solution provided that the increment of solutions during evolution is time-independent. Originally, the study of stationary measures dates back to the works of Hopf [28], Doob [9], Doob [12], Halmos [22, 23], Feller [13], and Harris and Robbins [25, 24], who contributed to the theory of discrete Markov processes from 1930s to 1950s. The study of invariant measure of fluid models dates back to Cruzeiro [8] for stochastic incompressible Navier-Stokes equations in 1989, by Galerkin approximation with dimensions $D \geq 2$. Flandoli [14] proved the existence of an invariant measure by the “remote start” method for 2-D incompressible Navier-Stokes equations in 1994. One year later, Flandoli-Gatarek [15] showed the existence of stationary solutions for 3-D incompressible Navier-Stokes equations by a different method with [8]. In 2002, Mattingly [41] proved the existence of exponentially attracting invariant measure with respect to initial data for the incompressible N-S equations. Later, Goldys-Maslowski [20] showed that transition measures of the 2-D stochastic Navier-Stokes equations converge exponentially fast to the corresponding invariant measures in the distance of total variation. Then, for the 3-D case, Da Prato and Debussche [46] constructed a transition semigroup for 3-D stochastic Navier-Stokes equations without the uniqueness, which allows for rather irregular solutions. Flandoli-Romito [17] used the classical Stroock-Varadhan type argument to find the almost-sure Markov

selection. The above works are for the incompressible case. For stochastic compressible Navier-Stokes equations, Breit-Feireisl-Hofmanová-Maslowski [6] proved the existence of stationary solutions. Compared to the Navier-Stokes equations, the regularity effect of viscosity is lost for the Euler system. Hofmanová-Zhu-Zhu [27] selected the dissipative global martingale solutions to the stochastic incompressible Euler system, and obtained the non-uniqueness of strong Markov solutions. Very recently, they [26] showed that stationary solutions to the Euler equations is a vanishing viscosities limit in the law of stationary analytically weak solutions to the Navier-Stokes equations. In terms of non-uniqueness studies, some scholars believe that a certain stochastic perturbation can provide a regularizing effect of the underlying PDE dynamics. For instance, Flandoli-Luo [16] showed that a noise of transport type prevents a vorticity blow-up in the incompressible Navier-Stokes equations. A linear multiplicative noise prevents the blow up of the velocity with high probability for the 3-D Euler system, which was shown by Glatt-Holtz-Vicol [19]. Gess-Souganidis [18] investigated the large-time behavior and established the existence of an invariant measure for stochastic scalar conservation laws, demonstrating that an algebraic decay rate in time holds. In their work, they introduced a particular type of noise that provided stronger regularization properties for the problem. Then, Dong-Zhang-Zhang [11] proved the existence of stationary solutions with multiplicative noise. For stochastic conservation laws, Da Prato-Gatarek studied the existence and uniqueness of invariant measures for the stochastic Burgers equation [47]. Da Prato-Zabczyk listed the basic theory of stationary solutions of general stochastic PDEs in view of invariant measures in [48]. Bedrossian-Liss [1] gave the existence of stationary measure for the stochastic ordinary differential equations with a nonlinear term. To the best of our knowledge, the stationary solutions of SEP have not been explored previously. For our SEP, the electrostatic potential term $\rho \nabla \Phi \, dt$ and the relaxation term $\frac{\rho \mathbf{u}}{\tau} \, dt$ are actually damping terms providing better regularity than the Euler equations. In this paper, we show the existence and uniqueness of invariant measure in a more regular space.

It is worth noting that the stationary solution we consider is in view of invariant measures. In this paper, the concepts of stationary solution for stochastically forced system (1) and steady state $(\bar{\rho}(\omega, x), \bar{\mathbf{u}}(\omega, x), \bar{\Phi}(\omega, x))$ for the following deterministic system (7) are distinguished. First, we establish the global existence and uniqueness of perturbed solutions around the steady state for the Euler-Poisson equations. Subsequently, we demonstrate the existence of stationary solutions and invariant measure based on the *a priori* energy estimates and weighted energy estimates.

We recall the steady state and recount the basic conclusion on the existence and uniqueness of $(\bar{\rho}(\omega, x), \bar{\mathbf{u}}(\omega, x), \bar{\Phi}(\omega, x))$. Within the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, the steady state $(\bar{\rho}(\omega, x), \bar{\mathbf{u}}(\omega, x), \bar{\Phi}(\omega, x))$ is assumed to adhere to the following equations:

$$\begin{cases} \nabla \cdot (\bar{\rho} \bar{\mathbf{u}}) = 0, \\ \nabla \cdot (\bar{\rho} \bar{\mathbf{u}} \otimes \bar{\mathbf{u}}) + \nabla P(\bar{\rho}) - \bar{\rho} \nabla \bar{\Phi} = -\frac{\bar{\rho} \bar{\mathbf{u}}}{\tau}, \\ \Delta \bar{\Phi} = \bar{\rho} - b(x). \end{cases} \quad (7)$$

For the deterministic steady state with insulating boundary condition, Guo-Strauss [21] gave a proof for existence and uniqueness of $(\bar{\rho}(x), \bar{\mathbf{u}}(x), \bar{\Phi}(x)) = (\bar{\rho}(x), 0, \bar{\Phi}(x))$. By substituting (7)₁ into (7)₂, and acting $\nabla \cdot$ on (7)₂, we have

$$\nabla \cdot (\bar{\rho} \bar{\mathbf{u}} \cdot \nabla \bar{\mathbf{u}}) + \Delta P(\bar{\rho}) - \nabla \cdot (\bar{\rho} \nabla \bar{\Phi}) = 0. \quad (8)$$

If $\bar{\mathbf{u}} = 0$, it reduces to

$$P'(\bar{\rho}) \triangle \bar{\rho} + P''(\bar{\rho}) |\nabla \bar{\rho}|^2 - \nabla \bar{\rho} \nabla \bar{\Phi} - \bar{\rho}(\bar{\rho} - b) = 0, \quad (9)$$

where $P'(\bar{\rho}) > 0 = |\bar{\mathbf{u}}|^2$ so that the equation of $\bar{\rho}$ given in (9) is uniformly elliptic. In this paper, we consider the subsonic case, i.e., the condition $P'(\rho) > |\mathbf{u}|^2$ holds. For every $\omega \in \Omega$, $(\bar{\rho}(\omega, x), 0, \bar{\Phi}(\omega, x)) = (\bar{\rho}(x), 0, \bar{\Phi}(x))$ is the unique solution of (7), which is called the steady state in this paper. We will denote $(\bar{\rho}(\omega, x), 0, \bar{\Phi}(\omega, x))$ by $(\bar{\rho}, 0, \bar{\Phi})$ for convenience in the following. The law of steady state is the Dirac measure $\delta_{\bar{\rho}} \times \delta_0 \times \delta_{\bar{\Phi}}$, see Appendix A. We conclude the following lemma for steady state. Here, $\bar{U}$ denotes the closed set of U .

Proposition 1.1. *Let $b(x) > 0$ in $\bar{U}$ and $P : (0, \infty) \rightarrow (0, \infty)$ be smooth with $P(0) = 0$. Then, there exists $(\bar{\rho}, \bar{\mathbf{u}}, \bar{\Phi})$, $\forall \omega \in \Omega$, a unique smooth steady-state solution of the insulating problem with Neumann boundary condition*

$$\frac{\partial \bar{\Phi}}{\partial \nu} \Big|_{\partial U} \equiv 0, \quad (10)$$

such that there holds

$$\bar{\rho} > \underline{\rho} > 0, \quad |\nabla \bar{\rho}| > 0, \quad \bar{\Phi} > 0, \quad \forall x \in \bar{U}, \quad \mathbb{P} \text{ a.s.}, \quad (11)$$

where $\underline{\rho}$ is a constant, and

$$\int_U \bar{\rho} \, dx = \int_U b(x) \, dx, \quad \mathbb{P} \text{ a.s.} \quad (12)$$

Let $Q(\rho)$ be such that $\nabla Q(\rho) = \nabla \Phi$ (cf. [21]). Then, the steady state satisfies

$$\nabla \bar{Q}(\bar{\rho}) = \nabla \bar{\Phi}, \quad \triangle \bar{\Phi} = \bar{\rho} - b(x). \quad (13)$$

We consider the solutions $(\rho, \mathbf{u}, \Phi)$ of the hydrodynamic system around the steady state $(\bar{\rho}, 0, \bar{\Phi})$, and we denote

$$\sigma = \rho - \bar{\rho}, \quad \phi = \Phi - \bar{\Phi}. \quad (14)$$

Our main result is on the existence of solutions near the steady state, and asymptotic stability for the insulating boundary condition.

We denote by $\|\cdot\|$, $\|\cdot\|_\infty$, and $\|\cdot\|_k$ the $L^2(U)$ -norm, $L^\infty(U)$ -norm, and $H^k(U)$ -norm, respectively. Let $\mathcal{L}(\cdot)$ be the law of random variables in $(\Omega, \mathcal{F}, \mathbb{P})$; see the definition in Appendix A. $L^{2m}(\Omega; C([0, T]; H^k(U)))$ is the space in which the $2m$ -th moment of the $C([0, T]; H^k(U))$ -norm of random variables is bounded. We state our main theorems as follows.

Theorem 1.2 (Global existence). *Let U be a smooth bounded domain in $\mathbb{R}^3$ and the pressure $P : (0, \infty) \rightarrow (0, \infty)$ be a smooth function, with $P(\cdot) > 0$ and $P'(\cdot) > 0$. Let $(\bar{\rho}, 0, \bar{\Phi})$ be the smooth steady state in Proposition 1.1, and*

$$\triangle \Phi_0 = \rho_0 - b(x). \quad (15)$$

Then, in $(\Omega, \mathcal{F}, \mathbb{P})$, there exists a unique global-in-time strong solution $(\rho, \mathbf{u}, \Phi)$ to the initial and boundary problem (1)-(5)-(6):

$$\rho, \mathbf{u} \in L^{2m}(\Omega; C([0, T]; H^3(U))), \quad \Phi \in L^{2m}(\Omega; C([0, T]; H^5(U))), \quad (16)$$

up to a modification, for any fixed integer $m \geq 2$ and $\forall T > 0$.

Moreover, for the small perturbation problem, the existence of invariant measure and decay rate hold.

Theorem 1.3 (Convergence to steady state). *Assume that the stochastic forces satisfy*

$$\begin{aligned} \sum a_k^2 &= 1, \quad |Y(\rho, \mathbf{u})| \leq C|\rho \mathbf{u}|, \\ \|\nabla_{\rho, \mathbf{u}} Y(\rho, \mathbf{u})\|_{L^\infty} &\leq C, \quad \|\nabla_{\rho, \mathbf{u}}^2 Y(\rho, \mathbf{u})\|_{L^\infty} \leq C. \end{aligned} \quad (17)$$

Here, $\nabla_{\rho, \mathbf{u}}$ denotes the differential operator with respect to ρ and $\mathbf{u}$. If there exists a constant $\varepsilon > 0$ such that the initial condition $(\rho_0, \mathbf{u}_0, \Phi_0)$ satisfies (15) and

$$\mathbb{E} \left[\left(\|\rho_0 - \bar{\rho}\|_3^2 + \|\mathbf{u}_0\|_3^2 + \|\nabla \Phi_0 - \nabla \bar{\Phi}\|^2 \right)^m \right] \leq \varepsilon^{2m}, \quad \forall m \geq 2, \quad (18)$$

then the decay rate and the existence of invariant measure hold:

1. There are positive constants C and α such that the expectation

$$\begin{aligned} &\mathbb{E} \left[\left(\sup_{s \in [0, t]} \left(\|\rho - \bar{\rho}\|_3^2 + \|\mathbf{u}\|_3^2 + \|\nabla \Phi - \nabla \bar{\Phi}\|^2 \right) \right)^m \right] \\ &\leq C e^{-\alpha m t} \mathbb{E} \left[\left(\|\rho_0 - \bar{\rho}\|_3^2 + \|\mathbf{u}_0\|_3^2 + \|\nabla \Phi_0 - \nabla \bar{\Phi}\|^2 \right)^m \right], \end{aligned} \quad (19)$$

holds, where C is independent of t and C is the m -th power of some constant.

2. The invariant measure generated by $\frac{1}{T} \int_0^T \mathcal{L}(\rho) \times \mathcal{L}(\mathbf{u}) \times \mathcal{L}(\Phi) dt$ is exactly the Dirac measure of steady state $(\bar{\rho}, 0, \bar{\Phi})$.

Remark 1.4. After passing to the limit $t \rightarrow \infty$ in (19), the stationary solution coincides with the steady state $\mathbb{P}$ a.s., since the m -th moment of their difference tends to zero.

Remark 1.5. If for any $\omega \in \Omega$,

$$\left(\|\rho_0 - \bar{\rho}\|_3^2 + \|\mathbf{u}_0\|_3^2 + \|\nabla \Phi_0 - \nabla \bar{\Phi}\|^2 \right) \leq \varepsilon^2, \quad (20)$$

then there exists some constant $\tilde{C}$ such that the asymptotic stability holds $\mathbb{P}$ a.s.:

$$\sup_{s \in [0, t]} \left(\|\rho - \bar{\rho}\|_3^2 + \|\mathbf{u}\|_3^2 + \|\nabla \Phi - \nabla \bar{\Phi}\|^2 \right) \leq 2\tilde{C} e^{-\alpha t} \varepsilon^2. \quad (21)$$

Actually, by Chebyshev's inequality (see Appendix A), it holds that

$$\begin{aligned} &\mathbb{P} \left[\left\{ \omega \in \Omega \mid \|\rho - \bar{\rho}\|_3^2 + \|\mathbf{u}\|_3^2 + \|\nabla \Phi - \nabla \bar{\Phi}\|^2 > 2\tilde{C} e^{-\alpha t} \varepsilon^2 \right\} \right] \\ &\leq \frac{\mathbb{E} \left[\left(\|\rho - \bar{\rho}\|_3^2 + \|\mathbf{u}\|_3^2 + \|\nabla \Phi - \nabla \bar{\Phi}\|^2 \right)^m \right]}{\left(2\tilde{C} e^{-\alpha t} \varepsilon^2 \right)^m} \\ &\leq \frac{\mathbb{E} \left[\tilde{C} e^{-\alpha t} \left(\|\rho_0 - \bar{\rho}\|_3^2 + \|\mathbf{u}_0\|_3^2 + \|\nabla \Phi_0 - \nabla \bar{\Phi}\|^2 \right)^m \right]}{\left(2\tilde{C} e^{-\alpha t} \varepsilon^2 \right)^m} = \frac{1}{2^m}. \end{aligned} \quad (22)$$

Letting $m \rightarrow \infty$, it holds that

$$\mathbb{P} \left[\left\{ \omega \in \Omega \mid \|\rho - \bar{\rho}\|_3^2 + \|\mathbf{u}\|_3^2 + \|\nabla \Phi - \nabla \bar{\Phi}\|^2 > 2\tilde{C} e^{-\alpha t} \varepsilon^2 \right\} \right] \rightarrow 0,$$

i.e.,

$\|\rho - \bar{\rho}\|_3^2 + \|\mathbf{u}\|_3^2 + \|\nabla \Phi - \nabla \bar{\Phi}\|^2 \leq 2\tilde{C} e^{-\alpha t} \varepsilon^2$ holds $\mathbb{P}$ a.s. for every $s \in [0, t]$.

Remark 1.6. The argument in this paper implies the same existence and asymptotic stability of solutions around the steady state for the 2-D system with insulating boundary conditions. Repeating the argument, by Sobolev's embedding, the existence of perturbed solutions and asymptotic stability of steady state for the 1-D system with insulating boundary conditions holds: ρ and $\mathbf{u}$ are in $L^{2m}(\Omega; C([0, T]; H^2(U)))$, $\Phi \in L^{2m}(\Omega; C([0, T]; H^4(U)))$ in $(\Omega, \mathcal{F}, \mathbb{P})$.

As mentioned before, the study of stochastic Euler-Poisson equations, totally from the existing studies for the deterministic case, is new and challenging. The idea of the proof is as follows. We first prove the local existence by Banach's fixed point theorem, then we establish the uniform energy estimates in time t to show the global existence of $(\sigma, \mathbf{u}, \phi)$. Furthermore, we prove the weighted energy estimates so that we can obtain the asymptotic stability for steady states with the insulating boundary conditions. The *a priori* estimates imply the tightness of approximate measures, which will converge to an invariant measure by Krylov-Bogoliubov's theorem in a complete probability space. The global existence does not require the small perturbation condition (17) and (18). However, the existence of invariant measure in Theorem 4.5 requires (17) and (18). From the weighted energy estimates, we then prove that the invariant measure for (1) is exactly the law of steady state, c.f. Section 4. This intricate relationship has not been uncovered in the asymptotic behavior analysis of the stochastic Navier-Stokes equations yet [41, 20].

Here, we explain in detail the main difficulties we face and the strategies we propose.

1. **No temporal solutions due to the stochastic term.** Since Brownian motion is at most Hölder- $\frac{1}{2}$ -continuous with respect to t and it is nowhere differentiable, we have neither $\frac{dW}{dt}$ nor $\frac{d(\rho\mathbf{u})}{dt}$. No temporal derivative is involved in the norm of solutions. Thus, in deterministic cases [21, 42], the spatial estimates bounded by the temporal derivatives estimates, such as

$$\begin{aligned} & \|(\rho - \bar{\rho})\|^2 + \|\nabla(\rho - \bar{\rho})\|^2 + \|\nabla \cdot \mathbf{u}\|^2 \\ & \leq C \left(\|u_t\|^2 + \|(\rho - \bar{\rho})_t\|^2 + \|u\|^2 + \|w\|^3 \right), \end{aligned} \quad (23)$$

do not apply to this stochastic case, where $\|\cdot\|$ means the temporal and spatial mixed derivatives. Consequently, the different energy estimates with the spatial and temporal mixed estimates are necessary in this paper. The spatial derivative estimates are based on Itô's formula. We also symmetrize the system compatible with the insulating boundary conditions to control the linear term and to facilitate the *a priori* estimates.

It is interesting that the noise in the form of (4) is in the higher order of $\mathbf{u}$ than being Lipschitz continuous on $\mathbf{u}$. This is reasonable when we consider the small perturbation around the steady state, which is different with most cases in which Lipschitz continuous coefficients give birth to well-posedness. In this case, the influence of stochastic forces has not been exaggerated too much.

2. **Weighted energy estimates on account of the estimates of the stochastic integral.** Recalling the 3-D deterministic case [21], for instance, based on the energy estimates, one can obtain the ordinary differential inequality (ODI). The ODI is then multiplied with the exponential function of t directly to facilitate the stability analysis. However, in this paper, in order to

estimate the stochastic term, we apply Burkholder-Davis-Gundy's inequality to the stochastic integral of the Wiener process. Then, the *a priori* estimate (151) is already in the form of time integrals rather than an ODI. Integration with respect to time twice does not imply the asymptotic stability. Consequently, direct acquisition of asymptotic stability becomes challenging. To overcome this obstacle, we employ the weighted energy estimates. Moreover, the weight is determined by the *a priori* estimates, which should be obtained first, cf. Section 3.1.

This paper is organized as follows. Section 2 is dedicated to establishing the global existence of solutions around the steady state. In Section 3, we investigate the asymptotic stability of semiconductor equations. Finally, in Section 4, we demonstrate the existence and property of invariant measures. Section 5 is the appendix, in which we provide an overview of stochastic analysis theories that are employed in this study.

2. Global existence of solutions. In this section, we first establish the local existence of strong solutions by Banach's fixed point theorem. Specifically, we derive the system of perturbed solutions in matrix notation by (1). In Step 1, we symmetrize it to simplify the energy estimates and proceed to linearize the system. In Step 2, following a standard procedure in view of Picard iteration, we establish the uniform estimates onto mapping. We utilize Itô's formula and Burkholder-Davis-Gundy's inequality to estimate the stochastic force. In Step 3, we demonstrate contraction. In Step 4, we get the *a priori* estimates in §2.2 so as to obtain the global existence of $(\rho - \bar{\rho}, \mathbf{u})$, or equivalently, $\rho, \mathbf{u} \in L^{2m}(\Omega; C([0, T]; H^3(U)))$ in §2.3. Step 5 concerns the proof of global existence.

In the form $(\sigma, \mathbf{u}, \phi) = (\rho - \bar{\rho}, \mathbf{u}, \Phi - \bar{\Phi})$, the hydrodynamic system becomes

$$\begin{cases} \sigma_t + \nabla \cdot ((\bar{\rho} + \sigma) \mathbf{u}) = 0, \\ d(\mathbf{u}) + ((\mathbf{u} \cdot \nabla) \mathbf{u} + \mathbf{u} + \nabla Q(\bar{\rho} + \sigma) - \nabla Q(\bar{\rho})) dt = \nabla \phi dt + \frac{\mathbb{F}}{\bar{\rho} + \sigma} dW, \\ \Delta \phi = \sigma. \end{cases} \quad (24)$$

Here, we view τ as a constant 1 without loss of generality for the stability analysis. In terms of components, by Taylor's expansion, we have

$$\begin{aligned} \nabla Q(\bar{\rho} + \sigma) - \nabla Q(\bar{\rho}) &= (Q(\bar{\rho} + \sigma) - Q(\bar{\rho}))_{,i} \\ &= Q'(\bar{\rho} + \sigma)(\bar{\rho} + \sigma)_{,i} - Q'(\bar{\rho})\bar{\rho}_{,i} \\ &= Q'(\bar{\rho} + \sigma)\sigma_{,i} + (Q'(\bar{\rho} + \sigma) - Q'(\bar{\rho}))\bar{\rho}_{,i} \\ &= Q'(\bar{\rho} + \sigma)\sigma_{,i} + Q''(\bar{\rho})\sigma\bar{\rho}_{,i} + (Q'(\bar{\rho} + \sigma) - Q'(\bar{\rho}) - Q''(\bar{\rho})\sigma)\bar{\rho}_{,i} \\ &:= Q'(\bar{\rho} + \sigma)\sigma_{,i} + Q''(\bar{\rho})\sigma\bar{\rho}_{,i} + h_i, \end{aligned} \quad (25)$$

where $(\cdot)_{,i}$ represents the derivative with respect to x_i , and

$$h_i = O(\sigma^2). \quad (26)$$

In term of components, there holds

$$\phi_{,i} = \Delta^{-1}\sigma_{,i}, \quad (27)$$

where Δ^{-1} is well-defined under condition (5). In matrix notation, denoting $\mathbf{w} = \begin{bmatrix} \sigma \\ \mathbf{u} \end{bmatrix}$, we write the system as

$$d\mathbf{w} + (\mathcal{A}^1\mathbf{w}_{,1} + \mathcal{A}^2\mathbf{w}_{,2} + \mathcal{A}^3\mathbf{w}_{,3} + \mathcal{B}\mathbf{w} + \mathcal{L}\mathbf{u}) dt = \mathcal{L}_\phi dt + f, \quad (28)$$

where

$$\mathcal{A}^1 = \begin{bmatrix} u^1 & \bar{\rho} + \sigma & 0 & 0 \\ Q'(\bar{\rho} + \sigma) & u^1 & 0 & 0 \\ 0 & 0 & u^1 & 0 \\ 0 & 0 & 0 & u^1 \end{bmatrix}, \quad (29)$$

$$\mathcal{A}^2 = \begin{bmatrix} u^2 & 0 & \bar{\rho} + \sigma & 0 \\ 0 & u^2 & 0 & 0 \\ Q'(\bar{\rho} + \sigma) & 0 & u^2 & 0 \\ 0 & 0 & 0 & u^2 \end{bmatrix}, \quad (30)$$

$$\mathcal{A}^3 = \begin{bmatrix} u^3 & 0 & 0 & \bar{\rho} + \sigma \\ 0 & u^3 & 0 & 0 \\ 0 & 0 & u^3 & 0 \\ Q'(\bar{\rho} + \sigma) & 0 & 0 & u^3 \end{bmatrix}, \quad (31)$$

$$\mathcal{B} = \begin{bmatrix} 0 & \bar{\rho}_{,1} & \bar{\rho}_{,2} & \bar{\rho}_{,3} \\ Q''(\bar{\rho})\bar{\rho}_{,1} & 0 & 0 & 0 \\ Q''(\bar{\rho})\bar{\rho}_{,2} & 0 & 0 & 0 \\ Q''(\bar{\rho})\bar{\rho}_{,3} & 0 & 0 & 0 \end{bmatrix}, \quad (32)$$

$$\mathcal{L}_{\mathbf{u}} = \begin{bmatrix} 0 \\ u^1 \\ u^2 \\ u^3 \end{bmatrix}, \quad \mathcal{L}_{\phi} = \begin{bmatrix} 0 \\ \Delta^{-1}\sigma_{,1} \\ \Delta^{-1}\sigma_{,2} \\ \Delta^{-1}\sigma_{,3} \end{bmatrix}, \quad f = - \begin{bmatrix} 0 \\ h(\sigma)_{,1} - \mathbb{F}^1 \mathrm{d} W \\ h(\sigma)_{,2} - \mathbb{F}^2 \mathrm{d} W \\ h(\sigma)_{,3} - \mathbb{F}^3 \mathrm{d} W \end{bmatrix}. \quad (33)$$

Step 1. Symmetrizing and Linearizing.

We define the symmetrizer

$$\mathcal{D} = \text{diag} [Q'(\bar{\rho} + \sigma), \bar{\rho} + \sigma, \bar{\rho} + \sigma, \bar{\rho} + \sigma] := \text{diag} [d_1, d_2, d_3, d_4].$$

Then, the system becomes

$$\mathcal{D} \mathrm{d} \mathbf{w} + \left(\tilde{\mathcal{A}}^1 \mathbf{w}_{,1} + \tilde{\mathcal{A}}^2 \mathbf{w}_{,2} + \tilde{\mathcal{A}}^3 \mathbf{w}_{,3} + \tilde{\mathcal{B}} \mathbf{w} + \tilde{\mathcal{L}}_{\mathbf{u}} \right) \mathrm{d} t = \tilde{\mathcal{L}}_{\phi} \mathrm{d} t + \tilde{f}, \quad (34)$$

where

$$\tilde{\mathcal{A}}^1 = \begin{bmatrix} u^1 Q'(\bar{\rho} + \sigma) & (\bar{\rho} + \sigma) Q'(\bar{\rho} + \sigma) & 0 & 0 \\ (\bar{\rho} + \sigma) Q'(\bar{\rho} + \sigma) & (\bar{\rho} + \sigma) u^1 & 0 & 0 \\ 0 & 0 & (\bar{\rho} + \sigma) u^1 & 0 \\ 0 & 0 & 0 & (\bar{\rho} + \sigma) u^1 \end{bmatrix}, \quad (35)$$

$$\tilde{\mathcal{A}}^2 = \begin{bmatrix} u^2 Q'(\bar{\rho} + \sigma) & 0 & (\bar{\rho} + \sigma) Q'(\bar{\rho} + \sigma) & 0 \\ 0 & (\bar{\rho} + \sigma) u^2 & 0 & 0 \\ (\bar{\rho} + \sigma) Q'(\bar{\rho} + \sigma) & 0 & (\bar{\rho} + \sigma) u^2 & 0 \\ 0 & 0 & 0 & (\bar{\rho} + \sigma) u^2 \end{bmatrix}, \quad (36)$$

$$\tilde{\mathcal{A}}^3 = \begin{bmatrix} u^3 Q'(\bar{\rho} + \sigma) & 0 & 0 & (\bar{\rho} + \sigma) Q'(\bar{\rho} + \sigma) \\ 0 & (\bar{\rho} + \sigma) u^3 & 0 & 0 \\ 0 & 0 & (\bar{\rho} + \sigma) u^3 & 0 \\ (\bar{\rho} + \sigma) Q'(\bar{\rho} + \sigma) & 0 & 0 & (\bar{\rho} + \sigma) u^3 \end{bmatrix}, \quad (37)$$

$$\tilde{\mathcal{B}} = \begin{bmatrix} 0 & \bar{\rho}_{,1} & \bar{\rho}_{,2} & \bar{\rho}_{,3} \\ Q''(\bar{\rho})\bar{\rho}_{,1} & 0 & 0 & 0 \\ Q''(\bar{\rho})\bar{\rho}_{,2} & 0 & 0 & 0 \\ Q''(\bar{\rho})\bar{\rho}_{,3} & 0 & 0 & 0 \end{bmatrix}, \quad (38)$$

$$\tilde{\mathcal{L}}_{\mathbf{u}} = (\bar{\rho} + \sigma) \begin{bmatrix} 0 \\ u^1 \\ u^2 \\ u^3 \end{bmatrix}, \quad \tilde{\mathcal{L}}_{\phi} = (\bar{\rho} + \sigma) \begin{bmatrix} 0 \\ \Delta^{-1}\sigma_{,1} \\ \Delta^{-1}\sigma_{,2} \\ \Delta^{-1}\sigma_{,3} \end{bmatrix}, \quad (39)$$

$$\tilde{f} = -(\bar{\rho} + \sigma) \begin{bmatrix} 0 \\ h(\sigma)_{,1} dt - \mathbb{F}^1 dW \\ h(\sigma)_{,2} dt - \mathbb{F}^2 dW \\ h(\sigma)_{,3} dt - \mathbb{F}^3 dW \end{bmatrix}. \quad (40)$$

2.1. Local existence. In this subsection, the main estimates for the stochastic forces are manipulated, considering the assumption of (17), for instance. Similar to the approach in [37, 45], we first linearize the system, and then we use Banach's fixed point theorem to get the local existence by the *a priori* energy estimates.

The linearized system is

$$\begin{aligned} & \mathcal{D}(\hat{\sigma}) d\mathbf{w} + \left(\tilde{\mathcal{A}}^1(\hat{\mathbf{w}}) \mathbf{w}_{,1} + \tilde{\mathcal{A}}^2(\hat{\mathbf{w}}) \mathbf{w}_{,2} + \tilde{\mathcal{A}}^3(\hat{\mathbf{w}}) \mathbf{w}_{,3} + \tilde{\mathcal{B}} \mathbf{w} \right) dt \\ &= -\tilde{\mathcal{L}}_{\hat{\mathbf{u}}}(\hat{\sigma}, \mathbf{u}) dt + \tilde{\mathcal{L}}_{\hat{\phi}}(\hat{\sigma}, \hat{\phi}) dt + \tilde{f}(\hat{\mathbf{w}}), \end{aligned} \quad (41)$$

where $\hat{\mathbf{w}} = \begin{bmatrix} \hat{\sigma} \\ \hat{\mathbf{u}} \end{bmatrix}$ is given, $\hat{\sigma} \in C([0, T]; H^3(U))$, and $\hat{\mathbf{u}} \in C([0, T]; H^3(U))$. We denote $M = \sup_{t \in [0, T]} \|\hat{\sigma}, \hat{\mathbf{u}}\|_3$.

Step 2. Estimates for the uniform upper bound.

By Itô's formula (see Appendix A), it holds that

$$\begin{aligned} & \int_U d\left(\frac{1}{2} \mathcal{D} \mathbf{w} \cdot \mathbf{w}\right) dx \\ &= \int_U \frac{1}{2} d\mathcal{D} \mathbf{w} \cdot \mathbf{w} dx + \int_U \mathcal{D} \mathbf{w} \cdot d\mathbf{w} dx + \int_U \mathcal{D} \mathbb{F} \cdot \mathbb{F} dx dt. \end{aligned} \quad (42)$$

Integrating

$$\begin{aligned} & \mathcal{D} d\mathbf{w} \cdot \mathbf{w} + \left(\tilde{\mathcal{A}}^1(\hat{\mathbf{w}}) \mathbf{w}_{,1} \cdot \mathbf{w} + \tilde{\mathcal{A}}^2(\hat{\mathbf{w}}) \mathbf{w}_{,2} \cdot \mathbf{w} + \tilde{\mathcal{A}}^3(\hat{\mathbf{w}}) \mathbf{w}_{,3} \cdot \mathbf{w} + \tilde{\mathcal{B}} \mathbf{w} \cdot \mathbf{w} \right) dt \\ &= -\tilde{\mathcal{L}}_{\hat{\mathbf{u}}} \cdot \mathbf{w} dt + \tilde{\mathcal{L}}_{\hat{\phi}} \cdot \mathbf{w} dt + \tilde{f} \cdot \mathbf{w} \end{aligned} \quad (43)$$

over the domain U , we gain

$$\begin{aligned} & \int_U \mathcal{D} \mathbf{w} \cdot d\mathbf{w} dx \\ &= - \int_U \left(\tilde{\mathcal{A}}^1 \mathbf{w}_{,1} \cdot \mathbf{w} + \tilde{\mathcal{A}}^2 \mathbf{w}_{,2} \cdot \mathbf{w} + \tilde{\mathcal{A}}^3 \mathbf{w}_{,3} \cdot \mathbf{w} + \tilde{\mathcal{B}} \mathbf{w} \cdot \mathbf{w} \right) dx dt \\ & \quad - \int_U \tilde{\mathcal{L}}_{\hat{\mathbf{u}}} \cdot \mathbf{w} dx dt + \tilde{\mathcal{L}}_{\hat{\phi}} \cdot \mathbf{w} + \int_U \nabla h(\hat{\sigma}) \cdot \mathbf{w} dx dt + \int_U \mathcal{D} \mathbb{F} dW \cdot \mathbf{w} dx. \end{aligned} \quad (44)$$

By integration by parts, we have

$$\begin{aligned} & \int_U \left(\tilde{\mathcal{A}}^1 \mathbf{w}_{,1} \cdot \mathbf{w} + \tilde{\mathcal{A}}^2 \mathbf{w}_{,2} \cdot \mathbf{w} + \tilde{\mathcal{A}}^3 \mathbf{w}_{,3} \cdot \mathbf{w} + \tilde{\mathcal{B}} \mathbf{w} \cdot \mathbf{w} \right) dx dt \\ &= \int_U \left(-\frac{1}{2} \left(\mathbf{w} \tilde{\mathcal{A}}_1^1 \mathbf{w} + \mathbf{w} \tilde{\mathcal{A}}_2^2 \mathbf{w} + \mathbf{w} \tilde{\mathcal{A}}_3^3 \mathbf{w} \right) + \tilde{\mathcal{B}} |\mathbf{w}|^2 \right) dx dt + \int_U \left(\mathbf{w} \tilde{\mathcal{A}}^j \mathbf{w} \right)_{,j} dx dt. \end{aligned} \quad (45)$$

On account of the insulated boundary condition $\hat{\mathbf{u}} \cdot \nu|_{\partial U} = 0$, it holds that

$$\begin{aligned} & \int_U \left(\mathbf{w} \tilde{\mathcal{A}}^j \mathbf{w} \right)_{,j} dx \\ &= \int_{\partial U} \left((\hat{\mathbf{u}} \cdot \nu) \left(Q'(\bar{\rho} + \hat{\sigma}) \sigma^2 + (\bar{\rho} + \hat{\sigma}) |\mathbf{u}|^2 + 2\hat{\rho} \mathbf{u} Q'(\bar{\rho} + \hat{\sigma}) (\bar{\rho} + \hat{\sigma}) \right) \right) dS \equiv 0. \end{aligned} \quad (46)$$

In summary, there holds

$$\begin{aligned} & \int_U d \left(\frac{1}{2} \mathcal{D} \mathbf{w} \cdot \mathbf{w} \right) dx \\ &= \int_U \frac{1}{2} d \mathcal{D} \mathbf{w} \cdot \mathbf{w} dx - \int_U \left(-\frac{1}{2} \left(\mathbf{w} \tilde{\mathcal{A}}_1^1 \mathbf{w} + \mathbf{w} \tilde{\mathcal{A}}_2^2 \mathbf{w} + \mathbf{w} \tilde{\mathcal{A}}_3^3 \mathbf{w} \right) + \tilde{\mathcal{B}} |\mathbf{w}|^2 \right) dx dt \\ & \quad + \int_U \left(-\tilde{\mathcal{L}}_{\hat{\mathbf{u}}} \cdot \mathbf{w} + \tilde{\mathcal{L}}_{\hat{\phi}} \cdot \mathbf{w} \right) dx dt + \int_U \nabla h(\hat{\sigma}) \cdot \mathbf{w} dx dt \\ & \quad + \int_U \mathcal{D} \mathbb{F} dW \cdot \mathbf{w} dx + \int_U \mathcal{D} \mathbb{F} \cdot \mathbb{F} dx dt. \end{aligned} \quad (47)$$

Direct calculation shows that

$$\begin{aligned} & -\frac{1}{2} \tilde{\mathcal{A}}_{,i}^i + \tilde{\mathcal{B}} - \text{diag} \left[-\frac{1}{2} (u^i Q'(\rho))_{,i}, -\frac{1}{2} (u^i \rho)_{,i}, -\frac{1}{2} (u^i \rho)_{,i}, -\frac{1}{2} (u^i \rho)_{,i} \right] \\ &= \begin{bmatrix} 0 & -\frac{1}{2} \{\rho q\}_{,1} + \bar{\rho}_{,1} q & -\frac{1}{2} \{\rho q\}_{,2} + \bar{\rho}_{,2} q & -\frac{1}{2} \{\rho q\}_{,3} + \bar{\rho}_{,3} q \\ -\frac{1}{2} \{\rho q\}_{,1} + \rho Q''(\bar{\rho}) \bar{\rho} & 0 & 0 & 0 \\ -\frac{1}{2} \{\rho q\}_{,2} + \rho Q''(\bar{\rho}) \bar{\rho} & 0 & 0 & 0 \\ -\frac{1}{2} \{\rho q\}_{,3} + \rho Q''(\bar{\rho}) \bar{\rho} & 0 & 0 & 0 \end{bmatrix} \end{aligned} \quad (48)$$

is anti-symmetric [21], where $q = Q'(\rho)$. Then, we estimate

$$\begin{aligned} & \int_U \left(-\frac{1}{2} \left(\mathbf{w} \tilde{\mathcal{A}}_1^1 \mathbf{w} + \mathbf{w} \tilde{\mathcal{A}}_2^2 \mathbf{w} + \mathbf{w} \tilde{\mathcal{A}}_3^3 \mathbf{w} \right) + \tilde{\mathcal{B}} |\mathbf{w}|^2 \right) dx dt \\ & \leq C \|\mathbf{w}\|^2 (\|\hat{\sigma}\|_3 + \|\hat{\mathbf{u}}\|_3) dt. \end{aligned} \quad (49)$$

Recalling (39), we have

$$\int_U \tilde{\mathcal{L}}_{\hat{\mathbf{u}}}(\hat{\sigma}, \mathbf{u}) \cdot \mathbf{w} dx dt = \int_U (\bar{\rho} + \hat{\sigma}) |\mathbf{u}|^2 dx dt \geq C \int_U \bar{\rho} |\mathbf{u}|^2 dx dt, \quad (50)$$

and

$$\begin{aligned} & \int_U \tilde{\mathcal{L}}_{\hat{\phi}}(\hat{\sigma}, \phi) \cdot \mathbf{w} dx dt \\ &= \int_U (\bar{\rho} + \hat{\sigma}) \nabla \phi \cdot \mathbf{u} dx dt = - \int_U \nabla \cdot ((\bar{\rho} + \hat{\sigma}) \mathbf{u}) \phi dx dt \\ &= \int_U \sigma_t \phi dx dt = \int_U (\Delta \phi)_t \phi dx dt = -d \int_U |\nabla \phi|^2 dx. \end{aligned} \quad (51)$$

For $\tilde{f}$, there holds

$$\int_0^t \int_U \tilde{f} \cdot \mathbf{w} \, dx = C \int_0^t \int_U \hat{\sigma}^2 \cdot \mathbf{u} \, dx \, dt + \left| \int_0^t \int_U (\bar{\rho} + \hat{\sigma}) \mathbb{F} \cdot \mathbf{u} \, dx \, dW \right|, \quad (52)$$

where

$$\int_0^t \int_U \hat{\sigma}^2 \cdot \mathbf{u} \, dx \, ds \leq \int_0^t \|\hat{\sigma}\|_2 \|\hat{\sigma}\| \|\mathbf{w}\| \, ds. \quad (53)$$

For $\int_0^t \int_U (\bar{\rho} + \hat{\sigma}) \mathbb{F} \cdot \mathbf{u} \, dx \, dW$, one can see the definition of the stochastic integral in Appendix A. Since $|\mathbb{F}(\hat{\sigma}, \hat{\mathbf{u}})|^2 \leq C |(\bar{\rho} + \hat{\sigma}) \hat{\mathbf{u}}|^4$, there holds

$$\int_U \mathcal{D}\mathbb{F} \cdot \mathbb{F} \, dx \, dt \leq C \|\hat{\mathbf{u}}\|^2 \left(\|\hat{\mathbf{u}}\|_3^2 \|\hat{\sigma}\|_3^4 \right) dt \leq CM^8 dt, \quad (54)$$

and

$$\begin{aligned} & \mathbb{E} \left[\left| \int_0^t \int_U \mathbb{F} \cdot \mathbf{u} \, dx \, dW \right|^m \right] \leq \mathbb{E} \left[\left(\int_0^t \left| \int_U \mathbb{F} \cdot \mathbf{u} \, dx \right|^2 ds \right)^{\frac{m}{2}} \right] \\ & \leq \mathbb{E} \left[\left(C \int_0^t \left| \int_U |(\bar{\rho} + \hat{\sigma}) \hat{\mathbf{u}}|^2 \mathbf{u} \, dx \right|^2 ds \right)^{\frac{m}{2}} \right] \\ & \leq \mathbb{E} \left[\left(C \sup_{s \in [0, t]} \|\mathbf{u}\|^2 \int_0^t \|(\bar{\rho} + \hat{\sigma})\|_3^4 \|\hat{\mathbf{u}}\|_3^4 ds \right)^{\frac{m}{2}} \right] \\ & \leq \delta_1^m \mathbb{E} \left[\left(\sup_{s \in [0, t]} \|\mathbf{u}\|^2 \right)^m \right] + C_{\delta_1}^m \mathbb{E} \left[\left(\int_0^t \|\hat{\mathbf{u}}\|_3^4 \|(\bar{\rho} + \hat{\sigma})\|_3^4 ds \right)^m \right], \end{aligned} \quad (55)$$

by Burkholder-Davis-Gundy's inequality (see Appendix A), where δ_1 is taken such that $\delta_1 \sup_{s \in [0, t]} \|\mathbf{u}\|^2$ can be balanced by the left side. We estimate

$$\begin{aligned} & \int_U \mathbf{w} \, (d\mathcal{D}) \mathbf{w} \, dx \\ &= \int_U \mathbf{w} \, (\text{diag} \{Q'(\bar{\rho} + \hat{\sigma})_t, (\bar{\rho} + \hat{\sigma})_t, (\bar{\rho} + \hat{\sigma})_t, (\bar{\rho} + \hat{\sigma})_t\}) \mathbf{w} \, dx \, dt \\ &= \int_U \left(Q''(\bar{\rho} + \hat{\sigma}) \hat{\sigma}_t \sigma^2 + \hat{\sigma}_t |\mathbf{u}|^2 \right) dx \, dt \\ &= \int_U (Q''(\bar{\rho}) + O(\hat{\sigma})) (-\nabla \cdot ((\bar{\rho} + \hat{\sigma}) \hat{\mathbf{u}})) \sigma^2 dx + \int_U (-\nabla \cdot ((\bar{\rho} + \hat{\sigma}) \hat{\mathbf{u}})) |\mathbf{u}|^2 dx \, dt \\ &\leq C \|\mathbf{w}\|^2 (\|\hat{\mathbf{u}}\|_2 + \|\hat{\sigma}\|_2 \|\hat{\mathbf{u}}\|_2 + \|\hat{\sigma}\|_2 \|\hat{\sigma}\|_3 \|\hat{\mathbf{u}}\|_2) dt, \end{aligned} \quad (56)$$

where O means the same order. In summary, there holds

$$\begin{aligned} & \mathbb{E} \left[\left(\sup_{s \in [0, t]} \int_0^s \left(\int_U \bar{\rho} |\mathbf{w}|^2 dx + \int_U |\nabla \phi|^2 dx \right) + c_1 \int_0^t \int_U \bar{\rho} |\mathbf{u}|^2 dx \, ds \right)^m \right] \\ & \leq \mathbb{E} \left[\left(C \int_0^t \left(\|\hat{\mathbf{u}}\|^2 (1 + \|\hat{\sigma}\|_{H^1}^2) + \|\mathbf{w}\|^2 \|\hat{\mathbf{w}}\| + \|\hat{\mathbf{w}}\|^2 \|\mathbf{w}\| \right) ds \right)^m \right] \\ & \quad + \mathbb{E} \left[\left(C \int_0^t \|\hat{\mathbf{u}}\|^4 \|(\bar{\rho} + \hat{\sigma})\|^4 ds \right)^m \right] \end{aligned} \quad (57)$$

$$\begin{aligned}
 & + \mathbb{E} \left[\left(C \int_0^t \|\mathbf{w}\|^2 (\|\hat{\mathbf{u}}\|_2 + \|\hat{\sigma}\|_2 \|\hat{\mathbf{u}}\|_2 + \|\hat{\sigma}\|_2 \|\hat{\sigma}\|_3 \|\hat{\mathbf{u}}\|_2) \, ds \right)^m \right] \\
 & \leq \mathbb{E} \left[\left(C \int_0^t (M + M^2 + M^4) \, ds + \int_0^t M \|\mathbf{w}\|^2 \, ds + \int_0^t M^2 \|\mathbf{w}\| \, ds \right)^m \right] \\
 & + \mathbb{E} \left[\left(C \int_0^t (M^4 + M^8) \, ds \right)^m \right] + C \mathbb{E} \left[\left(\int_0^t \|\mathbf{w}\|^2 (M + M^2 + M^3) \, ds \right)^m \right].
 \end{aligned}$$

Furthermore, for $\bar{\rho}$ with a positive lower bound, we have

$$\begin{aligned}
 & \mathbb{E} \left[\left(\sup_{s \in [0, t]} \int_0^s \left(\int_U |\mathbf{w}|^2 \, dx + \int_U |\nabla \phi|^2 \, dx \right) \, ds \right)^m \right] \\
 & \leq C_{M, m} \left(t^m + \mathbb{E} \left[\left(\int_0^t \|\mathbf{w}\|^2 \, ds \right)^m \right] \right),
 \end{aligned} \tag{58}$$

where $C_{M, m}$ is a constant depending on m, M . Similarly, we take higher-order derivatives of system (41) up to the third order, and we do the *a priori* estimates. There holds

$$\begin{aligned}
 & \mathbb{E} \left[\left(\sup_{s \in [0, t]} \int_0^s \left(\|\mathbf{w}\|_3^2 + \|\nabla \phi\|^2 \right) \, ds \right)^m \right] \\
 & \leq C_{M, m} \left(t^m + \mathbb{E} \left[\left(\int_0^t \left(\|\mathbf{w}\|_3^2 + \|\nabla \phi\|^2 \right) \, ds \right)^m \right] \right).
 \end{aligned} \tag{59}$$

By Grönwall's inequality, we have $\mathbf{w} \in L^{2m}(\Omega; C([0, T]; H^3(U)))$. More precisely,

$$\begin{aligned}
 & \mathbb{E} \left[\left(\sup_{s \in [0, t]} \left(\|\mathbf{w}\|_3^2 + \|\nabla \phi\|^2 \right) (s) \right)^m \right] \\
 & \leq \mathbb{E} \left[\left(\left(\|\mathbf{w}\|_3^2 + \|\nabla \phi\|^2 \right) (0) \right)^m \right] + C_{M, m} t^m \\
 & + \int_0^t \left(\mathbb{E} \left[\left(\left(\|\mathbf{w}\|_3^2 + \|\nabla \phi\|^2 \right) (0) \right)^m \right] + C_{M, m} t^m \right) C_{M, m} e^{\int_0^s C_{M, m} \, d\tau} \, ds \\
 & \leq \left(\mathbb{E} \left[\left(\left(\|\mathbf{w}_0\|_3^2 + \|\nabla \phi_0\|^2 \right) \right)^m \right] + C_{M, m} t^m \right) e^{C_{M, m} t}.
 \end{aligned} \tag{60}$$

From the estimates of the time shift of solutions, similar to (60), by applying the Kolmogorov-Centov theorem (see Appendix A), following the standard argument in stochastic analysis [5], we deduce the time continuity of $\mathbf{w}$ up to a modification in probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and we omit the details.

The iteration scheme is

$$\begin{aligned}
 & \mathcal{D}(\sigma_{n-1}) \, d\mathbf{w}_n + \\
 & \left(\tilde{\mathcal{A}}^1(\mathbf{w}_{n-1}) \mathbf{w}_{n,1} + \tilde{\mathcal{A}}^2(\mathbf{w}_{n-1}) \mathbf{w}_{n,2} + \tilde{\mathcal{A}}^3(\mathbf{w}_{n-1}) \mathbf{w}_{n,3} + \tilde{\mathcal{B}} \mathbf{w}_n \right) \, dt \\
 & = -L_{\mathbf{u}_{n-1}}(\sigma_{n-1}, \mathbf{u}_n) \, dt + \tilde{\mathcal{L}}_\phi(\sigma_{n-1}, \phi_n) \, dt + \tilde{f}(\mathbf{w}_{n-1}).
 \end{aligned} \tag{61}$$

By energy estimates (60), we take T_0 such that

$$e^{C_{M, m} T_0} \leq 2, \quad C_{M, m} T_0 \leq \mathbb{E} \left[\left(\left(\|\mathbf{w}_0\|_3^2 + \|\nabla \phi_0\|^2 \right) \right)^m \right], \tag{62}$$

if

$$\mathbb{E} \left[\left(\sup_{s \in [0, t]} \|\mathbf{w}_{n-1}(s)\|_3^2 \right)^m \right] \leq 4\mathbb{E} \left[\left(\left(\|\mathbf{w}_0\|_3^2 + \|\nabla \phi_0\|^2 \right) \right)^m \right], \quad (63)$$

and then

$$\mathbb{E} \left[\left(\sup_{s \in [0, t]} \|\mathbf{w}_n(s)\|_3^2 \right)^m \right] \leq 4\mathbb{E} \left[\left(\left(\|\mathbf{w}_0\|_3^2 + \|\nabla \phi_0\|^2 \right) \right)^m \right]. \quad (64)$$

Remark 2.1. For general stochastic forces without condition (17), there also holds

$$\begin{aligned} & \mathbb{E} \left[\left(\sup_{s \in [0, t]} \int_0^s d \left(\|\mathbf{w}\|_3^2 + \|\nabla \phi\|^2 \right) \right)^m \right] \\ & \leq C_{M,m} \left(t^m + \mathbb{E} \left[\left(\int_0^t \left(\|\mathbf{w}\|_3^2 + \|\nabla \phi\|^2 \right) ds \right)^m \right] \right). \end{aligned} \quad (65)$$

with another expression of the constant $C_{M,m}$. Thus, we get the local existence by Grönwall's inequality, similarly to the above statement.

Step 3. Contraction.

For $\|\mathbf{w}_n - \mathbf{w}_{n-1}\|_3$, we show that it is a Cauchy sequence. $(\mathbf{w}_n - \mathbf{w}_{n-1})$ satisfies

$$\begin{aligned} & \mathcal{D}(\sigma_{n-1}) (d\mathbf{w}_n - d\mathbf{w}_{n-1}) + (\mathcal{D}(\sigma_{n-1}) - \mathcal{D}(\sigma_{n-2})) d\mathbf{w}_{n-1} \\ & + \tilde{\mathcal{A}}^1(\mathbf{w}_{n-1}) (\mathbf{w}_{n,1} - \mathbf{w}_{n-1,1}) dt + \left(\tilde{\mathcal{A}}^1(\mathbf{w}_{n-1}) - \tilde{\mathcal{A}}^1(\mathbf{w}_{n-2}) \right) \mathbf{w}_{n-1,1} dt \\ & + \tilde{\mathcal{A}}^2(\mathbf{w}_{n-1}) (\mathbf{w}_{n,2} - \mathbf{w}_{n-1,2}) dt + \left(\tilde{\mathcal{A}}^2(\mathbf{w}_{n-1}) - \tilde{\mathcal{A}}^2(\mathbf{w}_{n-2}) \right) \mathbf{w}_{n-1,2} dt \\ & + \tilde{\mathcal{A}}^3(\mathbf{w}_{n-1}) (\mathbf{w}_{n,3} - \mathbf{w}_{n-1,3}) dt + \left(\tilde{\mathcal{A}}^3(\mathbf{w}_{n-1}) - \tilde{\mathcal{A}}^3(\mathbf{w}_{n-2}) \right) \mathbf{w}_{n-1,3} dt \\ & + \tilde{\mathcal{B}}(\bar{\mathbf{w}}) (\mathbf{w}_n - \mathbf{w}_{n-1}) dt \\ & = -\tilde{\mathcal{L}}_{\mathbf{u}}(\sigma_{n-1}, \mathbf{u}_n) dt + \tilde{\mathcal{L}}_{\mathbf{u}}(\sigma_{n-2}, \mathbf{u}_{n-1}) dt \\ & + \left(\tilde{\mathcal{L}}_{\phi}(\sigma_{n-1}, \phi_n) - \tilde{\mathcal{L}}_{\phi}(\sigma_{n-2}, \phi_{n-1}) \right) dt + \left(\tilde{f}(\mathbf{w}_{n-1}) - \tilde{f}(\mathbf{w}_{n-2}) \right). \end{aligned} \quad (66)$$

Then, we multiply the above formula with $(\mathbf{w}_n - \mathbf{w}_{n-1})$, and the estimates of the terms

$$\begin{aligned} & \tilde{\mathcal{A}}^1(\mathbf{w}_{n-1}) (\mathbf{w}_{n,1} - \mathbf{w}_{n-1,1}) dt + \tilde{\mathcal{A}}^2(\mathbf{w}_{n-1}) (\mathbf{w}_{n,2} - \mathbf{w}_{n-1,2}) dt \\ & + \tilde{\mathcal{A}}^3(\mathbf{w}_{n-1}) (\mathbf{w}_{n,3} - \mathbf{w}_{n-1,3}) dt + \tilde{\mathcal{B}}(\bar{\mathbf{w}}) (\mathbf{w}_n - \mathbf{w}_{n-1}) dt \end{aligned} \quad (67)$$

are similar to (45), (46), and (49), so we omit them here. We focus on the estimates of

$$\sum \left(\tilde{\mathcal{A}}^i(\mathbf{w}_{n-1}) - \tilde{\mathcal{A}}^i(\mathbf{w}_{n-2}) \right) \mathbf{w}_{n-1,i} dt, \quad (68)$$

and the right-hand side terms in (66). By the expression formula of $\tilde{\mathcal{A}}^i$, it holds that

$$\begin{aligned} & \int_0^1 \sum \left(\tilde{\mathcal{A}}^i(\mathbf{w}_{n-1}) - \tilde{\mathcal{A}}^i(\mathbf{w}_{n-2}) \right) \mathbf{w}_{n-1,i} \cdot (\mathbf{w}_n - \mathbf{w}_{n-1}) dx dt \\ & \leq C \|\mathbf{w}_n - \mathbf{w}_{n-1}\| \|\mathbf{w}_{n-1} - \mathbf{w}_{n-2}\| dt. \end{aligned} \quad (69)$$

Since

$$-\tilde{\mathcal{L}}_{\mathbf{u}}(\sigma_{n-1}, \mathbf{u}_n) dt + \tilde{\mathcal{L}}_{\mathbf{u}}(\sigma_{n-2}, \mathbf{u}_{n-1}) dt \quad (70)$$

$$\begin{aligned}
 &= \left(\tilde{\mathcal{L}}_{\mathbf{u}}(\sigma_{n-1}, \mathbf{u}_{n-1}) - \tilde{\mathcal{L}}_{\mathbf{u}}(\sigma_{n-1}, \mathbf{u}_n) \right) dt \\
 &\quad + \left(\tilde{\mathcal{L}}_{\mathbf{u}}(\sigma_{n-2}, \mathbf{u}_{n-1}) - \tilde{\mathcal{L}}_{\mathbf{u}}(\sigma_{n-1}, \mathbf{u}_{n-1}) \right) dt,
 \end{aligned}$$

we estimate

$$\begin{aligned}
 &\int_0^1 \left(-\tilde{\mathcal{L}}_{\mathbf{u}}(\sigma_{n-1}, \mathbf{u}_n) dt + \tilde{\mathcal{L}}_{\mathbf{u}}(\sigma_{n-2}, \mathbf{u}_{n-1}) \right) \cdot (\mathbf{w}_n - \mathbf{w}_{n-1}) dx dt \\
 &= - \int_0^1 (\bar{\rho} + \sigma_{n-1}) |\mathbf{u}_n - \mathbf{u}_{n-1}|^2 dx dt \\
 &\quad - \int_0^1 (\sigma_{n-1} - \sigma_{n-2}) \mathbf{u}_{n-1} \cdot (\mathbf{u}_n - \mathbf{u}_{n-1}) dx dt \\
 &\leq - \int_0^1 \frac{\bar{\rho}}{2} |\mathbf{u}_n - \mathbf{u}_{n-1}|^2 dx dt - \int_0^1 (\sigma_{n-1} - \sigma_{n-2}) \mathbf{u}_{n-1} \cdot (\mathbf{u}_n - \mathbf{u}_{n-1}) dx dt,
 \end{aligned} \tag{71}$$

where

$$\begin{aligned}
 &\int_0^1 (\sigma_{n-1} - \sigma_{n-2}) \mathbf{u}_{n-1} (\mathbf{u}_n - \mathbf{u}_{n-1}) dx dt \\
 &\leq C \|\mathbf{w}_n - \mathbf{w}_{n-1}\| \|\mathbf{w}_{n-1} - \mathbf{w}_{n-2}\| dt.
 \end{aligned} \tag{72}$$

Since

$$\begin{aligned}
 &\left(\tilde{\mathcal{L}}_{\phi}(\sigma_{n-1}, \phi_n) - \tilde{\mathcal{L}}_{\phi}(\sigma_{n-2}, \phi_{n-1}) \right) dt \\
 &= \left(\tilde{\mathcal{L}}_{\phi}(\sigma_{n-1}, \phi_n) - \tilde{\mathcal{L}}_{\phi}(\sigma_{n-1}, \phi_{n-1}) \right) dt \\
 &\quad + \left(\tilde{\mathcal{L}}_{\phi}(\sigma_{n-1}, \phi_{n-1}) - \tilde{\mathcal{L}}_{\phi}(\sigma_{n-2}, \phi_{n-1}) \right) dt,
 \end{aligned} \tag{73}$$

then we have

$$\begin{aligned}
 &\int_0^1 \left(\tilde{\mathcal{L}}_{\phi}(\sigma_{n-1}, \phi_n) - \tilde{\mathcal{L}}_{\phi}(\sigma_{n-2}, \phi_{n-1}) \right) \cdot (\mathbf{w}_n - \mathbf{w}_{n-1}) dx dt \\
 &= \int_0^1 \left(\tilde{\mathcal{L}}_{\phi}(\sigma_{n-1}, \phi_n) - \tilde{\mathcal{L}}_{\phi}(\sigma_{n-1}, \phi_{n-1}) \right) \cdot (\mathbf{w}_n - \mathbf{w}_{n-1}) dx dt \\
 &\quad + \int_0^1 \left(\tilde{\mathcal{L}}_{\phi}(\sigma_{n-1}, \phi_{n-1}) - \tilde{\mathcal{L}}_{\phi}(\sigma_{n-2}, \phi_{n-1}) \right) \cdot (\mathbf{w}_n - \mathbf{w}_{n-1}) dx dt.
 \end{aligned} \tag{74}$$

By the continuity equation, there holds

$$\begin{aligned}
 &\int_U \tilde{\mathcal{L}}_{\phi}(\sigma_{n-1}, \phi_n - \phi_{n-1}) \cdot (\mathbf{w}_n - \mathbf{w}_{n-1}) dx dt \\
 &= \int_U (\bar{\rho} + \sigma_{n-1}) \nabla (\phi_n - \phi_{n-1}) \cdot (\mathbf{u}_n - \mathbf{u}_{n-1}) dx dt \\
 &= - \int_U \nabla \cdot ((\bar{\rho} + \sigma_{n-1}) (\mathbf{u}_n - \mathbf{u}_{n-1})) (\phi_n - \phi_{n-1}) dx dt \\
 &= \int_U (\sigma_n - \sigma_{n-1})_t (\phi_n - \phi_{n-1}) dx dt \\
 &\quad + \int_U \nabla \cdot ((\sigma_{n-1} - \sigma_{n-2}) \mathbf{u}_{n-1}) (\phi_n - \phi_{n-1}) dx dt \\
 &= \int_U (\Delta (\phi_n - \phi_{n-1}))_t (\phi_n - \phi_{n-1}) dx dt
 \end{aligned} \tag{75}$$

$$\begin{aligned}
& + \int_U \nabla \cdot ((\sigma_{n-1} - \sigma_{n-2}) \mathbf{u}_{n-1}) (\phi_n - \phi_{n-1}) \, dx \, dt \\
& = -d \int_U |\nabla (\phi_n - \phi_{n-1})|^2 \, dx - \int_U (\sigma_{n-1} - \sigma_{n-2}) \mathbf{u}_{n-1} \cdot \nabla (\phi_n - \phi_{n-1}) \, dx \, dt \\
& \leq -d \int_U |\nabla (\phi_n - \phi_{n-1})|^2 \, dx + C \|\mathbf{w}_n - \mathbf{w}_{n-1}\| \|\mathbf{w}_{n-1} - \mathbf{w}_{n-2}\| \, dt,
\end{aligned}$$

and

$$\begin{aligned}
& \int_0^1 \left(\tilde{\mathcal{L}}_\phi(\sigma_{n-1}, \phi_{n-1}) - \tilde{\mathcal{L}}_\phi(\sigma_{n-2}, \phi_{n-1}) \right) \cdot (\mathbf{w}_n - \mathbf{w}_{n-1}) \, dx \, dt \\
& = \int_U (\sigma_{n-1} - \sigma_{n-2}) \nabla \phi_{n-1} \cdot (\mathbf{u}_n - \mathbf{u}_{n-1}) \, dx \, dt \\
& \leq C \|\mathbf{w}_n - \mathbf{w}_{n-1}\| \|\mathbf{w}_{n-1} - \mathbf{w}_{n-2}\| \, dt.
\end{aligned} \tag{76}$$

For the terms in $\tilde{f}$, similarly, we have

$$\begin{aligned}
& \int_0^1 ((\bar{\rho} + \sigma_{n-1}) \nabla h(\sigma_{n-1}) - (\bar{\rho} + \sigma_{n-2}) \nabla h(\sigma_{n-2})) \cdot (\mathbf{w}_n - \mathbf{w}_{n-1}) \, dx \, dt \\
& \leq C \|\mathbf{w}_n - \mathbf{w}_{n-1}\| \|\mathbf{w}_{n-1} - \mathbf{w}_{n-2}\| \, dt;
\end{aligned} \tag{77}$$

and

$$\begin{aligned}
& \mathbb{E} \left[\left| \int_0^t \int_0^1 (\mathbb{F}_{n-1} - \mathbb{F}_{n-2}) \, dW \cdot (\mathbf{w}_n - \mathbf{w}_{n-1}) \, dx \right|^m \right] \\
& = \mathbb{E} \left[\left| \int_0^t \int_0^1 (\mathbb{F}_{n-1} - \mathbb{F}_{n-2}) \cdot (\mathbf{w}_n - \mathbf{w}_{n-1}) \, dx \, dW \right|^m \right] \\
& \leq \mathbb{E} \left[\left| C \int_0^t \left| \int_0^1 (\mathbb{F}_{n-1} - \mathbb{F}_{n-2}) \cdot (\mathbf{w}_n - \mathbf{w}_{n-1}) \, dx \right|^2 \, ds \right|^{\frac{m}{2}} \right] \\
& \leq \mathbb{E} \left[\left| C \int_0^t \|\mathbf{w}_n - \mathbf{w}_{n-1}\|^2 \|\mathbf{w}_{n-1} - \mathbf{w}_{n-2}\|^2 \, ds \right|^{\frac{m}{2}} \right] \\
& \leq \mathbb{E} \left[\left(C \sup_{s \in [0, t]} \|\mathbf{w}_{n-1} - \mathbf{w}_{n-2}\|^2 \int_0^t \|\mathbf{w}_n - \mathbf{w}_{n-1}\|^2 \, ds \right)^{\frac{m}{2}} \right] \\
& \leq \mathbb{E} \left[\left(\delta_2 \sup_{s \in [0, t]} \|\mathbf{w}_{n-1} - \mathbf{w}_{n-2}\|^2 \right)^m \right] + \mathbb{E} \left[\left(\int_0^t C_{\delta_2} \|\mathbf{w}_n - \mathbf{w}_{n-1}\|^2 \, ds \right)^m \right].
\end{aligned} \tag{78}$$

By Itô's formula, we have

$$\begin{aligned}
& d(\mathcal{D}(\sigma_{n-1})(\mathbf{w}_n - \mathbf{w}_{n-1}) \cdot (\mathbf{w}_n - \mathbf{w}_{n-1})) \\
& = d\mathcal{D}(\sigma_{n-1})(\mathbf{w}_n - \mathbf{w}_{n-1}) \cdot (\mathbf{w}_n - \mathbf{w}_{n-1}) \\
& \quad + 2\mathcal{D}(\sigma_{n-1})(\mathbf{w}_n - \mathbf{w}_{n-1}) \cdot d(\mathbf{w}_n - \mathbf{w}_{n-1}) \\
& \quad + \mathcal{D}(\sigma_{n-1}) \langle d(\mathbf{w}_n - \mathbf{w}_{n-1}), d(\mathbf{w}_n - \mathbf{w}_{n-1}) \rangle,
\end{aligned} \tag{79}$$

where $\langle \cdot \rangle$ is the short abbreviation of $\langle \langle \cdot \rangle, \langle \cdot \rangle \rangle_{\mathcal{H}}^{\frac{1}{2}}$ for quadratic variation, and

$$\begin{aligned}
& \mathcal{D}(\sigma_{n-1}) \langle d(\mathbf{w}_n - \mathbf{w}_{n-1}), d(\mathbf{w}_n - \mathbf{w}_{n-1}) \rangle \\
& = (\bar{\rho} + \sigma_{n-1}) |\mathbb{F}_{n-1} - \mathbb{F}_{n-2}|^2 \, dt,
\end{aligned} \tag{80}$$

and

$$\int_0^1 \mathcal{D}(\sigma_{n-1}) \langle d(\mathbf{w}_n - \mathbf{w}_{n-1}), d(\mathbf{w}_n - \mathbf{w}_{n-1}) \rangle dx \leq C \|\mathbf{w}_{n-1} - \mathbf{w}_{n-2}\|^2 dt. \quad (81)$$

Hence, it holds that

$$\int_0^1 d(\mathcal{D}(\sigma_{n-1})(\mathbf{w}_n - \mathbf{w}_{n-1}) \cdot (\mathbf{w}_n - \mathbf{w}_{n-1})) dx \leq C \|\mathbf{w}_n - \mathbf{w}_{n-1}\|^2 dt. \quad (82)$$

Combining the above estimates, for some $m \geq 2$, we have

$$\begin{aligned} & \mathbb{E} \left[\left(\sup_{s \in [0, t]} \int_0^s d \|\mathbf{w}_n - \mathbf{w}_{n-1}\|^2 \right)^m \right] \\ & \leq \mathbb{E} \left[\int_0^t C \left(\|\mathbf{w}_n - \mathbf{w}_{n-1}\|^2 + \|\mathbf{w}_{n-1} - \mathbf{w}_{n-2}\|^2 \right. \right. \\ & \quad \left. \left. + \|\mathbf{w}_n - \mathbf{w}_{n-1}\| \|\mathbf{w}_{n-1} - \mathbf{w}_{n-2}\| \right) ds \right]^m \\ & \quad + \mathbb{E} \left[\left(\delta_2 \sup_{s \in [0, t]} \|\mathbf{w}_{n-1} - \mathbf{w}_{n-2}\|^2 \right)^m \right] + \mathbb{E} \left[\left(\int_0^t C_{\delta_2} \|\mathbf{w}_n - \mathbf{w}_{n-1}\|^2 ds \right)^m \right], \end{aligned} \quad (83)$$

where C depends on M . By Cauchy's inequality and Jensen's inequality, we have

$$\begin{aligned} & \mathbb{E} \left[\left(\sup_{s \in [0, t]} \|\mathbf{w}_n - \mathbf{w}_{n-1}\|^2 \right)^m \right] \\ & \leq \mathbb{E} \left[\left(\int_0^t C \left(\|\mathbf{w}_n - \mathbf{w}_{n-1}\|^2 + \|\mathbf{w}_{n-1} - \mathbf{w}_{n-2}\|^2 \right. \right. \right. \\ & \quad \left. \left. \left. + \|\mathbf{w}_n - \mathbf{w}_{n-1}\| \|\mathbf{w}_{n-1} - \mathbf{w}_{n-2}\| \right) ds \right)^m \right] \\ & \leq \mathbb{E} \left[\left(\int_0^t C \left(\|\mathbf{w}_n - \mathbf{w}_{n-1}\|^2 + \|\mathbf{w}_{n-1} - \mathbf{w}_{n-2}\|^2 \right) ds \right)^m \right] \\ & \leq \int_0^t \left(\mathbb{E} \left[\left(C_0 \|\mathbf{w}_n - \mathbf{w}_{n-1}\|^2 \right)^m \right] + \mathbb{E} \left[\left(C_0 \|\mathbf{w}_{n-1} - \mathbf{w}_{n-2}\|^2 \right)^m \right] \right) ds. \end{aligned} \quad (84)$$

The higher-order contraction estimates are proved similarly to the zeroth-order, with the same symmetrizing matrix and the important insulating boundary condition, and so the detailed proof is omitted here. In summary, we have

$$\begin{aligned} & \mathbb{E} \left[\left(\sup_{s \in [0, t]} \|\mathbf{w}_n - \mathbf{w}_{n-1}\|_3^2 \right)^m \right] \\ & \leq \int_0^t \left(\mathbb{E} \left[\left(C_0 \|\mathbf{w}_n - \mathbf{w}_{n-1}\|_3^2 \right)^m \right] + \mathbb{E} \left[\left(C_0 \|\mathbf{w}_{n-1} - \mathbf{w}_{n-2}\|_3^2 \right)^m \right] \right) ds. \end{aligned} \quad (85)$$

By Grönwall's inequality, we have

$$\begin{aligned} & \mathbb{E} \left[\left(\sup_{s \in [0, t]} \|\mathbf{w}_n - \mathbf{w}_{n-1}\|_3^2 \right)^m \right] \\ & \leq \mathbb{E} \left[\left(\sup_{s \in [0, t]} \|\mathbf{w}_{n-1} - \mathbf{w}_{n-2}\|_3^2 \right)^m \right] C_0^m t \\ & \quad + \int_0^t \mathbb{E} \left[\left(\sup_{s \in [0, \tau]} \|\mathbf{w}_{n-1} - \mathbf{w}_{n-2}\|_3^2 \right)^m \right] \tau C_0^{2r} e^{C_0^m \tau} d\tau \end{aligned} \quad (86)$$

$$\leq 3C_0^m \mathbb{E} \left[\left(\sup_{s \in [0, \tau]} \|\mathbf{w}_{n-1} - \mathbf{w}_{n-2}\|_3^2 \right)^m \right] t.$$

Let $T_1 \leq T_0$ and $3C_0^m T_1 < 1$, $e^{C_0^m T_1} \leq 2$. Then,

$$\mathbb{E} \left[\left(\sup_{s \in [0, t]} \|\mathbf{w}_n - \mathbf{w}_{n-1}\|_3 \right)^m \right] \leq a \mathbb{E} \left[\left(\sup_{s \in [0, t]} \|\mathbf{w}_{n-1} - \mathbf{w}_{n-2}\|_3 \right)^m \right], \quad a < 1, \quad (87)$$

where $a = 3C_0^m T_1$ with C_0 depending on the initial data by the onto mapping estimates. Hence, $\mathbf{w}_n$ is a Cauchy sequence. By Banach's fixed point theorem, there exists a unique solution $\mathbf{w}$ in $L^{2m}(\Omega; C([0, T_1]; H^3(U)))$. Since $\Delta\phi = \sigma$ holds, ϕ is also a unique solution in $L^{2m}(\Omega; C([0, T_1]; H^5(U)))$ up to a constant, with the boundary condition $\nabla\phi \cdot \nu = 0$.

By the proof of Theorem 5.2.9 in [35], $(\rho, \mathbf{u}, \Phi)$ is the unique strong solutions to SEP, where $\rho, \mathbf{u} \in C([0, T_1]; H^3(U))$ and $\Phi \in C([0, T_1]; H^5(U))$ hold $\mathbb{P}$ a.s. We give the definition of the local strong solution as follows.

Definition 2.2. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a fixed stochastic basis with a complete right-continuous filtration $\mathcal{F} = (\mathcal{F}_s)_{s \geq 0}$, and W be the fixed Wiener process. $(\rho, \mathbf{u}, \Phi)$ is called a strong solution to the initial and boundary value problem (1)-(5)-(6)-(15)-(4), if:

1. $(\rho, \mathbf{u}, \Phi)$ is adapted to the filtration $(\mathcal{F}_s)_{s \geq 0}$;
2. $\mathbb{P}[(\rho(0), \mathbf{u}(0), \Phi(0)) = (\rho_0, \mathbf{u}_0, \Phi_0)] = 1$;
3. the equation of continuity

$$\rho(t) = \rho_0 - \int_0^t \nabla \cdot (\rho \mathbf{u}) \, ds,$$

holds $\mathbb{P}$ a.s. for any $t \in [0, T_1]$;

4. the momentum equation

$$\begin{aligned} \mathbf{u}(t) = & \mathbf{u}_0 - \int_0^t (\mathbf{u} \cdot \nabla) \mathbf{u} \, ds - \int_0^t \frac{\nabla P(\rho)}{\rho} \, ds + \int_0^t \nabla \Phi \, ds - \int_0^t \mathbf{u} \, ds \\ & + \int_0^t \frac{\mathbb{F}(\rho, \mathbf{u})}{\rho} \, dW(s), \end{aligned} \quad (88)$$

holds $\mathbb{P}$ a.s. for any $t \in [0, T_1]$;

5. the electrostatic potential equation

$$\Delta\Phi = \rho - b, \quad (89)$$

holds $\mathbb{P}$ a.s. for any $t \in [0, T_1]$.

Remark 2.3. Reviewing the above proof, (87) holds for general stochastic forces without (17). Thus, the local existence also holds.

Step 4. Energy estimates.

2.2. Estimates up to third-order. In this subsection, we begin by symmetrizing the system. Then, we proceed with energy estimates up to the third order, taking stochastic forces under condition (17) for instance.

2.2.1. *Zero-order estimates.* For system (28), we define the energy

$$\mathcal{E}(t) = \int_U \frac{1}{2} \left(\bar{\rho} |\mathbf{u}|^2 + Q'(\bar{\rho}) \sigma^2 + |\nabla \phi|^2 \right) dx. \quad (90)$$

By Itô's formula, we have

$$\begin{aligned} & d \int_U \frac{1}{2} (\bar{\rho} + \sigma) |\mathbf{u}|^2 dx \\ &= \int_U \frac{1}{2} d(\bar{\rho} + \sigma) |\mathbf{u}|^2 dx + \int_U (\bar{\rho} + \sigma) \mathbf{u} \cdot d\mathbf{u} dx + \int_U \frac{1}{2} (\bar{\rho} + \sigma) |\mathbb{F}|^2 dt dx. \end{aligned} \quad (91)$$

But, here we will deal with $\bar{\rho} + \sigma$ and $\mathbf{u}$ together by considering the symmetrized system of $\mathbf{w}$. By Itô's formula, it holds that

$$\begin{aligned} & \int_U d \left(\frac{1}{2} \mathcal{D} \mathbf{w} \cdot \mathbf{w} \right) dx \\ &= \int_U \frac{1}{2} \mathbf{w} (d\mathcal{D}) \mathbf{w} dx + \int_U \mathcal{D} d\mathbf{w} \cdot \mathbf{w} dx + \int_U \frac{1}{2} \mathcal{D} \mathbb{F} \cdot \mathbb{F} dx dt, \end{aligned} \quad (92)$$

which is $d \int_U \frac{1}{2} \left(\bar{\rho} |\mathbf{u}|^2 + Q'(\bar{\rho}) \sigma^2 \right) dx$. Over the domain U , we integrate

$$\begin{aligned} & \mathcal{D} d\mathbf{w} \cdot \mathbf{w} + \left(\tilde{\mathcal{A}}^1 \mathbf{w}_{,1} \cdot \mathbf{w} + \tilde{\mathcal{A}}^2 \mathbf{w}_{,2} \cdot \mathbf{w} + \tilde{\mathcal{A}}^3 \mathbf{w}_{,3} \cdot \mathbf{w} + \tilde{\mathcal{B}} \mathbf{w} \cdot \mathbf{w} + \tilde{\mathcal{L}}_{\mathbf{u}} \cdot \mathbf{w} \right) dt \\ &= \tilde{\mathcal{L}}_{\phi} \cdot \mathbf{w} dt + \tilde{f} \cdot \mathbf{w}. \end{aligned} \quad (93)$$

Then, we have

$$\begin{aligned} & \int_U d \left(\frac{1}{2} \mathcal{D} \mathbf{w} \cdot \mathbf{w} \right) dx \\ &= \int_U \frac{1}{2} \mathbf{w} (d\mathcal{D}) \mathbf{w} dx + \int_U \mathcal{D} d\mathbf{w} \cdot \mathbf{w} dx + \int_U \frac{1}{2} \mathcal{D} \mathbb{F} \cdot \mathbb{F} dx dt \\ &= \int_U \frac{1}{2} \mathbf{w} (d\mathcal{D}) \mathbf{w} dx - \int_U \left(\tilde{\mathcal{A}}^1 \mathbf{w}_{,1} \cdot \mathbf{w} + \tilde{\mathcal{A}}^2 \mathbf{w}_{,2} \cdot \mathbf{w} + \tilde{\mathcal{A}}^3 \mathbf{w}_{,3} \cdot \mathbf{w} + \tilde{\mathcal{B}} \mathbf{w} \cdot \mathbf{w} \right) dx dt \\ &\quad - \int_U \tilde{\mathcal{L}}_{\mathbf{u}} \cdot \mathbf{w} dx dt + \int_U \tilde{\mathcal{L}}_{\phi} \cdot \mathbf{w} dx dt + \int_U \nabla h(\sigma) \cdot \mathbf{w} dx dt \\ &\quad + \int_U \mathcal{D} \mathbb{F} dW \cdot \mathbf{w} dx + \int_U \mathcal{D} \mathbb{F} \cdot \mathbb{F} dx dt. \end{aligned} \quad (94)$$

Direct calculation shows that

$$-\frac{1}{2} \tilde{\mathcal{A}}_{,i}^i + \tilde{\mathcal{B}} - \text{diag} \left[-\frac{1}{2} (u^i Q'(\rho))_{,i}, -\frac{1}{2} (u^i \rho)_{,i}, -\frac{1}{2} (u^i \rho)_{,i}, -\frac{1}{2} (u^i \rho)_{,i} \right]$$

is anti-symmetric [21]. Hence, we have

$$\begin{aligned} & \int_U \left(\tilde{\mathcal{A}}^1 \mathbf{w}_{,1} \cdot \mathbf{w} + \tilde{\mathcal{A}}^2 \mathbf{w}_{,2} \cdot \mathbf{w} + \tilde{\mathcal{A}}^3 \mathbf{w}_{,3} \cdot \mathbf{w} + \tilde{\mathcal{B}} \mathbf{w} \cdot \mathbf{w} \right) dx dt \\ &= \int_U \left(-\frac{1}{2} \left(\mathbf{w} \tilde{\mathcal{A}}_{,1}^1 \mathbf{w} + \mathbf{w} \tilde{\mathcal{A}}_{,2}^2 \mathbf{w} + \mathbf{w} \tilde{\mathcal{A}}_{,3}^3 \mathbf{w} \right) + \tilde{\mathcal{B}} |\mathbf{w}|^2 \right) dx dt + \int_U \left(\mathbf{w} \tilde{\mathcal{A}}^j \mathbf{w} \right)_{,j} dx dt. \end{aligned} \quad (95)$$

On account of the insulated boundary condition $\mathbf{u} \cdot \nu|_{\partial U} = 0$, it holds that

$$\int_U \left(\mathbf{w} \tilde{\mathcal{A}}^j \mathbf{w} \right)_{,j} dx = \int_{\partial U} \left((\mathbf{u} \cdot \nu) \left(Q'(\bar{\rho}) \sigma^2 + \rho |\mathbf{u}|^2 + 2\rho Q'(\bar{\rho}) \right) \right) dS \equiv 0. \quad (96)$$

Hence, it holds that

$$\int_U \left(-\frac{1}{2} \left(\mathbf{w} \tilde{\mathcal{A}}_1^1 \mathbf{w} + \mathbf{w} \tilde{\mathcal{A}}_2^2 \mathbf{w} + \mathbf{w} \tilde{\mathcal{A}}_3^3 \mathbf{w} \right) + \tilde{\mathcal{B}} |\mathbf{w}|^2 \right) dx dt \leq C \|\mathbf{w}\|_3^3 dt. \quad (97)$$

Recalling (39), there hold

$$\int_U \tilde{\mathcal{L}}_{\mathbf{u}} \cdot \mathbf{w} dx dt = \int_U (\bar{\rho} + \sigma) |\mathbf{u}|^2 dx dt \geq \int_U C \bar{\rho} |\mathbf{u}|^2 dx dt, \quad (98)$$

and

$$\begin{aligned} \int_U \tilde{\mathcal{L}}_{\phi} \cdot \mathbf{w} dx dt &= \int_U (\bar{\rho} + \sigma) \nabla \phi \cdot \mathbf{u} dx dt = - \int_U \nabla \cdot ((\bar{\rho} + \sigma) \mathbf{u}) \phi dx dt \\ &= \int_U \sigma_t \phi dx dt = \int_U (\Delta \phi)_t \phi dx dt = - \int_U |\nabla \phi|^2 dx. \end{aligned} \quad (99)$$

For the stochastic term, it holds that

$$\begin{aligned} \int_U \tilde{f} \cdot \mathbf{w} dx &= \int_U (O(\sigma^2) dt - \mathbb{F} dW) \cdot \mathbf{u} dx \\ &\leq \|\mathbf{w}\|_3^3 dt + \left| \int_U \mathbb{F} dW \cdot \mathbf{u} dx \right|. \end{aligned} \quad (100)$$

For $|\mathbb{F}| \leq C |\rho \mathbf{u}|^2$, we estimate

$$\begin{aligned} \mathbb{E} \left[\left| \int_0^t \int_U \mathbb{F} \cdot \mathbf{u} dx dW \right|^m \right] &\leq \mathbb{E} \left[\left(\int_0^t \left| C \int_U \mathbb{F} \cdot \mathbf{u} dx \right|^2 ds \right)^{\frac{m}{2}} \right] \\ &\leq \mathbb{E} \left[\left(C \int_0^t \left| \int_U |\bar{\rho} \mathbf{u}|^2 |\mathbf{u}|^2 \cdot \mathbf{u} dx \right|^2 ds \right)^{\frac{m}{2}} \right] \\ &\leq \mathbb{E} \left[\left(C \sup_{s \in [0, t]} \|\mathbf{u}\|^2 \int_0^t \|\mathbf{u}\|_3^4 ds \right)^{\frac{m}{2}} \right] \\ &\leq \delta_3^m \mathbb{E} \left[\left(\sup_{s \in [0, t]} \|\mathbf{u}\|^2 \right)^m \right] + C_{\delta_3}^m \mathbb{E} \left[\left(\int_0^t \|\mathbf{u}\|_3^3 ds \right)^m \right], \end{aligned} \quad (101)$$

where δ_3 is taken such that $\delta_3^m \mathbb{E} \left[\left(\sup_{s \in [0, t]} \|\mathbf{u}\|^2 \right)^m \right]$ can be balanced by the left side by the time continuity of solutions. Similarly, it holds that

$$\int_U \frac{1}{2} \mathcal{D} \mathbb{F} \cdot \mathbb{F} dx dt \leq C \|\mathbf{w}\|_3^3 dt. \quad (102)$$

Besides, there holds

$$\begin{aligned} &\int_U \mathbf{w} (d\mathcal{D}) \mathbf{w} dx \\ &= \int_U \mathbf{w} (\text{diag} \{ Q'(\bar{\rho} + \sigma)_t, (\bar{\rho} + \sigma)_t, (\bar{\rho} + \sigma)_t, (\bar{\rho} + \sigma)_t \}) \mathbf{w} dx dt \\ &= \int_U \left(Q''(\bar{\rho} + \sigma) \sigma_t \sigma^2 + \sigma_t |\mathbf{u}|^2 \right) dx dt \\ &= \int_U (Q''(\bar{\rho}) + O(\sigma)) (-\nabla \cdot ((\bar{\rho} + \sigma) \mathbf{u})) \sigma^2 dx dt + \int_U (-\nabla \cdot ((\bar{\rho} + \sigma) \mathbf{u})) |\mathbf{u}|^2 dx dt \end{aligned} \quad (103)$$

$$\leq C \|\mathbf{w}\|_3^3 \, dt.$$

In conclusion, as $\bar{\rho}$ has a positive lower bound, we have

$$\begin{aligned} & \mathbb{E} \left[\left(\sup_{s \in [0, t]} \int_0^s \left(\int_U |\mathbf{w}|^2 \, dx + \int_U |\nabla \phi|^2 \, dx \right) \, ds \right)^m \right] \\ & + \mathbb{E} \left[\left(c_2 \int_0^t \int_U |\mathbf{u}|^2 \, dx \, ds \right)^m \right] \\ & \leq \mathbb{E} \left[\left(C \int_0^t \|\mathbf{w}\|_3^3 \, ds \right)^m \right]. \end{aligned} \quad (104)$$

Next, we give the estimates of $\int_0^t \int_U \|\sigma\|^2 \, dx \, ds$. From the velocity equation (24), we have

$$\begin{aligned} & (\nabla Q(\bar{\rho} + \sigma) - \nabla Q(\bar{\rho})) \, dt \\ & = -d\mathbf{u} - ((\mathbf{u} \cdot \nabla) \mathbf{u} - \mathbf{u}) \, dt + \nabla \phi \, dt + \frac{\mathbb{F}}{\bar{\rho} + \sigma} \, dW, \end{aligned} \quad (105)$$

with

$$\nabla(Q(\bar{\rho} + \sigma) - Q(\bar{\rho})) = Q'(\bar{\rho} + \sigma) \nabla \sigma + Q''(\bar{\rho}) \sigma \nabla \bar{\rho} + \mathbf{h}, \quad (106)$$

where

$$h_i = O(\sigma^2). \quad (107)$$

We multiply equation (105) with $(\sigma, \sigma, \sigma)^T$. By integration by parts and the insulating boundary condition, due to the condition that $|\nabla \bar{\rho}| > 0$, the left side is

$$\int_U |Q''(\bar{\rho}) \nabla \bar{\rho}| |\sigma|^2 \, dx + \int_U O(\sigma^3) \, dx. \quad (108)$$

By Itô's formula, there holds

$$(du^i) \sigma = d(u^i \sigma) - u^i \, d\sigma, \quad (109)$$

where

$$- \int_0^t d \int_U (u^i \sigma) \, dx \leq \int_0^t d \left(\frac{1}{2} \|\sigma\|^2 + \frac{1}{2} \|u^i\|^2 \right). \quad (110)$$

By the continuity equation, it holds that

$$\int_0^t \int_U |u^i \, d\sigma| \, dx \leq C \int_0^t \|\mathbf{w}\|_3^3 \, ds. \quad (111)$$

For $-u \, dt$, we directly estimate

$$\int_0^t \int_U |-u^i \sigma| \, dx \, ds \leq \frac{\delta_4}{2} \int_0^t \|\sigma\|^2 \, ds + C_{\delta_4} \int_0^t \|u^i\|^2 \, ds, \quad (112)$$

where δ_4 is small such that $\delta_4 \int_0^t \|\sigma\|^2 \, ds$ can be balanced by the left side. For the term $\nabla \phi \, dt$ in (105), we estimate

$$\int_0^t \int_U |-\phi_{,i} \sigma| \, dx \, ds \leq \frac{\delta_4}{2} \int_0^t \|\sigma\|^2 \, ds + C_{\delta_4} \sup_{s \in [0, t]} \|\phi_{,i}\|^2. \quad (113)$$

For the stochastic term, since $|\mathbb{F}| \leq C|\rho \mathbf{u}|^2$, we estimate

$$\begin{aligned}
& \mathbb{E} \left[\left| \int_0^t \int_U \frac{\mathbb{F}^i}{\bar{\rho} + \sigma} dW \sigma dx \right|^m \right] \leq \mathbb{E} \left[\left| C \int_0^t \left| \int_U \frac{\mathbb{F}^i}{\bar{\rho} + \sigma} \sigma dx \right|^2 ds \right|^{\frac{m}{2}} \right] \\
& \leq \mathbb{E} \left[\left| C \int_0^t \left| \int_U |\bar{\rho} + \sigma| |\mathbf{u}|^2 \sigma dx \right|^2 ds \right|^{\frac{m}{2}} \right] \\
& \leq \mathbb{E} \left[\left| C \int_0^t \|\mathbf{u}\|^2 \|\bar{\rho}\sigma + \sigma^2\|_\infty^2 ds \right|^{\frac{m}{2}} \right] \\
& \leq \mathbb{E} \left[\left(\frac{1}{4} \sup_{s \in [0, t]} \|\mathbf{u}\|^2 \right)^m \right] + \mathbb{E} \left[\left(C \int_0^t \|\mathbf{u}\|^2 \|\sigma\|_\infty^2 ds \right)^m \right] \\
& \quad + \mathbb{E} \left[\left(\frac{1}{4} \sup_{s \in [0, t]} \|\mathbf{u}\|^2 \right)^m \right] + \mathbb{E} \left[\left(C \int_0^t \|\mathbf{u}\|^2 \|\sigma\|_\infty^4 ds \right)^m \right] \\
& \leq \mathbb{E} \left[\left(\frac{1}{2} \sup_{s \in [0, t]} \|\mathbf{u}\|^2 \right)^m \right] + \mathbb{E} \left[\left(C \int_0^t \|\mathbf{w}\|_3^3 ds \right)^m \right].
\end{aligned} \tag{114}$$

Therefore, we have

$$\begin{aligned}
& \mathbb{E} \left[\left(\int_0^t \|\sigma\|^2 ds \right)^m \right] \\
& \leq \mathbb{E} \left[\left(\int_0^t d \left(\frac{1}{2} \|\sigma\|^2 + \frac{1}{2} \|\mathbf{u}\|^2 \right) \right)^m \right] + \mathbb{E} \left[\left(\frac{1}{2} \sup_{s \in [0, t]} \|\mathbf{u}\|^2 \right)^m \right] \\
& \quad + \mathbb{E} \left[\left(C \sup_{s \in [0, t]} \|\nabla \phi\|^2 \right)^m \right] + \mathbb{E} \left[\left(C \int_0^t \|\mathbf{u}\|^2 ds \right)^m \right] + \mathbb{E} \left[\left(C \int_0^t \|\mathbf{w}\|_3^3 ds \right)^m \right].
\end{aligned} \tag{115}$$

Furthermore, we can give the estimate of $\mathbb{E} \left[\left(\int_0^t \|\nabla \phi\|^2 ds \right)^m \right]$. We multiply (105) with $\nabla \phi$ and integrate it over U . Then, we have

$$\begin{aligned}
& \int_U |\nabla \phi|^2 dx dt \\
& = - \int_U (\nabla Q(\bar{\rho} + \sigma) - \nabla Q(\bar{\rho})) \cdot \nabla \phi dx dt + \int_U d\mathbf{u} \cdot \nabla \phi dx \\
& \quad + \int_U ((\mathbf{u} \cdot \nabla) \mathbf{u} - \mathbf{u}) \cdot \nabla \phi dx dt - \int_U \frac{\mathbb{F}}{\bar{\rho} + \sigma} dW \cdot \nabla \phi dx.
\end{aligned} \tag{116}$$

From (106), by integration by parts and $\triangle \phi = \sigma$, we estimate

$$\begin{aligned}
& \int_0^t \int_U (\nabla Q(\bar{\rho} + \sigma) - \nabla Q(\bar{\rho})) \cdot \nabla \phi dx ds \\
& \leq C \left(\sup_{s \in [0, t]} \|\nabla \phi\|^2 + \int_0^t \|\sigma\|^2 ds + \int_0^t \|\sigma\|_3^3 ds \right).
\end{aligned} \tag{117}$$

By Itô's formula, there holds

$$(d\mathbf{u}) \nabla \phi = d(\mathbf{u} \nabla \phi) - \mathbf{u} d\nabla \phi, \tag{118}$$

where

$$-\int_0^t \int_U (\mathbf{u} \nabla \phi) \, dx \leq \int_0^t \left(\frac{1}{2} \|\nabla \phi\|^2 + \frac{1}{2} \|\mathbf{u}\|^2 \right) \, ds. \quad (119)$$

By the continuity equation, it holds that

$$\begin{aligned} \int_0^t \int_U |\mathbf{u} \, \nabla \phi| \, dx &= \int_0^t \int_U |\mathbf{u} \, \nabla \Delta^{-1} \sigma| \, dx \\ &= \int_0^t \int_U |\mathbf{u} \nabla \Delta^{-1} \sigma| \, dx \leq C \int_0^t \|\mathbf{w}\|_3^3 \, ds. \end{aligned} \quad (120)$$

It is clear that

$$\int_0^t \int_U ((\mathbf{u} \cdot \nabla) \mathbf{u}) \cdot \nabla \phi \, dx \, dt \leq C \int_0^t \|\mathbf{w}\|_3^3 \, ds. \quad (121)$$

For $-\mathbf{u} \, \nabla \phi$, we directly estimate

$$\int_0^t \int_U |-\mathbf{u} \cdot \nabla \phi| \, dx \, ds \leq \frac{1}{2} \int_0^t \|\nabla \phi\|^2 \, ds + \frac{1}{2} \int_0^t \|\mathbf{u}\|^2 \, ds. \quad (122)$$

For the stochastic term, since $|\mathbb{F}| \leq C |\rho \mathbf{u}|^2$ and $\Delta \phi = \sigma$, we estimate

$$\begin{aligned} \mathbb{E} \left[\left| \int_0^t \int_U \frac{\mathbb{F}}{\bar{\rho} + \sigma} \, dW \cdot \nabla \phi \, dx \right|^m \right] &\leq \mathbb{E} \left[\left| C \int_0^t \left| \int_U \frac{\mathbb{F}}{\bar{\rho} + \sigma} \cdot \nabla \phi \, dx \right|^2 \, ds \right|^{\frac{m}{2}} \right] \\ &\leq \mathbb{E} \left[\left| C \int_0^t \left| \int_U |\bar{\rho} + \sigma| |\mathbf{u}|^2 |\nabla \phi| \, dx \right|^2 \, ds \right|^{\frac{m}{2}} \right] \\ &\leq \mathbb{E} \left[\left| C \int_0^t \|\mathbf{u}\|^2 \|\bar{\rho} + \sigma\|_\infty^2 \|\nabla \phi\|_\infty^2 \, ds \right|^{\frac{m}{2}} \right] \\ &\leq \mathbb{E} \left[\left(\frac{1}{4} \sup_{s \in [0, t]} \|\mathbf{u}\|^2 \right)^m \right] + \mathbb{E} \left[\left(C \int_0^t \|\mathbf{u}\|^2 \|\sigma\|_1^2 \, ds \right)^m \right] \\ &\quad + \mathbb{E} \left[\left(\frac{1}{4} \sup_{s \in [0, t]} \|\mathbf{u}\|^2 \right)^m \right] + \mathbb{E} \left[\left(C \int_0^t \|\mathbf{u}\|^2 \|\sigma\|_\infty^2 \|\sigma\|_1^2 \, ds \right)^m \right] \\ &\leq \mathbb{E} \left[\left(\frac{1}{2} \sup_{s \in [0, t]} \|\mathbf{u}\|^2 \right)^m \right] + \mathbb{E} \left[\left(C \int_0^t \|\mathbf{w}\|_3^3 \, ds \right)^m \right]. \end{aligned} \quad (123)$$

Therefore, we have

$$\begin{aligned} &\mathbb{E} \left[\left(\int_0^t \|\nabla \phi\|^2 \, ds \right)^m \right] \\ &\leq \mathbb{E} \left[\left(\int_0^t \left(\frac{1}{2} \|\nabla \phi\|^2 + \frac{1}{2} \|\mathbf{u}\|^2 \right) \, ds \right)^m \right] + \mathbb{E} \left[\left(\frac{1}{2} \sup_{s \in [0, t]} \|\mathbf{u}\|^2 \right)^m \right] \\ &\quad + \mathbb{E} \left[\left(C \sup_{s \in [0, t]} \|\nabla \phi\|^2 \right)^m \right] + \mathbb{E} \left[\left(C \int_0^t \|\mathbf{w}\|^2 \, ds \right)^m \right] + \mathbb{E} \left[\left(C \int_0^t \|\mathbf{w}\|_3^3 \, ds \right)^m \right]. \end{aligned} \quad (124)$$

Multiplying a small constant to it, we add the zeroth-order estimates (104) such that

$$\begin{aligned} & \mathbb{E} \left[\left(\sup_{s \in [0, t]} \int_0^s d \left(\frac{1}{2} \|\sigma\|^2 + \|\mathbf{u}\|^2 + \frac{1}{2} \|\nabla \phi\|^2 \right) \right)^m \right] + \mathbb{E} \left[\left(\frac{1}{2} \sup_{s \in [0, t]} \|\mathbf{u}\|^2 \right)^m \right] \\ & + \mathbb{E} \left[\left(C \sup_{s \in [0, t]} \|\nabla \phi\|^2 \right)^m \right] + \mathbb{E} \left[\left(C \int_0^t \|\mathbf{w}\|^2 ds \right)^m \right] \end{aligned} \quad (125)$$

can be balanced by (104). Then, we obtain

$$\begin{aligned} & \mathbb{E} \left[\left(\sup_{s \in [0, t]} \int_0^s d \left(\|\mathbf{w}\|^2 + \|\nabla \phi\|^2 \right) + \zeta_5 \left(\int_0^t \|\mathbf{w}\|^2 + \|\nabla \phi\|^2 ds \right) \right)^m \right] \\ & \leq \mathbb{E} \left[\left(C \int_0^t \|\mathbf{w}\|_3^3 ds \right)^m \right]. \end{aligned} \quad (126)$$

where C depends on m .

2.2.2. *First-order estimates.* Taking the derivative of (34), we have

$$\begin{aligned} & \nabla (\mathcal{D} d\mathbf{w}) + \nabla \left(\left(\tilde{\mathcal{A}}^1 \mathbf{w}_{,1} + \tilde{\mathcal{A}}^2 \mathbf{w}_{,2} + \tilde{\mathcal{A}}^3 \mathbf{w}_{,3} \right) + \nabla (\tilde{\mathcal{B}} \mathbf{w}) + \nabla (\tilde{\mathcal{L}}_{\mathbf{u}}) \right) dt \\ & = \nabla (\tilde{\mathcal{L}}_{\phi}) dt + \nabla \tilde{f}. \end{aligned} \quad (127)$$

Recalling $\mathcal{D} = \text{diag} [Q'(\bar{\rho} + \sigma), \bar{\rho} + \sigma, \bar{\rho} + \sigma, \bar{\rho} + \sigma]$, we calculate

$$\begin{aligned} \nabla (\mathcal{D} d\mathbf{w}) &= \partial_j \begin{pmatrix} Q'(\bar{\rho} + \sigma) \sigma_t \\ (\bar{\rho} + \sigma) du^1 \\ (\bar{\rho} + \sigma) du^2 \\ (\bar{\rho} + \sigma) du^3 \end{pmatrix} = \begin{pmatrix} \partial_j Q'(\bar{\rho} + \sigma) \sigma_t + Q'(\bar{\rho} + \sigma) \partial_j \sigma_t \\ \partial_j (\bar{\rho} + \sigma) du^1 + (\bar{\rho} + \sigma) \partial_j du^1 \\ \partial_j (\bar{\rho} + \sigma) du^2 + (\bar{\rho} + \sigma) \partial_j du^2 \\ \partial_j (\bar{\rho} + \sigma) du^3 + (\bar{\rho} + \sigma) \partial_j du^3 \end{pmatrix} \\ &= \nabla \cdot \mathcal{D} d\mathbf{w} + \mathcal{D} \nabla d\mathbf{w}, \end{aligned} \quad (128)$$

$$\nabla (\tilde{\mathcal{A}}^i \mathbf{w}_{,i}) = \partial_l (\tilde{\mathcal{A}}_{jk}^i w_{k,i}) = \partial_l \tilde{\mathcal{A}}_{jk}^i w_{k,i} + \tilde{\mathcal{A}}_{jk}^i \partial_l w_{k,i} = \nabla \tilde{\mathcal{A}}^i \mathbf{w}_{,i} + \tilde{\mathcal{A}}^i \nabla \mathbf{w}_{,i}, \quad (129)$$

$$\nabla (\tilde{\mathcal{B}} \mathbf{w}) = \partial_l (\tilde{\mathcal{B}}_{jk} w_k) = \partial_l \tilde{\mathcal{B}}_{jk} w_k + \tilde{\mathcal{B}}_{jk} \partial_l w_k = \nabla \tilde{\mathcal{B}} \mathbf{w} + \tilde{\mathcal{B}} \nabla \mathbf{w}, \quad (130)$$

$$\nabla (\tilde{\mathcal{L}}_{\mathbf{u}}) = \begin{bmatrix} 0 \\ \nabla \mathbf{u} \end{bmatrix}, \quad (131)$$

$$\nabla (\tilde{\mathcal{L}}_{\phi}) = \begin{bmatrix} 0 \\ \nabla ((\bar{\rho} + \sigma) \nabla \phi) \end{bmatrix}, \quad (132)$$

and

$$\nabla (\tilde{f}) = \begin{bmatrix} 0 \\ \nabla (O(\sigma^2) - \mathbb{F} dW) \end{bmatrix} = \begin{bmatrix} 0 \\ O(\sigma \nabla \sigma) dt - \nabla \mathbb{F} dW \end{bmatrix}. \quad (133)$$

Hence, (127) is reduced to

$$\begin{aligned} & d\mathbf{w} \nabla \cdot \mathcal{D} + \mathcal{D} \nabla d\mathbf{w} + \left(\nabla \tilde{\mathcal{A}}^i \mathbf{w}_{,i} + \tilde{\mathcal{A}}^i \nabla \mathbf{w}_{,i} + \nabla \tilde{\mathcal{B}} \mathbf{w} + \tilde{\mathcal{B}} \nabla \mathbf{w} + \nabla \mathbf{u} \right) dt \\ & = \begin{bmatrix} 0 \\ \nabla ((\bar{\rho} + \sigma) \nabla \phi) \end{bmatrix} dt + \begin{bmatrix} 0 \\ O(\sigma \nabla \sigma) \end{bmatrix} dt - \begin{bmatrix} 0 \\ \nabla \mathbb{F} dW \end{bmatrix}. \end{aligned} \quad (134)$$

Multiplying (134) with $\nabla \mathbf{w}$ and integrating it on U , we have

$$\begin{aligned}
 & \int_U \frac{1}{2} \mathcal{D} \, d \nabla \mathbf{w} : \nabla \mathbf{w} \, dx + \int_U \tilde{\mathcal{A}}^i \partial_i (|\nabla \mathbf{w}|^2) \, dx \, dt \\
 & + \int_U \tilde{\mathcal{B}} |\nabla \mathbf{w}|^2 \, dx \, dt + \int_U |\nabla \mathbf{u}|^2 \, dx \, dt \\
 = & \int_U \left[\begin{smallmatrix} 0 \\ \nabla ((\bar{\rho} + \sigma) \nabla \phi) \end{smallmatrix} \right] : \nabla \mathbf{w} \, dx \, dt - \int_U \nabla \tilde{\mathcal{A}}^i \mathbf{w}_{,i} : \nabla \mathbf{w} \, dx \, dt \\
 & - \int_U \nabla \tilde{\mathcal{B}} \mathbf{w} : \nabla \mathbf{w} \, dx \, dt - \int_U d \mathbf{w} \nabla \cdot \mathcal{D} : \nabla \mathbf{w} \, dx + \int_U \left[\begin{smallmatrix} 0 \\ O(\sigma \nabla \sigma) \end{smallmatrix} \right] : \nabla \mathbf{w} \, dx \, dt \\
 & - \int_U \left[\begin{smallmatrix} 0 \\ \nabla \mathbb{F} \, dW \end{smallmatrix} \right] : \nabla \mathbf{w} \, dx.
 \end{aligned} \tag{135}$$

Since $\sigma_t = -\nabla \cdot ((\bar{\rho} + \sigma) \mathbf{u})$, we estimate

$$\begin{aligned}
 & \int_U \frac{1}{2} \mathcal{D}_t \nabla \mathbf{w} : \nabla \mathbf{w} \, dx \, dt \\
 = & \int_U \nabla \mathbf{w} \operatorname{diag} [(Q'(\bar{\rho} + \sigma))_t, (\bar{\rho} + \sigma)_t, (\bar{\rho} + \sigma)_t, (\bar{\rho} + \sigma)_t] \nabla \mathbf{w} \, dx \, dt \\
 = & \int_U \left(\sigma_t Q''(\bar{\rho} + \sigma) \sigma^2 + \sigma_t |\nabla \mathbf{u}|^2 \right) \, dx \, dt \\
 = & \int_U \left(-Q''(\bar{\rho} + \sigma) \nabla \cdot ((\bar{\rho} + \sigma) \mathbf{u}) \sigma^2 - \nabla \cdot ((\bar{\rho} + \sigma) \mathbf{u}) |\nabla \mathbf{u}|^2 \right) \, dx \, dt \\
 \leq & C \|\mathbf{w}\|_3^3 \, dt.
 \end{aligned} \tag{136}$$

Due to the boundary conditions $\mathbf{u} \cdot \nu = 0$, it follows that

$$\int_U \tilde{\mathcal{A}}^i \partial_i (|\nabla \mathbf{w}|^2) \, dx + \int_U \tilde{\mathcal{B}} |\nabla \mathbf{w}|^2 \, dx = 0, \tag{137}$$

and

$$\begin{aligned}
 & \int_0^t \int_U \left[\begin{smallmatrix} 0 \\ \nabla ((\bar{\rho} + \sigma) \nabla \phi) \end{smallmatrix} \right] : \nabla \mathbf{w} \, dx \, ds \\
 & - \int_0^t \left(\int_U \nabla \tilde{\mathcal{A}}^i \mathbf{w}_{,i} : \nabla \mathbf{w} \, dx + \int_U \nabla \tilde{\mathcal{B}} \mathbf{w} : \nabla \mathbf{w} \, dx \right) \, ds \\
 \leq & C \int_0^t \|\mathbf{w}\|_3^3 \, ds + \frac{\delta_5}{3} \sup_{s \in [0, t]} \|\nabla \mathbf{u}\|^2 + C_{\delta_5} \int_0^t \|\nabla \phi\|^2 \, ds,
 \end{aligned} \tag{138}$$

where $\int_0^t \|\nabla \phi\|^2 \, ds = \int_0^t \|\tilde{e}\|^2 \, ds$ can be bounded by $\int_0^t \|\mathbf{w}\|_3^3 \, ds$ from the zeroth-order energy estimates, δ_5 being determined later. Similarly, we estimate

$$\begin{aligned}
 & \mathbb{E} \left[\left| \int_0^t \int_U \nabla \mathbb{F} \, dW : \nabla \mathbf{w} \, dx \right|^m \right] \\
 \leq & \mathbb{E} \left[\left| \frac{\delta_5}{3} \sup_{s \in [0, t]} \|\mathbf{u}\|^2 \right|^m \right] + \mathbb{E} \left[\left| C_{\delta_5} \int_0^t \|\mathbf{w}\|_3^3 \, ds \right|^m \right].
 \end{aligned} \tag{139}$$

From (28), we have

$$- \int_U d \mathbf{w} \nabla \mathcal{D} : \nabla \mathbf{w} \, dx$$

$$\begin{aligned}
&= - \int_U (\mathcal{A}^1 \mathbf{w}_{,1} + \mathcal{A}^2 \mathbf{w}_{,2} + \mathcal{A}^3 \mathbf{w}_{,3} + \mathcal{B} \mathbf{w} + \mathcal{L} \mathbf{u}) \nabla \mathcal{D} : \nabla \mathbf{w} \, dx \, dt \\
&\quad + \int_U \mathcal{L} \phi \nabla \mathcal{D} : \nabla \mathbf{w} \, dx \, dt + \int_U (\mathcal{L} \phi + O(\sigma^2)) \nabla \mathcal{D} : \nabla \mathbf{w} \, dx \, dt \\
&\leq C \|\mathbf{w}\|_3^3 \, dt + \int_U \mathbb{F} \, dW \nabla \mathcal{D} : \nabla \mathbf{w} \, dx.
\end{aligned} \tag{140}$$

Similarly, we have

$$\begin{aligned}
&\mathbb{E} \left[\left| \int_0^t \int_U \mathbb{F} \, dW \nabla \mathcal{D} : \nabla \mathbf{w} \, dx \right|^m \right] \\
&\leq \mathbb{E} \left[\left(\frac{\delta_5}{3} \sup_{s \in [0,t]} \|\mathbf{u}\|^2 \right)^m \right] + \mathbb{E} \left[\left(C_{\delta_5} \int_0^t \|\mathbf{w}\|_3^3 \, ds \right)^m \right].
\end{aligned} \tag{141}$$

We take δ_5 such that $\frac{\delta_5}{3} \|\mathbf{u}\|^2$ and $\frac{\delta_5}{3} \|\nabla \mathbf{u}\|^2$ can be balanced by the left side of the energy estimates. Similar to the estimates for (124), we have the estimate of $\int_0^t \|\nabla \sigma\|^2 \, ds$. In conclusion, we have

$$\begin{aligned}
&\mathbb{E} \left[\left| \sup_{s \in [0,t]} \int_0^s \left(\frac{1}{2} \int_U \mathcal{D} \nabla \mathbf{w} : \nabla \mathbf{w} \, dx \right) \right|^m \right] + \mathbb{E} \left[\left| c_4 \int_0^t \int_U |\nabla \mathbf{w}|^2 \, dx \, ds \right|^m \right] \\
&\leq \mathbb{E} \left[\left| \int_0^t C \|\mathbf{w}\|_3^3 \, ds \right|^m \right],
\end{aligned} \tag{142}$$

where C is independent on t .

2.2.3. Second-order estimates. We write (28) in the form of components, and the i th equation is

$$\begin{aligned}
&d_i \, dw_i + \left(\left(\tilde{\mathcal{A}}^1 \right)_{ij} w_{j,1} + \left(\tilde{\mathcal{A}}^2 \right)_{ij} w_{j,2} + \left(\tilde{\mathcal{A}}^3 \right)_{ij} w_{j,3} + \left(\tilde{\mathcal{B}} \right)_{ij} w_j + d_i w_i \right) dt \\
&= (d_i \phi_{,i} + h(\sigma)_i) \, dt - \mathbb{F}_i \, dW.
\end{aligned} \tag{143}$$

Taking the second-order derivatives, we have

$$\begin{aligned}
&\partial_{kl}^2 \left(d_i \, dw_i + \left(\left(\tilde{\mathcal{A}}^1 \right)_{ij} w_{j,1} + \left(\tilde{\mathcal{A}}^2 \right)_{ij} w_{j,2} + \left(\tilde{\mathcal{A}}^3 \right)_{ij} w_{j,3} + \left(\tilde{\mathcal{B}} \right)_{ij} w_j + d_i w_i \right) dt \right) \\
&= \partial_{kl}^2 ((d_i \phi_{,i} + h(\sigma)_i) \, dt - \mathbb{F}_i \, dW).
\end{aligned} \tag{144}$$

Multiplying (144) with $\partial_k \partial_l w_i$ and integrating it over U , we have

$$\int_U d_i \partial_k \partial_l \, dw_i \partial_k \partial_l w_i \, dx = \int_U d_i \, d|\partial_k \partial_l w_i|^2 \, dx, \tag{145}$$

By the insulated boundary condition $\mathbf{u} \cdot \nu = 0$, for all i, j , there holds

$$\int_U \partial_k \partial_l w_j \left(-\frac{1}{2} \left(\left(\tilde{\mathcal{A}}^1 \right)_{ij,1} + \left(\tilde{\mathcal{A}}^2 \right)_{ij,2} + \left(\tilde{\mathcal{A}}^3 \right)_{ij,3} \right) + \left(\tilde{\mathcal{B}} \right)_{ij} \right) \partial_k \partial_l w_i \, dx = 0. \tag{146}$$

By (28), the integrals in the deterministic terms are bounded by $C \|\mathbf{w}\|_3^3$. The stochastic term is estimated as follows:

$$\mathbb{E} \left[\left| \int_0^t \int_U \partial_{kl}^2 \mathbb{F}_i \partial_k \partial_l w_i \, dx \, dW \right|^m \right] \tag{147}$$

$$\leq \mathbb{E} \left[\left(\delta_6 \sup_{s \in [0, t]} \|\nabla \mathbf{w}\|^2 \right)^m \right] + \mathbb{E} \left[\left(C_{\delta_6} \int_0^t \|\mathbf{w}\|_3^3 ds \right)^m \right],$$

where δ_6 is taken such that $\delta_6 \sup_{s \in [0, t]} \|\nabla \mathbf{w}\|^2$ can be obtained by the left side in first-

order estimates. Similar to estimates (124), we have estimates for $\int_0^t \int_U |\partial^2 \sigma|^2 dx ds$. Taking the sum over the indices $i = 1, 2, 3, 4$, we have

$$\begin{aligned} & \mathbb{E} \left[\left| \sup_{s \in [0, t]} \int_0^s d \left(\int_U \frac{1}{2} |\partial^2 \mathbf{w}|^2 dx \right) \right|^m \right] + \mathbb{E} \left[\left| c_5 \int_0^t \int_U |\partial^2 \mathbf{w}|^2 dx ds \right|^m \right] \\ & \leq \mathbb{E} \left[\left| \int_0^t C \|\mathbf{w}\|_3^3 ds \right|^m \right], \end{aligned} \quad (148)$$

with the assumption that $\bar{\rho}$ has a positive lower bound, where C is independent of t .

2.2.4. Third-order estimates. Considering the *third-order* estimates, we take an additional derivative of (144). Repeating the argument in subsection 2.2.3, we have

$$\begin{aligned} & \mathbb{E} \left[\left| \sup_{s \in [0, t]} \int_0^s d \left(\frac{1}{2} \int_U \mathcal{D} \nabla \mathbf{w} : \nabla \mathbf{w} dx \right) \right|^m \right] + \mathbb{E} \left[\left| c_4 \int_0^t \int_U |\nabla \mathbf{w}|^2 dx ds \right|^m \right] \\ & \leq \mathbb{E} \left[\left| \int_0^t C \|\mathbf{w}\|_3^3 ds \right|^m \right], \end{aligned} \quad (149)$$

where C is independent of t , and

$$\begin{aligned} & \mathbb{E} \left[\left| \sup_{s \in [0, t]} \int_0^s d \left(\int_U \frac{1}{2} |\partial^3 \mathbf{w}|^2 dx \right) \right|^m \right] + \mathbb{E} \left[\left| c_6 \int_0^t \int_U |\partial^3 \mathbf{w}|^2 dx ds \right|^m \right] \\ & \leq \mathbb{E} \left[\left| \int_0^t C \|\mathbf{w}\|_3^3 ds \right|^m \right], \end{aligned} \quad (150)$$

with the assumption that $\bar{\rho}$ has a positive lower bound.

Step 5. Global existence.

2.3. Global existence. In this subsection, we show the global existence for both cases on stochastic forces under (17) and general forces.

2.3.1. For stochastic forces under (17) and small perturbations for initial data (18). We combine the energy estimates up to the third-order. Then, the assumption that $\bar{\rho}$ has a positive lower bound leads to the following inequality:

$$\begin{aligned} & \mathbb{E} \left[\left| \sup_{s \in [0, t]} \left(\|\mathbf{w}\|_3^2(s) + \|\nabla \phi\|^2(s) \right) + \alpha \int_0^t \left(\|\mathbf{w}\|_3^2 + \|\nabla \phi\|^2 \right)(s) ds \right|^m \right] \\ & \leq \mathbb{E} \left[\left| C \int_0^t \|\mathbf{w}\|_3^3 ds \right|^m \right] + \mathbb{E} \left[\left(C \left(\|\mathbf{w}_0\|_3^2 + \|\nabla \phi_0\|^2 \right) \right)^m \right], \end{aligned} \quad (151)$$

where $\alpha \leq c_i, i = 1, \dots, 6$, and C depends on $\bar{\rho}$, m , and the domain U , but is independent of t . Since $\|\mathbf{w}\|_3$ is small, we have

$$\begin{aligned} & \mathbb{E} \left[\sup_{s \in [0, t]} \left(\|\mathbf{w}\|_3^2 + \|\nabla \phi\|^2 \right) + \alpha \int_0^t \left(\|\mathbf{w}\|_3^2 + \|\nabla \phi\|^2 \right) (s) \, ds - C \int_0^t \|\mathbf{w}\|_3^3 \, ds \right]^m \\ & \leq \mathbb{E} \left[\left(C \left(\|\mathbf{w}_0\|_3^2 + \|\nabla \phi_0\|^2 \right) \right)^m \right], \end{aligned} \quad (152)$$

and

$$\mathbb{E} \left[\left(\sup_{s \in [0, t]} \left(\|\mathbf{w}\|_3^2 + \|\nabla \phi\|^2 \right) \right)^m \right] \leq \mathbb{E} \left[\left(C \left(\|\mathbf{w}_0\|_3^2 + \|\nabla \phi_0\|^2 \right) \right)^m \right], \quad (153)$$

where C is independent on t . With the above uniform estimates for any time t , and the local existence on $[0, T_1]$, we can extend the existence to $[T_1, T_1 + \tilde{T}]$, and extend to any time $T_1 + k\tilde{T}$, $\forall k \in \mathbb{N}^+$. More specifically, for the estimate of onto mapping, if

$$\begin{aligned} \mathbb{E} \left[\left(\sup_{s \in [T_1, t]} \|\mathbf{w}_{n-1}(s)\|_3^2 \right)^m \right] & \leq 4\mathbb{E} \left[\left(\left(\|\mathbf{w}(T_1)\|_3^2 + \|\nabla \phi(T_1)\|^2 \right) \right)^m \right] \\ & \leq 4\mathbb{E} \left[\left(C \left(\|\mathbf{w}_0\|_3^2 + \|\nabla \phi_0\|^2 \right) \right)^m \right], \end{aligned} \quad (154)$$

then

$$\mathbb{E} \left[\left(\sup_{s \in [T_1, t]} \|\mathbf{w}_n(s)\|_3^2 \right)^m \right] \leq 4\mathbb{E} \left[\left(C \left(\|\mathbf{w}_0\|_3^2 + \|\nabla \phi_0\|^2 \right) \right)^m \right]. \quad (155)$$

Similarly, the contraction holds from T_1 to $T_1 + \tilde{T}$. Then, the existence is extended to $T_1 + k\tilde{T}$ for any $k \in \mathbb{N}^+$. In conclusion, we obtain the global existence of $\mathbf{w}$ and ϕ , which is equivalent to the global existence of strong solutions $(\rho, \mathbf{u}, \Phi)$ stated by the following proposition.

Proposition 2.4. *In $(\Omega, \mathcal{F}, \mathbb{P})$, there exists a unique global-in-time strong solution $(\rho, \mathbf{u}, \Phi)$ to (1):*

$$\rho, \mathbf{u} \in C([0, T]; H^3(U)), \Phi \in C([0, T]; H^5(U)), \forall T > 0, \quad (156)$$

up to a modification, where $m \geq 2$ is a constant.

2.3.2. For general stochastic forces. If the stochastic forces have linear growth in $\rho\mathbf{u}$, then the following energy estimates hold:

$$\begin{aligned} & \mathbb{E} \left[\left| \sup_{s \in [0, t]} \int_0^s \left(\|\mathbf{w}\|_3^2(s) + \|\nabla \phi\|^2(s) \right) \right. \right. \\ & \quad \left. \left. + \alpha \int_0^t \int_U \left(\|\mathbf{w}\|_3^2(s) + \|\nabla \phi\|^2(s) \right) \, dx \, ds \right|^m \right] \\ & \leq \mathbb{E} \left[\left| \int_0^t C \|\mathbf{w}\|_3^3 \, ds \right|^m \right] + \mathbb{E} \left[\left| \int_0^t C \|\mathbf{w}\|_3^2 \, ds \right|^m \right]. \end{aligned} \quad (157)$$

Without the small perturbation of initial data (18), we can use the generalized Grönwall's inequality to obtain

$$\mathbb{E} \left[\left(\sup_{s \in [0, t]} \left(\|\mathbf{w}\|_3^2 + \|\nabla \phi\|^2 \right) \right)^m \right] \leq \mathbb{E} \left[\left(C(t) \left(\|\mathbf{w}_0\|_3^2 + \|\nabla \phi_0\|^2 \right) \right)^m \right], \quad (158)$$

where $C(t)$ is increasing with respect to t . Similarly, if the stochastic forces have cubic growth in $\rho \mathbf{u}$, then the energy estimates become

$$\begin{aligned} & \mathbb{E} \left[\left| \sup_{s \in [0, t]} \int_0^s \mathrm{d} \left(\|\mathbf{w}\|_3^2(s) + \|\nabla \phi\|^2(s) \right) \right. \right. \\ & \quad \left. \left. + \alpha \int_0^t \int_U \left(\|\mathbf{w}\|_3^2(s) + \|\nabla \phi\|^2(s) \right) \mathrm{d} x \mathrm{d} s \right|^m \right] \\ & \leq \mathbb{E} \left[\left| \int_0^t C \|\mathbf{w}\|_3^3 \mathrm{d} s \right|^m \right] + \mathbb{E} \left[\left| \int_0^t C \|\mathbf{w}\|_3^4 \mathrm{d} s \right|^m \right]. \end{aligned} \quad (159)$$

By the generalized Grönwall's inequality, there also holds (158). Hence, for the smooth Y in (4) such that Y can be bounded by the homogeneous polynomials, the estimates of (158) hold as well. For the estimate of onto mapping, for any fixed T , $t \in [0, T]$, if

$$\begin{aligned} \mathbb{E} \left[\left(\sup_{s \in [T_1, t]} \|\mathbf{w}_{n-1}(s)\|_3^2 \right)^m \right] & \leq 4 \mathbb{E} \left[\left(\left(\|\mathbf{w}(T_1)\|_3^2 + \|\nabla \phi(T_1)\|^2 \right) \right)^m \right] \\ & \leq 4 \mathbb{E} \left[\left(C(T) \left(\|\mathbf{w}_0\|_3^2 + \|\nabla \phi_0\|^2 \right) \right)^m \right], \end{aligned} \quad (160)$$

then

$$\mathbb{E} \left[\left(\sup_{s \in [T_1, t]} \|\mathbf{w}_n(s)\|_3^2 \right)^m \right] \leq 4 \mathbb{E} \left[\left(C(T) \left(\|\mathbf{w}_0\|_3^2 + \|\nabla \phi_0\|^2 \right) \right)^m \right]. \quad (161)$$

Thus, we extend the local existence on $[0, T_1]$ to $[0, T_1 + \tilde{T}]$ and to $[0, T_1 + k\tilde{T}]$, $\forall k \in \mathbb{N}^+$. By Zorn's lemma, the global existence holds.

3. Asymptotic stability of solutions. In this section, we consider the stability under the assumptions of (17) and (18). The *a priori* estimates (151) show the stability of solutions around the steady state. However, (151) is insufficient for investigating the decay rate since the *a priori* estimates are already in the form of time integrals rather than a differential inequality. Integrating twice with respect time might not be wise as it could lead to disappearance of the favorable temporal properties. The asymptotic decay of solution is then derived from the following weighted estimates up to the second-order. To manipulate the weighted energy estimates for stochastic system, we need to multiply $\mathrm{d} \left(\frac{1}{2} \mathcal{D} \mathbf{w} \cdot \mathbf{w} \right)$ directly with $e^{\alpha t}$ first, where α is in (151). Then, we integrate it with respect to x , t , and ω to estimate the time integral.

3.1. Weighted decay estimates.

3.1.1. Zeroth-order weighted estimates. We multiply (94) with $e^{\alpha t}$. Then, we have

$$\begin{aligned} & \int_U e^{\alpha t} \mathrm{d} \left(\frac{1}{2} \mathcal{D} \mathbf{w} \cdot \mathbf{w} \right) \mathrm{d} x \\ & = e^{\alpha t} \int_U \frac{1}{2} \mathbf{w} (\mathrm{d} \mathcal{D}) \mathbf{w} \mathrm{d} x \\ & \quad - e^{\alpha t} \int_U \left(\tilde{\mathcal{A}}^1 \mathbf{w}_{,1} \cdot \mathbf{w} + \tilde{\mathcal{A}}^2 \mathbf{w}_{,2} \cdot \mathbf{w} + \tilde{\mathcal{A}}^3 \mathbf{w}_{,3} \cdot \mathbf{w} + \tilde{\mathcal{B}} \mathbf{w} \cdot \mathbf{w} \right) \mathrm{d} x \mathrm{d} t \end{aligned} \quad (162)$$

$$\begin{aligned}
& -e^{\alpha t} \int_U \tilde{\mathcal{L}}_{\mathbf{u}} \cdot \mathbf{w} \, dx \, dt + e^{\alpha t} \int_U \tilde{\mathcal{L}}_{\phi} \cdot \mathbf{w} \, dx \, dt + e^{\alpha t} \int_U \nabla h(\sigma) \cdot \mathbf{w} \, dx \, dt \\
& + e^{\alpha t} \int_U \mathcal{D}\mathbb{F} \, dW \cdot \mathbf{w} \, dx + e^{\alpha t} \int_U \mathcal{D}\mathbb{F} \cdot \mathbb{F} \, dx \, dt.
\end{aligned}$$

From the estimates of the zeroth-order estimates in subsection 2.2, we conclude the following estimates, omitting detailed calculation:

$$\int_U e^{\alpha t} \left(-\frac{1}{2} \left(\mathbf{w} \tilde{\mathcal{A}}_1^1 \mathbf{w} + \mathbf{w} \tilde{\mathcal{A}}_2^2 \mathbf{w} + \mathbf{w} \tilde{\mathcal{A}}_3^3 \mathbf{w} \right) + \tilde{\mathcal{B}} |\mathbf{w}|^2 \right) dx \, dt \leq C e^{\alpha t} \|\mathbf{w}\|_3^3 \, dt; \quad (163)$$

$$e^{\alpha t} \int_U \tilde{\mathcal{L}}_{\mathbf{u}} \cdot \mathbf{w} \, dx \, dt \geq e^{\alpha t} \int_U C \bar{\rho} |\mathbf{u}|^2 \, dx \, dt \geq \alpha e^{\alpha t} \int_U C \bar{\rho} |\mathbf{u}|^2 \, dx \, dt; \quad (164)$$

$$e^{\alpha t} \int_U \tilde{\mathcal{L}}_{\phi} \cdot \mathbf{w} \, dx \, dt = -e^{\alpha t} \int_U |\nabla \phi|^2 \, dx; \quad (165)$$

$$e^{\alpha t} \int_U \mathbf{w} \, (d\mathcal{D}) \mathbf{w} \, dx \leq C e^{\alpha t} \|\mathbf{w}\|_3^3 \, dt; \quad (166)$$

$$e^{\alpha t} \int_U \frac{1}{2} \mathcal{D}\mathbb{F} \cdot \mathbb{F} \, dx \, dt \leq C e^{\alpha t} \|\mathbf{w}\|_3^3 \, dt. \quad (167)$$

For the estimates of the stochastic integral, it holds that

$$e^{\alpha t} \int_U \tilde{f} \cdot \mathbf{w} \, dx \leq e^{\alpha t} \|\mathbf{w}\|_3^3 + e^{\alpha t} \left| \int_U \mathbb{F} \, dW \cdot \mathbf{u} \, dx \right|. \quad (168)$$

For $|\mathbb{F}| \leq C |\rho \mathbf{u}|^2$,

$$\begin{aligned}
& \mathbb{E} \left[\left| \int_0^t e^{\alpha s} \int_U \mathbb{F} \cdot \mathbf{u} \, dx \, dW \right|^m \right] \leq \mathbb{E} \left[\left(C \int_0^t e^{2\alpha s} \left| \int_U \mathbb{F} \cdot \mathbf{u} \, dx \right|^2 \, ds \right)^{\frac{m}{2}} \right] \\
& \leq \mathbb{E} \left[\left(C \int_0^t e^{2\alpha s} \left| \int_U |\bar{\rho} \mathbf{u}|^2 |\mathbf{u}| \, dx \right|^2 \, ds \right)^{\frac{m}{2}} \right] \\
& \leq \mathbb{E} \left[\left(C \sup_{s \in [0, t]} \|\mathbf{u}\|^2 \int_0^t e^{2\alpha s} \|\mathbf{u}\|_3^4 \, ds \right)^{\frac{m}{2}} \right] \\
& \leq e^{\alpha m t} \mathbb{E} \left[\left(\sup_{s \in [0, t]} \|\mathbf{u}\|^2 \right)^m \right] + e^{\alpha m t} \mathbb{E} \left[\left(C \int_0^t \|\mathbf{u}\|_3^3 \, ds \right)^m \right] \\
& \leq e^{\alpha m t} \mathbb{E} \left[\left(C \int_0^t \|\mathbf{u}\|_3^3 \, ds \right)^m \right], \quad (169)
\end{aligned}$$

where the last inequality holds due to the zeroth-order estimates in subsection 2.2, and C is a general constant. In summary, as $\bar{\rho}$ has a positive lower bound, we have

$$\begin{aligned}
& \mathbb{E} \left[\left(\int_0^t e^{\alpha s} \, d \left(\int_U |\mathbf{w}|^2 \, dx + \int_U |\nabla \phi|^2 \, dx \right) \right)^m \right] + \mathbb{E} \left[\left(\int_0^t e^{\alpha s} \int_U |\mathbf{u}|^2 \, dx \, ds \right)^m \right] \\
& \leq e^{\alpha m t} \mathbb{E} \left[\left(C \int_0^t \|\mathbf{w}\|_3^3 \, ds \right)^m \right]. \quad (170)
\end{aligned}$$

Next, we give the estimates of $\int_0^t e^{\alpha s} \int_U \|\sigma\|^2 dx ds$. From the velocity equation (24), we have

$$\begin{aligned} & e^{\alpha t} (\nabla Q(\bar{\rho} + \sigma) - \nabla Q(\bar{\rho})) dt \\ &= -e^{\alpha t} d\mathbf{u} - e^{\alpha t} ((\mathbf{u} \cdot \nabla) \mathbf{u} - \mathbf{u}) dt + e^{\alpha t} \nabla \phi dt + e^{\alpha t} \frac{\mathbb{F}}{\bar{\rho} + \sigma} dW, \end{aligned} \quad (171)$$

with

$$\nabla(Q(\bar{\rho} + \sigma) - Q(\bar{\rho})) = Q'(\bar{\rho} + \sigma) \nabla \sigma + Q''(\bar{\rho}) \sigma \nabla \bar{\rho} + \mathbf{h},$$

where

$$h_i = O(\sigma^2). \quad (172)$$

We multiply equation (171) with $(\sigma, \sigma, \sigma)^T$. The left side of (171) is

$$e^{\alpha t} \int_U |Q''(\bar{\rho}) \nabla \bar{\rho}| |\sigma|^2 dx + e^{\alpha t} \int_U O(\sigma^3) dx. \quad (173)$$

By Itô's formula,

$$e^{\alpha t} (du^i) \sigma = e^{\alpha t} d(u^i \sigma) - e^{\alpha t} u^i d\sigma, \quad (174)$$

where

$$-\int_0^t e^{\alpha s} d \int_U (u^i \sigma) dx \leq \int_0^t e^{\alpha s} d \left(\frac{1}{2} \|\sigma\|^2 + \frac{1}{2} \|u^i\|^2 \right). \quad (175)$$

By the continuity equation, it holds that

$$\int_0^t e^{\alpha s} \int_U |u^i d\sigma| dx \leq C \int_0^t e^{\alpha s} \|\mathbf{w}\|_3^3 ds. \quad (176)$$

For $-\mathbf{u} dt$, we directly estimate

$$\int_0^t \int_U e^{\alpha s} |-u^i \sigma| dx ds \leq \frac{\delta_4}{2} \int_0^t e^{\alpha s} \|\sigma\|^2 ds + C_{\delta_4} \int_0^t \|u^i\|^2 ds, \quad (177)$$

where δ_4 is small such that $\delta_4 \int_0^t e^{\alpha s} \|\sigma\|^2 ds$ can be balanced by the left side. For the term $\nabla \phi dt$ in (105), we estimate

$$\int_0^t \int_U e^{\alpha s} |-\phi_{,i} \sigma| dx ds \leq \frac{\delta_4}{2} \int_0^t e^{\alpha s} \|\sigma\|^2 ds + C_{\delta_4} e^{\alpha t} \sup_{s \in [0, t]} \|\phi_{,i}\|^2. \quad (178)$$

For the stochastic term, since $|\mathbb{F}| \leq C |\rho \mathbf{u}|^2$, we estimate

$$\begin{aligned} & \mathbb{E} \left[\left| \int_0^t e^{\alpha s} \int_U \frac{\mathbb{F}^i}{\bar{\rho} + \sigma} dW \sigma dx \right|^m \right] \leq \mathbb{E} \left[\left| C \int_0^t e^{\alpha s} \left| \int_U \frac{\mathbb{F}^i}{\bar{\rho} + \sigma} \sigma dx \right|^2 ds \right|^{\frac{m}{2}} \right] \\ & \leq \mathbb{E} \left[\left| C \int_0^t e^{\alpha s} \left| \int_U |\bar{\rho} + \sigma| |\mathbf{u}|^2 \sigma dx \right|^2 ds \right|^{\frac{m}{2}} \right] \\ & \leq \mathbb{E} \left[\left| C \int_0^t e^{\alpha s} \|\mathbf{u}\|^2 \|\bar{\rho} \sigma + \sigma^2\|_\infty^2 ds \right|^{\frac{m}{2}} \right] \\ & \leq e^{\alpha m t} \mathbb{E} \left[\left(\frac{1}{4} \sup_{s \in [0, t]} \|\mathbf{u}\|^2 \right)^m \right] + \mathbb{E} \left[\left(C \int_0^t e^{\alpha s} \|\mathbf{u}\|^2 \|\sigma\|_\infty^2 ds \right)^m \right] \end{aligned} \quad (179)$$

$$\begin{aligned}
& + e^{\alpha m t} \mathbb{E} \left[\left(\frac{1}{4} \sup_{s \in [0, t]} \|\mathbf{u}\|^2 \right)^m \right] + \mathbb{E} \left[\left(C \int_0^t e^{\alpha s} \|\mathbf{u}\|^2 \|\sigma\|_\infty^4 \, ds \right)^m \right] \\
& \leq e^{\alpha m t} \mathbb{E} \left[\left(\frac{1}{2} \sup_{s \in [0, t]} \|\mathbf{u}\|^2 \right)^m \right] + \mathbb{E} \left[\left(C \int_0^t e^{\alpha s} \|\mathbf{w}\|_3^3 \, ds \right)^m \right] \\
& \leq e^{\alpha m t} \mathbb{E} \left[\left(C \int_0^t \|\mathbf{w}\|_3^3 \, ds \right)^m \right] + \mathbb{E} \left[\left(C \int_0^t e^{\alpha s} \|\mathbf{w}\|_3^3 \, ds \right)^m \right].
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
& \mathbb{E} \left[\left(\int_0^t e^{\alpha s} \|\sigma\|^2 \, ds \right)^m \right] \\
& \leq \mathbb{E} \left[\left(\int_0^t e^{\alpha s} \, ds \left(\frac{1}{2} \|\sigma\|^2 + \frac{1}{2} \|\mathbf{u}\|^2 \right) \right)^m \right] \\
& \quad + e^{\alpha m t} \mathbb{E} \left[\left(C \int_0^t \|\mathbf{w}\|_3^3 \, ds \right)^m \right] + \mathbb{E} \left[\left(C \int_0^t e^{\alpha s} \|\mathbf{w}\|_3^3 \, ds \right)^m \right] \\
& \leq \mathbb{E} \left[\left(\int_0^t e^{\alpha s} \, ds \left(\frac{1}{2} \|\sigma\|^2 + \frac{1}{2} \|\mathbf{u}\|^2 \right) \right)^m \right] + e^{\alpha m t} \mathbb{E} \left[\left(C \int_0^t \|\mathbf{w}\|_3^3 \, ds \right)^m \right].
\end{aligned} \tag{180}$$

Similarly, the estimates for $\mathbb{E} \left[\left(\int_0^t e^{\alpha s} \|\nabla \phi\|^2 \, ds \right)^m \right]$ hold:

$$\begin{aligned}
& \mathbb{E} \left[\left(\int_0^t e^{\alpha s} \|\nabla \phi\|^2 \, ds \right)^m \right] \\
& \leq \mathbb{E} \left[\left(\int_0^t e^{\alpha s} \, ds \left(\frac{1}{2} \|\nabla \phi\|^2 + \frac{1}{2} \|\mathbf{u}\|^2 \right) \right)^m \right] \\
& \quad + e^{\alpha m t} \mathbb{E} \left[\left(C \int_0^t \|\mathbf{w}\|_3^3 \, ds \right)^m \right] + \mathbb{E} \left[\left(C \int_0^t e^{\alpha s} \|\mathbf{w}\|_3^3 \, ds \right)^m \right] \\
& \leq \mathbb{E} \left[\left(\int_0^t e^{\alpha s} \, ds \left(\frac{1}{2} \|\nabla \phi\|^2 + \frac{1}{2} \|\mathbf{u}\|^2 \right) \right)^m \right] + e^{\alpha m t} \mathbb{E} \left[\left(C \int_0^t \|\mathbf{w}\|_3^3 \, ds \right)^m \right].
\end{aligned} \tag{181}$$

Multiplying a small constant with (180) and (181), we add the zeroth-order estimates (104) such that

$$\mathbb{E} \left[\left(\int_0^t e^{\alpha s} \, ds \left(\frac{1}{2} \|\sigma\|^2 + \|\mathbf{u}\|^2 + \frac{1}{2} \|\nabla \phi\|^2 \right) \right)^m \right] \tag{182}$$

can be balanced by (104). Then, we obtain

$$\begin{aligned}
& \mathbb{E} \left[\left(\int_0^t e^{\alpha s} \, ds \left(\|\mathbf{w}\|^2 + \|\nabla \phi\|^2 \right) + \alpha \int_0^t e^{\alpha s} \left(\|\mathbf{w}\|^2 + \|\nabla \phi\|^2 \right) \, ds \right)^m \right] \\
& \leq e^{\alpha m t} \mathbb{E} \left[\left(C \int_0^t \|\mathbf{w}\|_3^3 \, dt \right)^m \right].
\end{aligned} \tag{183}$$

3.1.2. *First-order weighted estimates.* Multiplying (134) by $e^{\alpha t} \nabla \mathbf{w}$ and integrating it over U , we can repeat the argument from subsection 3.1.1 to obtain

$$\begin{aligned} & \mathbb{E} \left[\left| \int_0^t e^{\alpha s} \mathrm{d} \left(\int_U \mathcal{D} \nabla \mathbf{w} : \nabla \mathbf{w} \mathrm{d} x \right) \right|^m \right] + \mathbb{E} \left[\left| \int_0^t \alpha e^{\alpha s} \int_U |\nabla \mathbf{w}|^2 \mathrm{d} x \mathrm{d} s \right|^m \right] \\ & \leq \mathbb{E} \left[\left| e^{\alpha t} \int_0^t C \|\mathbf{w}\|_3^3 \mathrm{d} s \right|^m \right]. \end{aligned} \quad (184)$$

3.1.3. *Second-order weighted estimates.* Similarly, we multiply (144) with $e^{\alpha t} \partial^2 \mathbf{w}$, and then integrate it on U . Repeating the procedure in subsection 3.1.1, we have

$$\begin{aligned} & \mathbb{E} \left[\left| \int_0^t e^{\alpha s} \mathrm{d} \left(\int_U |\partial^2 \mathbf{w}|^2 \mathrm{d} x \right) \right|^m \right] + \mathbb{E} \left[\left| \int_0^t \alpha e^{\alpha s} \int_U |\partial^2 \mathbf{w}|^2 \mathrm{d} x \mathrm{d} s \right|^m \right] \\ & \leq \mathbb{E} \left[\left| e^{\alpha t} \int_0^t C \|\mathbf{w}\|_3^3 \mathrm{d} s \right|^m \right]. \end{aligned} \quad (185)$$

3.1.4. *Third-order weighted estimates.* Considering the *third-order* weighted estimates, repeating the procedure in subsection 3.1.1, we have

$$\begin{aligned} & \mathbb{E} \left[\left| \int_0^t e^{\alpha s} \mathrm{d} \left(\int_U |\partial^3 \mathbf{w}|^2 \mathrm{d} x \right) \right|^m \right] + \mathbb{E} \left[\left| \int_0^t \alpha e^{\alpha s} \int_U |\partial^3 \mathbf{w}|^2 \mathrm{d} x \mathrm{d} s \right|^m \right] \\ & \leq \mathbb{E} \left[\left| e^{\alpha t} \int_0^t C \|\mathbf{w}\|_3^3 \mathrm{d} s \right|^m \right]. \end{aligned} \quad (186)$$

3.2. **Asymptotic stability.** Combining the weighted estimates in the previous subsections, we obtain

$$\begin{aligned} & \mathbb{E} \left[\left| \int_0^t e^{\alpha s} \mathrm{d} \left(\|\mathbf{w}\|_3^2 + \|\nabla \phi\|^2 \right) + \int_0^t \alpha e^{\alpha s} \left(\|\mathbf{w}\|_3^2 + \|\nabla \phi\|^2 \right) \mathrm{d} s \right|^m \right] \\ & \leq \mathbb{E} \left[\left| e^{\alpha t} \int_0^t C \|\mathbf{w}\|_3^3 \mathrm{d} s \right|^m \right]. \end{aligned} \quad (187)$$

Therefore, we have

$$\begin{aligned} & \mathbb{E} \left[\left| e^{\alpha t} \left(\|\mathbf{w}\|_3^2 + \|\nabla \phi\|^2 \right) \right|^m \right] \\ & \leq \mathbb{E} \left[\left| \left(\|\mathbf{w}_0\|_3^2 + \|\nabla \phi_0\|^2 \right) \right|^m \right] + \mathbb{E} \left[\left| e^{\alpha t} \int_0^t C \|\mathbf{w}\|_3^3 \mathrm{d} s \right|^m \right]. \end{aligned} \quad (188)$$

Since $\|\mathbf{w}_0\|_3^2 + \|\nabla \phi_0\|^2$ is small, we have

$$\begin{aligned} & \mathbb{E} \left[\left| e^{\alpha t} \left(\|\mathbf{w}\|_3^2 + \|\nabla \phi\|^2 \right) - e^{\alpha t} \int_0^t C \|\mathbf{w}\|_3^3 \mathrm{d} s \right|^m \right] \\ & \leq \mathbb{E} \left[\left| \left(\|\mathbf{w}_0\|_3^2 + \|\nabla \phi_0\|^2 \right) \right|^m \right]. \end{aligned} \quad (189)$$

We estimate

$$e^{\alpha t} \int_0^t \|\mathbf{w}\|_3^3 \mathrm{d} s \leq e^{\alpha t} t \sup_{s \in [0, t]} \|\mathbf{w}\|_3^3 \leq e^{\frac{3\alpha t}{2}} \sup_{s \in [0, t]} \|\mathbf{w}\|_3^3. \quad (190)$$

Therefore, we obtain the asymptotic decay estimates

$$\mathbb{E} \left[\left| \sup_{s \in [0, t]} \left(\|\mathbf{w}\|_3^2 + \|\nabla \phi\|^2 \right) \right|^m \right] \leq e^{-\alpha m t} \mathbb{E} \left[\left| C \left(\|\mathbf{w}_0\|_3^2 + \|\nabla \phi_0\|^2 \right) \right|^m \right], \quad (191)$$

on the account that $\|\mathbf{w}_0\|_3^2 + \|\nabla \phi_0\|^2$ is sufficiently small, where $m \geq 2$.

4. Invariant measures. The law generated by the initial data $\mathbf{z}_0 := (\rho_0, \mathbf{u}_0, \Phi_0)$ in probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is denoted by $\mathcal{L}(\mathbf{z}_0)$. We denote $\mathcal{H} := H^3(U) \times H^3(U) \times H^5(U)$. With the initial data $\mathbf{z}_0 := (\rho_0, \mathbf{u}_0, \Phi_0) \in \mathcal{H}$ and the assumptions of (17) and (18), SEP system (1) admits a unique martingale solution

$$\mathbf{z}(t, x, \omega) := (\rho, \mathbf{u}, \Phi) \in \mathcal{H}. \quad (192)$$

Let $\mathcal{S}_t$ be the transition semigroup [48]

$$\mathcal{S}_t \psi(\mathbf{z}_0) = \mathbb{E}[\psi(\mathbf{z}((t, \mathbf{z}_0)))], \quad t \geq 0, \quad (193)$$

where ψ is the bounded function on $\mathcal{H}$, i.e., $\psi \in C_b(\mathcal{H})$. For $t \geq 0$, $\mathcal{S}(t, \mathbf{z}_0, \Gamma)$ is the transition function

$$\mathcal{S}(t, \mathbf{z}_0, \Gamma) := \mathcal{S}_t(\mathbf{z}_0, \Gamma) = \mathcal{S}_t \chi_\Gamma(\mathbf{z}_0) = \mathcal{L}(\mathbf{z}(t, \mathbf{z}_0))(\Gamma), \quad \mathbf{z}_0 \in \mathcal{H}, \Gamma \in \mathcal{B}(\mathcal{H}). \quad (194)$$

For $\mathbf{v}_0 := (\rho_0 - \bar{\rho}, \mathbf{u}_0, \Phi_0 - \bar{\Phi})$ in probability space $(\Omega, \mathcal{F}, \mathbb{P})$, the perturbed system (24) admits a unique martingale solution

$$\mathbf{v}(t, x, \omega) := (\rho - \bar{\rho}, \mathbf{u}, \Phi - \bar{\Phi}) \in \mathcal{H}. \quad (195)$$

$\tilde{\mathcal{S}}_t$ is the transition semigroup

$$\tilde{\mathcal{S}}_t \psi(\mathbf{v}_0) = \mathbb{E}[\psi(\mathbf{v}((t, \mathbf{v}_0)))], \quad t \geq 0, \quad (196)$$

where ψ is the bounded function on $\mathcal{H}$, i.e., $\psi \in C_b(\mathcal{H})$. The transition function for the perturbed system (24) is denoted by $\tilde{\mathcal{S}}(t, \mathbf{z}_0, \Gamma)$.

We give the definition of the stationary solution for (1).

Definition 4.1. A martingale solution $(\rho; \mathbf{u}; \Phi)$ to system (1) under the initial boundary conditions (5)-(6) is called stationary, provided that the transition function $(\mathcal{S}_\tau \rho, \mathcal{S}_\tau \mathbf{u}, \mathcal{S}_\tau \Phi)$ on $C([0, T]; H^3(U)) \times C([0, T]; H^3(U)) \times C([0, T]; H^5(U))$ is independent of $\tau \geq 0$.

Let $\mathcal{M}(\mathcal{H})$ be the space of all bounded measures on $(\mathcal{H}, \mathcal{B}(\mathcal{H}))$. For any $\psi \in C_b(\mathcal{H})$ and any $\mu \in \mathcal{M}(\mathcal{H})$, we set

$$\langle \psi, \mu \rangle = \int_{\mathcal{H}} \psi(x) \mu(dx). \quad (197)$$

For $t \geq 0$, $\mu \in \mathcal{M}(\mathcal{H})$, $\mathcal{S}_t^*$ acts on $\mathcal{M}(\mathcal{H})$ by

$$\mathcal{S}_t^* \mu(\Gamma) = \int_{\mathcal{H}} \mathcal{S}(t, x, \Gamma) \mu(dx), \quad \Gamma \in \mathcal{B}(\mathcal{H}). \quad (198)$$

Moreover, there holds

$$\langle \psi, \mathcal{S}_t^* \mu \rangle = \langle \mathcal{S}_t \psi, \mu \rangle, \quad \forall \psi \in C_b(\mathcal{H}), \quad \mu \in \mathcal{M}(\mathcal{H}). \quad (199)$$

Particularly, for the perturbed system (24) and $\mathbf{v}_0 := (\rho_0 - \bar{\rho}, \mathbf{u}_0, \Phi_0 - \bar{\Phi})$ in probability space $(\Omega, \mathcal{F}, \mathbb{P})$, there holds $\mathcal{S}_t^* \mathcal{L}(\mathbf{v}_0) = \mathcal{L}(\mathbf{v}(t, \mathbf{v}_0))$. In other words,

$$(\mathcal{S}_t \psi) \mathcal{L}(\mathbf{v}_0) = \mathbb{E}[\psi(\mathbf{v}(t))], \quad (200)$$

where $\psi \in C_b(\mathcal{H})$.

Definition 4.2. A measure μ in $\mathcal{M}(\mathcal{H})$ is said to be an invariant (stationary) measure if

$$P_t^* \mu = \mu, \quad \forall t > 0. \quad (201)$$

The Dirac measure centered at the steady state $(\bar{\rho}, 0, \bar{\Phi})$ is the invariant measure for (7) since it stays unchanged after the action of the transition semigroup for (7).

For $\mathbf{z}_0 \in \mathcal{H}$ and $T > 0$, the formula

$$\frac{1}{T} \int_0^T \mathcal{S}_t(\mathbf{z}_0, \Gamma) dt = R_T(\mathbf{z}_0, \Gamma), \quad \Gamma \in \mathcal{B}(\mathcal{H}), \quad (202)$$

defines a probability measure. For any $\nu \in \mathcal{M}(H)$, $R_T^* \nu$ is defined as follows:

$$R_T^* \nu(\Gamma) = \int_{\mathcal{H}} R_T(x, \Gamma) \nu(dx), \quad \Gamma \in \mathcal{B}(\mathcal{H}). \quad (203)$$

For any $\psi \in C_b(\mathcal{H})$, there holds

$$\langle R_T^* \nu, \psi \rangle = \frac{1}{T} \int_0^T \langle \mathcal{S}_t^* \nu, \psi \rangle dt. \quad (204)$$

$\mathcal{S}_t$ is a Feller semigroup provided that, for arbitrary $\psi \in C_b(\mathcal{H})$, the function

$$[0, +\infty) \times \mathcal{H}, \quad (t, x) \mapsto \mathcal{S}_t \psi(x) \quad (205)$$

is continuous. Since the solution is continuous and unique, we do not need the Markov selection as in [17, 27].

The method of constructing an invariant measure described in the following theorem is due to Krylov-Bogoliubov [38].

Theorem 4.3. *If for some $\nu \in \mathcal{M}(\mathcal{H})$ and some sequence $T_n \uparrow +\infty$, $R_{T_n}^* \nu \rightarrow \mu$ weakly as $n \rightarrow \infty$, then μ is an invariant measure for the Feller semigroup $\tilde{\mathcal{S}}_t, t \geq 0$.*

The following lemma is obtained similarly to [6], and we provide a proof for the convenience of the readers. $\mathbf{v}_t^{\mathbf{v}_0}$ represents the stochastic process initiated from $\mathbf{v}_0$ for the sake of expediency.

Lemma 4.4. *The SEP (24) defines a Feller-Markov process, i.e., $\tilde{\mathcal{S}}_t : C_b(\mathcal{H}) \rightarrow C_b(\mathcal{H})$, and*

$$\mathbb{E} [\psi(\mathbf{v}_{t+s}^{\mathbf{v}_0}) | \mathcal{F}_t] = (\tilde{\mathcal{S}}_s \psi)(\mathbf{v}_t^{\mathbf{v}_0}), \quad \forall \mathbf{v}_0 \in \mathcal{H}, \quad \psi \in C_b(\mathcal{H}), \quad \forall t, s > 0, \quad (206)$$

Proof. From the continuity of solutions, it is easy to see the Feller property that $\mathcal{S}_t : C_b(\mathcal{H}) \rightarrow C_b(\mathcal{H})$ is continuous. For the Markov property, it suffices to prove

$$\mathbb{E} [\psi(\mathbf{v}_{t+s}^{\mathbf{v}_0}) X] = \mathbb{E} [\mathcal{S}_s \psi(\mathbf{v}_t^{\mathbf{v}_0}) X], \quad (207)$$

where $X \in \mathcal{F}_t$.

Let $\mathbf{D}$ be any $\mathcal{F}_t$ -measurable random variable. We denote $\mathbf{D}_n = \sum_{i=1}^n \mathbf{D}^i \mathbf{1}_{\Omega^i}$, where $\mathbf{D}^i \in H$ are deterministic and $(\Omega^i) \subset \mathcal{F}_t$ is a collection of disjoint sets such that $\bigcup_i \Omega^i = \Omega$. $\mathbf{D}_n \rightarrow \mathbf{D}$ in $\mathcal{H}$ implies $\mathcal{S}_t \varphi(\mathbf{D}_n) \rightarrow \mathcal{S}_t \varphi(\mathbf{D})$ in $\mathcal{H}$. For every deterministic $\mathbf{D} \in \mathcal{F}_t$, the random variable $\mathbf{v}_{t,t+s}^{\mathbf{D}}$ depends only on the increments of the Brownian motion $W_{t+s} - W_t$, and hence it is independent of $\mathcal{F}_t$. Therefore, it holds that

$$\mathbb{E} [\psi(\mathbf{v}_{t,t+s}^{\mathbf{D}}) X] = \mathbb{E} [\psi(\mathbf{v}_{t,t+s}^{\mathbf{D}})] \mathbb{E}[X], \quad \forall \mathbf{D} \in \mathcal{F}_t. \quad (208)$$

Since $\mathbf{v}_{t,t+s}^{\mathbf{D}}$ has the same law as $\mathbf{v}_s^{\mathbf{D}}$ by uniqueness, we have

$$\mathbb{E} [\psi (\mathbf{v}_{t,t+s}^{\mathbf{D}}) X] = \mathbb{E} [\psi (\mathbf{v}_s^{\mathbf{D}})] \mathbb{E}[X] = \mathcal{S}_s \psi(\mathbf{D}) \mathbb{E}[X] = \mathbb{E} [\mathcal{S}_s \psi(\mathbf{D}) X]. \quad (209)$$

Thus, there holds

$$\mathbb{E} [\varphi (\mathbf{v}_{t,t+s}^{\mathbf{D}}) X] = \mathbb{E} [(\mathcal{S}_s \varphi) (\mathbf{D}) X] \quad (210)$$

for every $\mathbf{D}$. By uniqueness, we have

$$\mathbf{v}_{t+s}^{\mathbf{v}_0} = \mathbf{v}_{t,t+s}^{\mathbf{v}_t}, \quad \mathbb{P} \quad \text{a.s.}, \quad (211)$$

which completes the proof. $\square$

We shall prove the tightness of the law

$$\left\{ \frac{1}{T} \int_0^T \mathcal{L}(\mathbf{w}(t)) \times \mathcal{L}(\phi(t)) dt, \quad T > 0 \right\}, \quad (212)$$

so as to apply Krylov-Bogoliubov's theorem.

Theorem 4.5. *There exists an invariant measure for system (24).*

Proof. From the energy estimates of global existence, we know that

$$\mathbb{E} \left[\left(\sup_{t \in [0, T]} \|\mathbf{w}(t)\|_3^2 \right)^m \right] \leq \mathbb{E} \left[\left(C \left(\|\mathbf{w}_0\|_3^2 + \|\nabla \phi_0\|^2 \right) \right)^m \right]. \quad (213)$$

The sets

$$B_L := \{ \mathbf{w}(t) \in H^3(U) \mid \|\mathbf{w}(t)\|_3 \leq L \}, \quad L > 0, \quad (214)$$

are compact in $C^1(U)$. Consequently, there holds

$$\begin{aligned} \frac{1}{T} \int_0^T \mathcal{L}(\mathbf{w}(t)) (B_L^c) dt &= \frac{1}{T} \int_0^T \mathbb{P}[\{\|\mathbf{w}(t)\|_3 > L\}] dt \\ &\leq \frac{1}{L^{2m} T} \int_0^T \mathbb{E} [\|\mathbf{w}(t)\|_3^{2m}] dt \\ &\leq \frac{1}{L^{2m}} \mathbb{E} \left[\left(C \left(\|\mathbf{w}_0\|_3^2 + \|\nabla \phi_0\|^2 \right) \right)^m \right] \\ &\rightarrow 0, \quad \text{as } L \rightarrow +\infty. \end{aligned} \quad (215)$$

This gives the tightness of $\frac{1}{T} \int_0^T \mathcal{L}(\mathbf{w}(t)) dt$. The tightness of $\frac{1}{T} \int_0^T \mathcal{L}(\phi(t)) dt$ is obtained similarly due to the energy estimate

$$\mathbb{E} \left[\left(\sup_{t \in [0, T]} \|\nabla \phi(t)\|^2 \right)^m \right] \leq \mathbb{E} \left[\left(C \left(\|\mathbf{w}_0\|_3^2 + \|\nabla \phi_0\|^2 \right) \right)^m \right]. \quad (216)$$

Hence, the tightness of (212) holds. Therefore, there exists an invariant measure by Krylov-Bogoliubov's theorem. $\square$

Remark 4.6. In the above proof, we need that the constant in energy estimate (213) is independent of T . That is the reason why we assume (17) and (18).

(1) defines a Feller-Markov process as well, similarly to (24). Since $(\bar{\rho}, \bar{\mathbf{u}}, \bar{\phi})$ is smooth, by the uniqueness of solutions, $\frac{1}{T} \int_0^T \mathcal{L}(\rho) \times \mathcal{L}(\mathbf{u}) \times \mathcal{L}(\Phi) ds$ is also a tight measure, which generates an invariant measure. Actually, for compact sets

$$B_{\rho, L} = \{ \rho \in H^3(U) \mid \|\rho\|_3 \leq L \}, \quad L > 0, \quad (217)$$

in $C^1(U)$, there holds

$$\begin{aligned} \frac{1}{T} \int_0^T \mathcal{L}(\rho)(B_L^c) dt &= \frac{1}{T} \int_0^T \mathbb{P}[\{\|\rho\|_3 > L\}] dt \\ &\leq \frac{1}{L^{2m}T} \int_0^T \mathbb{E}[\|\rho\|_3^{2m}] dt \\ &\leq \frac{1}{L^{2m}} C \left(\mathbb{E}[\|\rho_0\|_3^{2m}] + \mathbb{E}[\|\bar{\rho}\|_3^{2m}] \right) \\ &\rightarrow 0, \text{ as } L \rightarrow +\infty. \end{aligned} \quad (218)$$

We also care about what the limit of $\frac{1}{T} \int_0^T \mathcal{L}(\rho) \times \mathcal{L}(\mathbf{u}) \times \mathcal{L}(\Phi) dt$ is.

Theorem 4.7. *The invariant measure generated by $\frac{1}{T} \int_0^T \mathcal{L}(\rho) \times \mathcal{L}(\mathbf{u}) \times \mathcal{L}(\Phi) dt$, for system (1), is the Dirac measure of the steady state $(\bar{\rho}, 0, \bar{\Phi})$. That is, the limit*

$$\lim_{T \rightarrow +\infty} \frac{1}{T} \int_0^T \mathcal{L}(\rho) \times \mathcal{L}(\mathbf{u}) \times \mathcal{L}(\Phi) dt = \delta_{\bar{\rho}} \times \delta_0 \times \delta_{\bar{\Phi}} \quad (219)$$

holds weakly.

Proof. For any $\psi \in C_b(H^3)$, we have

$$\begin{aligned} \lim_{T \rightarrow +\infty} \frac{1}{T} \int_0^T \langle \mathcal{L}(\rho), \psi \rangle dt &= \lim_{T \rightarrow +\infty} \frac{1}{T} \int_0^T \mathbb{E}[\psi(\rho)] dt \\ &= \lim_{T \rightarrow +\infty} \frac{1}{T} \int_0^T (\mathbb{E}[\psi(\rho) - \psi(\bar{\rho})] + \mathbb{E}[\psi(\bar{\rho})]) dt. \end{aligned} \quad (220)$$

We claim that $\lim_{T \rightarrow +\infty} \frac{1}{T} \int_0^T \mathbb{E}[\psi(\rho) - \psi(\bar{\rho})] ds = 0$. Actually, we separate Ω into

$$\Omega_t = \left\{ \psi(\rho) - \psi(\bar{\rho}) \leq \frac{1}{\sqrt{t}} \right\}, \quad t > 0, \quad (221)$$

and Ω_t^c . Then, there holds

$$\begin{aligned} \mathbb{E}[\psi(\rho) - \psi(\bar{\rho})] &= \int_{\Omega} (\psi(\rho) - \psi(\bar{\rho})) \mathbb{P}(d\omega) \\ &= \int_{\Omega \cap \Omega_t} (\psi(\rho) - \psi(\bar{\rho})) \mathbb{P}(d\omega) + \int_{\Omega \cap \Omega_t^c} (\psi(\rho) - \psi(\bar{\rho})) \mathbb{P}(d\omega) \\ &:= I_1 + I_2. \end{aligned} \quad (222)$$

For I_1 , it holds that

$$\lim_{T \rightarrow +\infty} \frac{1}{T} \int_0^T \int_{\Omega \cap \Omega_t} (\psi(\rho) - \psi(\bar{\rho})) \mathbb{P}(d\omega) dt \leq \lim_{T \rightarrow +\infty} \frac{1}{T} \int_0^T \frac{1}{\sqrt{t}} dt = 0. \quad (223)$$

For I_2 , by the weighted energy estimates and Chebyshev's inequality, there holds

$$\begin{aligned} \int_{\Omega \cap \Omega_t^c} (\psi(\rho) - \psi(\bar{\rho})) \mathbb{P}(d\omega) &\leq \int_{\Omega \cap \Omega_t^c} (|\psi(\rho)| + |\psi(\bar{\rho})|) \mathbb{P}(d\omega) \\ &\leq C \int_{\Omega \cap \Omega_t^c} (\|\rho\|_3 + \|\bar{\rho}\|_3) \mathbb{P}(d\omega) \leq C \mathbb{P} \left[\left\{ \psi(\rho) - \psi(\bar{\rho}) > \frac{1}{\sqrt{t}} \right\} \right] \\ &\leq C \frac{\mathbb{E}[|\psi(\rho) - \psi(\bar{\rho})|^{2m}]}{\left(\frac{1}{\sqrt{t}}\right)^{2m}} \leq C t^m e^{-\gamma m t} \mathbb{E}[|\rho_0 - \bar{\rho}|^{2m}]. \end{aligned} \quad (224)$$

Hence, we have

$$\begin{aligned} & \lim_{T \rightarrow +\infty} \frac{1}{T} \int_0^T \int_{\Omega \cap \Omega_t^c} (\psi(\rho) - \psi(\bar{\rho})) \mathbb{P}(\mathrm{d}\omega) \mathrm{d}t \\ & \leq \lim_{T \rightarrow +\infty} C \frac{1}{T} \int_0^T t^m e^{-\gamma m t} \mathrm{d}t = 0. \end{aligned} \quad (225)$$

Therefore, there holds

$$\lim_{T \rightarrow +\infty} \frac{1}{T} \int_0^T \langle \mathcal{L}(\rho), \psi \rangle \mathrm{d}t = \lim_{T \rightarrow +\infty} \frac{1}{T} \int_0^T \mathbb{E}[\psi(\bar{\rho})] \mathrm{d}t = \mathbb{E}[\psi(\bar{\rho})] = \langle \delta_{\bar{\rho}}, \psi \rangle. \quad (226)$$

A similar calculation shows that

$$\lim_{T \rightarrow +\infty} \int_0^T \frac{1}{T} \mathcal{L}(\mathbf{u}) \mathrm{d}t = \delta_0; \quad (227)$$

and

$$\lim_{T \rightarrow +\infty} \int_0^T \frac{1}{T} \mathcal{L}(\Phi) \mathrm{d}t = \delta_{\bar{\Phi}}. \quad (228)$$

This completes the proof by the tightness of joint distributions. $\square$

Appendix A. We provide an overview of the fundamental theory concerning stochastic analysis. Let E be a separable Banach space and $\mathcal{B}(E)$ be the σ -field of its Borel subsets, respectively. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a stochastic basis. A filtration $\mathcal{F} = (\mathcal{F}_t)_{t \in \mathbf{T}}$ is a family of σ -algebras on Ω indexed by $\mathbf{T}$ such that $\mathcal{F}_s \subseteq \mathcal{F}_t \subseteq \mathcal{F}$, $s \leq t$, $s, t \in \mathbf{T}$. $(\Omega, \mathcal{F}, \mathbb{P})$ is also called a filtered space. We first list some definitions.

1. **E -valued random variables.** [48] For $(\Omega, \mathcal{F})$ and $(E, \mathcal{E})$ being two measurable spaces, a mapping X from Ω into E such that the set $\{\omega \in \Omega : X(\omega) \in A\} = \{X \in A\}$ belongs to $\mathcal{F}$ for arbitrary $A \in \mathcal{E}$, is called a measurable mapping or a random variable from $(\Omega, \mathcal{F})$ into $(E, \mathcal{E})$ or an E -valued random variable.
2. **Strongly measurable operator valued random variables.** [48] Let $\mathcal{U}$ and $\mathcal{H}$ be two separable Hilbert spaces which can be infinite-dimensional, and denote by $L(\mathcal{U}, \mathcal{H})$ the set of all linear bounded operators from $\mathcal{U}$ into $\mathcal{H}$. A functional operator $\Psi(\cdot)$ from Ω into $L(\mathcal{U}, \mathcal{H})$ is said to be strongly measurable, if, for arbitrary $X \in \mathcal{U}$, the function $\Psi(\cdot)X$ is measurable, as a mapping from $(\Omega, \mathcal{F})$ into $(\mathcal{H}, \mathcal{B}(\mathcal{H}))$. Let $\mathcal{L}$ be the smallest σ -field of subsets of $L(\mathcal{U}, \mathcal{H})$ containing all sets of the form

$$\{\Psi \in L(\mathcal{U}, \mathcal{H}) : \Psi X \in A\}, \quad X \in \mathcal{U}, \quad A \in \mathcal{B}(\mathcal{H}). \quad (229)$$

Then, $\Psi : \Omega \rightarrow L(\mathcal{U}, \mathcal{H})$ is a strongly measurable mapping from $(\Omega, \mathcal{F})$ into $(L(\mathcal{U}, \mathcal{H}), \mathcal{L})$.

3. **Law of a random variable.** For an E -valued random variable $X : (\Omega, \mathcal{F}) \rightarrow (E, \mathcal{E})$, we denote by $\mathcal{L}[X]$ the law of X on E , that is, $\mathcal{L}[X]$ is the probability measure on $(E, \mathcal{E})$ given by

$$\mathcal{L}[X](A) = \mathbb{P}[X \in A], \quad A \in \mathcal{E}. \quad (230)$$

4. **Stochastic process.** [48] A stochastic process X is defined as an arbitrary family $X = \{X_t\}_{t \in \mathbf{T}}$ of E -valued random variables X_t , $t \in \mathbf{T}$. X is also regarded as a mapping from Ω into a Banach space like $C([0, T]; E)$ or $L^p = L^p(0, T; E)$, $1 \leq p < +\infty$, by associating $\omega \in \Omega$ with the trajectory $X(\cdot, \omega)$.

5. **Cylindrical Wiener Process valued in Hilbert space.** [48] A $\mathcal{U}$ -valued stochastic process $W(t), t \geq 0$, is called a cylindrical Wiener process if
 - $W(0) = 0$;
 - W has continuous trajectories;
 - W has independent increments;
 - the distribution of $(W(t) - W(s))$ is $\mathcal{N}(0, (t - s))$, $0 \leq s \leq t$.
6. **Adapted stochastic process.** A stochastic process X is $\mathcal{F}$ -adapted if X_t is $\mathcal{F}_t$ -measurable for every $t \in \mathbf{T}$;
7. **Martingale.** The E -valued process X is called integrable provided that $\mathbb{E}[\|X_t\|] < +\infty$ for every $t \in \mathbf{T}$. An integrable and adapted E -valued process $X_t, t \in \mathbf{T}$, is a martingale if
 - X is adapted;
 - $X_s = \mathbb{E}[X_t | \mathcal{F}_s]$, for arbitrary $t, s \in \mathbf{T}$, $0 \leq s \leq t$.
8. **Stopping time.** On $(\Omega, \mathcal{F}, \mathbb{P})$, a random time is a measurable mapping $\tau : \Omega \rightarrow \mathbf{T} \cup \infty$. A random time is a stopping time if $\{\tau \leq t\} \in \mathcal{F}_t$ for every $t \in \mathbf{T}$. For a process X and a subset V of the state space, we define the hitting time of X in V as

$$\tau_V(\omega) = \inf \{t \in \mathbf{T} | X_t(\omega) \in V\}. \quad (231)$$

If X is a continuous adapted process, and V is closed, then τ_V is a stopping time.

9. **Modification.** A stochastic process Y is called a modification or a version of X if

$$\mathbb{P}[\{\omega \in \Omega : X(t, \omega) \neq Y(t, \omega)\}] = 0 \quad \text{for all } t \in \mathbf{T}. \quad (232)$$

10. **Progressive measurability.** In $(\Omega, \mathcal{F}, \mathbb{P})$, the stochastic process X is progressively measurable or simply progressively measurable, if, for $\omega \in \Omega$, $(\omega, s) \mapsto X(s, \omega)$, $s \leq t$ is $\mathcal{F}_t \otimes \mathcal{B}(\mathbf{T} \cap [0, t])$ -measurable for every $t \in \mathbf{T}$.
11. **Progressive measurability of continuous functions.** Let $X(t), t \in [0, T]$, be a stochastically continuous and adapted process with values in a separable Banach space E . Then, X has a progressively measurable modification.
12. **Cross quadratic variation.** Fixing a number $T > 0$, we denote by $\mathcal{M}_T^2(E)$ the space of all E -valued continuous, square integrable martingales M , such that $M(0) = 0$. If $M \in \mathcal{M}_T^2(\mathbb{R}^1)$, then there exists a unique increasing predictable process $\langle M(\cdot) \rangle$, starting from 0, such that the process

$$M^2(t) - \langle M(\cdot) \rangle, \quad t \in [0, T] \quad (233)$$

is a continuous martingale. The process $\langle M(\cdot) \rangle$ is called the quadratic variation of M . If $M_1, M_2 \in \mathcal{M}_T^2(\mathbb{R}^1)$, then the process

$$\langle M_1(t), M_2(t) \rangle = \frac{1}{4} [\langle (M_1 + M_2)(t) \rangle - \langle (M_1 - M_2)(t) \rangle] \quad (234)$$

is called the cross quadratic variation of M_1, M_2 . It is a unique, predictable process with trajectories of bounded variation, starting from 0 such that

$$M_1(t)M_2(t) - \langle M_1(t), M_2(t) \rangle, \quad t \in [0, T] \quad (235)$$

is a continuous martingale.

For $M \in \mathcal{M}_T^2(\mathcal{H})$, where $\mathcal{H}$ is Hilbert space, the quadratic variation is defined by

$$\langle M(t) \rangle = \sum_{i,j=1}^{\infty} \langle M_i(t), M_j(t) \rangle e_i \otimes e_j, \quad t \in [0, T], \quad (236)$$

as an integrable adapted process, where $M_i(t)$ and $M_j(t)$ are in $\mathcal{M}_T^2(\mathbb{R}^1)$. For $a \in \mathcal{H}_1, b \in \mathcal{H}_2$, $a \otimes b$ denotes a linear operator from $\mathcal{H}_2$ into $\mathcal{H}_1$ given by the formula

$$(a \otimes b)x = a\langle b, x \rangle_{\mathcal{H}_2}, \quad x \in \mathcal{H}_2. \quad (237)$$

We define a cross quadratic variation for $M^1 \in \mathcal{M}_T^2(\mathcal{H}_1)$, $M^2 \in \mathcal{M}_T^2(\mathcal{H}_2)$ where $\mathcal{H}_1$ and $\mathcal{H}_2$ are two Hilbert spaces. Namely we define

$$\langle M^1(t), M^2(t) \rangle = \sum_{i,j=1}^{\infty} \langle M_i^1(t), M_j^2(t) \rangle e_i^1 \otimes e_j^2, \quad t \in [0, T], \quad (238)$$

where $\{e_i^1\}$ and $\{e_j^2\}$ are complete orthonormal bases in $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively.

13. **Stochastic integral.** Let W be the Wiener process. Let $\Psi(t), t \in [0, T]$, be a measurable Hilbert–Schmidt operators in $L(\mathcal{U}, \mathcal{H})$. $\Psi(t)$ is set in the space $\mathcal{L}_2$ such that

$$\mathbb{E} \left[\int_0^t \|\Psi(s)\|_{\mathcal{L}_2}^2 ds \right] := \mathbb{E} \int_0^t \langle \Psi(s), \Psi^*(s) \rangle_{\mathcal{H}} ds < +\infty, \quad (239)$$

where $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ means the inner product in $\mathcal{H}$. For the stochastic integral $\int_0^t \Psi dW$, there holds

$$\mathbb{E} \left[\left(\int_0^t \Psi dW \right)^2 \right] = \mathbb{E} \left[\int_0^t \|\Psi(s)\|_{\mathcal{L}_2}^2 ds \right]. \quad (240)$$

Furthermore, the following properties hold

- Linearity: $\int (a\Psi_1 + b\Psi_2) dW = a \int \Psi_1 dW + b \int \Psi_2 dW$ for constants a and b ;
- Stopping property: $\int 1_{\{\cdot \leq \tau\}} \Psi dW = \int \Psi dM^\tau = \int_0^{\wedge \tau} \Psi dW$;
- Itô-isometry: for every t ,

$$\mathbb{E} \left[\left(\int_0^t \Psi dW \right)^2 \right] = \mathbb{E} \left[\int_0^t \|\Psi(s)\|_{\mathcal{L}_2}^2 ds \right]. \quad (241)$$

14. **Dirac measure.** Let $(E, \mathcal{B}(E))$ be a measurable space. Given $x \in E$, the Dirac measure δ_x at x is the measure defined by

$$\delta_x(A) := \begin{cases} 1, & x \in A \\ 0, & x \notin A \end{cases} \quad (242)$$

for each measurable set $A \subseteq E$. In this paper, there holds

$$\delta_{\bar{\rho}} = \mathcal{L}[\bar{\rho}](A) = \mathbb{P}[\{\omega \in \Omega | \bar{\rho}(x) \in A\}] = 1.$$

15. **Tightness of measures.** [2] Let E be a Hausdorff space, and let $\mathcal{E}$ be a σ -algebra on E . Let $\mathcal{M}$ be a collection of measures defined on $\mathcal{E}$. The collection

$\mathcal{M}$ is called tight, if, for any $\varepsilon > 0$, there is a compact subset K_ε of E such that, for all measures $\mu \in \mathcal{M}$,

$$|\mu|(E \setminus K_\varepsilon) < \varepsilon, \quad (243)$$

where $|\mu|$ is the total variation measure of μ . More specially, for probability measures μ , (243) can be written as

$$\mu(K_\varepsilon) > 1 - \varepsilon. \quad (244)$$

We list some important theorems in stochastic analysis.

1. **Itô's formula.** [34, 48] Assume that Ψ is an $\mathcal{L}_2$ -valued process stochastically integrable in $[0, T]$, φ being a $\mathcal{H}$ -valued predictable process Bochner integrable on $[0, T]$, $\mathbb{P}$ -a.s., and $X(0)$ being a $\mathcal{F}_0$ -measurable $\mathcal{H}$ -valued random variable. Then, the following process

$$X(t) = X(0) + \int_0^t \varphi(s) ds + \int_0^t \Psi(s) dW(s), \quad t \in [0, T] \quad (245)$$

is well defined. Assume that a function $F : [0, T] \times \mathcal{H} \rightarrow \mathbb{R}^1$, and its partial derivatives F_t, F_x, F_{xx} , are uniformly continuous on bounded subsets of $[0, T] \times \mathcal{H}$. Under the above conditions, $\mathbb{P}$ -a.s., for all $t \in [0, T]$,

$$\begin{aligned} F(t, X(t)) &= F(0, X(0)) + \int_0^t \langle F_x(s, X(s)), \Psi(s) dW(s) \rangle_{\mathcal{H}} \\ &\quad + \int_0^t \{F_t(s, X(s)) + \langle F_x(s, X(s)), \varphi(s) \rangle_{\mathcal{H}}\} ds \\ &\quad + \int_0^t \frac{1}{2} F_{xx}(s, X(s)) \|\Psi(s)\|_{\mathcal{L}_2}^2 ds. \end{aligned} \quad (246)$$

Applying the above formula for $F = \langle x, x \rangle_{\mathcal{H}}$, we have Itô's formula for $\langle X, X \rangle_{\mathcal{H}}$. Then, by

$$\langle X, Y \rangle_{\mathcal{H}} = \frac{\langle X + Y, X + Y \rangle_{\mathcal{H}} - \langle X - Y, X - Y \rangle_{\mathcal{H}}}{4} \quad (247)$$

in Hilbert space, the following Itô's formula holds for X and Y in form of (245),

$$\begin{aligned} &\langle X, Y \rangle_{\mathcal{H}} \\ &= \langle X_0, Y_0 \rangle_{\mathcal{H}} + \int \langle X, dY \rangle_{\mathcal{H}} + \int \langle Y, dX \rangle_{\mathcal{H}} + \int d \langle \langle X, Y \rangle, \langle X, Y \rangle \rangle_{\mathcal{H}}^{\frac{1}{2}} \\ &= \langle X_0, Y_0 \rangle_{\mathcal{H}} + \int \langle X, dY \rangle_{\mathcal{H}} + \int \langle Y, dX \rangle_{\mathcal{H}} + \langle \langle X, Y \rangle, \langle X, Y \rangle \rangle_{\mathcal{H}}^{\frac{1}{2}}, \end{aligned} \quad (248)$$

where $\langle X, Y \rangle$ means the cross quadratic variation of X and Y defined above.

2. **Chebyshev's inequality.** Let Y be a random variable in probability space $(\Omega, \mathcal{F}, \mathbb{P})$, $\varepsilon > 0$. For every $0 < r < \infty$, Chebyshev's inequality reads

$$\mathbb{P}\{|Y| \geq \varepsilon\} \leq \frac{1}{\varepsilon^r} \mathbb{E}[|Y|^r]. \quad (249)$$

3. **Burkholder-Davis-Gundy's inequality.** [7, 48] Let M be a continuous local martingale in $\mathcal{H}$. Let $M^* = \max_{0 \leq s \leq t} |M(s)|$, for any $m \geq 1$. $\langle M \rangle_T$ denotes

the quadratic variation stopped by T . Then there exist constants K^m and K_m such that

$$K_m \mathbb{E}[(\langle M \rangle_T)^m] \leq \mathbb{E}[(M_T^*)^{2m}] \leq K^m \mathbb{E}[(\langle M \rangle_T)^m], \quad (250)$$

for every stopping time T . For $m \geq 1$, $K^m = \left(\frac{2m}{2m-1}\right)^{\frac{2m(2m-2)}{2}}$, which is equivalent to e^m as $m \rightarrow \infty$. Specifically, for every $m \geq 1$, and for every $t \geq 0$, there holds

$$\mathbb{E} \left[\sup_{s \in [0, t]} \left| \int_0^s \Psi(s) dW(s) \right|^{2m} \right] \leq K^m \left(\mathbb{E} \left[\int_0^t \|\Psi(s)\|_{\mathcal{L}_2}^2 ds \right] \right)^m \quad (251)$$

4. **Stochastic Fubini theorem.** Assume that $(E, \mathcal{E})$ is a measurable space and let

$$\Psi : (t, \omega, x) \rightarrow \Psi(t, \omega, x)$$

be a measurable mapping from $(\Omega_T \times E, \mathcal{B}(\Omega_T) \times \mathcal{B}(E))$ into $(\mathcal{L}^2, \mathcal{B}(\mathcal{L}^2))$. Moreover, assume that

$$\int_E \left[\mathbb{E} \int_0^T \langle \Psi(s), \Psi^*(s) \rangle_{\mathcal{H}} dt \right]^{\frac{1}{2}} \mu(dx) < +\infty. \quad (252)$$

Then, the following equality holds $\mathbb{P}$ -a.s.

$$\int_E \left[\int_0^T \Psi(t, x) dW(t) \right] \mu(dx) = \int_0^T \left[\int_E \Psi(t, x) \mu(dx) \right] dW(t). \quad (253)$$

5. **Kolmogorov-Centov's continuity theorem.** [35, 48] Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and $\bar{X}$ be a process on $[0, T]$ with values in a complete metric space $(E, \mathcal{E})$. Suppose that

$$\mathbb{E} [|\bar{X}_t - \bar{X}_s|^a] \leq C |t - s|^{1+b}, \quad (254)$$

for every $s < t \leq T$ and some strictly positive constants a, b and C . Then, $\bar{X}$ admits a continuous modification X , $\mathbb{P}[\{X_t = \bar{X}_t\}] = 1$ for every t , and X is locally Hölder continuous for every exponent $0 < \gamma < \frac{b}{a}$, namely,

$$\mathbb{P} \left[\left\{ \omega : \sum_{0 < t-s < h(\omega), t, s \leq T} \frac{|X_t(\omega) - X_s(\omega)|}{|t-s|^\gamma} \leq \delta \right\} \right] = 1, \quad (255)$$

where $h(\omega)$ is an strictly positive random variable a.s., and the constant satisfies $\delta > 0$.

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