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Optimal convergence rate to nonlinear diffusion waves for Euler equations with critical overdamping

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ABSTRACT

This paper is concerned with the Cauchy problem for the one-dimensional compressible Euler equations with critical time-dependent overdamping. In this critical case, the damped Euler system is proved to admit a unique global solution that converges time-asymptotically to the corresponding nonlinear diffusion wave. Particularly, the optimal convergence rate in logarithmic form is first derived by the technical time-weighted method.

1. Introduction

In this paper, we consider a mathematical model for the compressible flow through porous media, which is presented as the 1D compressible Euler equations with time-dependent damping:

$$\begin{cases} v_t - u_x = 0, \\ u_t + p(v)_x = -\frac{\mu}{(1+t)^\lambda} u, \quad (x,t) \in \mathbb{R} \times (t_0, +\infty), \\ (v,u)(x,t_0) = (v_0,u_0)(x) \to (v_\pm, u_\pm) \quad \text{as } x \to \pm\infty. \end{cases} \tag{1.1}$$

Here $v > 0$ is the specific volume, u is the velocity, $p(v)$ is the pressure with $p(v) > 0$ and $p'(v) < 0$. A physical example is $p(v) = v^{-\gamma}$ for the polytropic gas satisfying the γ-law with $\gamma \geq 1$. The external term $-\frac{\mu}{(1+t)^\lambda} u$ with parameters $\mu > 0$ and $\lambda \in \mathbb{R}$ represents the damping effect. All $v_\pm > 0$ and $u_\pm$ are constant

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states. Subsequent to the current studies [1–4], we further consider the critical case $\lambda=-1$, of course, a challenging case as we know.

When $\lambda=0$, Hsiao and Liu [5] first investigated how the damping effect makes the structure of solutions of (1.1) to be different from that of Euler equations. Although there are two basic characters for the damped Euler equations, one is the hyperbolic structure, the other is the damping structure, Hsiao and Liu recognized that the damping effect plays an essential role in determining the dynamics of damped Euler equations, and proved that the solutions of (1.1) with $\lambda=0$ tend time-asymptotically to the shifted nonlinear diffusion waves at algebraic rates. Since then, Nishihara etc. [6,7] improved the convergence rates to be optimal when the initial perturbations are in L^2 and L^1, respectively. After that, Mei [8] found that the best asymptotic profiles are actually the solutions of nonlinear diffusion equations with some specified initial data, and obtained better convergence rates. See also late [9] for the compressible adiabatic flow.

When $\lambda>0$, the damping gradually degenerates with time. Pan [10,11] showed that the condition $\lambda=1$ and $\mu=2$ is the critical threshold of global existence and finite-time blow-up for (1.1). Particularly, (1.1) has globally smooth solutions when $\lambda=1$ with $\mu>2$ or $0\le\lambda<1$ with $\mu>0$. When $\lambda=1$ with $0\le\mu\le 2$ or $\lambda>1$ with $\mu\ge 0$, the solutions of (1.1) may blow up in finite time. Hou etc. [12] and Hou and Yin [13] generalized these results to the 3D case. By introducing a kind of maximum principle for the corresponding Riemann invariants in 1D, Chen etc. [1] further investigated the global existence of solutions with large initial values for $0\le\lambda<1$, and the blow-up problem for small initial values for $\lambda\ge 1$. However, these results did not include the asymptotic profiles of the global solutions. Recently, Cui etc. [2] and Li etc. [3] independently proved that the global solutions of (1.1) time-asymptotically converge to nonlinear diffusion waves when $0<\lambda<1$. Geng etc. [14] showed that the asymptotic profiles in the critical case $\lambda=1$ and $\mu>2$ are actually the solutions of the linear damped wave equations.

When $\lambda<0$, system (1.1) is the Euler equations with overdamping. Recalling the scaling analysis carried out in [4], the asymptotic profiles of solutions to (1.1) should be nonlinear diffusion waves for any $\lambda\in(-1,0)$. However, the approach in [4] based on constructing polynomial time weights cannot be applied to the critical case $\lambda=-1$. In this paper, we show that, in this critical case, the system (1.1) still admits nonlinear diffusion phenomenon, and first derive the optimal convergence rate in logarithmic form by constructing logarithmic time weights.

In the critical case $\lambda=-1$, the system (1.1) reduces to

$$\begin{cases} v_t-u_x=0,\\ u_t+p(v)_x=-(1+t)u,\\ (v,u)(x,t_0)=(v_0,u_0)(x)\to(v_\pm,u_\pm)\quad \text{as } x\to\pm\infty,\ v_+\neq v_-. \end{cases}\tag{1.2}$$

Here, for simplicity, we set $\mu=1$. The corresponding nonlinear diffusion system is

$$\begin{cases} \bar v_t-\bar u_x=0,\\ p(\bar v)_x=-(1+t)\bar u,\quad (x,t)\in\mathbb{R}\times\mathbb{R}_+,\\ (\bar v,\bar u)(x,t)\to(v_\pm,0)\quad \text{as } x\to\pm\infty, \end{cases}\tag{1.3}$$

which admits nonlinear diffusion waves in the form $(\bar v,\bar u)(x,t)=(\bar v,\bar u)\big(x/\sqrt{\ln(1+t)}\big)$ up to a shift.

To show the convergence of the solutions (v,u) toward diffusion waves, we use the framework of [5] to first construct correction functions $(\hat v,\hat u)(x,t)$ to eliminate the gap between the original solutions and the diffusion waves at far fields. Set

$$\hat u(x,t):=m(x)e^{\frac12[1-(1+t)^2]}\ \text{and}\ \hat v(x,t):=(u_--u_+)m_0(x)\int_t^{+\infty}e^{\frac12[1-(1+s)^2]}ds,$$

where $m(x)=u_-+(u_+-u_-)\int_{-\infty}^x m_0(y)dy$, $m_0(x)\ge 0$, $m_0(x)\in C_0^\infty(\mathbb{R})$, and $\int_{-\infty}^{+\infty}m_0(x)dx=1$. The correction functions $(\hat v,\hat u)$ satisfy

$$\begin{cases} \hat v_t-\hat u_x=0,\\ \hat u_t=-(1+t)\hat u, \end{cases}\quad \text{and}\quad (\hat v,\hat u)\to(0,u_\pm)\ \text{as } x\to\pm\infty.\tag{1.4}$$

It then follows from (1.2)–(1.4) that

$$\begin{cases}(v-\bar v-\hat v)_t-(u-\bar u-\hat u)_x=0,\\(u-\bar u-\hat u)_t+(p(v)-p(\bar v))_x=-(1+t)(u-\bar u-\hat u)-\bar u_t,\end{cases}\tag{1.5}$$

where $(\bar v,\bar u)$ are the shifted diffusion waves $(\bar v,\bar u)(x+x_0,t)$, and the shift x_0 is determined by $x_0=\frac{1}{v_+-v_-}\int_{-\infty}^{+\infty}(v_0(x)-\bar v(x,t_0)-\hat v(x,t_0))dx$. Therefore $\int_{-\infty}^{+\infty}(v(x,t)-\bar v(x+x_0,t)-\hat v(x,t))dx=0$. We now define

$$V(x,t):=\int_{-\infty}^{x}\big[v(y,t)-\bar v(y+x_0,t)-\hat v(y,t)\big]dy,\ \ z(x,t):=u(x,t)-\bar u(x+x_0,t)-\hat u(x,t),$$
$$V_0(x):=\int_{-\infty}^{x}\big[v_0(y)-\bar v(y+x_0,t_0)-\hat v(y,t_0)\big]dy,\quad z_0(x):=u_0(x)-\bar u(x+x_0,t_0)-\hat u(x,t_0).$$

Then, (1.5) becomes

$$\begin{cases}V_t=z,\\z_t+\big(p'(\bar v)V_x\big)_x=-(1+t)z+F-G_x,\\(V,z)(x,t_0)=(V_0,z_0)(x),\end{cases}\tag{1.6}$$

where $F:=(1+t)^{-1}p(\bar v)_{xt}-(1+t)^{-2}p(\bar v)_x$ and $G:=p(V_x+\bar v+\hat v)-p(\bar v)-p'(\bar v)V_x$.

Denote by $H^m(\mathbb{R})$ the usual Sobolev space with norm $\|f\|_m:=\sum_{k=0}^m\|\partial_x^kf\|$ and $\|f\|:=\|f\|_{L^2(\mathbb{R})}$. Set $\delta:=|v_+-v_-|+|u_+-u_-|$. Our main result is as follows.

Theorem 1.1. *Suppose that $\|V_0\|_{H^3}+\|z_0\|_{H^2}$ and δ are sufficiently small. Then system (1.2) has a unique global solution $(v,u)(x,t)$, which converges to the nonlinear diffusion wave $(\bar v,\bar u)(x,t)$ at the following rates:*

$$\|(v-\bar v)(\cdot,t)\|_{L^\infty}=O(1)\ln^{-\frac34}(1+t),\quad \|(u-\bar u)(\cdot,t)\|_{L^\infty}=O(1)(1+t)^{-1}\ln^{-\frac54}(1+t).\tag{1.7}$$

Remark 1.1. The convergence rates are optimal when the initial perturbations are in L^2. Actually, it is easy to see that $w(x,t)=\frac{1}{2\sqrt{\pi\ln(1+t)}}\int\exp(-\frac{(x-y)^2}{4\ln(1+t)})w_0(y)dy$ is the solution of the heat equation $(1+t)w_t-w_{xx}=0$ with initial value $w(x,0)=w_0(x)$. Then, for $t>1$,

$$|w_x(x,t)|=\frac{1}{2\sqrt{\pi\ln(1+t)}}\left|\int e^{-\frac{(x-y)^2}{4\ln(1+t)}}\left(\frac{x-y}{2\ln(1+t)}\right)w_0(y)dy\right|=O(1)\ln^{-\frac34}(1+t)\|w_0\|_{L^2},$$
$$|w_t(x,t)|=O(1)(1+t)^{-1}\ln^{-\frac54}(1+t)\|w_0\|_{L^2}.$$

2. Proof of Theorem 1.1

We first present some elementary properties of the nonlinear diffusion wave. From (1.3), we get

$$\bar v_t=-(1+t)^{-1}p(\bar v)_{xx},\quad p'(\bar v)<0.\tag{2.1}$$

If we define $\tilde v(x,t):=\bar v(x,\ln(1+t))$, then $\tilde v$ satisfies $\tilde v_t=-p(\tilde v)_{xx}$. Thus, using the classical results for $\tilde v$ (see [2–5]), one can easily see that (2.1) has a unique (up to a shift) monotone self-similar solution $\bar v(x,t)=\bar v\Big(\frac{x}{\sqrt{\ln(1+t)}}\Big)$, which satisfies

$$\|\partial_x^k\bar v(\cdot,t)\|^2=O(1)\big|v_+-v_-\big|^2\ln^{-\frac{2k-1}{2}}(1+t),\quad k=1,2,3;\tag{2.2}$$
$$\|\partial_x^k\bar v_t(\cdot,t)\|^2=O(1)\big|v_+-v_-\big|^2(1+t)^{-2}\ln^{-\frac{2k+3}{2}}(1+t),\quad k=0,1,2,3;\tag{2.3}$$
$$\|\partial_x^k\bar v_{tt}(\cdot,t)\|^2=O(1)\big|v_+-v_-\big|^2(1+t)^{-4}\ln^{-\frac{2k+3}{2}}(1+t),\quad k=0,1,2.\tag{2.4}$$

By (1.6), it is easy to see that V satisfies

$$\begin{cases} V_{tt} + (1+t)V_t + \big(p'(\bar{v})V_x\big)_x = F - G_x, \\ (V, V_t)|_{t=t_0} = (V_0, z_0)(x). \end{cases} \tag{2.5}$$

For Eq. (2.5), we have the following result.

Proposition 2.1. *Under the assumptions of Theorem 1.1, Eq. (2.5) has a unique global solution satisfying $V \in C^k\left([t_0,\infty); H^{3-k}(\mathbb{R})\right)$ for $t_0 > 0$, $k = 0, 1, 2$, and the estimate*

$$\begin{aligned} &\sum_{k=0}^{3} \ln^k(1+t)\|\partial_x^k V(\cdot,t)\|^2 + (1+t)^2 \left(\sum_{k=0}^{1} \ln^{2+k}(1+t)\|\partial_x^k V_t(\cdot,t)\|^2 + \ln^2(1+t)\|\partial_x^2 V_t(\cdot,t)\|^2 \right) \\ &\quad + \int_{t_0}^{t} \left[\sum_{k=1}^{3} (1+s)^{-1}\ln^{k-1}(1+s)\|\partial_x^k V(\cdot,s)\|^2 + \sum_{k=0}^{2} (1+s)\ln^{1+k}(1+s)\|\partial_x^k V_t(\cdot,s)\|^2 \right] ds \\ &\le C\big(\|V_0\|_{H^3}^2 + \|z_0\|_{H^2}^2 + \delta\big). \end{aligned} \tag{2.6}$$

Proof. The local well-posedness of Eq. (2.5) is standard. To construct the global solution of (2.5), we only need to establish the following a priori estimate. Let $T \in (t_0, \infty]$. We define

$$\begin{aligned} N(T)^2 := \sup_{t_0 \le t \le T} \Bigg\{ &\sum_{k=0}^{3} \ln^k(1+t)\|\partial_x^k V(\cdot,t)\|^2 + \sum_{k=0}^{1} (1+t)^2 \ln^{2+k}(1+t)\|\partial_x^k V_t(\cdot,t)\|^2 \\ &+ (1+t)^2 \ln^2(1+t)\|\partial_x^2 V_t(\cdot,t)\|^2 \Bigg\}. \end{aligned}$$

A priori estimate: Assume that $V \in C^k\left([t_0,T]; H^{3-k}\right)$ is a solution of (2.5). Then there exists an $\varepsilon > 0$ independent of T such that if $N(T) + \delta \le \varepsilon$, then V satisfies (2.6) for any $t \in [t_0, T]$. The proof of this a priori estimate is divided into several steps.

Step 1. Multiplying (2.5) by $2V_t + (1+t)^{-1}V$, we get

$$\begin{aligned} &\frac{d}{dt}\int \left[V_t^2 - p'(\bar{v})V_x^2 + (1+t)^{-1}VV_t + \frac{1}{2}(1+t)^{-2}V^2 + \frac{1}{2}V^2 \right] \\ &+ \int \left[(2(1+t) - (1+t)^{-1})V_t^2 - (p'(\bar{v})(1+t)^{-1} - p''(\bar{v})\bar{v}_t)V_x^2 + (1+t)^{-3}V^2 \right] \\ &= \int (F - G_x)(2V_t + (1+t)^{-1}V). \end{aligned} \tag{2.7}$$

Noting $-p'(\bar{v}) > C_0$ and $V_t^2 + (1+t)^{-1}VV_t + \frac{1}{2}(1+t)^{-2}V^2 \ge C_1 V_t^2 + C_2(1+t)^{-2}V^2$ for $C_i > 0 (i = 0, 1, 2)$, we get

$$\begin{aligned} &\int (V^2 + V_x^2 + V_t^2) + \int_{t_0}^{t}\int \left[(1+s)^{-3}V^2 + (1+s)^{-1}V_x^2 + (1+s)V_t^2 \right] \\ &\le C\left(\|V_0\|_1^2 + \|z_0\|^2\right) + C\int_{t_0}^{t}\int |F - G_x \parallel 2V_t + (1+t)^{-1}V|. \end{aligned} \tag{2.8}$$

We next estimate the nonlinear terms. By the Gagliardo–Nirenberg inequality,

$$\begin{aligned} \|V_x(\cdot,t)\|_{L^\infty} &\le C\|V_x(\cdot,t)\|_{L^2}^{\frac{1}{2}}\|V_{xx}(\cdot,t)\|_{L^2}^{\frac{1}{2}} \le CN(t)\ln^{-\frac{3}{4}}(1+t), \\ \|V_{xx}(\cdot,t)\|_{L^\infty} &\le C\|V_{xx}(\cdot,t)\|_{L^2}^{\frac{1}{2}}\|V_{xxx}(\cdot,t)\|_{L^2}^{\frac{1}{2}} \le CN(t)\ln^{-\frac{5}{4}}(1+t). \end{aligned}$$

Noting $|p'(V_x+\bar v+\hat v)-p'(\bar v)|\le C|V_x+\hat v|$ and $|\hat v(x,t)|+|\hat v_x(x,t)|\le C\delta e^{-C(1+t)^2}(m_0(x)+|m_0'(x)|)$ and using (2.2) and (2.3), we have

$$\begin{aligned}&\int_{t_0}^t\int|F-G_x\parallel 2V_t+(1+t)^{-1}V|\\&\le C\int_{t_0}^t\int\big((1+s)^{-1}(|\bar v_t\bar v_x|+|\bar v_{xt}|)+(1+s)^{-2}|\bar v_x|+|V_x+\hat v\parallel V_{xx}+\bar v_x|+|\hat v_x|\big)\\&\quad\cdot|2V_t+(1+t)^{-1}V|\\&\le C(N(t)+\delta)\int_{t_0}^t\int\left[(1+s)^{-1}V_x^2+(1+s)V_t^2\right]+C\delta.\end{aligned}$$

Substituting this inequality into (2.8) gives

$$\int(V^2+V_x^2+V_t^2)+\int_{t_0}^t\int\left[(1+s)^{-1}V_x^2+(1+s)V_t^2\right]\le C\left(\|V_0\|_1^2+\|z_0\|^2+\delta\right). \tag{2.9}$$

Multiplying (2.5) by $\ln(1+t)V_t$ and using (2.9), we get

$$\begin{aligned}&\|V(\cdot,t)\|^2+\ln(1+t)\|(V_x,V_t)(\cdot,t)\|^2+\int_{t_0}^t\Big[(1+s)^{-1}\|V_x(\cdot,s)\|^2\\&\quad+(1+s)\ln(1+s)\|V_t(\cdot,s)\|^2\Big]ds\le C\left(\|V_0\|_1^2+\|z_0\|^2+\delta\right).\end{aligned} \tag{2.10}$$

Step 2. Differentiating (2.5) in x yields

$$V_{xtt}+(1+t)V_{xt}+\left(p'(\bar v)V_{xx}\right)_x=-\left(p''(\bar v)\bar v_xV_x\right)_x+F_x-G_{xx}. \tag{2.11}$$

Multiplying (2.11) by $\ln(1+t)(2V_{xt}+(1+t)^{-1}V_x)$, we have

$$\begin{aligned}&\int\ln(1+t)(V_{xx}^2+V_{xt}^2)+\int_{t_0}^t\int\ln(1+s)\left[(1+s)^{-1}V_{xx}^2+(1+s)V_{xt}^2\right]\\&\le C\left(\|V_0\|_2^2+\|z_0\|_1^2+\delta\right).\end{aligned} \tag{2.12}$$

Then, multiplying (2.11) by $\ln^2(1+t)V_{xt}$ and using (2.12), we get

$$\begin{aligned}&\ln^2(1+t)\|(V_{xx},V_{xt})(\cdot,t)\|^2+\int_{t_0}^t\Big[(1+s)^{-1}\ln(1+s)\|V_{xx}(\cdot,s)\|^2\\&\quad+(1+s)\ln^2(1+s)\|V_{xt}(\cdot,s)\|^2\Big]ds\le C\left(\|V_0\|_2^2+\|z_0\|_1^2+\delta\right).\end{aligned} \tag{2.13}$$

Step 3. Multiplying (2.11) by $-\ln^k(1+t)(2V_{xxxt}+(1+t)^{-1}V_{xxx})$ for $k=0,1,2$, we have

$$\begin{aligned}&\ln^2(1+t)\|(V_{xxx},V_{xxt})(\cdot,t)\|^2+\int_{t_0}^t\Big[(1+s)^{-1}\ln^2(1+s)\|V_{xxx}(\cdot,s)\|^2\\&\quad+(1+s)\ln^2(1+s)\|V_{xxt}(\cdot,s)\|^2\Big]ds\le C\big(\|V_0\|_3^2+\|z_0\|_2^2+\delta\big).\end{aligned} \tag{2.14}$$

Then, multiplying (2.11) by $-\ln^3(1+t)V_{xxxt}$ and using (2.14), we get

$$\begin{aligned}&\ln^3(1+t)\|(V_{xxx},V_{xxt})(\cdot,t)\|^2+\int_{t_0}^t\Big[(1+s)^{-1}\ln^2(1+s)\|V_{xxx}(\cdot,s)\|^2\\&\quad+(1+s)\ln^3(1+s)\|V_{xxt}(\cdot,s)\|^2\Big]ds\le C\big(\|V_0\|_3^2+\|z_0\|_2^2+\delta\big).\end{aligned} \tag{2.15}$$

Step 4. Differentiating (2.5) in t yields

$$V_{ttt}+V_t+(1+t)V_{tt}+\left(p'(\bar v)V_{xt}\right)_x=F_t-G_{xt}-\left(p''(\bar v)\bar v_tV_x\right)_x. \tag{2.16}$$

Multiplying (2.16) by $(\beta+t)^2\ln^k(1+t)V_{tt}$ with $\beta>1,\ k=0,1$, we get

$$\begin{aligned}&(1+t)^2\ln(1+t)\|(V_t,V_{tt},V_{xt})(\cdot,t)\|^2+\int_{t_0}^t(1+s)^3\ln(1+s)\|V_{tt}(\cdot,s)\|^2ds\\&\le C\big(\|V_0\|_2^2+\|z_0\|_1^2+\delta\big).\end{aligned}\tag{2.17}$$

Noting $(2.5)_1$ yields $(1+t)^2V_t^2\le V_{tt}^2+\left|\big(p'(\bar v)V_x\big)_x\right|^2+|F|^2+|G_x|^2$, it follows from (2.2), the estimates of Steps 1–3, and (2.17) that

$$(1+t)^2\ln^2(1+t)\|V_t(\cdot,t)\|^2\le C\big(\|V_0\|_2^2+\|z_0\|_1^2+\delta\big).\tag{2.18}$$

Step 5. Multiplying (2.16) by $-(\beta+t)^2\ln^k(1+t)V_{xxtt}$ with $\beta>1,\ k=0,1,2$, we get

$$\begin{aligned}&\int(1+t)^2\ln^2(1+t)(V_{xt}^2+V_{xtt}^2+V_{xxt}^2)+\int_{t_0}^t\int(1+s)^3\ln^2(1+s)V_{xtt}^2\\&\le C\big(\|V_0\|_3^2+\|z_0\|_2^2+\delta\big).\end{aligned}\tag{2.19}$$

Noting (2.11) gives $(1+t)^2V_{xt}^2\le V_{xtt}^2+\left|\big(p'(\bar v)V_{xx}\big)_x\right|^2+\left|\,(p''(\bar v)\bar v_xV_x)_x\,\right|^2+|F_x|^2+|G_{xx}|^2$, it follows from (2.2) and the estimates of Steps 1–4 that

$$(1+t)^2\ln^3(1+t)\|V_{xt}(\cdot,t)\|^2\le C\big(\|V_0\|_3^2+\|z_0\|_2^2+\delta\big).\tag{2.20}$$

The estimate (2.6) follows from (2.10), (2.13), (2.15), and (2.18)–(2.20). (2.6) guarantees that $N(t)$ is small for all $t>t_0$ if $N(t_0)$ is small enough. Applying the classical extension argument, one can get the global well-posedness of Eq. (2.5). □

Proof of Theorem 1.1. By Proposition 2.1, system (1.2) has a unique global solution. We next show the convergence rates. By (2.6) and the Sobolev inequality $\|f\|_{L^\infty}\le\sqrt{2}\|f\|_{L^2}^{\frac12}\|f_x\|_{L^2}^{\frac12}$, we have

$$\begin{aligned}\|V_x(\cdot,t)\|_{L^\infty}&\le C\ln^{-1/4}(1+t)\cdot\ln^{-1/2}(1+t)=O(1)\ln^{-3/4}(1+t),\\\|V_t(\cdot,t)\|_{L^\infty}&\le C(1+t)^{-1/2}\ln^{-1/2}(1+t)\cdot(1+t)^{-1/2}\ln^{3/4}(1+t)\\&=O(1)(1+t)^{-1}\ln^{-5/4}(1+t).\end{aligned}$$

Hence the convergence rates in (1.7) hold. □

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