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NONLINEAR STABILITY OF SHOCK PROFILES TO BURGERS EQUATION WITH CRITICAL FAST DIFFUSION AND SINGULARITY*

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Abstract. In this paper, we propose the first framework to study Burgers equation featuring critical fast diffusion in the form of $u_t + f(u)_x = (\ln u)_{xx}$. The solution possesses a strong singularity when $u = 0$, hence bringing technical challenges. The main purpose of this paper is to investigate the asymptotic stability of viscous shocks, particularly those with shock profiles vanishing at the far field $x = +\infty$. To overcome the singularity, we introduce some weight functions and show the nonlinear stability of shock profiles through the weighted energy method.

Key words. nonlinear stability, shock profiles, Burgers equation, singularity, weighted energy estimates

MSC codes. 35B35, 35L65, 76N10

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1. Introduction.

Burgers equation is a nonlinear PDE deeply rooted in fluid mechanics. It has appeared in a multitude of physical phenomena, such as jet flow [25], molecular interface growth [21], and traffic flow [50]. With the growing diversity of physical phenomena, various variants of the Burgers equation have emerged. Among them, a notable class with nonlinear diffusion can be expressed in porous media flow,

$$u_t + f(u)_x = \left(|u|^{m-1}u\right)_{xx}, \tag{1.1}$$

where u denotes the fluid velocity and f is a sufficiently smooth function. $m > 0$ is a parameter which determines whether the fluid is slow diffusion when $m > 1$ or fast diffusion when $0 < m < 1$. If $u \geq 0$, then (1.1) can be written by

$$u_t + f(u)_x = (u^m)_{xx}. \tag{1.2}$$

When $m = 1$, (1.2) serves as a prototype for various variants of the Burgers equation and was initially proposed by Burgers [2, 3] to investigate the turbulence

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phenomena arising from the interaction between convection and diffusion in fluids, generating a wealth of intriguing stability results in the field. Historically, the first result on the stability of shock profiles for the Cauchy problem of (1.2) was due to Il'in and Oleinik [15]. They utilized the maximum principle to prove that the solution of (1.2) asymptotically converges to the solution of the Riemann problem for the inviscid Burgers equation as $t \to +\infty$. Subsequently, Sattinger [47] showed the stability based on spectral analysis. Later, Kawashima and Matsumura [22] and Nishihara [44] proved the stability as well as convergence rates by the L^2-energy method. Furthermore, Kim and Tzavaras [24] considered viscous shocks and N-waves of the classical Burgers equation ($f(u) = u^2/2$). The metastability of N-waves have been investigated in [1, 39] using linear stability analysis and a dynamical system approach. Recently, Kang and Vasseur [18] introduced the relative entropy method to show an L^2 contraction property of general perturbations of shock waves. This powerful technique does not make use of the antiderivative variables and was adopted by Kang, Vasseur, and Wang to the multidimensional case [19], where they obtained the L^2 contraction of planar shock waves. It is worth mentioning that all these stability results need the convexity of the flux f.

When the flux $f(u)$ is nonconvex, the stability of shock waves was investigated by Kawashima and Matsumura [23], including the system case, and the convergence rates were achieved by Mei [40] and Matsumura and Nishihara [37], respectively. Very recently, Huang, Wang, and Zhang [14] exploited the compatibility between the relative entropy method for shock waves and the direct energy method for rarefaction waves and obtained the stability of composite waves of degenerate Oleinik shock and rarefaction waves. In consideration of the boundary effect, Liu and Yu [30] were the first to study the asymptotic behavior of the solutions toward the stationary shock waves on a half line. Then it was also discussed in [28, 29] for shock waves and rarefaction waves on the half line and bounded interval. In [10], Hashimoto is concerned with stability of the stationary solution of Burgers equation in exterior domains on multidimensional spaces. These results are free of singularities. See also the significant contributions by Weinberger [51], Jones, Gardner, and Kapitula [17], Freistühler and Serre [5], Howard [12, 11, 13], and Engler [4] and the references therein.

Regarding the system of conservation laws, the stability of shock profiles was first independently contributed by Matsumura and Nishihara [36] and Goodman [9], then significantly developed by Liu [27], Szepessy and Xin [48], Fries [6], Matsumura and Mei [35, 34], Mei and Nishihara [41], Mascia and Zumbrun [33], Liu and Zeng [31, 32], and Kang, Vasseur, and Wang [20]; see also the references therein.

Compared with the large number of results available for scalar viscous conservation laws (1.2) with $m = 1$, fundamental questions, such as that of the global well-posedness of (1.2) with $0 < m < 1$ or $m > 1$, still remain poorly understood and specifically for the critical fast diffusion case $(\frac{u_x}{u})_x = (\ln u)_{xx}$. The primary obstacles here are the occurrence of singularity for $(u^m)_{xx} = m(\frac{u_x}{u^{1-m}})_x$ as $0 < m < 1$ and for $(\ln u)_{xx}$ when $u = 0$ and the formation of solutions with sharp corners caused by the degeneracy for $m > 1$, respectively. These pose inherent technical difficulties, preventing many classical results and conclusions. Furthermore, the nonlinear viscous term $(u^m)_{xx}$ with $m \neq 1$ of (1.2), while better aligned with the physical context, introduces complications in developing estimates compared to the linear viscous term u_{xx}.

In this paper, we are mainly interested in the following Burgers equation with the critical fast diffusion:

$$\begin{cases} u_t - (\ln u)_{xx} + f(u)_x = 0, & x \in \mathbb{R},\ t > 0, \\ u(x,0) = u_0(x), & x \in \mathbb{R}, \end{cases} \tag{1.3}$$

where f under consideration is suitably smooth and the initial data $u_0(x) \geq 0$ satisfy

$$u_0(x) \to u_\pm \text{ as } x \to \pm\infty \text{ with } 0 = u_+ < u_-. \tag{1.4}$$

The main task of the paper is to show the asymptotic stability of viscous shocks with singular state $u_+ = 0$. Namely, we shall prove that

(i) the Cauchy problem (1.3)–(1.4) admits a unique monotone shock profile $U(x-st)$ connecting 0 and u_-, which satisfies $U_z(z) < 0$ for all $z \in \mathbb{R}$ (see Theorem 2.1);

(ii) the unique shock profile U obtained above is asymptotically stable. Actually, we show that if the initial value u_0 is a small perturbation of the shock profile U in some precise topological sense, then the solution of (1.3)–(1.4) will converge to the spatially shifted U pointwise as time tends to infinity (see Theorem 2.3).

As far as the authors know, this is the first result on the global well-posedness and asymptotic dynamics of the system (1.3)–(1.4) for fast diffusion. In contrast to the previous studies, our work takes into account two aspects. On the one hand, we consider the critical case $(\ln u)_{xx}$ of fast diffusion, that is, $m = 0$ of $(\frac{u_x}{u^{1-m}})_x$. On the other hand, we address the situation where the velocity approaches 0 as $z \to +\infty$ and successfully resolve the challenges brought by the singular nature of the problem. Below we shall briefly present the strategies used to solve our problem.

Given the particular structure of (1.3), we can deduce that a shock profile U, if it exists, must meet both the Rankine–Hugoniot (R-H) condition and the generalized shock condition. On the contrary, the result (i) can be obtained by a first-order differential equation (ODE) satisfied by U, employing the R-H condition and shock condition together with the inverse function theorem, requiring only a moderate level of analytical effort. For the asymptotic stability of U stated in (ii), our primary focus lies in handling the previously mentioned singularity $(u_x/u)_x$, which is resolved by employing the technique of taking antiderivatives and the method of weighted energy estimates. However, all weight functions were carefully choosen to also possess the singularities, which results in the appearance of nonzero boundary terms when performing integration by parts. To overcome this barrier, we apply the cutoff technique in the estimates. This is the outline of our approach, and the precise procedures will be presented in section 4 below.

We would like to point out that this paper is the first attempt to study fast diffusion with singularity for Burgers equations, in particular, in the critical fast diffusion case of $m = 0$. Furthermore, as a supplement to this study, Xu et al. [52] consider the noncritical fast diffusion case with $0 < m < 1$ by the same setting as shown in this work. So the study [52] can be regarded as a supplementary work to ours, but it does not cover the critical case that we consider here.

Before concluding this section, we mention some other works comparable to the current work. When $f(u)_x$ of (1.2) is replaced by u^p, which corresponds to the reaction-diffusion equation, there are some results regarding blowup with nonliner diffusion. Galaktionov et al. [7] proved that the critical blow-up index is $m+2$ with slow diffusion; that is, the solution of the porous media equation blows up in finite time for any nonnegative and nontrivial initial value when $1 < p < m+2$. Subsequently, Qi [45] and Mochizuki and Mukai [42] extended the above results to fast diffusion, and the critical extinction index is m, pointed out by Li and Wu [26]. Furthermore, there have also been studies on the blowup [8, 46] and extinction properties [49, 53] of solutions for the Cauchy problem to the p-Laplace equation; these were later expanded to more general non-Newtonian polytropic filtration equations [16, 54].

The rest of this paper is arranged as follows. In section 2, we state our main results on the existence and nonlinear stability of shock wave solutions to the viscous conservation laws (1.3). In section 3, the existence of shock profiles will be discussed. In section 4, we show the details of weighted energy estimates and prove the nonlinear stability results with $u_+ = 0$.

Notations. Before proceeding, we introduce some notations for convenience. In this paper, the symbol C represents a generic positive constant that may vary from one line to another. The integrals $\int_{\mathbb{R}} f(x)\mathrm{d}x$ and $\int_0^t \int_{\mathbb{R}} f(x,\tau)\mathrm{d}x\mathrm{d}\tau$ will be abbreviated as $\int f(x)$ and $\int_0^t \int f(x,\tau)$, respectively. $H^k(\mathbb{R})$ is the usual kth order Sobolev space defined on $\mathbb{R}$ with norm $\|f\|_{H^k(\mathbb{R})} := (\sum_{j=0}^k \|\partial_x^j f\|^2_{L^2(\mathbb{R})})^{1/2}$. $H^k_w(\mathbb{R})$ denotes the weighted space of measurable functions f such that $\sqrt{w}\partial_x^j f \in L^2$ for $0 \le j \le k$ with norm $\|f\|_{H^k_w(\mathbb{R})} := (\sum_{j=0}^k \int w(x)|\partial_x^j f|^2 dx)^{1/2}$. For simplicity, we denote $\|\cdot\| := \|\cdot\|_{L^2(\mathbb{R})}$, $\|\cdot\|_w := \|\cdot\|_{L^2_w(\mathbb{R})}$, $\|\cdot\|_k := \|\cdot\|_{H^k(\mathbb{R})}$, and $\|\cdot\|_{k,w} := \|\cdot\|_{H^k_w(\mathbb{R})}$. When two positive functions $f(x)$ and $g(x)$ satisfy $cg(x) \le f(x) \le Cg(x)$ with some positive constants c and C for x in an interval I, or $x \to a$, then we simply write

$$f(x) \sim g(x) \quad \text{for } x \in I, \text{ or } x \to a.$$

2. Preliminaries and main results.

Let $u_\pm$ be the state constants such that $0 = u_+ < u_-$, and let s be the velocity of shocks. The shock profile of (1.3) with (1.4) is a nonconstant smooth solution $u(x,t) = U(x - st)$ satisfying

$$\begin{cases} -sU_z - \left(\dfrac{U_z}{U}\right)_z + f(U)_z = 0, \quad z \in \mathbb{R}, \\ U(+\infty) = 0, \; U(-\infty) = u_-, \end{cases} \tag{2.1}$$

with $z = x - st$, which connects 0 and u_-.

We first state the sufficient and necessary conditions for the existence of the shock profile $U(x - st)$ to system (1.3) as follows.

THEOREM 2.1 (existence of shock profiles). *Let $u_- > u_+ := 0$, and let*

$$g(u) \triangleq f(u) - f(u_\pm) - s(u - u_\pm). \tag{2.2}$$

Suppose that $f \in C^{\max\{k_\pm+1,3\}}(\mathbb{R})$, $k_\pm \ge 0$, and $g'(u_\pm) = \cdots = g^{(k_\pm)}(u_\pm) = 0$ with $g^{(k_\pm+1)}(u_\pm) \ne 0$.

(i) *If (1.3) admits a shock profile $U(x - st)$ connecting 0 and u_-, then u_- and s must satisfy the R-H condition*

$$s = \frac{f(0) - f(u_-)}{0 - u_-} \tag{2.3}$$

and the generalized shock condition

$$g(u) < 0, \quad \text{for } u \in (0, u_-). \tag{2.4}$$

(ii) *Conversely, suppose that (2.3) and (2.4) hold. Then there exists a shock profile $U(x - st)$ of (1.3) connecting 0 and u_-. The shock profile $U(z)$ is unique up to a shift in z and satisfies*

$$U_z(z) < 0 \quad \forall z \in \mathbb{R}. \tag{2.5}$$

Moreover,

$$\begin{cases} U(z) \sim |z|^{-1} & \text{as } z \to +\infty \\ u_- - U(z) \sim \mathrm{e}^{-\lambda_- |z|} & \text{as } z \to -\infty \end{cases} \quad \text{if } f'(0) < s < f'(u_-), \tag{2.6}$$

$$\begin{cases} U(z) \sim |z|^{-\frac{1}{1+k_+}} & \text{as } z \to +\infty \\ u_- - U(z) \sim \mathrm{e}^{-\lambda_- |z|} & \text{as } z \to -\infty \end{cases} \quad \text{if } f'(0) = s < f'(u_-), \tag{2.7}$$

and

$$\begin{cases} U(z) \sim |z|^{-1} & \text{as } z \to +\infty \\ u_- - U(z) \sim |z|^{-\frac{1}{k_-}} & \text{as } z \to -\infty \end{cases} \quad \text{if } f'(0) < s = f'(u_-), \tag{2.8}$$

where $\lambda_- = u_-(f'(u_-) - s)$.

Remark 2.2. It is noted that the generalized shock condition (2.4) implies the Lax degenerate entropy condition

$$f'(0) \le s \le f'(u_-). \tag{2.9}$$

In fact, when we fix $u_+ = 0$ in (2.2), we have

$$\begin{aligned} g(U) &= f(U) - f(0) - s(U - 0) \\ &= \left(\frac{f(U) - f(0)}{U - 0} - s \right)(U - 0), \end{aligned}$$

which, in combination with $g(U) < 0$ and $U > 0$ for $U \in (0, u_-)$, yields

$$\frac{f(U) - f(0)}{U - 0} - s < 0. \tag{2.10}$$

Taking the limit of (2.10) as U tends to 0, we then arrive at

$$f'(0) \le s. \tag{2.11}$$

Applying the same procedure for u_- in (2.2) and combining with (2.11), one obtains (2.9).

The Lax degenerate entropy condition (2.9) includes the following three cases: the nondegenerate shock condition

$$f'(0) < s < f'(u_-) \tag{2.12}$$

and the degenerate shock conditions

$$f'(0) = s < f'(u_-) \tag{2.13}$$

and

$$f'(0) < s = f'(u_-). \tag{2.14}$$

We thus categorize our discussion on stability into three different situations.

Given a shock profile $U(x - st)$, we expect $U(x - st + x_0)$ to be the asymptotic profile of the original solution $u(x, t)$ to the Cauchy problem (1.3) with given initial data $u_0(x)$ as $t \to +\infty$. We may determine the shift x_0 from the perturbed equation around $U(x-st+x_0)$:

$$(u-U)_t + (f(u)-f(U))_x - \left(\frac{u_x}{u} - \frac{U_x}{U}\right)_x = 0.$$

Integrating the above equation with respect to x over $(-\infty,+\infty)$, we formally have

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{-\infty}^{+\infty} (u(x,t) - U(x-st+x_0))\,\mathrm{d}x = 0,$$

which implies that

$$\int_{-\infty}^{+\infty} (u(x,t) - U(x-st+x_0))\,dx = \int_{-\infty}^{+\infty} (u_0(x) - U(x+x_0))\,\mathrm{d}x. \tag{2.15}$$

Since it is expected that $U(x-st+x_0)$ is the asymptotic profile of $u(x,t)$ as $t \to +\infty$, we thus heuristically expect

$$\int_{-\infty}^{+\infty} (u_0(x) - U(x+x_0))\,\mathrm{d}x = 0. \tag{2.16}$$

Namely,

$$\begin{aligned}
0 &= \int_{-\infty}^{+\infty} (u_0(x) - U(x+x_0))\,\mathrm{d}x \\
&= \int_{-\infty}^{+\infty} (u_0(x) - U(x))\,\mathrm{d}x - \int_{-\infty}^{+\infty} (U(x+x_0) - U(x))\,\mathrm{d}x \\
&= \int_{-\infty}^{+\infty} (u_0(x) - U(x))\,\mathrm{d}x - \int_{-\infty}^{+\infty}\int_0^{x_0} U'(x+\eta)\mathrm{d}\eta\mathrm{d}x \\
&= \int_{-\infty}^{+\infty} (u_0(x) - U(x))\,\mathrm{d}x - \int_0^{x_0}\int_{-\infty}^{+\infty} U'(x+\eta)\mathrm{d}x\mathrm{d}\eta \\
&= \int_{-\infty}^{+\infty} (u_0(x) - U(x))\,\mathrm{d}x - \int_0^{x_0} (0-u_-)\,\mathrm{d}\eta \\
&= \int_{-\infty}^{+\infty} (u_0(x) - U(x))\,\mathrm{d}x + x_0 u_-.
\end{aligned}$$

This gives

$$x_0 := -\frac{1}{u_-}\int_{-\infty}^{+\infty} (u_0(x) - U(x))\,\mathrm{d}x. \tag{2.17}$$

Let us define

$$\phi_0(x) = \int_{-\infty}^{x} (u_0(y) - U(y+x_0))\,\mathrm{d}y, \tag{2.18}$$

and thus $\phi_0(+\infty) = 0$. Our stability of shock profile $U(z+x_0)$ with singularity at $u_+ = 0$ is stated as follows.

THEOREM 2.3 (stability of shock profiles). *Let $U(z)$ be a shock profile obtained in Theorem* 2.1. *Assume that $u_0(x)-U(x)\in L^1(\mathbb{R})$. Define x_0 and $\phi_0(x)$ by* (2.17) *and* (2.18). *Then the following hold:*

(i) *When $f'(0)<s<f'(u_-)$, there exists a positive constant ϵ_1 such that if $\|\phi_0\|+\|u_0-U\|_{1,w_1}\le\epsilon_1$ with*

$$w_1(U(z+x_0))=U^{-2}(z+x_0)\sim\begin{cases}1+z^2, & z>0,\\ 1, & z\le 0,\end{cases}\tag{2.19}$$

then the Cauchy problem (1.3)–(1.4) *has a unique global solution $u(x,t)$ satisfying*

$$u-U\in C\left([0,\infty);H^1_{w_1}\right)\cap L^2\left((0,\infty);L^2_{w_2}\right),\quad (u-U)_x\in L^2\left((0,\infty);H^1_{w_3}\right),$$

where $U=U(x-st+x_0)$ is the shifted shock profile and the weight functions $w_i(U)$, $i=2,3$, are defined by

$$w_2(U(z+x_0))=U^{-1}(z+x_0)\sim\begin{cases}\sqrt{1+z^2}, & z>0,\\ 1, & z\le 0,\end{cases}\tag{2.20}$$

$$w_3(U(z+x_0))=U^{-3}(z+x_0)\sim\begin{cases}1+z^3, & z>0,\\ 1, & z\le 0,\end{cases}\tag{2.21}$$

and moreover

$$\sup_{x\in\mathbb{R}}|u(x,t)-U(x-st+x_0)|\to 0\qquad \text{as } t\to+\infty.\tag{2.22}$$

(ii) *When $f'(0)=s<f'(u_-)$, there exists a positive constant ϵ_2 such that if $\|\phi_0\|_{w_4}+\|u_0-U\|_{1,w_1}\le\epsilon_2$ with*

$$w_4(U(z+x_0))=\frac{U\,(U-u_-)}{g(U)}\sim\begin{cases}\left(1+z^{k_+}\right)^{\frac{1}{1+k_+}}, & z>0,\\ 1, & z\le 0,\end{cases}\tag{2.23}$$

$$w_1(U(z+x_0))=U^{-2}(z+x_0)\sim\begin{cases}\left(\sqrt{1+z^2}\right)^{\frac{2}{1+k_+}}, & z>0,\\ 1, & z\le 0,\end{cases}\tag{2.24}$$

then the Cauchy problem (1.3)–(1.4) *has a unique global solution $u(x,t)$ satisfying*

$$u-U\in C\left([0,\infty);H^1_{w_1}\right)\cap L^2\left((0,\infty);L^2_{w_5}\right),\quad (u-U)_x\in L^2\left((0,\infty);H^1_{w_3}\right),$$

where the weight function $w_5(U)$ is defined by

$$w_5(U(z+x_0))=\frac{U-u_-}{g(U)}\sim\begin{cases}\sqrt{1+z^2}, & z>0,\\ 1, & z\le 0,\end{cases}\tag{2.25}$$

$$w_3(U(z+x_0))=U^{-3}(z+x_0)\sim\begin{cases}\left(\sqrt{1+z^2}\right)^{\frac{3}{1+k_+}}, & z>0,\\ 1, & z\le 0,\end{cases}\tag{2.26}$$

and moreover

$$\sup_{x\in\mathbb{R}}|u(x,t)-U(x-st+x_0)|\to 0,\qquad \text{as } t\to+\infty.\tag{2.27}$$

(iii) *When $f'(0) < s = f'(u_-)$, there exists a positive constant ϵ_3 such that if $\|\phi_0\|_{w_4} + \|u_0 - U\|_{1,w_1} \le \epsilon_3$ with*

$$w_4(U(z+x_0)) = \frac{U\,(U-u_-)}{g(U)} \sim \begin{cases} 1, & z>0, \\ \sqrt{1+z^2}, & z\le 0, \end{cases} \tag{2.28}$$

$$w_1(U(z+x_0)) = U^{-2}(z+x_0) \sim \begin{cases} 1+z^2, & z>0, \\ 1, & z\le 0, \end{cases} \tag{2.29}$$

then the Cauchy problem (1.3)–(1.4) *has a unique global solution $u(x,t)$ satisfying*

$$u - U \in C\left([0,\infty); H^1_{w_1}\right) \cap L^2\left((0,\infty); L^2_{w_5}\right), \quad (u-U)_x \in L^2\left((0,\infty); H^1_{w_3}\right),$$

where the weight function $w_5(U)$ is defined by

$$w_5(U(z+x_0)) = \frac{U-u_-}{g(U)} \sim \sqrt{1+z^2}, \quad z\in\mathbb{R}, \tag{2.30}$$

$$w_3(U(z+x_0)) = U^{-3}(z+x_0) \sim \begin{cases} 1+z^3, & z>0, \\ 1, & z\le 0, \end{cases} \tag{2.31}$$

and moreover

$$\sup_{x\in\mathbb{R}} |u(x,t) - U(x-st+x_0)| \to 0 \quad \text{as } t\to+\infty. \tag{2.32}$$

3. Existence of the shock profile. This section is devoted to showing that there exists a shock profile $U(x-st)$ connecting 0 and u_- if and only if u_- and s satisfy the R-H condition (2.3) and the generalized shock condition (2.4); along this line, we prove Theorem 2.1.

Proof of Theorem 2.1. To prove (i), we suppose that (1.3) admits a shock profile $U(x-st)$ connecting 0 and u_-. Therefore, $U(z)$ with $z = x-st$ must satisfy (2.1) and $U_z(z) < 0$ due to $u_- > 0$. Formally, we expect that $\lim_{z\to\pm\infty} \frac{U_z}{U}(z) = 0$. Then integrating (2.1) in z over $\mathbb{R}$ yields

$$-s\,(0-u_-) + f\,(0) - f\,(u_-) = 0,$$

which implies the R-H condition (2.3). Additionally, integrating (2.1) in z over $(z,+\infty)$, we have

$$-sU - \frac{U_z}{U} + f(U) = a, \tag{3.1}$$

where a is a constant. Since $U(\pm\infty) = u_\pm$ and $\lim_{z\to\pm\infty}\frac{U_z}{U}(z) = 0$, letting $z\to\pm\infty$ in (3.1), we see easily that

$$a = -su_\pm + f\,(u_\pm)\,.$$

This along with (3.1) gives

$$U_z = U\left[f(U) - f(u_\pm) - s(U-u_\pm)\right]. \tag{3.2}$$

Define

$$h(U) \triangleq U\left[f(U) - f(u_\pm) - s(U - u_\pm)\right]. \tag{3.3}$$

Thus, $h(U) = Ug(U)$, where $g(U)$ is given in (2.2). From standard ODE theory, (3.2) with $h(u_\pm) = 0$ admits a smooth solution $U(z)$ satisfying $U(\pm\infty) = u_\pm$ if and only if

$$h(U) < 0 \quad \forall U \in (0, u_-).$$

Here and hereafter, $(0, u_-)$ denotes the open interval for u or U. Consequently, the generalized shock condition (2.4) holds due to $U > 0$ for $U \in (0, u_-)$. This completes the proof of (i).

Conversely, we suppose that (2.3) and (2.4) hold. By virtue of (2.4), (3.2), and (3.3), we only need to find the global solution of the following ODE:

$$\frac{\mathrm{d}z}{\mathrm{d}U} = \frac{1}{h(U)}. \tag{3.4}$$

In this case, we can solve (3.4) in the form

$$z + \text{constant} = \int_{u_\star}^{U} \frac{1}{h(y)} \mathrm{d}y \triangleq \Lambda(U),$$

where $u_\star = (u_+ + u_-)/2$. Since for any given $U \in (0, u_-)$ the function $\frac{1}{h(y)}$ is integrable over $(u_\star, U(z))$ and $\Lambda'(U) = \frac{1}{h(U)} < 0$ due to $h(U) = Ug(U)$ and (2.4), we know that $\int_{u_\star}^{U} \frac{1}{h(y)} \mathrm{d}y$ is finite and monotonically decreasing for any $U \in (0, u_-)$. It remains to check the limiting values $U(\pm\infty) = u_\pm$. When $f'(0) < s < f'(u_-)$, we have $|g(U)| \sim |U - u_\pm|$ as $U \to u_\pm$ because of $g'(u_\pm) \neq 0$. Therefore, $h(U) < 0$ and $|h(U)| = O(1)U^2$ as $U \to 0$, and

$$\int_{u_\star}^{0} \frac{1}{h(y)} \mathrm{d}y = +\infty. \tag{3.5}$$

As $U \to u_-$, it becomes apparent that

$$\int_{u_\star}^{u_-} \frac{1}{h(y)} \mathrm{d}y = -\infty \tag{3.6}$$

because of $h(U) \sim U - u_-$ as $U \to u_-$. When $s = f'(0)$ or $s = f'(u_-)$, since $g'(u_\pm) = \cdots = g^{(k_\pm)}(u_\pm) = 0$ with $g^{(k_\pm+1)}(u_\pm) \neq 0$, we have $|g(U)| \sim |U - u_\pm|^{1+k_\pm}$ for $k_\pm \geq 1$ as $U \to u_\pm$, and thus (3.5) and (3.6) are satisfied. Therefore, according to the inverse function theorem, there exists a unique continuous function Λ^{-1} such that

$$U = \Lambda^{-1}(z - z_\star) \quad \forall z \in (-\infty, +\infty).$$

Thus, the existence of the shock profile $U(z)$ is proved. Equation (2.5) follows from (2.4) and (3.2).

Next, we are going to show the convergence rate for $U(z)$ as $z \to \pm\infty$. When $f'(0) < s < f'(u_-)$, we get from (3.3) that

$$U_z = Uh(U) \sim U^2\left(f'(0) - s\right) \quad \text{as } z \to +\infty, \text{i.e., } U \to 0. \tag{3.7}$$

A direct calculation gives

$$U \sim \left[C_1 - (f'(0) - s)z\right]^{-1} \text{ for } z > 0.$$

Here $C_1 > 0$ is a bounded constant due to $U(+\infty) = 0$, which implies the convergence rate $|z|^{-1}$ for $U(z) \to 0$ as $z \to +\infty$. When $z \to -\infty$, we have

$$U_z \sim u_- \left(f'(u_-) - s\right)(U - u_-), \tag{3.8}$$

and hence

$$u_- - U \sim C_2 \mathrm{e}^{u_-(f'(u_-)-s)z} \text{ for } z < 0,$$

where C_2 is a positive constant due to $U < u_-$ for $U \in (0, u_-)$. This implies that the convergence rate for $U(z) \to u_-$ as $z \to -\infty$ is $\mathrm{e}^{-\lambda_-|z|}$ with $\lambda_- = u_-(f'(u_-) - s)$. In particular, when $s = f'(0)$, we have $|g(U)| \sim |U|^{1+k_+}$ for $k_+ \geq 1$ as $z \to +\infty$ as discussed above, and this implies the convergence rate $|z|^{-\frac{1}{1+k_+}}$ as $z \to +\infty$. Similarly, when $s = f'(u_-)$, we have $|g(U)| \sim |U - u_-|^{1+k_-}$ for $k_- \geq 1$ as $z \to -\infty$, and the convergence rate for $U(z) \to u_-$ is $|z|^{-\frac{1}{k_-}}$ as $z \to -\infty$. Thus, the proof of Theorem 2.1 is complete. ∎

Remark 3.1. By virtue of $u_- > 0$, if $f'(0) < s < f'(u_-)$, then it is easy to see from (3.7) and (3.8) that

$$|U_z(z)| \leq CU^2(z) \quad \text{for all } z \in (-\infty, +\infty). \tag{3.9}$$

Recalling that $U_z = Ug(U)$ and $|g(U)| \sim |U|^{1+k_+}$ for $k_+ \geq 1$ as $z \to +\infty$ if $s = f'(0)$, then we arrive at

$$|U_z(z)| \leq C\,|U(z) - 0|^{2+k_+} \leq CU^2(z) \quad \text{for } z > 0. \tag{3.10}$$

Similarly, if $s = f'(u_-)$ since $|g(U)| \sim |U - u_-|^{1+k_-}$ for $k_- \geq 1$ as $z \to -\infty$, then we have

$$|U_z(z)| \leq Cu_-\,|U(z) - u_-|^{1+k_-} \quad \text{for } z < 0. \tag{3.11}$$

4. Nonlinear stability. In this section, we prove Theorem 2.3 and hence establish the nonlinear stability of the shock profile of (1.3)–(1.4). One may observe that the system (1.3) exhibits a singularity in the vicinity of $z = +\infty$; hence, we shall devise new strategies to address this challenge. Before proceeding, we first use the technique of the antiderivative to reformulate the problem.

4.1. Reformulation of the problem. Let $U(x - st)$ be the shock profile obtained in Theorem 2.1. According to (2.15) and (2.16), we have

$$\int_{-\infty}^{+\infty} \left(u(x,t) - U\left(x - st + x_0\right)\right) dx = \int_{-\infty}^{+\infty} \left(u_0(x) - U\left(x + x_0\right)\right) dx = 0. \tag{4.1}$$

We thus decompose the solution of (1.3) as

$$u(x,t) = U\left(x - st + x_0\right) + \phi_z(z,t), \tag{4.2}$$

where $z = x - st$, that is,

$$\phi(z,t) = \int_{-\infty}^{z} \left(u(y,t) - U\left(y - st + x_0\right)\right) \mathrm{d}y \tag{4.3}$$

for all $z \in \mathbb{R}$ and $t \geq 0$. It then follows from (4.1) that

$$\phi(\pm\infty, t) = 0 \quad \text{for all } t > 0. \tag{4.4}$$

Substituting (4.2) into (1.3), using (2.1) and (4.4), and integrating the system with respect to z, the problem (1.3) is reduced to

$$\phi_t = s\phi_z + (\ln(U+\phi_z) - \ln U)_z - (f(U+\phi_z) - f(U)),$$

which is rewritten as

$$\phi_t + g'(U)\phi_z - \left(\frac{\phi_z}{U}\right)_z = F + G_z, \tag{4.5}$$

where

$$F \equiv -(f(U+\phi_z) - f(U) - f'(U)\phi_z) \tag{4.6}$$

and

$$G \equiv \ln(U+\phi_z) - \ln U - \frac{\phi_z}{U}. \tag{4.7}$$

The initial perturbation of ϕ is thus given by

$$\phi(z,0) = \int_{-\infty}^{z} (u_0(y) - U(y+x_0))\,dy \text{ with } \phi(\pm\infty,0) = 0. \tag{4.8}$$

We search for solutions of the system (4.5) in the following space:

$$\begin{aligned} X(0,T) := \{\phi(z,t) \mid \phi \in C([0,T]; L^2 \cap L^2_{w_4}), \phi_z \in C([0,T]; H^1_{w_1}) \\ \cap L^2((0,T); L^2_{w_2} \cap L^2_{w_5}), \phi_{zz} \in L^2((0,T); H^1_{w_3})\}, \end{aligned} \tag{4.9}$$

where the weight functions w_i, $i = 1, \cdots, 5$, are given by

$$\begin{aligned} &w_1(U) = U^{-2}, \quad w_2(U) = U^{-1}, \quad w_3(U) = U^{-3}, \\ &w_4(U) = \frac{U(U-u_-)}{g(U)}, \quad w_5(U) = \frac{U-u_-}{g(U)} \end{aligned} \tag{4.10}$$

with $U = U(z+x_0)$. Define

$$N(t) := \sup_{\tau\in[0,t]} \left(\|\phi(\cdot,\tau)\|_{w_4} + \|\phi_z(\cdot,\tau)\|_{1,w_1}\right). \tag{4.11}$$

It is easy to see the following: (1) If $f'(0) < s < f'(u_-)$, then we have $|g(U)| \sim |U - u_-|$ as $U \to u_-$ and $|g(U)| \sim U$ as $U \to 0$, and then $w_4(U) \sim C$ for some constant $C > 0$. (2) If $f'(0) = s < f'(u_-)$, then one has $|g(U)| \sim |U|^{1+k_+}$ for $k_+ \geq 1$ as $U \to 0$, and then $w_4(U) = \frac{U(U-u_-)}{g(U)} \sim U^{-k_+}$ as $z \to +\infty$. (3) If $f'(0) < s = f'(u_-)$, $|g(U)| \sim |U-u_-|^{1+k_-}$ for $k_- \geq 1$ as $U \to u_-$, then we have $w_4(U) = \frac{U(U-u_-)}{g(U)} \sim |U-u_-|^{-k_-}$ as $z \to -\infty$. Thus, in any case, we get $w_4(U) \geq 1$. Clearly, if $\phi_z \in L^2_{w_1}$, then $\phi_z \in L^2$ since $w_1(U) \geq 1$. Then, by the Sobolev embedding inequality $H^1(R) \hookrightarrow C^0(R)$ (see [41]), one has

$$\sup_{\tau\in[0,t]} \|\phi(\cdot,\tau)\|_{L^\infty} \leq CN(t) \tag{4.12}$$

and

$$\|\sqrt{w_1(U)}\phi_z(\cdot,t)\|^2_{L^\infty} \leq C\|\sqrt{w_1(U)}\phi_z(\cdot,t)\|^2_1. \tag{4.13}$$

Thanks to $|U_z(z)|\le CU^2(z)$ for all $z\in(-\infty,+\infty)$, we get

$$\begin{aligned}\|\sqrt{w_1(U)}\phi_z\|_1^2=\|\phi_z/U\|_1^2&\le\int\frac{\phi_z^2}{U^2}+2\int\frac{\phi_{zz}^2}{U^2}+2\int\frac{U_z^2}{U^2}\frac{\phi_z^2}{U^2}\\&\le C\left(\int\frac{\phi_z^2}{U^2}+\int\frac{\phi_{zz}^2}{U^2}\right)\\&=C\|\phi_z\|_{1,w_1}^2.\end{aligned}$$

This along with (4.11) and (4.13) gives rise to

$$\sup_{\tau\in[0,t]}\|\sqrt{w_1}\phi_z(\cdot,\tau)\|_{L^\infty}\le CN(t). \tag{4.14}$$

Theorem 2.3 is a consequence of the following theorem.

Theorem 4.1. *Suppose that $\phi_0\in L^2(\mathbb{R})\cap L^2_{w_4}(\mathbb{R})$ and that $\phi_{0z}\in H^1_{w_1}(\mathbb{R})$. Then there exists a positive constant δ_1 such that if $N(0)\le\delta_1$, then the Cauchy problem* (4.5)–(4.7) *with* (4.8) *has a unique global solution $\phi\in X(0,\infty)$ satisfying*

$$\begin{aligned}&\|\phi(\cdot,t)\|^2+\|\phi(\cdot,t)\|_{w_4}^2+\|\phi_z(\cdot,t)\|_{1,w_1}^2\\&\quad+\int_0^t\left(\|\phi_z(\cdot,\tau)\|_{w_2}^2+\|\phi_z(\cdot,\tau)\|_{w_5}^2+\|\phi_{zz}(\cdot,\tau)\|_{1,w_3}^2\right)\\&\le C\left(\|\phi_0\|^2+\|\phi_0\|_{w_4}^2+\|\phi_{0z}\|_{1,w_1}^2\right)\le CN^2(0)\end{aligned} \tag{4.15}$$

for any $t\in[0,\infty)$, where w_i, $i=1,\cdots,5$, are defined by (4.10). *Moreover, ϕ_z tends to 0 in the supremum norm as $t\to+\infty$, that is,*

$$\sup_{x\in\mathbb{R}}|\phi_z(z,t)|\to0\qquad\text{as } t\to+\infty. \tag{4.16}$$

The global existence of ϕ, as stated in Theorem 4.1, can be proved by the weighted energy method based on local existence with the a priori estimates provided below [22].

Proposition 4.2 (local existence). *Suppose that $\phi_0\in L^2(\mathbb{R})\cap L^2_{w_4}(\mathbb{R})$ and that $\phi_{0z}\in H^1_{w_1}(\mathbb{R})$. For any $\delta_0>0$, there exists a positive constant T_0 depending on δ_0 such that if $N(0)\le\delta_0$, then the problem* (4.5)–(4.7) *with* (4.8) *has a unique solution $\phi\in X(0,T_0)$ satisfying $N(t)\le 2N(0)$ for any $0\le t\le T_0$.*

Proposition 4.3 (a priori estimates). *Let ϕ be a solution in $X(0,T)$ obtained in Proposition* 4.2 *for a positive constant T. Then there exists a positive constant δ_2 independent of T such that if*

$$N(t)\le\delta_2 \text{ for all } 0\le t\le T, \tag{4.17}$$

then the estimate (4.15) *holds for $t\in[0,T]$.*

Proposition 4.2 can be proved in the standard way (see, e.g., [43]). So we omit its proof. However, it is crucial to establish the a priori estimates as stated in Proposition 4.3, which will be proved in the next subsection.

4.2. Basic estimates and stability theorem. We first derive the basic estimates, which play the key role in our proof.

LEMMA 4.4. *Under the a priori assumption* (4.17)*, if $N(T)$ is sufficiently small, then there exists a constant $C>0$ independent of T such that*

$$|F| \le C\phi_z^2, \quad |G| \le C\frac{\phi_z^2}{U^2}, \tag{4.18}$$

$$|F_z| \le C\left(|U_z|\phi_z^2 + |\phi_z||\phi_{zz}|\right), \tag{4.19}$$

$$|G_z| \le C\left(\frac{|U_z|}{U^3}\phi_z^2 + \frac{|\phi_z||\phi_{zz}|}{U^2}\right), \tag{4.20}$$

$$|G_{zz}| \le C\left[\left(\frac{U_z^2}{U^4} + \frac{|U_{zz}|}{U^3}\right)\phi_z^2 + \frac{\phi_{zz}^2}{U^2} + \frac{|U_z|}{U^3}|\phi_z||\phi_{zz}| + \frac{|\phi_z||\phi_{zzz}|}{U^2}\right]. \tag{4.21}$$

Proof. From the formulations of F and G given in (4.6) and (4.7), respectively, a direct calculation gives

$$\begin{aligned} F_z &= -f'(U+\phi_z)(U_z+\phi_{zz}) + f'(U)U_z + f''(U)U_z\phi_z + f'(U)\phi_{zz} \\ &= -\left(f'(U+\phi_z) - f'(U) - f''(U)\phi_z\right)U_z - \left(f'(U+\phi_z) - f'(U)\right)\phi_{zz}, \end{aligned}$$

$$\begin{aligned} G_z &= \frac{1}{(U+\phi_z)}(U_z+\phi_{zz}) - \frac{U_z}{U} - \frac{\phi_{zz}}{U} - \left(-\frac{U_z}{U^2}\right)\phi_z \\ &= \left[\frac{1}{(U+\phi_z)} - \frac{1}{U} - \left(-\frac{1}{U^2}\right)\phi_z\right]U_z + \left[\frac{1}{(U+\phi_z)} - \frac{1}{U}\right]\phi_{zz}, \end{aligned}$$

and

$$\begin{aligned} G_{zz} &= \left[\frac{(U_z+\phi_{zz})}{U+\phi_z} - \frac{U_z}{U} - \frac{\phi_{zz}}{U} + \frac{U_z}{U^2}\phi_z\right]_z \\ &= \left[-\frac{1}{(U+\phi_z)^2} - \left(-\frac{1}{U^2}\right) - \frac{2\phi_z}{U^3}\right]U_z^2 - \frac{\phi_{zz}^2}{(U+\phi_z)^2} + \left(\frac{1}{U+\phi_z} - \frac{1}{U}\right)\phi_{zzz} \\ &\quad + 2\left[-\frac{1}{(U+\phi_z)^2} - \left(-\frac{1}{U^2}\right)\right]U_z\phi_{zz} + \left[\frac{1}{U+\phi_z} - \frac{1}{U} - \left(-\frac{\phi_z}{U^2}\right)\right]U_{zz}, \end{aligned}$$

where, thanks to $f \in C^{\max\{k_\pm+1,3\}}(\mathbb{R})$, $\|\phi_z(\cdot,t)/U\|_{L^\infty} \le CN(t)$ from (4.12), and Taylor's expansion, it is straightforward to imply (4.18), (4.19), (4.20), and (4.21). ◻

LEMMA 4.5. *Let the assumptions of Proposition* 4.3 *hold. If $N(T)$ is sufficiently small, then there exists a constant $C>0$ independent of T such that*

$$\|\phi(\cdot,t)\|_{w_4}^2 + \int_0^t \|\phi_z(\cdot,\tau)\|_{w_5}^2 \le C\|\phi_0\|_{w_4}^2 \tag{4.22}$$

for any $t \in [0,T]$.

Proof. Multiplying (4.5) by $w_4(U)\phi(z,t)$ with $w_4(U) = \frac{U(U-u_-)}{g(U)}$ gives

$$\begin{aligned} &\left(\frac{1}{2}w_4(U)\phi^2\right)_t + \left[\frac{1}{2}(w_4 g)'(U)\phi^2 - \frac{w_4(U)}{U}\phi\phi_z - w_4(U)G\phi\right]_z + \frac{w_4(U)}{U}\phi_z^2 \\ &\quad - \frac{1}{2}(w_4 g)''(U)U_z\phi^2 = w_4(U)F\phi - w_4'(U)U_zG\phi - w_4(U)G\phi_z. \end{aligned} \tag{4.23}$$

To handle the singularity of the equation at $z=+\infty$, we take a smooth cutoff function $\eta(z)$ satisfying

$$\eta(z)=1 \text{ for } |z|<L,\ \eta(z)=0 \text{ for } |z|>2L,\ 0\le\eta(z)\le 1, \text{ and } |\eta_z(z)|\le\frac{C}{L} \text{ for } z\in\mathbb{R}. \tag{4.24}$$

Multiplying (4.23) by the cutoff function $\eta^2(z)$, integrating the result, and noting that $(w_4g)''(U)=2$, we have

$$\begin{aligned}\frac{1}{2}\int_{-2L}^{2L} w_4(U)\phi^2\eta^2 &+\int_0^t\int_{-2L}^{2L}|U_z|\phi^2\eta^2+\int_0^t\int_{-2L}^{2L}\frac{w_4(U)}{U}\phi_z^2\eta^2\\ &=\frac{1}{2}\int_{-2L}^{2L} w_4(U)\phi_0^2\eta^2+\int_0^t\int_{-2L}^{2L}\left[w_4(U)F\phi\eta^2-w_4'(U)U_zG\phi\eta^2-w_4(U)G\phi_z\eta^2\right]\\ &\quad+\int_0^t\int_{-2L}^{2L}2\eta\eta_z(z)\left(\frac{1}{2}(w_4g)'(U)\phi^2-\frac{w_4(U)}{U}\phi\phi_z-w_4(U)G\phi\right).\end{aligned} \tag{4.25}$$

We next estimate the last term of (4.25). Since $\phi\in X(0,T)$ defined in (4.9), we know that

$$\int_{-2L}^{2L} w_4(U)\phi^2\le\int w_4(U)\phi^2\le N^2(t) \text{ for any } t\in[0,T].$$

A direct calculation gives $(w_4g)'(U)=2U-u_-$. We then deduce that

$$\begin{aligned}\int_0^t\int_{-2L}^{2L}(w_4g)'(U)\phi^2\eta\eta_z(z)&\le C\int_0^t\int_{-2L}^{2L}\phi^2|\eta_z(z)|\\ &\le\frac{C}{L}\int_0^t\int_{-2L}^{2L}w_4(U)\phi^2\le\frac{CTN^2(T)}{L}.\end{aligned} \tag{4.26}$$

By the Cauchy–Schwarz inequality,

$$\begin{aligned}&\left|2\int_{-2L}^{2L}\frac{w_4(U)}{U}\phi\phi_z\eta\eta_z(z)\right|\\ &\quad\le\int_{-2L}^{2L}w_4(U)\phi^2\frac{|\eta_z(z)|^{\frac{3}{2}}}{U}+\int_{-2L}^{2L}\frac{w_4(U)}{U}\phi_z^2\eta^2|\eta_z(z)|^{\frac{1}{2}}.\end{aligned} \tag{4.27}$$

By Theorem 2.1, we have (1) if $f'(0)<s\le f'(u_-)$, then $U(z+x_0)\sim|z+x_0|^{-1}\sim|z|^{-1}$ as $z\to+\infty$, which implies that $\frac{|\eta_z(z)|^{\frac{3}{2}}}{U}\le\frac{C}{\sqrt{L}}$, and (2) if $f'(0)=s<f'(u_-)$, then $U(z+x_0)\sim|z+x_0|^{-\frac{1}{1+k_+}}\sim|z|^{-\frac{1}{1+k_+}}$ as $z\to+\infty$, which gives $\frac{|\eta_z(z)|^{\frac{3}{2}}}{U}\le\frac{C}{L^{\frac{3}{2}-\frac{1}{1+k_+}}}\le\frac{C}{\sqrt{L}}$. In any case, we have

$$\frac{|\eta_z(z)|^{\frac{3}{2}}}{U}\le\frac{C}{\sqrt{L}}. \tag{4.28}$$

Thus,

$$\int_{-2L}^{2L}w_4(U)\phi^2\frac{|\eta_z(z)|^{\frac{3}{2}}}{U}\le\frac{C}{\sqrt{L}}\int_{-2L}^{2L}w_4(U)\phi^2\le\frac{CN^2(t)}{\sqrt{L}}. \tag{4.29}$$

In addition,

$$\int_{-2L}^{2L}\frac{w_4(U)}{U}\phi_z^2\eta^2|\eta_z(z)|^{\frac{1}{2}}\le\frac{C}{\sqrt{L}}\int_{-2L}^{2L}\frac{w_4(U)}{U}\phi_z^2\eta^2. \tag{4.30}$$

Substituting (4.29)–(4.30) into (4.27) leads to

$$\left|2\int_0^t\int_{-2L}^{2L}\frac{w_4(U)}{U}\phi\phi_z\eta\eta_z(z)\right|\le\frac{CTN^2(T)}{\sqrt{L}}+\frac{C}{\sqrt{L}}\int_0^t\int_{-2L}^{2L}\frac{w_4(U)}{U}\phi_z^2\eta^2. \tag{4.31}$$

By (4.18) and (4.14), one obtains

$$\begin{aligned}\left|\int_0^t\int_{-2L}^{2L}2\eta\eta_z(z)w_4(U)G\phi\right|&\le C\int_0^t\int_{-2L}^{2L}\frac{w_4(U)}{U}|\phi\phi_z\eta\eta_z(z)|\cdot\frac{|\phi_z|}{U}\\&\le CN(t)\int_0^t\int_{-2L}^{2L}\frac{w_4(U)}{U}|\phi\phi_z\eta\eta_z(z)|\\&\le\frac{CTN^3(T)}{\sqrt{L}}+\frac{CN(T)}{\sqrt{L}}\int_0^t\int_{-2L}^{2L}\frac{w_4(U)}{U}\phi_z^2\eta^2,\end{aligned} \tag{4.32}$$

where we have used (4.31) in the last inequality. Substituting (4.26)–(4.32) into (4.25), we derive

$$\begin{aligned}&\frac{1}{2}\int_{-2L}^{2L}w_4(U)\phi^2\eta^2+\int_0^t\int_{-2L}^{2L}|U_z|\phi^2\eta^2+\left(1-\frac{C+CN(T)}{\sqrt{L}}\right)\int_0^t\int_{-2L}^{2L}\frac{w_4(U)}{U}\phi_z^2\eta^2\\&\quad\le\frac{1}{2}\int_{-2L}^{2L}w_4(U)\phi_0^2\eta^2+\int_0^t\int_{-2L}^{2L}\left[w_4(U)F\phi\eta^2-w_4'(U)U_zG\phi\eta^2-w_4(U)G\phi_z\eta^2\right]\\&\qquad+\frac{CTN^2(T)}{\sqrt{L}}.\end{aligned}$$

Now passing the limit $L\to+\infty$ yields

$$\begin{aligned}&\int w_4(U)\phi^2+\int_0^t\int|U_z|\phi^2+\int_0^t\int\frac{w_4(U)}{U}\phi_z^2\\&\quad\le C\left(\int w_4(U)\phi_0^2+\int_0^t\int|w_4(U)F\phi|+\int_0^t\int|w_4'(U)U_zG\phi|+\int_0^t\int|w_4(U)G\phi_z|\right).\end{aligned} \tag{4.33}$$

By the first inequality of (4.18) and $\|\phi(\cdot,t)\|_{L^\infty}\le CN(t)$, we deduce that

$$\int_0^t\int|w_4(U)F\phi|\le CN(t)\int_0^t\int w_4(U)\phi_z^2\le CN(t)\int_0^t\int\frac{w_4(U)}{U}\phi_z^2. \tag{4.34}$$

In view of the second inequality of (4.18) and $\|\phi(\cdot,t)\|_{L^\infty}\le CN(t)$, we get

$$\begin{aligned}\int_0^t\int|w_4'(U)U_zG\phi|&\le CN(t)\int_0^t\int\frac{|w_4'(U)||U_z|}{U^2}\phi_z^2\\&=CN(t)\int_0^t\int\frac{w_4(U)}{U}\phi_z^2\cdot\left|\frac{w_4'(U)}{w_4(U)}\right|\left|\frac{U_z}{U}\right|.\end{aligned} \tag{4.35}$$

We can prove that the following holds in the cases where both the nondegenerate shock condition and the degenerate shock condition are satisfied:

$$\left|\frac{w_4'(U)}{w_4(U)}\right|\left|\frac{U_z}{U}\right|\le C\quad\forall z\in(-\infty,\infty). \tag{4.36}$$

First, consider the case $f'(0) \le s < f'(u_-)$. Since $|g(U)| \sim |U|^{1+k_+}$ with $k_+ \ge 0$ as $z \to +\infty$, one obtains

$$\begin{aligned}\left|\frac{w_4'(U)}{w_4(U)}\right| &= \left|\frac{(2U-u_-)\,g(U) - U\,(U-u_-)\,g'(U)}{U\,(U-u_-)\,g(U)}\right| \\ &\le C\frac{|U|^{1+k_+}\,(|2U-u_-|+|U-u_-|)}{|U|^{2+k_+}\,|U-u_-|} \\ &\le \frac{C}{U},\end{aligned} \tag{4.37}$$

and when $z \to -\infty$, $|g(U)| \sim |U-u_-|$, we have

$$\begin{aligned}\left|\frac{w_4'(U)}{w_4(U)}\right| &= \left|\frac{(2U-u_-)\,g(U) - U\,(U-u_-)\,g'(U)}{U\,(U-u_-)\,g(U)}\right| \\ &\le C\frac{|U-u_-|\,(|2U-u_-|+|U|)}{|U-u_-|^2\,|U|} \\ &\le \frac{C}{|U-u_-|}.\end{aligned} \tag{4.38}$$

On the other hand, based on (3.2) and (3.3), for any $k_+ \ge 0$,

$$\left|\frac{U_z}{U}\right| \le C\,|U|^{k_++1} \ \text{ as } z\to+\infty \text{ and } \left|\frac{U_z}{U}\right| \le C\,|U-u_-| \ \text{ as } z \to -\infty.$$

This along with (4.37) and (4.38) yields

$$\left|\frac{w_4'(U)}{w_4(U)}\right|\left|\frac{U_z}{U}\right| \le C\,|U-0|^{k_+} \le C \qquad \text{as } z\to+\infty \tag{4.39}$$

and

$$\left|\frac{w_4'(U)}{w_4(U)}\right|\left|\frac{U_z}{U}\right| \le C \qquad \text{as } z\to-\infty. \tag{4.40}$$

If $f'(0) < s = f'(u_-)$, then (4.39) still holds as $z \to +\infty$. As $z\to-\infty$,

$$\begin{aligned}\left|\frac{w_4'(U)}{w_4(U)}\right| &\le C\frac{|U-u_-|^{1+k_-}\,(|2U-u_-|+U)}{|U-u_-|^{2+k_-}\,U} \\ &\le \frac{C}{|U-u_-|}.\end{aligned}$$

Thanks to (3.11), it implies that for any $k_- \ge 1$,

$$\left|\frac{w_4'(U)}{w_4(U)}\right|\left|\frac{U_z}{U}\right| \le C\,|U-u_-|^{k_-} \le C \quad \text{as } z\to-\infty. \tag{4.41}$$

Therefore, by (4.39), (4.40), and (4.41), we obtain (4.36). In view of (4.35) and (4.36), we further obtain

$$\int_0^t\int |w_4'(U)U_zG\phi| \le CN(t)\int_0^t\int \frac{w_4(U)}{U}\phi_z^2. \tag{4.42}$$

Finally, by virtue of (4.6) and $\|\phi_z(\cdot,t)/U\|_{L^\infty} \le CN(t)$, the last term of (4.33) can be estimated as

$$\int_0^t\int |w_4(U)G\phi_z| \le C\int_0^t\int \frac{w_4(U)}{U^2}|\phi_z^3| \le CN(t)\int_0^t\int\frac{w_4(U)}{U}\phi_z^2. \tag{4.43}$$

Adding (4.34), (4.42), and (4.43), we can derive

$$\int w_4(U)\phi^2 + \int_0^t \int |U_z|\phi^2 + (1 - CN(t)) \int_0^t \int \frac{w_4(U)}{U}\phi_z^2 \le C \int w_4(U)\phi_0^2.$$

The proof of Lemma 4.5 is complete provided that $CN(t) \le 1/2$. ∎

Remark 4.6. Note that when $f'(0) < s < f'(u_-)$, $w_4(U) \sim C$, we also have the estimate for ϕ that

$$\|\phi(\cdot,t)\|^2 + \int_0^t \|\phi_z(\cdot,\tau)\|_{w_2}^2 \le C\|\phi_0\|^2 \text{ for any } t \in [0,T]. \tag{4.44}$$

When $f'(0) = s < f'(u_-)$, since $|g(U)| \sim |U|^{1+k_+}$ as $U \to 0$ for $k_+ \ge 1$, $w_4(U) \sim |U|^{-k_+}$ and $\frac{w_4(U)}{U} \sim |U|^{-k_+-1}$, which along with (4.22) implies that

$$\|\phi(\cdot,t)\|^2 + \int_0^t \|\phi_z(\cdot,\tau)\|_{w_2}^2 \le \|\phi(\cdot,t)\|_{w_4}^2 + \int_0^t \|\phi_z(\cdot,\tau)\|_{w_5}^2 \le C\|\phi_0\|_{w_4}^2. \tag{4.45}$$

Next, we give the estimate of the first-order derivative of ϕ.

LEMMA 4.7. *Let the assumptions of Proposition* 4.3 *hold. If $N(T)$ is sufficiently small, then there exists a constant $C > 0$ independent of T such that*

$$\|\phi_z(\cdot,t)\|_{w_1}^2 + \int_0^t \|\phi_{zz}(\cdot,\tau)\|_{w_3}^2 \le C\left(\|\phi_0\|^2 + \|\phi_0\|_{w_4}^2 + \|\phi_{0z}\|_{w_1}^2\right) \tag{4.46}$$

for any $t \in [0,T]$.

Proof. Multiplying (4.5) by $(-\frac{\phi_z}{U^2})_z\eta^2$, where $\eta^2(z)$ is the cutoff function given by (4.24), integrating the result, and noting that

$$\begin{aligned} -\left(\frac{\phi_z}{U}\right)_z\left(-\frac{\phi_z}{U^2}\right)_z &= \left(\frac{\phi_{zz}}{U} - \frac{U_z\phi_z}{U^2}\right)\left(\frac{\phi_{zz}}{U^2} - \frac{2U_z}{U^3}\phi_z\right) \\ &= \frac{\phi_{zz}^2}{U^3} - 3\frac{U_z}{U^4}\phi_z\phi_{zz} + 2\frac{U_z^2}{U^5}\phi_z^2, \end{aligned}$$

we have

$$\begin{aligned} &\frac{1}{2}\int_{-2L}^{2L} \frac{\phi_z^2}{U^2}\eta^2 + \int_0^t \int_{-2L}^{2L} \frac{\phi_{zz}^2}{U^3}\eta^2 + 2\int_0^t \int_{-2L}^{2L} \frac{U_z^2}{U^5}\phi_z^2\eta^2 \\ &= \frac{1}{2}\int_{-2L}^{2L} \frac{\phi_{0z}^2}{U^2}\eta^2 + 3\int_0^t \int_{-2L}^{2L} \frac{U_z}{U^4}\phi_z\phi_{zz}\eta^2 + \int_0^t \int_{-2L}^{2L} g'(U)\phi_z\left(\frac{\phi_z}{U^2}\right)_z\eta^2 \\ &\quad - \int_0^t \int_{-2L}^{2L} F\left(\frac{\phi_z}{U^2}\right)_z\eta^2 - \int_0^t \int_{-2L}^{2L} G_z\left(\frac{\phi_z}{U^2}\right)_z\eta^2 - 2\int_0^t \int_{-2L}^{2L} \eta\eta_z\frac{\phi_t\phi_z}{U^2}. \end{aligned} \tag{4.47}$$

We next estimate the last term of (4.47). By (4.5), a direct calculation gives

$$\begin{aligned} |\phi_t| &= \left|\left(-g'(U) - \frac{U_z}{U^2}\right)\phi_z + \frac{\phi_{zz}}{U} + F + G_z\right| \\ &\le \left|g'(U) + \frac{U_z}{U^2}\right||\phi_z| + \frac{|\phi_{zz}|}{U} + |F| + |G_z|. \end{aligned}$$

Thanks to Remark 3.1, which provides $|U_z(z)| \le CU^2(z)$ for all $z \in (-\infty,\infty)$, it then follows from (4.20) that

$$|G_z| \le C\left(\frac{\phi_z^2}{U} + \frac{|\phi_z||\phi_{zz}|}{U^2}\right).$$

Hence, by (4.18), we obtain

$$|\phi_t| \le C\left(|\phi_z| + \frac{|\phi_{zz}|}{U} + \phi_z^2 + \frac{\phi_z^2}{U} + \frac{|\phi_z||\phi_{zz}|}{U^2}\right)$$
$$\le C\left(|\phi_z| + \frac{|\phi_{zz}|}{U} + \frac{\phi_z^2}{U} + \frac{|\phi_z||\phi_{zz}|}{U^2}\right).$$

Therefore,

$$(4.48)\quad \begin{aligned} 2\left|\int_0^t\int_{-2L}^{2L}\eta\eta_z\frac{\phi_t\phi_z}{U^2}\right| &\le C\int_0^t\int_{-2L}^{2L}\left(|\phi_z| + \frac{|\phi_{zz}|}{U} + \frac{\phi_z^2}{U} + \frac{|\phi_z||\phi_{zz}|}{U^2}\right)\frac{|\phi_z|}{U^2}\eta|\eta_z| \\ &\le C\int_0^t\int_{-2L}^{2L}\left(\frac{\phi_z^2}{U^2} + \frac{|\phi_z||\phi_{zz}|}{U^3}\right)\eta|\eta_z|, \end{aligned}$$

where we have used the fact that $\|\phi_z(\cdot,t)/U\|_{L^\infty} \le CN(t) \le C$ in the last inequality. Since $\phi \in X(0,T)$ defined by (4.9), we know that

$$\int_{-2L}^{2L}\frac{\phi_z^2}{U^2} \le \int\frac{\phi_z^2}{U^2} \le N^2(t) \text{ for any } t\in[0,T].$$

Thus,

$$(4.49)\quad \int_0^t\int_{-2L}^{2L}\frac{\phi_z^2}{U^2}\eta|\eta_z| \le \frac{C}{L}\int_0^t\int_{-2L}^{2L}\frac{\phi_z^2}{U^2} \le \frac{CTN^2(T)}{L}.$$

By (4.28), we deduce that

$$(4.50)\quad \begin{aligned} \int_0^t\int_{-2L}^{2L}\frac{|\phi_z||\phi_{zz}|}{U^3}\eta|\eta_z| &\le \int_0^t\int_{-2L}^{2L}\frac{\phi_{zz}^2}{U^3}\eta^2|\eta_z|^{\frac12} + \int_0^t\int_{-2L}^{2L}\frac{\phi_z^2}{U^2}\frac{|\eta_z|^{\frac32}}{U} \\ &\le \frac{C}{\sqrt{L}}\int_0^t\int_{-2L}^{2L}\frac{\phi_{zz}^2}{U^3}\eta^2 + \frac{C}{\sqrt{L}}\int_0^t\int_{-2L}^{2L}\frac{\phi_z^2}{U^2} \\ &\le \frac{C}{\sqrt{L}}\int_0^t\int_{-2L}^{2L}\frac{\phi_{zz}^2}{U^3}\eta^2 + \frac{CTN^2(T)}{\sqrt{L}}. \end{aligned}$$

Adding (4.49) and (4.50) with (4.48), the last term of (4.47) is estimated as

$$(4.51)\quad 2\left|\int_0^t\int_{-2L}^{2L}\eta\eta_z\frac{\phi_t\phi_z}{U^2}\right| \le \frac{C}{\sqrt{L}}\int_0^t\int_{-2L}^{2L}\frac{\phi_{zz}^2}{U^3}\eta^2 + \frac{CTN^2(T)}{\sqrt{L}}.$$

Substituting (4.51) into (4.47), we obtain

$$\begin{aligned} &\frac12\int_{-2L}^{2L}\frac{\phi_z^2}{U^2}\eta^2 + \left(1-\frac{C}{\sqrt{L}}\right)\int_0^t\int_{-2L}^{2L}\frac{\phi_{zz}^2}{U^3}\eta^2 + 2\int_0^t\int_{-2L}^{2L}\frac{U_z^2}{U^5}\phi_z^2\eta^2 \\ &\quad\le \frac12\int_{-2L}^{2L}\frac{\phi_{0z}^2}{U^2}\eta^2 + 3\int_0^t\int_{-2L}^{2L}\frac{U_z}{U^4}\phi_z\phi_{zz}\eta^2 + \int_0^t\int_{-2L}^{2L}g'(U)\phi_z\left(\frac{\phi_z}{U^2}\right)_z\eta^2 \\ &\qquad - \int_0^t\int_{-2L}^{2L}F\left(\frac{\phi_z}{U^2}\right)_z\eta^2 - \int_0^t\int_{-2L}^{2L}G_z\left(\frac{\phi_z}{U^2}\right)_z\eta^2 + \frac{CTN^2(T)}{\sqrt{L}}. \end{aligned}$$

Now passing the limit $L\to+\infty$ yields

$$(4.52)\quad \begin{aligned} \int\frac{\phi_z^2}{U^2} + \int_0^t\int\frac{\phi_{zz}^2}{U^3} &\le \int\frac{\phi_{0z}^2}{U^2} + C\int_0^t\int\frac{|U_z|}{U^4}|\phi_z\phi_{zz}| + C\left|\int_0^t\int g'(U)\phi_z\left(\frac{\phi_z}{U^2}\right)_z\right| \\ &\quad + C\left|\int_0^t\int F\left(\frac{\phi_z}{U^2}\right)_z\right| + C\left|\int_0^t\int G_z\left(\frac{\phi_z}{U^2}\right)_z\right|. \end{aligned}$$

In view of Remark 3.1, we get from the Cauchy–Schwarz inequality that

$$C\int_0^t\int\frac{|U_z|}{U^4}|\phi_z\phi_{zz}|\le\frac14\int\frac{\phi_{zz}^2}{U^3}+C\int\frac{U_z^2}{U^5}\phi_z^2\le\frac14\int\frac{\phi_{zz}^2}{U^3}+C\int\frac{\phi_z^2}{U}. \tag{4.53}$$

Similarly, thanks to $|g'(U)|\le C$ and $|g''(U)|\le C$ due to (3.3) and $f\in C^{\max\{k_\pm+1,3\}}(\mathbb{R})$,

$$\begin{aligned}C\left|\int_0^t\int g'(U)\phi_z\left(\frac{\phi_z}{U^2}\right)_z\right|&=C\left|\int_0^t\int\left(g'(U)\frac{\phi_z\phi_{zz}}{U^2}-g'(U)\frac{2U_z}{U^3}\phi_z^2\right)\right|\\&\le\frac14\int_0^t\int\frac{\phi_{zz}^2}{U^3}+C\int_0^t\int\frac{\phi_z^2}{U}.\end{aligned}$$

For the last two terms on the right-hand side of (4.52), by virtue of $\|\phi_z(\cdot,t)\|_{L^\infty}\le C\|\phi_z(\cdot,t)/U\|_{L^\infty}\le CN(t)$, Remark 3.1, and the Cauchy–Schwarz inequality, it follows from (4.18) and (4.20) that

$$\begin{aligned}C\left|\int_0^t\int F\left(\frac{\phi_z}{U^2}\right)_z\right|&=C\left|\int_0^t\int F\left(\frac{\phi_{zz}}{U^2}-\frac{2U_z}{U^3}\phi_z\right)\right|\\&\le C\left(\int_0^t\int\frac{\phi_z^2|\phi_{zz}|}{U^2}+\int_0^t\int\frac{|U_z|}{U^3}|\phi_z|^3\right)\\&\le CN(t)\int_0^t\int\frac{\phi_{zz}^2}{U^3}+C\int_0^t\int\frac{\phi_z^2}{U}\end{aligned}\tag{4.54}$$

and

$$\begin{aligned}C\left|\int_0^t\int G_z\left(\frac{\phi_z}{U^2}\right)_z\right|&=C\left|\int_0^t\int G_z\left(\frac{\phi_{zz}}{U^2}-\frac{2U_z}{U^3}\phi_z\right)\right|\\&\le C\left(\int_0^t\int\frac{|U_z|}{U^5}\phi_z^2|\phi_{zz}|+\int_0^t\int\frac{U_z^2}{U^6}|\phi_z|^3+\int_0^t\int\frac{|\phi_z|\phi_{zz}^2}{U^4}\right)\\&\le CN(t)\left(\int_0^t\int\frac{|\phi_z||\phi_{zz}|}{U^2}+\int\frac{\phi_z^2}{U}+\int\frac{\phi_{zz}^2}{U^3}\right)\\&\le CN(t)\int\frac{\phi_{zz}^2}{U^3}+C\int\frac{\phi_z^2}{U}.\end{aligned}\tag{4.55}$$

Substituting (4.53)–(4.55) into (4.52), by (4.22), (4.44), and (4.45), we have

$$\int\frac{\phi_z^2}{U^2}+\left(\frac12-CN(t)\right)\int_0^t\int\frac{\phi_{zz}^2}{U^3}\le C\left(\int\frac{\phi_{0z}^2}{U^2}+\int\phi_0^2+\int w_4(U)\phi_0^2\right).$$

Thus, (4.46) is derived if $CN(t)\le1/4$, and we finish the proof of Lemma 4.7. ∎

To close the a priori estimates, we give the estimate of the second-order derivative of ϕ.

LEMMA 4.8. *Let the assumptions of Proposition* 4.3 *hold. If $N(T)$ is sufficiently small, then there exists a constant $C>0$ independent of T such that*

$$\|\phi_{zz}(\cdot,t)\|_{w_1}^2+\int_0^t\|\phi_{zzz}(\cdot,\tau)\|_{w_3}^2\le C\left(\|\phi_0\|^2+\|\phi_0\|_{w_4}^2+\|\phi_{0z}\|_{1,w_1}^2\right)\tag{4.56}$$

for any $t\in[0,T]$.

Proof. We differentiate (4.5) with respect to z to get

$$\phi_{zt} + g''(U)U_z\phi_z + g'(U)\phi_{zz} - \left(\frac{\phi_z}{U}\right)_{zz} = F_z + G_{zz}. \tag{4.57}$$

Multiplying (4.57) by $\left(-\frac{\phi_{zz}}{U^2}\right)_z \eta^2$, where $\eta^2(z)$ is the cutoff function, integrating the result, and noting that

$$\begin{aligned} -\left(\frac{\phi_z}{U}\right)_{zz}\left(-\frac{\phi_{zz}}{U^2}\right)_z &= \left(\frac{\phi_{zzz}}{U} - \frac{2U_z}{U^2}\phi_{zz} + \frac{2U_z^2}{U^3}\phi_z - \frac{U_{zz}}{U^2}\phi_z\right)\left(\frac{\phi_{zzz}}{U^2} - \frac{2U_z}{U^3}\phi_{zz}\right) \\ &= \frac{\phi_{zzz}^2}{U^3} - \frac{4U_z}{U^4}\phi_{zz}\phi_{zzz} + \frac{4U_z^2}{U^5}\phi_{zz}^2 + \left(\frac{2U_z^2}{U^5} - \frac{U_{zz}}{U^4}\right)\phi_z\phi_{zzz} \\ &\quad - 2\left(\frac{2U_z^3}{U^6} - \frac{U_zU_{zz}}{U^5}\right)\phi_z\phi_{zz}, \end{aligned}$$

we have

$$\begin{aligned} &\frac{1}{2}\int_{-2L}^{2L}\frac{\phi_{zz}^2}{U^2}\eta^2 + \int_0^t\int_{-2L}^{2L}\frac{\phi_{zzz}^2}{U^3}\eta^2 + 4\int_0^t\int_{-2L}^{2L}\frac{U_z^2}{U^5}\phi_{zz}^2\eta^2 \\ &= \frac{1}{2}\int_{-2L}^{2L}\frac{\phi_{0zz}^2}{U^2}\eta^2 + 4\int_0^t\int_{-2L}^{2L}\frac{U_z}{U^4}\phi_{zz}\phi_{zzz}\eta^2 - \int_0^t\int_{-2L}^{2L}\left(\frac{2U_z^2}{U^5} - \frac{U_{zz}}{U^4}\right)\phi_z\phi_{zzz}\eta^2 \\ &\quad + 2\int_0^t\int_{-2L}^{2L}\left(\frac{2U_z^3}{U^6} - \frac{U_zU_{zz}}{U^5}\right)\phi_z\phi_{zz}\eta^2 + \int_0^t\int_{-2L}^{2L} g'(U)\phi_{zz}\left(\frac{\phi_{zz}}{U^2}\right)_z\eta^2 \\ &\quad + \int_0^t\int_{-2L}^{2L} g''(U)U_z\phi_z\left(\frac{\phi_{zz}}{U^2}\right)_z\eta^2 - \int_0^t\int_{-2L}^{2L}(F_z + G_{zz})\left(\frac{\phi_{zz}}{U^2}\right)_z\eta^2 \\ &\quad - 2\int_0^t\int_{-2L}^{2L}\eta\eta_z\frac{\phi_{zt}\phi_{zz}}{U^2}. \end{aligned} \tag{4.58}$$

We next estimate the last term of (4.58). By (4.57), a direct calculation yields

$$\begin{aligned} |\phi_{zt}| &= \left|\left(-g''(U)U_z + \frac{2U_z^2}{U^3} - \frac{U_{zz}}{U^2}\right)\phi_z - \left(g'(U) + \frac{2U_z}{U^2}\right)\phi_{zz} + \frac{\phi_{zzz}}{U} + F_z + G_{zz}\right| \\ &\le \left|\left(-g''(U)U_z + \frac{2U_z^2}{U^3} - \frac{U_{zz}}{U^2}\right)\right||\phi_z| + \left|\left(g'(U) + \frac{2U_z}{U^2}\right)\right||\phi_{zz}| + \frac{|\phi_{zzz}|}{U} + |F_z| + |G_{zz}|. \end{aligned} \tag{4.59}$$

In view of (3.2), one obtains

$$\begin{aligned} U_{zz} &= U_z\left[f(U) - f(0) - s(U-0)\right] + U(f'(U) - s)U_z \\ &= \frac{U_z^2}{U} + (f'(U) - s)UU_z, \end{aligned}$$

which, in combination with (3.9), implies, when $f'(0) < s < f'(u_-)$, that

$$|U_{zz}(z)| \le C\left(\frac{|U_z(z)|}{U(z)} + U(z)\right)|U_z(z)| \le CU^3(z) \quad \text{for all } z \in (-\infty, +\infty) \tag{4.60}$$

and, with (3.10), for $f'(0) = s$, that

$$|U_{zz}(z)| \le C\left(\frac{|U_z(z)|}{U(z)} + U(z)\right)|U_z(z)| \le CU^3(z), \quad \text{as } z \to +\infty. \tag{4.61}$$

Now by (4.60), (4.61), and $|U_z(z)| \le CU^2(z)$ for all $z \in (-\infty, \infty)$, we get from (4.19) and (4.21) that

$$|F_z| \le C\left(U^2\phi_z^2 + |\phi_z||\phi_{zz}|\right) \tag{4.62}$$

and

$$|G_{zz}| \le C\left(\phi_z^2 + \frac{\phi_{zz}^2}{U^2} + \frac{|\phi_z||\phi_{zz}|}{U} + \frac{|\phi_z||\phi_{zzz}|}{U^2}\right). \tag{4.63}$$

Thus, by (4.59), we have

$$|\phi_{zt}| \le C\left(|\phi_z| + |\phi_{zz}| + \frac{|\phi_{zzz}|}{U} + \phi_z^2 + \frac{\phi_{zz}^2}{U^2} + \frac{|\phi_z||\phi_{zz}|}{U} + \frac{|\phi_z||\phi_{zzz}|}{U^2}\right),$$

and hence

$$\begin{aligned} &2\left|\int_0^t \int_{-2L}^{2L} \eta\eta_z \frac{\phi_{zt}\phi_{zz}}{U^2}\right| \\ &\quad \le C\int_0^t \int_{-2L}^{2L} \left(\frac{|\phi_z||\phi_{zz}|}{U^2} + \frac{\phi_{zz}^2}{U^2} + \frac{|\phi_{zzz}||\phi_{zz}|}{U^3}\right)\eta|\eta_z| + C\int_0^t \int_{-2L}^{2L} \frac{\phi_{zz}^3}{U^4}\eta|\eta_z| \\ &\qquad + C\int_0^t \int_{-2L}^{2L} \left(\frac{|\phi_z||\phi_{zz}|}{U} + \frac{\phi_{zz}^2}{U^2} + \frac{|\phi_{zzz}||\phi_{zz}|}{U^3}\right)\eta|\eta_z| \cdot \frac{|\phi_z|}{U} \\ &\quad \le C\int_0^t \int_{-2L}^{2L} \left(\frac{|\phi_z||\phi_{zz}|}{U^2} + \frac{\phi_{zz}^2}{U^2} + \frac{|\phi_{zzz}||\phi_{zz}|}{U^3}\right)\eta|\eta_z| + C\int_0^t \int_{-2L}^{2L} \frac{\phi_{zz}^3}{U^4}\eta|\eta_z|, \end{aligned} \tag{4.64}$$

where we have used $\|\phi_z(\cdot,t)/U\|_{L^\infty} \le CN(t) \le C$ and $\frac{|\phi_z||\phi_{zz}|}{U} \le C\frac{|\phi_z||\phi_{zz}|}{U^2}$ in the last inequality. Since $\phi \in X(0,T)$ defined by (4.9), one has

$$\int_{-2L}^{2L} \frac{\phi_{zz}^2}{U^2} \le \int \frac{\phi_{zz}^2}{U^2} \le N^2(t) \text{ for any } t \in [0,T].$$

Therefore, thanks to the Cauchy–Schwarz inequality, we derive

$$\int_0^t \int_{-2L}^{2L} \frac{|\phi_z\phi_{zz}|}{U^2}\eta|\eta_z| \le \frac{C}{L}\int_0^t \int_{-2L}^{2L} \left(\frac{\phi_{zz}^2}{U^2} + \frac{\phi_z^2}{U^2}\right) \le \frac{CTN^2(T)}{L}, \tag{4.65}$$

$$\int_0^t \int_{-2L}^{2L} \frac{\phi_{zz}^2}{U^2}\eta|\eta_z| \le \frac{C}{L}\int_0^t \int_{-2L}^{2L} \frac{\phi_{zz}^2}{U^2} \le \frac{CTN^2(T)}{L},$$

and

$$\begin{aligned} \int_0^t \int_{-2L}^{2L} \frac{|\phi_{zz}||\phi_{zzz}|}{U^3}\eta|\eta_z| &\le \int_0^t \int_{-2L}^{2L} \frac{\phi_{zzz}^2}{U^3}\eta^2|\eta_z|^{\frac{1}{2}} + \int_0^t \int_{-2L}^{2L} \frac{\phi_{zz}^2}{U^2}\frac{|\eta_z|^{\frac{3}{2}}}{U} \\ &\le \frac{C}{\sqrt{L}}\int_0^t \int_{-2L}^{2L} \frac{\phi_{zzz}^2}{U^3}\eta^2 + \frac{C}{\sqrt{L}}\int_0^t \int_{-2L}^{2L} \frac{\phi_{zz}^2}{U^2} \\ &\le \frac{C}{\sqrt{L}}\int_0^t \int_{-2L}^{2L} \frac{\phi_{zzz}^2}{U^3}\eta^2 + \frac{CTN^2(T)}{\sqrt{L}}, \end{aligned} \tag{4.66}$$

where we have used (4.28) in the second equality. Furthermore, by $\|\phi_{zz}(\cdot,t)/U\|_{L^2} \leq N(t)$, $|\eta_z| \leq \frac{C}{L}$, and Hölder's inequality, the last term of (4.64) can be estimated as

$$\begin{aligned}(4.67)\quad \int_0^t\int_{-2L}^{2L}\frac{|\phi_{zz}|^3}{U^4}\eta|\eta_z| &\leq C\int_0^t L^{-\frac{1}{4}}\|U^{-\frac{3}{2}}\phi_{zz}\eta\|_{L^\infty}\|U^{-1}\phi_{zz}\|_{L^2}\|U^{-\frac{3}{2}}\phi_{zz}|\eta_z|^{\frac{3}{4}}\|_{L^2}\\ &\leq CN(t)\int_0^t L^{-\frac{1}{4}}\|U^{-\frac{3}{2}}\phi_{zz}\eta\|_{L^\infty}\|U^{-\frac{3}{2}}\phi_{zz}|\eta_z|^{\frac{3}{4}}\|_{L^2}.\end{aligned}$$

Note that

$$\begin{aligned}\frac{\phi_{zz}^2}{U^3}\eta^2 = \int_{-\infty}^z\left(\frac{\phi_{zz}^2}{U^3}\eta^2\right)_z &= \int_{-2L}^z\left(\frac{2\phi_{zz}\phi_{zzz}}{U^3}\eta^2 - \frac{3U_z}{U^4}\phi_{zz}^2\eta^2 + 2\frac{\phi_{zz}^2}{U^3}\eta\eta_z\right)\\ &\leq C\int_{-2L}^{2L}\left(\frac{\phi_{zzz}^2}{U^3}+\frac{\phi_{zz}^2}{U^3}\right)\eta^2 + C\int_{-2L}^{2L}\frac{\phi_{zz}^2}{U^3}\eta_z^2,\end{aligned}$$

which implies that

$$\begin{aligned}\|U^{-\frac{3}{2}}\phi_{zz}\eta\|_{L^\infty} &\leq C\left(\|U^{-\frac{3}{2}}\phi_{zz}\eta\|_{L^2}^2 + \|U^{-\frac{3}{2}}\phi_{zzz}\eta\|_{L^2}^2 + \|U^{-\frac{3}{2}}\phi_{zz}|\eta_z|\|_{L^2}^2\right)^{\frac{1}{2}}\\ &\leq C\left(\|U^{-\frac{3}{2}}\phi_{zz}\eta\|_{L^2}^2 + \|U^{-\frac{3}{2}}\phi_{zzz}\eta\|_{L^2}^2\right)^{\frac{1}{2}} + C\|U^{-\frac{3}{2}}\phi_{zz}|\eta_z|\|_{L^2}\\ &\leq C\left(\|U^{-\frac{3}{2}}\phi_{zz}\eta\|_{L^2}^2 + \|U^{-\frac{3}{2}}\phi_{zzz}\eta\|_{L^2}^2\right)^{\frac{1}{2}} + CL^{-\frac{1}{4}}\|U^{-\frac{3}{2}}\phi_{zz}|\eta_z|^{\frac{3}{4}}\|_{L^2}.\end{aligned}$$

We thus get from (4.67) and the Cauchy–Schwarz inequality that

$$\begin{aligned}(4.68)\quad &\int_0^t\int_{-2L}^{2L}\frac{|\phi_{zz}|^3}{U^4}\eta|\eta_z|\\ &\leq CN(t)\int_0^t L^{-\frac{1}{4}}\left(\|U^{-\frac{3}{2}}\phi_{zz}\eta\|_{L^2}^2 + \|U^{-\frac{3}{2}}\phi_{zzz}\eta\|_{L^2}^2\right)^{\frac{1}{2}}\cdot\|U^{-\frac{3}{2}}\phi_{zz}|\eta_z|^{\frac{3}{4}}\|_{L^2}\\ &\quad + CN(t)\int_0^t L^{-\frac{1}{2}}\|U^{-\frac{3}{2}}\phi_{zz}|\eta_z|^{\frac{3}{4}}\|_{L^2}^2\\ &\leq CN(t)\int_0^t L^{-\frac{1}{2}}\int_{-2L}^{2L}\left(\frac{\phi_{zzz}^2}{U^3}+\frac{\phi_{zz}^2}{U^3}\right)\eta^2 + CN(t)\int_0^t\int_{-2L}^{2L}\frac{\phi_{zz}^2}{U^2}\frac{|\eta_z|^{\frac{3}{2}}}{U}\\ &\leq \frac{CN(T)}{\sqrt{L}}\int_0^t\int_{-2L}^{2L}\left(\frac{\phi_{zzz}^2}{U^3}+\frac{\phi_{zz}^2}{U^3}\right)\eta^2 + \frac{CN(T)}{\sqrt{L}}\int_0^t\int_{-2L}^{2L}\frac{\phi_{zz}^2}{U^2}\\ &\leq \frac{CN(T)}{\sqrt{L}}\int_0^t\int_{-2L}^{2L}\left(\frac{\phi_{zzz}^2}{U^3}+\frac{\phi_{zz}^2}{U^3}\right)\eta^2 + \frac{CTN^3(T)}{\sqrt{L}},\end{aligned}$$

where we have used $L^{-\frac{1}{2}} \leq C$ in the second inequality and (4.28) in the penultimate inequality. Combining (4.65)–(4.66) and (4.68) with (4.64), we arrive at

$$(4.69)\quad 2\left|\int_0^t\int_{-2L}^{2L}\eta\eta_z\frac{\phi_{zt}\phi_{zz}}{U^2}\right| \leq \frac{C+CN(T)}{\sqrt{L}}\int_0^t\int_{-2L}^{2L}\frac{\phi_{zzz}^2}{U^3}\eta^2 + \frac{CN(T)}{\sqrt{L}}\int_0^t\int_{-2L}^{2L}\frac{\phi_{zz}^2}{U^3} + \frac{CTN^3(T)}{\sqrt{L}}.$$

Substituting (4.69) into (4.58), one obtains

$$
\begin{aligned}
&\frac{1}{2}\int_{-2L}^{2L}\frac{\phi_{zz}^2}{U^2}\eta^2+\left(1-\frac{C+CN(T)}{\sqrt{L}}\right)\int_0^t\int_{-2L}^{2L}\frac{\phi_{zzz}^2}{U^3}\eta^2+4\int_0^t\int_{-2L}^{2L}\frac{U_z^2}{U^5}\phi_{zz}^2\eta^2\\
&=\frac{1}{2}\int_{-2L}^{2L}\frac{\phi_{0zz}^2}{U^2}\eta^2+4\int_0^t\int_{-2L}^{2L}\frac{U_z}{U^4}\phi_{zz}\phi_{zzz}\eta^2-\int_0^t\int_{-2L}^{2L}\left(\frac{2U_z^2}{U^5}-\frac{U_{zz}}{U^4}\right)\phi_z\phi_{zzz}\eta^2\\
&\quad+2\int_0^t\int_{-2L}^{2L}\left(\frac{2U_z^3}{U^6}-\frac{U_zU_{zz}}{U^5}\right)\phi_z\phi_{zz}\eta^2+\int_0^t\int_{-2L}^{2L}g'(U)\phi_{zz}\left(\frac{\phi_{zz}}{U^2}\right)_z\eta^2\\
&\quad+\int_0^t\int_{-2L}^{2L}g''(U)U_z\phi_z\left(\frac{\phi_{zz}}{U^2}\right)_z\eta^2-\int_0^t\int_{-2L}^{2L}F_z\left(\frac{\phi_{zz}}{U^2}\right)_z\eta^2-\int_0^t\int_{-2L}^{2L}G_{zz}\left(\frac{\phi_{zz}}{U^2}\right)_z\eta^2\\
&\quad+\frac{CTN^2(T)}{\sqrt{L}}+\frac{CN(T)}{\sqrt{L}}\int_0^t\int_{-2L}^{2L}\frac{\phi_{zz}^2}{U^3}\eta^2.
\end{aligned}
$$

Now passing the limit $L\to+\infty$, by (4.46), we obtain

(4.70)
$$
\begin{aligned}
&\frac{1}{2}\int\frac{\phi_{zz}^2}{U^2}+\int_0^t\int\frac{\phi_{zzz}^2}{U^3}+4\int_0^t\int\frac{U_z^2}{U^5}\phi_{zz}^2\\
&\le\frac{1}{2}\int\frac{\phi_{0zz}^2}{U^2}+4\left|\int_0^t\int\frac{U_z}{U^4}\phi_{zz}\phi_{zzz}\right|+\left|\int_0^t\int\left(\frac{2U_z^2}{U^5}-\frac{U_{zz}}{U^4}\right)\phi_z\phi_{zzz}\right|\\
&\quad+2\left|\int_0^t\int\left(\frac{2U_z^3}{U^6}-\frac{U_zU_{zz}}{U^5}\right)\phi_z\phi_{zz}\right|+\left|\int_0^t\int g'(U)\phi_{zz}\left(\frac{\phi_{zz}}{U^2}\right)_z\right|\\
&\quad+\left|\int_0^t\int g''(U)U_z\phi_z\left(\frac{\phi_{zz}}{U^2}\right)_z\right|+\left|\int_0^t\int F_z\left(\frac{\phi_{zz}}{U^2}\right)_z\right|+\left|\int_0^t\int G_{zz}\left(\frac{\phi_{zz}}{U^2}\right)_z\right|.
\end{aligned}
$$

We next estimate the terms on the right-hand side of (4.70). In view of (4.60), (4.61), and $|U_z(z)|\le CU^2(z)$ for all $z\in(-\infty,\infty)$, we deduce that

$$
\begin{aligned}
&4\left|\int_0^t\int\frac{U_z}{U^4}\phi_{zz}\phi_{zzz}\right|+\left|\int_0^t\int\left(\frac{2U_z^2}{U^5}-\frac{U_{zz}}{U^4}\right)\phi_z\phi_{zzz}\right|\\
&\quad\le C\left(\int_0^t\int\frac{|\phi_{zz}\phi_{zzz}|}{U^2}+\int_0^t\int\frac{|\phi_z\phi_{zzz}|}{U}\right)\\
&\quad\le\frac{1}{4}\int_0^t\int\frac{\phi_{zzz}^2}{U^3}+C\left(\int_0^t\int\frac{\phi_{zz}^2}{U}+\int_0^t\int U\phi_z^2\right)
\end{aligned}
\tag{4.71}
$$

and

$$
2\left|\int_0^t\int\left(\frac{2U_z^3}{U^6}-\frac{U_zU_{zz}}{U^5}\right)\phi_z\phi_{zz}\right|\le C\left(\int_0^t\int\phi_{zz}^2+\int_0^t\int\phi_z^2\right).
\tag{4.72}
$$

Similarly, thanks to (3.3), (3.9), (3.10), and $f\in C^{\max\{k_\pm+1,3\}}(\mathbb{R})$,

$$
\begin{aligned}
\left|\int_0^t\int g'(U)\phi_{zz}\left(\frac{\phi_{zz}}{U^2}\right)_z\right|&=\left|\int_0^t\int g'(U)\phi_{zz}\left(\frac{\phi_{zzz}}{U^2}-\frac{2U_z}{U^3}\phi_{zz}\right)\right|\\
&\le C\left(\int_0^t\int\frac{|\phi_{zz}\phi_{zzz}|}{U^2}+\int_0^t\int\frac{\phi_{zz}^2}{U}\right)\\
&\le\frac{1}{4}\int_0^t\int\frac{\phi_{zzz}^2}{U^3}+C\int_0^t\int\frac{\phi_{zz}^2}{U}
\end{aligned}
$$

and

$$
\begin{aligned}
\left|\int_0^t\int g''(U)U_z\phi_z\left(\frac{\phi_{zz}}{U^2}\right)_z\right| &= \left|\int_0^t\int g''(U)U_z\phi_z\left(\frac{\phi_{zzz}}{U^2}-\frac{2U_z}{U^3}\phi_{zz}\right)\right|\\
&\le C\left(\int_0^t\int|\phi_z\phi_{zzz}|+\int_0^t\int|\phi_z\phi_{zz}|\right)\\
&\le \frac{1}{4}\int_0^t\int\frac{\phi_{zzz}^2}{U^3}+C\left(\int_0^t\int\phi_{zz}^2+\int_0^t\int\phi_z^2\right).
\end{aligned}
$$

For the last second term on the right-hand side of (4.70), noting that $\|\phi_z(\cdot,t)\|_{L^\infty}\le \|\phi_z(\cdot,t)/U\|_{L^\infty}\le CN(t)$, it follows from (4.62) that

$$
\begin{aligned}
\left|\int_0^t\int F_z\left(\frac{\phi_{zz}}{U^2}\right)_z\right| &= \left|\int_0^t\int F_z\left(\frac{\phi_{zzz}}{U^2}-\frac{2U_z}{U^3}\phi_{zz}\right)\right|\\
&\le C\int_0^t\int\left(U^2\phi_z^2+|\phi_z||\phi_{zz}|\right)\left(\frac{|\phi_{zzz}|}{U^2}+\frac{|\phi_{zz}|}{U}\right)\\
&\le CN(t)\int_0^t\int\left(|\phi_z||\phi_{zzz}|+|\phi_z||\phi_{zz}|+\frac{|\phi_{zz}||\phi_{zzz}|}{U^2}+\frac{\phi_{zz}^2}{U}\right)\\
&\le CN(t)\int_0^t\int\frac{\phi_{zzz}^2}{U^3}+C\left(\int_0^t\int\frac{\phi_{zz}^2}{U}+\int_0^t\int\phi_z^2\right).
\end{aligned}
$$

Thanks to (4.62), the last term on the right-hand side of (4.70) can be estimated as

$$
\begin{aligned}
\left|\int_0^t\int G_{zz}\left(\frac{\phi_{zz}}{U^2}\right)_z\right| &= \left|\int_0^t\int G_{zz}\left(\frac{\phi_{zzz}}{U^2}-\frac{2U_z}{U^3}\phi_{zz}\right)\right|\\
&\le C\Bigg[\int_0^t\int\frac{\phi_{zz}^2|\phi_{zzz}|}{U^4}+\int_0^t\int\frac{|\phi_{zz}|^3}{U^3}+\int_0^t\int\frac{\phi_z^2|\phi_{zzz}|}{U^2}+\int_0^t\int\frac{\phi_z^2|\phi_{zz}|}{U}\\
&\quad+\int_0^t\int\frac{|\phi_z||\phi_{zz}||\phi_{zzz}|}{U^3}+\int_0^t\int\frac{|\phi_z|\phi_{zz}^2}{U^2}+\int_0^t\int\frac{|\phi_z|\phi_{zzz}^2}{U^4}\Bigg]\\
&=: I_1+I_2+I_3+I_4+I_5+I_6+I_7.
\end{aligned}
$$

By virtue of $\|\phi_{zz}(\cdot,t)/U\|_{L^2}\le N(t)$ and Hölder's inequality, one has

$$
\begin{aligned}
I_1 &\le C\int_0^t\|U^{-\frac{3}{2}}\phi_{zz}\|_{L^\infty}\|U^{-1}\phi_{zz}\|_{L^2}\|U^{-\frac{3}{2}}\phi_{zzz}\|_{L^2}\\
&\le CN(t)\int_0^t\left(\|U^{-\frac{3}{2}}\phi_{zz}\|_{L^2}^2+\|U^{-\frac{3}{2}}\phi_{zzz}\|_{L^2}^2\right)^{\frac{1}{2}}\|U^{-\frac{3}{2}}\phi_{zzz}\|_{L^2}\\
&\le CN(t)\int_0^t\int\frac{\phi_{zzz}^2}{U^3}+C\int_0^t\int\frac{\phi_{zz}^2}{U^3}
\end{aligned}
$$

and

$$
\begin{aligned}
I_2 &\le C\int_0^t\|U^{-\frac{3}{2}}\phi_{zz}\|_{L^\infty}\|\phi_{zz}\|_{L^2}\|U^{-\frac{3}{2}}\phi_{zz}\|_{L^2}\\
&\le CN(t)\int_0^t\left(\|U^{-\frac{3}{2}}\phi_{zzz}\|_{L^2}^2+\|U^{-\frac{3}{2}}\phi_{zz}\|_{L^2}^2\right)^{\frac{1}{2}}\|U^{-\frac{3}{2}}\phi_{zz}\|_{L^2}\\
&\le CN(t)\int_0^t\int\frac{\phi_{zzz}^2}{U^3}+C\int\frac{\phi_{zz}^2}{U^3},
\end{aligned}
$$

where we have used $\|\phi_{zz}(\cdot,t)\|_{L^2} \le C\|\phi_{zz}(\cdot,t)/U\|_{L^2} \le N(t)$ in the second inequality. Furthermore, we utilize (3.9), (3.10), and the Cauchy–Schwarz inequality to get

(4.73)
$$
\begin{aligned}
&I_3 + I_4 + I_5 + I_6 + I_7 \\
&\le CN(t)\left(\int_0^t\int \frac{|\phi_z||\phi_{zzz}|}{U} + \int_0^t\int |\phi_z||\phi_{zz}| + \int_0^t\int \frac{|\phi_{zz}||\phi_{zzz}|}{U^2} + \int_0^t\int \frac{\phi_{zz}^2}{U} + \int_0^t\int \frac{\phi_{zzz}^2}{U^3}\right) \\
&\le CN(t)\int_0^t\int \frac{\phi_{zzz}^2}{U^3} + C\left(\int_0^t\int \frac{\phi_{zz}^2}{U} + \int_0^t\int U\phi_z^2\right),
\end{aligned}
$$

where we have used the fact that $\|\phi_z(\cdot,t)/U\|_{L^\infty} \le CN(t)$ in the first inequality. Noting that $U \le C \le \frac{C}{U} \le \frac{C}{U^3}$, combining (4.71)–(4.73), (4.22), (4.44), and (4.46), we get

$$
\begin{aligned}
&\int \frac{\phi_{zz}^2}{U^2} + \left(\frac{1}{4} - CN(t)\right)\int_0^t\int \frac{\phi_{zzz}^2}{U^3} \\
&\quad\le C\left(\int \frac{\phi_{0zz}^2}{U^2} + \int \frac{\phi_{0z}^2}{U^2} + \int \phi_0^2 + \int w_4(U)\phi_0^2 + \int_0^t\int \frac{\phi_{zz}^2}{U^3} + \int_0^t\int \frac{\phi_z^2}{U}\right) \\
&\quad\le C\left(\int \frac{\phi_{0zz}^2}{U^2} + \int \frac{\phi_{0z}^2}{U^2} + \int \phi_0^2 + \int w_4(U)\phi_0^2\right).
\end{aligned}
$$

We then obtain (4.56) provided that $N(t)$ is suitably small and finish the proof of Lemma 4.8. ∎

Proof of Proposition 4.3. The desired estimate (4.15) follows from (4.22), (4.44), (4.46), and (4.56). ∎

Before proving the main result, we present a well-known result for convenience [38].

LEMMA 4.9. *If a measurable function q defined on $(0,+\infty)$ satisfies*

$$\int_0^{+\infty} (|q(t)| + |q'(t)|)dt < +\infty,$$

then $q(t) \to 0$ as $t \to \infty$.

Proof of Theorem 4.1. Recalling the proof of Lemmas 4.5–4.8, we can select a positive constant δ_2 such that $\delta_2 \le \delta_0$, where δ_0 is taken arbitrarily and mentioned in Proposition 4.2. Let $\delta_1 = \min\{\delta_2/2, \delta_2/2C\}$, and let $N(0) \le \delta_1$. Since $N(0) \le \delta_1 \le \delta_2/2 < \delta_0$, Proposition 4.2 guarantees the existence of a unique solution $\phi \in X(0,T_0)$ for the system (4.5) satisfying $N(t) \le 2N(0) \le 2\delta_1 \le 2\delta_2/2 = \delta_2$ for any $0 \le t \le T_0$. Subsequently, by employing Proposition 4.3, we establish that $N(T_0) \le CN(0) \le C\delta_1 \le C\delta_2/2C = \delta_2/2$. Now considering the system (4.5) with the "initial data" at T_0 and utilizing Proposition 4.2 once again, we obtain a unique solution $\phi \in X(T_0, 2T_0)$, eventually on the interval $[0, 2T_0]$, satisfying $N(t) \le 2N(T_0) \le 2\delta_2/2 = \delta_2$ for any $0 \le t \le 2T_0$. Then, applying Proposition 4.3 with $T = 2T_0$ again, we deduce that $N(2T_0) \le CN(0) \le C\delta_1 \le C\delta_2/2C = \delta_2/2 < \delta_0$. Hence, by repeating this continuation process, we can derive a unique global solution $\phi \in X(0,\infty)$ that satisfies the estimate (4.15) for all $t \in [0,\infty)$.

To complete the proof, it remains to show (4.16). We denote $q(t) \triangleq \|\phi_z(\cdot,t)\|_1^2$. Given the established estimate (4.15) for all $t \in [0,\infty)$, it is easy to see that

$$\int_0^\infty \|\phi_z(\cdot,\tau)\|_1^2 \mathrm{d}\tau \le \int_0^\infty \left(\|\phi_z(\cdot,\tau)\|_{w_2}^2 + \|\phi_z(\cdot,\tau)\|_{w_5}^2 + \|\phi_{zz}(\cdot,\tau)\|_{w_3}^2\right)\mathrm{d}\tau \le CN^2(0) \le C,$$

which implies that $q(t) \in L^1(0,\infty)$. Furthermore, by combining (4.15) with (4.57), a direct calculation yields

$$\begin{aligned}\int_0^\infty |q'(\tau)|\mathrm{d}\tau &= \int_0^\infty \left|\frac{\mathrm{d}}{\mathrm{d}\tau}\|\phi_z(\cdot,\tau)\|_1^2\right|\mathrm{d}\tau\\ &= \int_0^\infty \left|\frac{\mathrm{d}}{\mathrm{d}\tau}\int\left(\phi_z^2+\phi_{zz}^2\right)\right|\mathrm{d}\tau\\ &= 2\int_0^\infty \left|\int\left(\phi_z\phi_{zt}+\phi_{zz}\phi_{zzt}\right)\right|\mathrm{d}\tau\\ &\le C\int_0^\infty\int\left(\frac{\phi_z^2}{U}+\frac{\phi_{zz}^2}{U^3}+\frac{\phi_{zzz}^2}{U^3}\right)\mathrm{d}\tau\\ &\le C,\end{aligned}$$

where

$$\begin{aligned}\phi_{zzt} &= -g'''(U)U_z^2\phi_z - g''(U)U_{zz}\phi_z - 2g''(U)U_z\phi_{zz}\\ &\quad - g'(U)\phi_{zzz} + F_{zz} + \left(\left(\frac{\phi_z}{U}\right)_{zz} + G_{zz}\right)_z\end{aligned}$$

and

$$\int \phi_{zz}\left(\left(\frac{\phi_z}{U}\right)_{zz}+G_{zz}\right)_z = -\int\phi_{zzz}\left(\left(\frac{\phi_z}{U}\right)_{zz}+G_{zz}\right)$$

have been used. Then it follows from Lemma 4.9 that

$$\|\phi_z(\cdot,t)\|_1^2 \to 0 \quad \text{as } t\to+\infty.$$

This along with the Hölder's inequality implies that

$$\begin{aligned}\phi_z^2(z,t) &= 2\int_{-\infty}^z \phi_z\phi_{zz}(y,t)dy\\ &\leqslant 2\left(\int_{-\infty}^\infty \phi_z^2 dy\right)^{1/2}\left(\int_{-\infty}^\infty\phi_{zz}^2dy\right)^{1/2}\\ &\leqslant C\left\|\phi_z(\cdot,t)\right\|_1 \to 0 \quad \text{as } t\to+\infty.\end{aligned}$$

Therefore, (4.16) is proved, and the proof of Theorem 4.1 is complete. ∎

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